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Preface to CAG special issues in honor of Karen Uhlenbeck's 75th birthday

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All of us in geometric analysis admire Karen with all our heart. When the American Mathematical Society asked me who is eligible for the Abel Prize, I instantly thought of Karen. She is a pioneering scholar and many of her studies have inspired and laid the groundwork for future generations.

Karen is one of the most outstanding mathematicians of our time. She has done pioneering works on minimal surfaces, harmonic maps, Yang–Mills theory, nonlinear wave equation and integrable systems, which have shaped the field of geometric analysis over the past 40 years. Her work has had a huge impact on differential geometry, partial differential equations, topology, and mathematical physics.

One of Karen's most influential contributions is minimal surface theory, the mathematical theory behind the shapes of bubbles that wrap up to achieve the minimum area. The breakthrough paper of Sacks and Uhlenbeck on the existence of minimal immersions of two-spheres is one of the most influential works in the field of geometric analysis. In the construction of quantum cohomology and Floer homology, the compactness argument for pseudoholomorphic curves is essentially due to Sacks–Uhlenbeck. In higher dimensions, Karen did profound work with Schoen on the regularity of energy minimizing harmonic maps.

Another major contribution of Karen is on gauge field theory. While the paper of Atiyah–Hitchin–Singer laid the algebraic and geometric foundation for self-dual connections, the analytic foundation was laid by Uhlenbeck who established the removable singularity theorem and compactness theorem for Yang–Mills connections. This paved the way for application of gauge theory to geometry and topology and eventually led to the groundbreaking work of Taubes and Donaldson on 4-dimensional topology.

Karen, in a series of papers with Chuu-Lian Terng, created a whole new and active research field on harmonic maps and integrable systems. The algebraic study of harmonic maps first appeared in papers by Calabi and Chern on minimal surfaces in spheres. But they were developed more fully only after the birth of gauge theory, where harmonic maps from two-dimensional domains are toy models. In a seminal work around 1990, Karen had an

in-depth study of the algebraic structure of the moduli space of harmonic mappings from two-dimensional domains into Grassmannians, establishing their amazing connections to integrable systems.

It was a great honor for me to collaborate with Karen on the existence of Hermitian Yang–Mills connections on stable holomorphic vector bundles. I was interested to generalize the notion of Kähler–Einstein metrics to bundles in 1978. Motivated by the works of Bogomolov, I suspected that its existence has something to do with Mumford stability of the bundle, which was found to be necessary by a later work of Martin Lübke. But I did not make enough progress until I met Karen who had been interested in Yang–Mills bundles for quite a while, where she proved some of the most fundamental theorems in this subject from the point of view of partial differential equations. At my suggestion, we started to work on the problem of existence on stable holomorphic bundles.

In 1981, I was invited to give a plenary talk for the London Mathematical Society annual meeting in Wales. There I met Hitchin, who told me that he has a brilliant student Simon Donaldson who was able to prove the existence of Hermitian Yang–Mills connections on any stable holomorphic bundle which can be deformed into the tangent bundle on the K3 surface. It certainly was a good progress on the problem that Karen and I were working, although I did not manage to meet the brilliant rising star at that time.

Karen came to visit me in Princeton several times and we worked extensively on this exciting problem. I remember that she came to give the Morse lecture at IAS. She gave a beautiful talk on her works on both minimal spheres and gauge theory. Both of the works laid solid foundations of the future development of geometric analysis.

Karen stayed in my home for a few days and we worked intensively. The regularity estimate for the work that we did on Hermitian Yang–Mills was tough. The procedure in our paper remained to be the only one until recently when Karen and Penny Smith found a beautiful regularity theory.

We hope that these two special issues in honor of Karen's 75th birthday could convey a sense of her long lasting influence on the mathematical landscape. We would like to thank all the authors and the referees for their contributions.

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