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Monopole Floer homology and the spectral geometry of three-manifolds

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We refine some classical estimates in Seiberg-Witten theory, and discuss an application to the spectral geometry of three-manifolds. We show that for any Riemannian metric on a rational homology three-sphere Y, the first eigenvalue of the Hodge Laplacian on coexact one-forms is bounded above explicitly in terms of the Ricci curvature, provided that Y is not an L-space (in the sense of Floer homology). The latter is a computable purely topological condition, and holds in a variety of examples. Performing the analogous refinement in the case of manifolds with $b_1 > 0$, we provide a gauge-theoretic proof of an inequality of Brock and Dunfield relating the Thurston and L^2 norms of hyperbolic three-manifolds, first proved using minimal surfaces.

We discuss a relation between gauge theory, Floer homology, and spectral geometry. In particular, we will be interested in the study of the spectrum of the Hodge Laplacian acting on differential forms on a compact Riemannian three-manifold.

The study of the interactions between the spectrum of the Laplacian on functions and the geometry of the underlying space dates back to at least Weyl [27]. The subject has been popularized again in the '60s by the famous papers [13] and [20], and is by now a fairly well understood topic. Much less is known about the spectrum on forms, except in the case of surfaces in which (by the Hodge decomposition) it is determined entirely by the spectrum on functions. In this sense, the first interesting case is that of three-manifolds, for which the spectrum on forms is determined by the spectrum on functions and the spectrum on coexact 1-forms. In the present paper, the quantity we are interested in is the least eigenvalue on coexact 1-forms, which we will denote by λ_1^*.

A classical upper bound for λ_1^* (which holds in every dimension) can be provided in terms of the sectional curvatures, their covariant derivatives and the injectivity radius [7], while some lower bounds can be exhibited in special cases using the Mayer-Vietoris arguments introduced in [19] (see also [8]).

In this paper, we will focus on dimension three. Given a rational homology sphere Y, we will provide an upper bound on λ_1^* purely in terms of the Ricci curvature, provided an extra topological assumption (which is gauge-theoretic in nature). In the following, we denote by $\tilde{s}(p)$ the sum of the two least eigenvalues of the Ricci curvature at the point p.

Theorem 1. *Let Y be a rational homology sphere of dimension three which is not an L-space. Then for every Riemannian metric on Y the upper bound*

$$\lambda_1^* \leq -\inf_{p\in Y} \tilde{s}(p) \tag{1}$$

holds. In particular, a lower bound on the Ricci curvature implies an upper bound on λ_1^*.

An L-space Y is a rational homology sphere Y for which the reduced monopole Floer homology $HM(Y)$ vanishes (see [14]). This condition is a topological invariant. An alternative definition is the vanishing of the reduced Heegaard Floer homology $HF_{\mathrm{red}}(Y)$ [21]: these conditions are equivalent via the isomorphism between the two theories (see [17], [5] and subsequent papers). Examples of L-spaces include spherical space forms [14] and branched double covers of alternating knots [23]. In general, the condition of being an L-space is quite well understood, and algorithmically computable [24]. Among rational homology spheres which are *not* L-spaces (so that the main result of the paper applies to them) we have the following classes of examples:

- any *integral* homology sphere obtained by a surgery on a knot in S^3 other than S^3 itself and the Poincaré homology sphere [9];
- any rational homology sphere obtained by surgery on a knot K is S^3 such that the Alexander polynomial of K has a coefficient different than ± 1 [22]. More generally, there are many restrictions on knots which admit an L-space surgery (see for example [11]);
- any rational homology sphere that admits a co-orientable taut foliation [15].

An intriguing conjecture [3] states that in fact an irreducible rational homology sphere is an L-space if and only if it does not admit a co-orientable taut foliation, if and only if its fundamental group is not left-orderable. This has been recently verified for graph manifolds [10].

In a different direction, we also provide an alternative proof of an interesting result from [4] (which was in turn inspired by [1]). Recall that the first cohomology $H^1(Y;\mathbb{R})$ of an oriented hyperbolic three-manifold Y comes with two natural norms: the Thurston norm $\|\cdot\|_{Th}$ [25], and the harmonic norm with respect to the hyperbolic metric $\|\cdot\|_{L^2}$. We then have the following.

Theorem 2 (Theorem 1.3 **of** [4]**).** *For every oriented closed hyperbolic three-maniifold Y, the inequality*

$$\frac{\pi}{\sqrt{\mathrm{vol}(Y)}}\|\cdot\|_{Th} \leq \|\cdot\|_{L^2} \tag{2}$$

between norms holds.

The authors also show that the inequality is qualitatively sharp (see Theorem 1.5 in [4]). Their proof relies on the theory of minimal surfaces in hyperbolic three-manifolds, and in particular an idea of Uhlenbeck [26]. As pointed out in [4], an inequality weaker than (2) is a direct consequence of the main result of [16], which asserts that for a closed oriented irreducible three-manifold Y, for any class $\alpha \in H^2(Y;\mathbb{R})$ the identity

$$|\alpha| = 4\pi \cdot \sup_h \|\alpha\|_{L^2(h)}/\|s_h\|_{L^2(h)} \tag{3}$$

holds. Here $|\cdot|$ denotes the dual Thurston norm, h varies along all Riemannian metrics on Y, and s_h is the scalar curvature. Recall that while for a general manifold the dual Thurston norm can attain the value ∞, in the case of hyperbolic three-manifolds it is a genuine norm, as it follows from the fact that hyperbolic three-manifolds are atoroidal. Passing to duals, this is equivalent to the fact that for a given $\phi \in H^1(Y;\mathbb{R})$ we have the identity

$$4\pi\|\phi\|_{Th} = \inf_h \|\phi\|_{L^2(h)}\|s_h\|_{L^2(h)}.$$

In our case of interest, by taking h to be the hyperbolic metric, which has scalar curvature -6, we obtain the bound

$$\frac{2\pi}{3\sqrt{\mathrm{vol}(Y)}}\|\cdot\|_{Th} \leq \|\cdot\|_{L^2}.$$

We will provide a gauge-theoretic proof of the sharper inequality (2). Furthermore, while the argument of [4] relies extensively on the assumption on sectional curvatures at the basis of Uhlenbeck's observation, we will see that our approach only requires a condition on the Ricci curvature.

The proof of both results involves a refinement of some well-known estimates for the solutions of the Seiberg-Witten equations. This is inspired by the estimates in the four-dimensional case involving the self-dual Weyl curvature discussed in [18]. The key idea in our case is to exploit the classical Bochner formula connecting the Hodge Laplacian and the Bochner Laplacian on 1-forms in terms of the Ricci curvature.

Proof of Theorem 1. We follow the conventions of [14]. A sufficient condition for a rational homology sphere Y to be an L-space is the existence of a pair consisting of a metric and perturbation which is admissible and for which the Seiberg-Witten equations do not admit irreducible solutions for any spin^c structure. We start by investigating this condition in detail in the case of the unperturbed equations, and discussed the perturbed case in the end. Fix a Riemannian metric for which

$$\lambda_1^* + \inf_{p\in Y}\tilde{s}(p) > 0 \tag{4}$$

and a spin^c structure $\mathfrak{s}$, and consider a solution (B,Ψ) of the equations

$$\begin{aligned}\frac{1}{2}\rho(F_{B^t}) - (\Psi\Psi^*)_0 &= 0\\ D_B\Psi &= 0.\end{aligned}$$

Then we have the identity

$$\Delta|\Psi|^2 = 2\langle\Psi, \nabla_B^*\nabla_B\Psi\rangle - 2|\nabla_B\Psi|^2 = -|\Psi|^4 - \frac{1}{2}s|\Psi|^2 - 2|\nabla_B\Psi|^2, \tag{5}$$

where we used the Weitzenböck formula

$$D_B^2\Psi = \nabla_B^*\nabla_B\Psi + \frac{1}{2}\rho(F_{B^t})\cdot\Psi + \frac{s}{4}\Psi.$$

We can now multiply (5) by $|\Psi|^2$, integrate over the manifold, and obtain by Green's identity

$$\int|\Psi|^6 + \frac{1}{2}s|\Psi|^4 + 2|\Psi|^2|\nabla_B\Psi|^2 = -\int|\Psi|^2\Delta|\Psi|^2 = -\int|d|\Psi|^2|^2 \leq 0. \tag{6}$$

The key idea is to get a better bound on the third term on the left hand side. To do this, recall that for a 1-form ξ the classical Bochner identity

$$(d+d^*)^2\xi = \nabla^*\nabla\xi + \mathrm{Ric}(\xi,\cdot)$$

holds, see for example [2]. Suppose now that ξ is coclosed. Integrating by parts, we obtain the inequality

$$\int |\nabla\xi|^2 = \int |(d+d^*)\xi|^2 + \int -\mathrm{Ric}(\xi,\xi) \geq \int (\lambda_1^* - m)|\xi|^2, \tag{7}$$

where $m(p)$ is the maximum eigenvalue of the Ricci curvature at p. We consider the variational definition of the first eigenvalue: as $b_1(Y) = 0$, there are not non-trivial harmonic 1-forms, so that we have

$$\int |(d+d^*)\xi|^2 \geq \lambda_1^* \int |\xi|^2. \tag{8}$$

We now apply this last inequality to the 1-form $\xi = \rho^{-1}(\Psi\Psi^*)_0$, which is coclosed because its Hodge star is a multiple of the curvature (recall that on a three-manifold, for any 1-form we have $\rho(\alpha) = -\rho(*\alpha)$). For this 1-form, recalling that we are using the inner product on $i\mathfrak{su}(2)$ given by $\mathrm{tr}(a^*b)/2$ (which makes the Clifford multiplication ρ an isometry), we have

$$|\nabla\xi|^2 \leq |\Psi|^2|\nabla_B\Psi|^2, \quad |\xi|^2 = \frac{1}{4}|\Psi|^4,$$

hence substituting in (6) we obtain

$$\int |\Psi|^6 + \frac{1}{2}(\lambda_1^* + \tilde{s})|\Psi|^4 \leq 0.$$

Here by definition $\tilde{s} = s - m$. Now, our assumption (4) implies that Ψ is identically zero, so that the Seiberg-Witten equations do not have irreducible solutions. Finally, as the quantity $\lambda_1^* + \tilde{s}$ is by assumption everywhere strictly positive, the same result holds for a small admissible perturbation of the equations (as those constructed in [14]), so that Y is an L-space. □

Remark. Of course (1) does not hold for spherical space forms, but one can also construct an example of a Riemannian three-manifold with $\inf\tilde{s} < 0$ for which (1) does not hold. For example, the Hantzsche-Wendt manifold is the unique rational homology three-sphere admitting a flat metric. We can choose a perturbation of the metric for which $\inf\tilde{s} < 0$. For perturbation small enough (in the C^∞ sense for example), condition (1) will still be false (see for example [6]).

Proof of Theorem 2. We follow closely the proof of the main result of [16] (beware that the conventions for the Seiberg-Witten equations in that paper are slightly different than [14]). Recall that the unit ball of the dual Thurston norm is a polytope P [25]. Combining deep work of Gabai, Thurston, Eliashberg and Taubes, it is shown in [16] that this polytope P coincides with the convex hull of monopole classes (so that in particular it has integral vertices). Let $\alpha \in H^2(Y;\mathbb{R})$ be a vertex of P. We know that in the corresponding spin^c structure there will be a solution (B, Ψ) to the Seiberg-Witten equations with $\Psi \neq 0$ for any metric, and in particular for the one we are working with. In particular, inequality (6) holds. The main difference is that now $*F_{B^t}$ is not necessarily coexact. Nevertheless, we can use the less refined inequality

$$\int |\nabla\xi|^2 \geq -\int m|\xi|^2$$

and obtain as above

$$\int |\Psi|^6 + \frac{1}{2}\tilde{s}|\Psi|^4 \leq 0.$$

Applying Hölder's inequality we get

$$\int |\Psi|^6 \leq \int -\frac{1}{2}\tilde{s}|\Psi|^4 \leq \left(\int |\tilde{s}/2|^3\right)^{1/3} \left(\int |\Psi|^6\right)^{2/3}$$

so that (as $\Psi \neq 0$) we have

$$\int |\tilde{s}/2|^3 \geq \int |\Psi|^6.$$

Then Hölder's inequality together with the Seiberg-Witten equations imply that

$$\mathrm{vol}(Y)^{1/3} \left(\int |\Psi|^6\right)^{2/3} \geq \int |\Psi|^4 = \int |F_{B^t}|^2,$$

so that putting the pieces together

$$\frac{1}{4}\mathrm{vol}(Y)^{1/3}\|\tilde{s}^2\|^2_{L^3(h)} \geq \int |F_{B^t}|^2 \geq 4\pi^2\|\alpha\|^2_{L^2(h)} \tag{9}$$

as F_{B^t} represents the class $-2\pi i\alpha$. As for a hyperbolic three-manifold $\tilde{s} = -4$, we get

$$\|\alpha\|^2_{L^2(h)} \leq \frac{\mathrm{vol}(Y)}{\pi^2}.$$

As this holds for all classes α which are vertices of the dual Thurston polytope, we get the inequality between the norms

$$\|\cdot\|^2_{L^2} \leq \frac{\mathrm{vol}(Y)}{\pi^2}|\cdot|^2.$$

Passing to dual norms and taking square roots the result follows. □

We conclude by discussing the extremal cases. In [12] it is shown that a metric for which the identity (3) is realized is very constrained: among the other features, the geometry is forced to be $\mathbb{R} \times \mathbb{H}$, so that in particular Y is Seifert. This follows from the fact that the curvature F_{B^t} is parallel. In our case, we can instead conclude if a manifold Y admits a metric for which (9) is an equality, then it must fiber over the circle. Indeed, if we are in the extremal case, inequality (7) implies that the curvature F_{B^t} is harmonic, so that in particular $*F_{B^t}$ is a closed form. Having an equality in equation (6) implies that $|\Psi|$ is a non-zero constant, so that by the Seiberg-Witten equations $*F_{B^t}$ is never zero. As $*F_{B^t}$ is (up to constants) the Poincaré dual to an integral form, by integrating it we obtain the required fibration to S^1.

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