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# A new geometric flow over Kähler manifolds

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In this paper, we introduce a geometric flow for Kähler metrics $\omega_t$ coupled with closed $(1,1)$-forms $\alpha_t$ on a compact Kähler manifold, whose stationary solution is a constant scalar curvature Kähler (cscK) metric, coupled with a harmonic $(1,1)$-form. We establish the long-time existence, i.e., assuming the initial $(1,1)$-form $\alpha$ is nonnegative, then the flow exists as long as the norm of the Riemannian curvature tensors are bounded.



## 1. Introduction

Let $(M, \omega)$ be a closed Kähler manifold of complex dimension $n$. The study of Kähler metrics with constant scalar curvature (cscK metric) in the Kähler

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class $[\omega]$ was initiated by Calabi [4, 5], by considering the Calabi-Futaki invariant of the Kähler class and the Calabi energy functional

$$\mathcal{C}_\omega(\varphi) := \int_M R^2_{\omega_\varphi}\,\omega^n_\varphi$$

on the space $\mathcal{H}_\omega := \{\varphi \in C^\infty(M) | \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$ of Kähler potentials. Here $R_{\omega_\varphi}$ is the scalar curvature of $\omega_\varphi$. The Euler-Lagrange equation for the Calabi functional is

$$\overline{\nabla}\,\overline{\nabla} R_{\omega_\varphi} = 0.$$

Solutions are the so called extremal Kähler metrics. Since the gradient vector field $\operatorname{grad} R_{\omega_\varphi}$ is holomorphic, if the manifold $X$ does not admit nontrivial holomorphic vector fields, then the extremal Kähler metric equation reduces to $R_{\omega_\varphi} =$ constant. We call such metrics cscK for constant scalar curvature Kähler. The search for cscK metrics is one of the central problems in Kähler geometry and there has been extensive study on this and related problems.

Calabi [4] proposed a parabolic flow, the Calabi flow:

$$\partial_t \omega_t = \sqrt{-1}\partial\bar{\partial} R_{\omega_t}, \quad \omega_0 = \omega,$$

with its stationary solution a cscK metric. Observe that this is a fully nonlinear fourth order partial differential equation. In [7], Chen and He derived the long-time existence provided the Ricci curvature is uniformly bounded. The interested reader may refer to [18, 19, 22, 23, 26, 28, 30, 35, 36, 39] for more recent development on the Calabi flow and the list is by no means to be complete.

The cscK metric is unique up to holomophic automorphisms [11, 13]. On the other hand, Yau conjectured that the existence of cscK metrics is equivalent to a certain geometric stability in geometric invariant theory (cf. [15, 37, 42]). One particularly important class of cscK metrics are Kähler-Einstein metrics. When $c_1(M) < 0$, the existence of Kähler-Einstein metrics was independently proven by Aubin [1] and Yau [41]. When $c_1(M) = 0$, the existence of Kähler-Einstein metrics was proved by Yau [41]. In this case the Kähler-Einstein metrics are Ricci-flat, and are called Calabi-Yau metrics. When $c_1(M) > 0$, the existence of Kähler-Einstein metrics was recently independently proved by Chen-Donaldson-Sun [8–10] and Tian [37], under some stability conditions. Besides Kähler-Einstein metrics, Donaldson proved the existence of the cscK metrics on toric surfaces under some stability conditions [17].

We now introduce a new parabolic flow to study the existence problem of cscK metrics. This flow is a modified Kähler-Ricci flow coupled with a heat flow for $(1,1)$-forms, whose stationary solution is a cscK metric, coupled with a harmonic $(1,1)$-forms. The short-time existence follows from the standard theory of strictly parabolic equations and we establish the long-time existence, i.e., if the initial $(1,1)$-form is nonnegative and the norm of the Riemannian curvature tensors is uniformly bounded, then the flow exists for all time.

Let $(M,\omega)$ be a closed Kähler manifold and $\alpha$ be a closed Hermitian $(1,1)$-form. Consider the flow

$$\partial_t\omega_t = -\rho(\omega_t) - \omega_t + \alpha_t, \quad \partial_t\alpha_t = \overline{\Box}_{\omega_t}\alpha_t, \quad (\omega_0,\alpha_0) = (\omega,\alpha), \tag{1.1}$$

where $\rho(\omega_t)$ denotes the Ricci form of $\omega_t$ and $\overline{\Box}_{\omega_t}$ denotes the complex Hodge-Laplace operator. Note that this flow is parallel to the pseudo-Calabi flow studied by Chen-Zheng [14].

The short-time existence of (1.1) follows from the standard parabolic theory. In fact, the solution still exists for short time when $(M,\omega)$ is only assumed to be Hermitian and $\alpha$ is not necessarily closed. Namely, if $(M,\omega)$ is a closed Hermitian manifold with a Hermitian $(1,1)$-form $\alpha$, the generalized flow of (1.1) defined by

$$\begin{aligned} \partial_t\omega_t &= -\hat{\rho}^{(2)}(\omega_t) - \omega_t + \alpha_t, \\ \partial_t\alpha_t &= \frac{1}{2}\Delta_{\mathrm{HL},\omega_t,\mathbf{R}}\alpha_t, \quad (\omega_0,\alpha_0) = (\omega,\alpha), \end{aligned} \tag{1.2}$$

exists for short time, where $\hat{\rho}^{(2)}(\omega_t)$ is the second Ricci-Chern form (see (2.22)) and $\Delta_{\mathrm{HL},\omega_t,\mathbf{R}} = dd^*_{\omega_t} + d^*_{\omega_t}d$ denotes the Riemannian Hodge-Laplace operator on $(M,\omega_t)$.

**Proposition 1.1.** *Suppose that $(M,\omega)$ is a closed Hermitian manifold and $\alpha$ is a Hermitian $(1,1)$-form on $M$. Then the flow (1.2) has a unique solution $(\omega_t,\alpha_t)$ on $[0,T)$ for some $T\in(0,\infty]$.*

If $\omega$ is Kähler and $\alpha$ is closed, then the flow (1.2) is exactly the flow (1.1). Suppose that $-2\pi c_1(M)+[\alpha]\in\mathcal{K}_M$, where $\mathcal{K}_M$ is the Kähler cone of $M$. Assume $\omega\in -2\pi c_1(M)+[\alpha]$. One can show that the solution $\omega_t$ also lies in the class $-2\pi c_1(M)+[\alpha]$ by Corollary 3.4. In addition, one can show that $\alpha_t\in[\alpha]$. By $\partial\bar{\partial}$-lemma, one writes

$$\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t \quad \text{and} \quad \alpha_t = \alpha + \sqrt{-1}\partial\bar{\partial}f_t$$

for smooth functions $\varphi_t$ and $f_t$ on $M$. We further choose a smooth volume form $\Omega$ on $M$ such that

$$\omega = \alpha + \sqrt{-1}\partial\bar{\partial}\log\Omega. \tag{1.3}$$

Then the flow (1.1) or (1.2) is equivalent to the following parabolic equations on scalar functions.

**Proposition 1.2.** *Under the assumption $\omega \in -2\pi c_1(M) + [\alpha]$, the flow (1.1) is equivalent to the parabolic complex Monge-Ampère equation coupled with a heat equation*

$$\partial_t\varphi_t = \log\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n}{\Omega} - \varphi_t + f_t, \quad \partial_t f_t = \Delta_{\omega_t} f_t + \mathrm{tr}_{\omega_t}\alpha, \tag{1.4}$$

*with $(\varphi_0, f_0) = (0, 0)$, where $\Delta_{\omega_t}$ stands for the complex Laplacian on $M$.*

Note that if $c_1(M) < 0$ and $\alpha = 0$, then the flow (1.1) is the normalized Kähler-Ricci flow on $M$. Thus the flow (1.1) converges exponentially to the negative Kähler-Einstein metric coupled with $\alpha_\infty = 0$ (cf. [1][6][41]). The main theorem of this paper is the long time existence of equation (1.1).

**Theorem 1.3.** *Suppose that $(M, \omega)$ is a closed Kähler manifold and $\alpha$ is a closed nonnegative $(1,1)$-form such that*

$$\omega \in -2\pi c_1(M) + [\alpha].$$

*Let $(\omega_t, \alpha_t)$ be the solution to the flow (1.1) on the maximal time interval $[0, T)$ for $T < \infty$ with the initial condition $(\omega, \alpha)$. Then*

$$\limsup_{t\to T}\max_M |\mathrm{Rm}_{\omega_t}|_{\omega_t} = \infty.$$

**Corollary 1.4.** *Assume that $\alpha$ is a closed nonnegative $(1,1)$-form such that*

$$\omega \in -2\pi c_1(M) + [\alpha].$$

*Let $(\omega_t, \alpha_t)$ be the solution to the flow (1.1) for $t \in [0, T)$ with the initial condition $(\omega, \alpha)$. Suppose that the Ricci curvature of $\omega_t$ and $|\alpha_t|_{\omega_t}$ are uniformly bounded on $[0, T)$. Then the solution $(\omega_t, \alpha_t)$ can be extended past time $T$.*

The motivation to study the flow (1.1) is its connection to cscK metrics. Suppose that $(\omega_\infty, \alpha_\infty)$ is a stationary solution to flow (1.1). Then, in particular, $\alpha_\infty$ is a harmonic $(1,1)$-form with respect to $\omega_\infty$. This implies that $\mathrm{tr}_{\omega_\infty}\alpha_\infty$ =constant, and therefore, $R_{\omega_\infty}$ = constant by equation (1.1).

To end the introduction, we remark that the nonnegativity assumption on the $(1,1)$-form $\alpha$ is not essential. Suppose we seek a cscK metric in the class $[\omega]$. As $\omega$ is Kähler, there exists a positive number $\delta > 0$ such that

$$[\omega] + \delta 2\pi c_1(M) \geq 0. \tag{1.5}$$

Hence there exists a closed nonnegative $(1,1)$-form $\hat{\alpha}$ satisfying

$$\omega \in -\delta 2\pi c_1(M) + [\hat{\alpha}] \in \mathcal{K}_M. \tag{1.6}$$

Now we can consider the flow

$$\partial_t \omega_t = -\delta\rho(\omega_t) - \omega_t + \alpha_t, \quad \partial_t \alpha_t = \Delta_{L,\omega_t}\alpha_t, \quad (\omega_0, \alpha_0) = (\omega, \hat{\alpha}), \tag{1.7}$$

which is a renormalized flow to (1.1). The stationary solution to (1.7) is still the pair of a cscK metric and a harmonic $(1,1)$-form. By the same argument, we can prove the similar existence result as Theorem 1.3.

We give an outline of the present paper. We review the basic notions in Kähler geometry and Chern connection in Section 2. In Section 3 we prove Proposition 1.1 for closed Hermitian manifolds. In Section 4, we prove higher derivatives estimates and then Theorem 1.3 under the assumption that the initial $(1,1)$-form is nonnegative.

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## 2. Basic complex geometry

We fix the notations in this section. Let $(M, g)$ be a Kähler manifold of complex dimension $n$ and let $(z^i)_{1\leq i\leq n}$ be holomorphic local coordinates on $M$. Write $\partial_i = \frac{\partial}{\partial z^i}$, $\overline{\partial_i} = \partial_{\bar{i}} = \frac{\partial}{\partial \bar{z}^i}$, and let

$$g_{AB} := g\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right), \quad A, B \in \{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}.$$

Since $g$ is invariant with respect to the complex structure of $M$, it follows that

$$g_{i\bar{j}} = \overline{g_{\bar{i}j}} = g_{\bar{j}i}, \quad g_{ij} = 0 = g_{\bar{i}\bar{j}}$$

and then the Hermitian metric $g$ takes the form

$$g = 2g_{i\bar{j}}dz^i \otimes dz^{\bar{j}}. \tag{2.1}$$

The fundamental form $\omega_g$ associated to $g$ is a real $(1,1)$-form given by

$$\omega_g = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}} \tag{2.2}$$

and the real cohomology class

$$[\omega_g] \in H^{1,1}_{\mathbf{R}}(M) \subset H^2_{\mathbf{R}}(M)$$

is called the *Kähler class* of $\omega_g$.

The complex linear extensions of the Levi-Civita connection $\nabla_g$, the Riemannian curvature tensor $\mathrm{Rm}_g$, and the Ricci curvature $\mathrm{Rc}_g$ are still denoted as $\nabla_g$, $\mathrm{Rm}_g$, and $\mathrm{Rc}_g$, respectively. The Christoffel symbols of the Levi-Civita connection are given by

$$(\nabla_g)_{\frac{\partial}{\partial z^A}}\frac{\partial}{\partial z^B} := \Gamma^C_{AB}\frac{\partial}{\partial z^C}, \quad A,B,C \in \{1,\ldots,n,\bar{1},\ldots,\bar{n}\},$$

and one can check that they are zero unless all the indices are unbarred or all the indices are barred. The Riemannian curvature tensor field $\mathrm{Rm}_g$ are defined by

$$\mathrm{Rm}_g\left(\frac{\partial}{\partial z^A},\frac{\partial}{\partial z^B}\right)\frac{\partial}{\partial z^C} := R^D_{ABC}\frac{\partial}{\partial z^D}, \quad A,B,C,D \in \{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$$

and

$$R_{ABCD} := g\left(\mathrm{Rm}_g\left(\frac{\partial}{\partial z^A},\frac{\partial}{\partial z^B}\right)\frac{\partial}{\partial z^C},\frac{\partial}{\partial z^D}\right),$$

The straightforward calculation indicates that the only non-vanishing components are $R^{\ell}_{i\bar{j}k}, R^{\bar{\ell}}_{i\bar{j}\bar{k}}, R^{\ell}_{\bar{i}jk}$ and $R^{\bar{\ell}}_{\bar{i}j\bar{k}}$ and $R_{i\bar{j}k\bar{\ell}}, R_{i\bar{j}\bar{k}\ell}, R_{\bar{i}jk\bar{\ell}}, R_{\bar{i}j\bar{k}\ell}$. By taking the trace, the non-vanishing components in

$$R_{AB} := \mathrm{Rc}_g\left(\frac{\partial}{\partial z^A},\frac{\partial}{\partial z^B}\right), \quad A,B \in \{1,\ldots,n,\bar{1},\ldots,\bar{n}\},$$

are $R_{i\bar{j}}$ and $R_{\bar{i}j}$satisfying $R_{i\bar{j}} = \overline{R_{\bar{j}\bar{i}}}$. The straightforward calculation yields

$$R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det(\mathfrak{g}), \tag{2.3}$$

where $\mathfrak{g} := (g_{k\bar{\ell}})_{1\le k,\ell\le n}$ stands for the matrix of $g_{k\bar{\ell}}$. The Ricci form $\rho_g$ associated to $g$ is a real closed $(1,1)$-form defined by

$$\rho_g = \sqrt{-1}R_{i\bar{j}}dz^i \wedge dz^{\bar{j}} = -\sqrt{-1}\partial\bar{\partial}\log\det(\mathfrak{g}). \tag{2.4}$$

The real de Rham cohomology class $[\rho_g/2\pi]$ is the first Chern class $c_1(M)$ of $M$. The (complex) scalar curvature is defined to be

$$R_g := R_{i\bar{j}}g^{\bar{j}i}, \tag{2.5}$$

which is the one-half of the Riemannian scalar curvature of $g$.

### 2.1. Hodge-Laplace operators

Suppose now $(M,g)$ is a closed complex manifold of complex dimension $n$. For any two (unordered) $(p,q)$-forms

$$\begin{aligned}\alpha &= \frac{1}{p!q!}\alpha_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge dz^{\bar{j}_1}\wedge dz^{\bar{j}_q},\\ \beta &= \frac{1}{p!q!}\beta_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}dz^{i_1}\wedge dz^{i_p}\wedge dz^{\bar{j}_1}\wedge\cdots\wedge dz^{\bar{j}_q},\end{aligned}$$

we define the *global inner product* of $\alpha$ and $\beta$ (cf. [24]) by

$$\langle\langle\alpha,\beta\rangle\rangle_g := \int_M \langle\alpha,\beta\rangle_g dV_g = \int_M \langle\alpha,\beta\rangle\frac{\omega_g^n}{n!}. \tag{2.6}$$

where

$$\langle\alpha,\beta\rangle_g := \frac{1}{p!q!}\alpha_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}\bar{\beta}^{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}, \tag{2.7}$$

and

$$\bar{\beta}^{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} := g^{\bar{k}_1 i_1}\cdots g^{\bar{k}_p i_p}g^{\bar{j}_1\ell_1}\cdots g^{\bar{j}_q\ell_q}\overline{\beta_{k_1\cdots k_p\bar{\ell}_1\cdots\bar{\ell}_q}}.$$

Let $\partial_g^* : \Omega^{p,q}(M)\to\Omega^{p-1,q}(M)$ be the adjoint operator of $\partial$. For any $p\ge 1$, $\partial_g^*$ is defined by

$$\langle\langle\partial_g^*\beta,\alpha\rangle\rangle_g = \langle\langle\beta,\partial\alpha\rangle\rangle_g, \tag{2.8}$$

for $\alpha\in\Omega^{p-1,q}(M)$ and $\beta\in\Omega^{p,q}(M)$. $\bar{\partial}_g^* : \Omega^{p,q}(M)\to\Omega^{p,q-1}(M)$ for $q\ge 1$ can be defined in the same manner.

The *complex Hodge-Laplace operators* of $g$ are given by

$$\Box_g := -\left(\partial\partial_g^* + \partial_g^*\partial\right), \quad \overline{\Box}_g := -\left(\bar\partial\bar\partial_g^* + \bar\partial_g^*\bar\partial\right). \tag{2.9}$$

The principal parts of $\Box_g, \overline{\Box}_g$ are both given by $g^{i\bar{j}}\partial_i\partial_{\bar{j}}$, and hence the partial differential operators $\Box_g, \overline{\Box}_g$ are strictly elliptic. More precisely, as showed in [24], for any $(p,q)$-form $\alpha$ we have

$$\overline{\Box}_g\alpha = g^{i\bar{j}}\partial_i\partial_{\bar{j}}\alpha + A^i(g,g^{-1})\partial_i\alpha + B^{\bar{j}}(g,g^{-1})\partial_{\bar{j}}\alpha + C(g,g^{-1})\alpha, \tag{2.10}$$

where $A^i(g,g^{-1}), B^{\bar{j}}(g,g^{-1}), C(g,g^{-1})$ are polynomials of $g, g^{-1}$ and their partial derivatives.

When $(M,g)$ is a Kähler manifold, it is well-known that $\Box_g = \overline{\Box}_g = \frac{1}{2}\Delta_{\mathrm{HL},g,\mathbf{R}}$, where $\Delta_{\mathrm{HL},g,\mathbf{R}} = dd_g^* + d_g^*d$ is the Riemannian Hodge-Laplace operator on $(M,g)$. In general, we have

$$\Delta_{\mathrm{HL},g,\mathbf{R}} = \Box_g + \overline{\Box}_g + \text{lower order terms}. \tag{2.11}$$

The following lemma can be found in the standard textbooks (cf. [24]).

**Lemma 2.1.** *Let $\alpha = \sqrt{-1}\sum_{1\le i,j\le n}\alpha_{i\bar{j}}dz^i\wedge dz^{\bar{j}}$ be a real $(1,1)$-form. Then*

$$(\bar\partial^*\alpha)^{\bar\ell} = -\left(\frac{\partial}{\partial z^k} + \frac{\partial\log\det(\mathfrak{g})}{\partial z^k}\right)\alpha^{\bar\ell k}$$

The following calculation of Hodge-Laplace operator can be found in standard textbooks (cf. [24]).

**Lemma 2.2.** *Let $\alpha = \sqrt{-1}\sum_{1\le i,j\le n}\alpha_{i\bar{j}}dz^i\wedge dz^{\bar{j}}$ be a real $(1,1)$-form. Then*

$$\overline{\Box}_g\alpha = \sqrt{-1}\sum_{1\le i,j\le n}\left(\nabla^{\bar\ell}\nabla_{\bar\ell}\alpha_{i\bar{j}} + g^{\bar{q}k}g^{\bar\ell p}R_{k\bar{j}i\bar\ell}\alpha_{p\bar{q}} - g^{\bar\ell k}R_{k\bar{j}}\alpha_{i\bar\ell}\right)dz^i\wedge dz^{\bar{j}}.$$

*We may rewrite the complex Laplace operator as:*

$$\overline{\Box}_g\alpha = \sqrt{-1}\sum_{1\le i,j\le n}\Bigg[\Delta_g\alpha_{i\bar{j}} + g^{\bar\ell p}g^{\bar{q}k}R_{k\bar{j}i\bar\ell}\alpha_{p\bar{q}} - \frac{1}{2}g^{\bar\ell k}\left(3R_{k\bar{j}}\alpha_{i\bar\ell} + R_{i\bar\ell}\alpha_{k\bar{j}}\right)\Bigg]dz^i\wedge dz^{\bar{j}}$$

*and it reads in the local coordinates:*

$$\overline{\Box}_g\alpha_{i\bar{j}} = \Delta_g\alpha_{i\bar{j}} + g^{\bar\ell p}g^{\bar{q}k}R_{k\bar{j}i\bar\ell}\alpha_{p\bar{q}} - \frac{1}{2}g^{\bar\ell k}\left(3R_{k\bar{j}}\alpha_{i\bar\ell} + R_{i\bar\ell}\alpha_{k\bar{j}}\right). \tag{2.12}$$

The following proposition is important in the later argument.

**Proposition 2.3.** *Let* $(M,g)$ *be a closed Kähler manifold and let* $\alpha = \sqrt{-1}\sum_{1\le i,j\le n}\alpha_{i\bar{j}}dz^i\wedge dz^{\bar{j}}$ *be a closed Hermitian* $(1,1)$*-form. Then*

$$\overline{\Box}_g\alpha = \sqrt{-1}\partial\bar{\partial}(\mathrm{tr}_g\alpha), \tag{2.13}$$

*where* $\mathrm{tr}_g\alpha = g^{i\bar{j}}\alpha_{i\bar{j}}$*. As a consequence,* $\overline{\Box}_g\alpha$ *is d-exact.*

*Proof.* We calculate $\overline{\Box}_g\alpha$ and $\sqrt{-1}\partial\bar{\partial}(\mathrm{tr}_g\alpha)$ in a normal coordinate. Making use of Lemma 2.1,

$$(\bar{\partial}^*\alpha)^{\bar{\ell}} = -\left(\frac{\partial}{\partial z^k} + \frac{\partial\log\det(\mathfrak{g})}{\partial z^k}\right)(g^{\bar{\ell}p}g^{k\bar{q}}\alpha_{p\bar{q}}) = -(\mathrm{I}) - (\mathrm{II}),$$

where

$$(\mathrm{I}) = -\frac{\partial g_{s\bar{t}}}{\partial z^k}g^{p\bar{t}}g^{s\bar{\ell}}g^{k\bar{q}}\alpha_{p\bar{q}} - \frac{\partial g_{s\bar{t}}}{\partial z^k}g^{p\bar{\ell}}g^{s\bar{q}}g^{k\bar{t}}\alpha_{p\bar{q}} + g^{\bar{\ell}p}g^{k\bar{q}}\frac{\partial\alpha_{p\bar{q}}}{\partial z^k},$$

and

$$(\mathrm{II}) = g^{s\bar{t}}\frac{\partial g_{s\bar{t}}}{\partial z^k}g^{\bar{\ell}p}g^{k\bar{q}}\alpha_{p\bar{q}}.$$

Therefore,

$$\begin{aligned}
(\overline{\Box}_g\alpha)_{i\bar{j}} &= -\partial_{\bar{j}}\left(g_{i\bar{\ell}}(\bar{\partial}^*\alpha)^{\bar{\ell}}\right)\\
&= \partial_{\bar{j}}\left(-g_{i\bar{\ell}}\frac{\partial g_{s\bar{t}}}{\partial z^k}g^{p\bar{t}}g^{s\bar{\ell}}g^{k\bar{q}}\alpha_{p\bar{q}} - g_{i\bar{\ell}}\frac{\partial g_{s\bar{t}}}{\partial z^k}g^{p\bar{\ell}}g^{s\bar{q}}g^{k\bar{t}}\alpha_{p\bar{q}}\right)\\
&\quad + \partial_{\bar{j}}\left(g_{i\bar{\ell}}g^{\bar{\ell}p}g^{k\bar{q}}\frac{\partial\alpha_{p\bar{q}}}{\partial z^k} + g_{i\bar{\ell}}g^{s\bar{t}}\frac{\partial g_{s\bar{t}}}{\partial z^k}g^{\bar{\ell}p}g^{k\bar{q}}\alpha_{p\bar{q}}\right)\\
&= -\sum_{1\le p\le n}\frac{\partial^2 g_{i\bar{p}}}{\partial z^k\partial z^{\bar{j}}}\alpha_{p\bar{k}} - \sum_{1\le q\le n}\frac{\partial^2 g_{q\bar{k}}}{\partial z^k\partial z^{\bar{j}}}\alpha_{i\bar{q}}\\
&\quad + \sum_{1\le k\le n}\frac{\partial^2\alpha_{i\bar{k}}}{\partial z^k\partial z^{\bar{j}}} + \sum_{1\le s\le n}\frac{\partial^2 g_{s\bar{s}}}{\partial z^k\partial z^{\bar{j}}}\alpha_{i\bar{k}}\\
&= -\sum_{1\le p,k\le n}\frac{\partial^2 g_{k\bar{p}}}{\partial z^i\partial z^{\bar{j}}}\alpha_{p\bar{k}} + \sum_{1\le k\le n}\frac{\partial^2\alpha_{k\bar{k}}}{\partial z^i\partial z^{\bar{j}}},
\end{aligned}$$

where the third equality uses the properties of normal coordinates and the fourth equality uses the closedness of $g$ and $\alpha$ and the second and fourth terms cancel each other among terms in front of the fourth equality by the closedness of $g$. On the other hand,

$$\begin{aligned}\sqrt{-1}\partial_i\partial_{\bar{j}}(\mathrm{tr}_g\alpha) &= \partial_i\left(-g^{p\bar{\ell}}g^{k\bar{q}}\frac{\partial g_{p\bar{q}}}{\partial z^{\bar{j}}}\alpha_{k\bar{\ell}} + g^{k\bar{\ell}}\frac{\partial\alpha_{k\bar{\ell}}}{\partial z^{\bar{j}}}\right)\\ &= -\sum_{1\le k,\ell\le n}\frac{\partial^2 g_{\ell\bar{k}}}{\partial z^i\partial z^{\bar{j}}}\alpha_{k\bar{\ell}} + \sum_{1\le k\le n}\frac{\partial^2\alpha_{k\bar{k}}}{\partial z^i\partial z^{\bar{j}}}.\end{aligned}$$

The proposition is proved. □

### 2.2. Chern connections on Hermitian manifolds

Let $(M, g)$ be a Hermitian manifold of complex dimension $n$. A *Chern connection* $\widehat{\nabla}$ is a unique connection on the holomorphic tangent bundle $T^{1,0}M$ such that $\widehat{\nabla}$ is compatible with the Hermitian structure and the holomorphic structure.

The non-vanishing components of Chern connection are $\widehat{\Gamma}^k_{ij}$ and $\widehat{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}}$, where

$$\widehat{\Gamma}^k_{ij} = g^{\bar{\ell}k}\partial_i g_{j\bar{\ell}}, \quad \widehat{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\widehat{\Gamma}^k_{ij}}. \tag{2.14}$$

The torsion $\widehat{T}$ of the Chern connection is defined by

$$\widehat{T}(X, Y) := \widehat{\nabla}_X Y - \widehat{\nabla}_Y X - [X, Y], \quad X, Y \in \Omega^{1,0}(M). \tag{2.15}$$

Its components $\widehat{T}^k_{ij}$ defined by $\widehat{T}(\partial_i, \partial_j) = \widehat{T}^k_{ij}\partial_k$ can be computed according to the formula

$$\widehat{T}^k_{ij} = \widehat{\Gamma}^k_{ij} - \widehat{\Gamma}^k_{ji}. \tag{2.16}$$

If we set $\widehat{T}_{ij\bar{k}} := g_{\ell\bar{k}}\widehat{T}^\ell_{ij}$, then

$$\widehat{T}_{ij\bar{k}} = \partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}. \tag{2.17}$$

The curvature of the Chern connection is

$$\widehat{R}_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}g_{k\bar{\ell}} + g^{\bar{q}p}\partial_i g_{k\bar{q}}\partial_{\bar{j}}g_{p\bar{\ell}}. \tag{2.18}$$

The *first Ricci-Chern curvature* defined by

$$\widehat{R}^{(1)}_{i\bar{j}} := g^{\bar{\ell}k}\widehat{R}_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}\log\det(\mathfrak{g}) \tag{2.19}$$

represents the first Chern class of $M$ (up to a constant). The *second Ricci-Chern curvature* is defined by

$$\widehat{R}^{(2)}_{i\bar{j}} := g^{\bar{\ell}k}\widehat{R}_{k\bar{\ell}i\bar{j}}. \tag{2.20}$$

Those two Ricci-Chern curvatures are related by

$$\widehat{R}^{(1)}_{i\bar{j}} - \widehat{R}^{(2)}_{i\bar{j}} = \widehat{\nabla}^{\bar{k}}\widehat{T}_{\bar{k}\bar{j}i} - \widehat{\nabla}_{\bar{j}}\widehat{\mathbf{T}}_i, \tag{2.21}$$

where $\widehat{T}_{\bar{k}\bar{j}i} = \overline{\widehat{T}_{kj\bar{i}}}$ and $\hat{\mathbf{T}}_i = \sum_{1\le j\le n}\widehat{T}^j_{ij}$. In particular, if $g$ is Kähler, then $\widehat{R}^{(1)}_{i\bar{j}} = \widehat{R}^{(2)}_{i\bar{j}}$. Set

$$\widehat{\rho}^{(1)} := \sqrt{-1}\widehat{R}^{(1)}_{i\bar{j}}dz^i \wedge dz^{\bar{j}}, \quad \widehat{\rho}^{(2)} := \sqrt{-1}\widehat{R}^{(2)}_{i\bar{j}}dz^i \wedge dz^{\bar{j}}. \tag{2.22}$$

Note that both $\widehat{\rho}^{(1)}$ and $\widehat{\rho}^{(2)}$ are real $(1,1)$-forms on $M$.

## 3. A geometric flow and cscK metrics

Let $M$ be a closed complex manifold with a fundamental form

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}.$$

Then $g := g_{i\bar{j}}dz^i \otimes dz^{\bar{j}}$ defines a Hermitian metric on $T^{1,0}M$. In this setting, all quantities associated to $g$ will be written as quantities associated to $\omega$. For example, the Ricci-Chern forms of $g$ are written as $\widehat{\rho}^{(\cdot)}(\omega)$.

Suppose $\alpha$ is a fixed Hermitian $(1,1)$-form on $M$. In this section we study the flow (1.2) starting with a initial pair $(\omega, \alpha)$ on $M$. In particular, if $(\omega, \alpha)$ is a closed pair (i.e., $d\omega = d\alpha = 0$), we can show that $(\omega_t, \alpha_t)$ is also a closed pair. Thus the flow preserves the closedness condition.

### 3.1. A flow on the triple $(M, \omega, \alpha)$ - short time existence

Consider the flow

$$\begin{aligned} \partial_t\omega_t &= -\widehat{\rho}^{(2)}(\omega_t) - \omega_t + \alpha_t, \\ \partial_t\alpha_t &= \frac{1}{2}\Delta_{\mathrm{HL},\omega_t,\mathbf{R}}\alpha_t, \quad (\omega_0, \alpha_0) = (\omega, \alpha). \end{aligned} \tag{3.1}$$

We say that a family of pairs $(\omega_t, \alpha_t)$ is a solution to (3.1), if

(i) $\omega_t$ is a fundamental form for each $t$,

(ii) $\alpha_t$ is a Hermitian $(1,1)$-form, and

(iii) $(\omega_t, \alpha_t)$ satisfies the equation (3.1).

The first result of this paper is the short-time existence and uniqueness of (3.1).

**Theorem 3.1.** *Suppose $M$ is a closed complex manifold with a fundamental form $\omega$ and $\alpha$ is a Hermitian $(1,1)$-form on $M$. Then the flow (3.1) has a unique solution $(\omega_t, \alpha_t)$ on $[0,T)$ for some $T > 0$.*

*Proof.* Let us denote by $\mathscr{H}$ and $\mathscr{H}_+$ the space of all Hermitian $(1,1)$-forms and the space of all positive Hermitian $(1,1)$-forms, respectively. Define an operator $\mathcal{F} : \mathscr{H}_+ \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ by

$$\mathcal{F}(\omega, \alpha) := \left(-\widehat{\rho}^{(2)}(\omega) - \omega + \alpha, \frac{1}{2}\Delta_{\mathrm{HL},\omega,\mathbf{R}}\alpha\right).$$

Writing $\alpha = \sqrt{-1}\alpha_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ and using (2.18), (2.20), (2.22), (2.11), (2.12), we obtain

$$\begin{aligned} -\widehat{\rho}^{(2)}(\omega) &= \sqrt{-1}\left(g^{\bar{\ell}k}\partial_k\partial_{\bar{\ell}}g_{i\bar{j}} - g^{\bar{\ell}k}g^{\bar{q}p}\partial_k g_{i\bar{q}}\partial_{\bar{\ell}}g_{p\bar{j}}\right) dz^i \wedge dz^{\bar{j}}, \\ \frac{1}{2}\Delta_{\mathrm{HL},\omega,\mathbf{R}}\alpha &= \sqrt{-1}\left(g^{\bar{\ell}k}\partial_k\partial_{\bar{\ell}}\alpha_{i\bar{j}} + \text{lower order terms of } \alpha\right) dz^i \wedge dz^{\bar{j}}. \end{aligned}$$

Consequently,

$$\mathcal{F}(\omega, \alpha) = g^{\bar{\ell}k}\partial_k\partial_{\bar{\ell}}(\omega, \alpha) + \text{lower order terms of } (\omega, \alpha);$$

thus the operator $\mathcal{F}$ is strictly elliptic and therefore the flow equation (3.1) is a system of strictly parabolic partial differential equations. The standard theory gives the short-time existence and uniqueness. □

As a corollary, we have the following theorem in the Kähler case.

**Theorem 3.2.** *Suppose that $(M, \omega)$ is a closed Kähler manifold with a closed Hermitian $(1,1)$-form $\alpha$. Then the solution $(\omega_t, \alpha_t)_{t\in[0,T)}$ to (3.1) is closed, i.e. $d\omega_t = d\alpha_t = 0$. Furthermore, the flow (3.1) reduces to*

$$\partial_t\omega_t = -\rho(\omega_t) - \omega_t + \alpha_t, \quad \partial_t\alpha_t = \overline{\Box}_{\omega_t}\alpha_t, \quad (\omega_0, \alpha_0) = (\omega, \alpha). \tag{3.2}$$

*Proof.* The proof follows similar calculation as in the Kähler-Ricci flow. Let $(\omega_t, \alpha_t)_{t\in[0,T)}$ be the unique solution to (3.1). Then

$$\partial_t(d\alpha_t) = d\partial_t\alpha_t = \frac{1}{2}d\Delta_{\mathrm{HL},\omega_t,\mathbf{R}}\alpha_t = \frac{1}{2}\Delta_{\mathrm{HL},\omega_t,\mathbf{R}}(d\alpha_t), \quad d\alpha_0 = d\alpha = 0.$$

By the uniqueness, we must have $d\alpha_t = 0$ for all $t \in [0, T)$. On the other hand, applying the exterior differentiation operator $d$ to the evolution equation of $\omega_t$, we arrive at

$$\partial_t(d\omega_t) = -d\widehat{\rho}^{(2)}(\omega_t) - d\omega_t, \quad d\omega_0 = d\omega = 0.$$

To prove the closedness of $\omega_t$, we use the similar argument in [27] (see the proof of Theorem 7.1). Since $d = \partial + \bar{\partial}$ and $\bar{\partial}\omega_t = \overline{\partial\omega_t}$ ($\omega_t$ is real), it suffices to prove $\partial\omega_t = 0$. In local holomorphic coordinates we have

$$\partial_t g_{i\bar{j}} = -\widehat{R}^{(2)}_{i\bar{j}} - g_{i\bar{j}} + \alpha_{i\bar{j}},$$

where

$$\omega_t = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}}, \widehat{\rho}^{(2)}(\omega_t) = \sqrt{-1}\widehat{R}^{(2)}_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$$

and

$$\alpha_t = \sqrt{-1}\alpha_{i\bar{j}}dz^i \wedge dz^{\bar{j}}$$

(without the subscript $t$ in the components). According to (2.17),

$$\partial\omega_t = \sqrt{-1}\sum_{1<i<j\leq n}\sum_{1\leq k\leq n}\widehat{T}_{ij\bar{k}}dz^i \wedge dz^j \wedge dz^{\bar{k}}.$$

We need only to show that $\widehat{T}_{ij\bar{k}} = 0$ along the flow equation (3.1). To achieve this, we compute the evolution equation of $\widehat{T}_{ij\bar{k}}$. Since $d\alpha_t = 0$, we obtain $\partial_i\alpha_{j\bar{k}} = \partial_j\alpha_{i\bar{k}}$ for any indices $i, j, k$. Therefore,

$$\begin{aligned}\partial_t\widehat{T}_{ij\bar{k}} &= \partial_t(\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}) \\ &= \partial_i\left(-\widehat{R}^{(2)}_{j\bar{k}} - g_{j\bar{k}} + \alpha_{j\bar{k}}\right) - \partial_j\left(-\widehat{R}^{(2)}_{i\bar{k}} - g_{i\bar{k}} + \alpha_{i\bar{k}}\right) \\ &= -\left(\partial_i\widehat{R}^{(2)}_{j\bar{k}} - \partial_j\widehat{R}^{(2)}_{i\bar{k}}\right) + \widehat{T}_{ij\bar{k}}.\end{aligned}$$

For operators $\mathcal{P}, \mathcal{Q}$, the symbol $\mathcal{P} \sim \mathcal{Q}$ means that $\mathcal{Q}$ is the principal part of $\mathcal{Q}$. Define an operator by

$$\mathcal{P}(\omega_t)_{ij\bar{k}} := -\left(\partial_i\widehat{R}^{(2)}_{j\bar{k}} - \partial_j\widehat{R}^{(2)}_{i\bar{k}}\right).$$

Using (2.19) and (2.21) we get

$$-\widehat{R}^{(2)}_{j\bar{k}} = \partial_j\partial_{\bar{k}}\log\det(\mathfrak{g}_t) + \widehat{\nabla}^{\bar{\ell}}\widehat{T}_{\overline{\ell k}j} - \widehat{\nabla}_{\bar{k}}\widehat{\mathbf{T}}_j$$

and thus

$$\begin{aligned}\mathcal{P}(\omega_t)_{ij\bar{k}} &= \partial_i\left(\partial_j\partial_{\bar{k}}\log\det(\mathfrak{g}_t) + \widehat{\nabla}^{\bar{\ell}}\widehat{T}_{\overline{\ell k}j} - \widehat{\nabla}_{\bar{k}}\widehat{\mathbf{T}}_j\right)\\ &\quad - \partial_j\left(\partial_i\partial_{\bar{k}}\log\det(\mathfrak{g}_t) + \widehat{\nabla}^{\bar{\ell}}\widehat{T}_{\overline{\ell k}i} - \widehat{\nabla}_{\bar{k}}\widehat{\mathbf{T}}_i\right)\\ &= \partial_i\left(g^{\bar{\ell}p}\widehat{\nabla}_p\widehat{T}_{\overline{\ell k}j}\right) - \partial_j\left(g^{\bar{\ell}p}\widehat{\nabla}_p\widehat{T}_{\overline{\ell k}i}\right) - \partial_i\widehat{\nabla}_{\bar{k}}\widehat{\mathbf{T}}_j + \partial_j\widehat{\nabla}_{\bar{k}}\widehat{\mathbf{T}}_i\\ &\sim g^{\bar{\ell}p}\left(\partial_i\partial_p\widehat{T}_{\overline{\ell k}j} - \partial_j\partial_p\widehat{T}_{\overline{\ell k}i}\right) - \partial_i\partial_{\bar{k}}\widehat{\mathbf{T}}_j + \partial_j\partial_{\bar{k}}\widehat{\mathbf{T}}_i.\end{aligned}$$

Using (2.17), we get

$$\partial_i\partial_p\widehat{T}_{\overline{\ell k}j} = \partial_i\partial_p(\partial_{\bar{\ell}}g_{j\bar{k}} - \partial_{\bar{k}}g_{j\bar{\ell}}) = \partial_p\partial_{\bar{\ell}}\widehat{T}_{ij\bar{k}} + \partial_p\partial_{\bar{\ell}}\partial_j g_{i\bar{k}} - \partial_i\partial_p\partial_{\bar{k}}g_{j\bar{\ell}}.$$

By switching the indices $i, j$, we obtain

$$\partial_j\partial_p\widehat{T}_{\overline{\ell k}i} = \partial_j\partial_p(\partial_{\bar{\ell}}g_{i\bar{k}} - \partial_{\bar{k}}g_{i\bar{\ell}}).$$

On the other hand,

$$\begin{aligned}-\partial_i\partial_{\bar{k}}\widehat{\mathbf{T}}_j + \partial_j\partial_{\bar{k}}\widehat{\mathbf{T}}_i &= -\partial_i\partial_{\bar{k}}\widehat{T}^{\ell}_{j\ell} + \partial_j\partial_{\bar{k}}\widehat{T}^{\ell}_{i\ell}\\ &= -\partial_i\partial_{\bar{k}}\left(g^{\bar{p}\ell}\partial_j g_{\ell\bar{p}} - g^{\bar{p}\ell}\partial_\ell g_{j\bar{p}}\right)\\ &\quad + \partial_j\partial_{\bar{k}}\left(g^{\bar{p}\ell}\partial_i g_{\ell\bar{p}} - g^{\bar{p}\ell}\partial_\ell g_{i\bar{p}}\right)\\ &\sim g^{\bar{p}\ell}\left(-\partial_i\partial_{\bar{k}}\partial_j g_{\ell\bar{p}} + \partial_i\partial_{\bar{k}}\partial_\ell g_{j\bar{p}} + \partial_j\partial_{\bar{k}}\partial_i g_{\ell\bar{p}} - \partial_j\partial_{\bar{k}}\partial_\ell g_{i\bar{p}}\right)\\ &= g^{\bar{p}\ell}\left(\partial_i\partial_{\bar{k}}\partial_\ell g_{j\bar{p}} - \partial_j\partial_{\bar{k}}\partial_\ell g_{i\bar{p}}\right).\end{aligned}$$

Plugging those expression into $\mathcal{P}(\omega_t)_{ij\bar{k}}$, we obtain

$$\mathcal{P}(\omega_t)_{ij\bar{k}} \sim g^{\bar{\ell}p}\partial_p\partial_{\bar{\ell}}\widehat{T}_{ij\bar{k}}.$$

Consequently, the system of partial differential equations

$$\partial_t\widehat{T}_{ij\bar{k}} = \mathcal{P}(\omega_t)_{ij\bar{k}} + \widehat{T}_{ij\bar{k}}$$

is strictly parabolic. When $\widehat{T}_{ij\bar{k}} \equiv 0$, we must have $\widehat{R}^{(1)}_{j\bar{k}} = \widehat{R}^{(2)}_{j\bar{k}}$ and then $\mathcal{P}(\omega_t)_{ij\bar{k}} = 0$. Thus 0 is a solution of the above system because $d\omega = 0$. By the uniqueness, we get $\widehat{T}_{ij\bar{k}} = 0$ and hence $\partial\omega_t = 0$. Taking the complex conjugate we prove that $d\omega_t = 0$ for any $t \in [0, T)$.

Finally, the equation (3.2) immediately follows from (3.1), since $\overline{\square}_{\omega_t} = \frac{1}{2}\Delta_{\mathrm{HL},\omega_t,\mathbf{R}}$ when $\omega_t$ is Kähler. □

**Remark 3.3.** The "Hermitian" condition in Theorem 3.1 and Theorem 3.2 is not necessary. In fact, we can always modify any $(1,1)$-form $\alpha$ to get a Hermitian $(1,1)$-form $\alpha^\dagger$.

### 3.2. Stationary solutions and cscK metrics

Let us now assume that $(M, \omega)$ is a closed Kähler manifold of complex dimension $n$. The *Kähler cone* $\mathcal{K}_M$ is the open convex cone consisting of all real cohomology classes in $H^{1,1}_{\mathbf{R}}(M)$ that can be represented by smooth closed positive $(1,1)$-forms. Namely,

$$\mathcal{K}_M := \{\Lambda \in H^{1,1}_{\mathbf{R}}(M) : \Lambda = [\eta] \text{ for some closed } (1,1)\text{-form } \eta > 0\}. \tag{3.3}$$

We also consider the set

$$\mathcal{K}'_M := \{\Lambda \in H^{1,1}_{\mathbf{R}}(M) : -2\pi c_1(M) + \Lambda \in \mathcal{K}_M\}. \tag{3.4}$$

**Corollary 3.4.** *Suppose that* $(M, \omega)$ *is a closed Kähler manifold with a closed* $(1,1)$*-form* $\alpha$ *satisfying*

(i) $[\alpha] \in \mathcal{K}'_M$, *and*

(ii) $\omega \in -2\pi c_1(M) + [\alpha]$.

*The the solution* $(\omega_t, \alpha_t)_{t\in[0,T)}$ *to (3.2) is closed and* $\omega_t \in [\omega] = -2\pi c_1(M) + [\alpha]$.

*Proof.* The closedness of solutions was proved in Theorem 3.2. Taking the cohomology class of (3.2) we have

$$\partial_t[\omega_t] = -2\pi c_1(M) - [\omega_t] + [\alpha_t] = -2\pi c_1(M) - [\omega_t] + [\alpha],$$

where we use Proposition 2.3 to deduce $[\alpha_t] = [\alpha]$. Therefore

$$[\omega_t] = -2\pi c_1(M) + [\alpha] + e^{-t}([\omega] + 2\pi c_1(M) - [\alpha]).$$

From assumption (ii), we have $[\omega_t] = -2\pi c_1(M) + [\alpha]$. □

**Theorem 3.5.** *If $(M,\omega)$ is a closed Kähler manifold with a closed Hermitian $(1,1)$-form $\alpha$, then the stationary solution of equation (3.2) is a cscK metric coupled with a harmonic $(1,1)$-form.*

*Proof.* Let $(\omega_\infty,\alpha_\infty)$ be the stable solution to (3.2). Then they satisfy

$$\rho(\omega_\infty) = -\omega_\infty + \alpha_\infty, \quad \overline{\Box}_{\omega_\infty}\alpha_\infty = 0.$$

Namely, $\alpha_\infty$ is a harmonic $(1,1)$-form. Hence, $\mathrm{tr}_{\omega_\infty}\alpha_\infty$ is constant by Proposition 2.3. Consequently,

$$R(\omega_\infty) = -n + \mathrm{tr}_{\omega_\infty}\alpha_\infty = \text{constant}.$$

Namely, $\omega_\infty$ is a cscK metric. □

### 3.3. Equivalent scalar equations

Let $(\omega_t,\alpha_t)_{t\in[0,T)}$ be the solution to the flow equation (3.2) given by Theorem 3.2. As $\overline{\Box}_{\omega_t}\alpha_t$ is $d$-exact by Proposition 2.3, one has $\partial_t[\alpha_t] = 0$, implying $\alpha_t \in [\alpha]$. By $\partial\bar\partial$-lemma, there exists a smooth function $f_t$ such that

$$\alpha_t = \alpha + \sqrt{-1}\partial\bar\partial f_t. \tag{3.5}$$

Plugging this into $\partial_t\alpha_t = \overline{\Box}_{\omega_t}\alpha_t$, making use of $\overline{\Box}_{\omega_t}\alpha_t = \partial\bar\partial\mathrm{tr}_{\omega_t}\alpha_t$ and getting rid of $\partial\bar\partial$, one obtains

$$\partial_t f_t = \Delta_{\omega_t} f_t + \mathrm{tr}_{\omega_t}\alpha,$$

with $f_t$ satisfying $f_0 = 0$ and the normalization $\int_M \partial_t f_t\omega_t^n = n[\alpha]\wedge[\omega]^{n-1}$.

Assume $\omega \in -2\pi c_1(M) + [\alpha]$. Then there exists a smooth volume form $\Omega$ on $M$ such that

$$\omega = \alpha + \sqrt{-1}\partial\bar\partial\log\Omega. \tag{3.6}$$

Since $\omega_t \in [\omega]$, we may write $\omega_t = \omega + \sqrt{-1}\partial\bar\partial\varphi_t$. Making use of (3.5), then the equation

$$\partial_t\omega_t = -\rho(\omega_t) - \omega_t + \alpha_t$$

is equivalent to the parabolic Monge-Ampere equation

$$\partial_t\varphi_t = \log\frac{(\omega + \sqrt{-1}\partial\bar\partial\varphi_t)^n}{\Omega} - \varphi_t + f_t$$

by the standard deduction as in the Kähler-Ricci flow. Proposition 1.2 is thus proved.

## 4. Higher order derivatives estimates and the long-time existence

In this section we first collect evolution equations for related geometric quantities and then derive the long time existence for the flow (3.2).

### 4.1. Evolution equations

To simplify notions, we always raise or lower the indices. For example,

$$R^{\bar{i}}{}_{\bar{j}k}{}^{\ell} := g^{\bar{i}p} g^{\bar{q}\ell} R_{p\bar{j}k\bar{q}}.$$

**Proposition 4.1.** *Along the flow (3.2), we have*

$$\begin{aligned}(4.1)\quad \partial_t R_{i\bar{j}k\bar{\ell}} = {} & \Delta_{g_t} R_{i\bar{j}k\bar{\ell}} + R_{i\bar{p}q\bar{\ell}} R^{\bar{p}}{}_{\bar{j}k}{}^{q} - R_{i\bar{p}k\bar{q}} R^{\bar{p}}{}_{\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} + R_{i\bar{j}p\bar{q}} R^{\bar{q}p}{}_{k\bar{\ell}} \\ & - \frac{1}{2}\left(R_{i\bar{p}} R^{\bar{p}}{}_{\bar{j}k\bar{\ell}} + R_{p\bar{j}} R_{i}{}^{p}{}_{k\bar{\ell}} + R_{k\bar{p}} R_{i\bar{j}}{}^{\bar{p}}{}_{\bar{\ell}} + R_{p\bar{\ell}} R_{i\bar{j}k}{}^{p}\right) - R_{i\bar{j}k\bar{\ell}} \\ & + R_{i\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{k\bar{q}} - \nabla_i \nabla_{\bar{j}} \alpha_{k\bar{\ell}}.\end{aligned}$$

*Proof.* Combining

$$\partial_t g^{i\bar{j}} = -g^{\bar{\ell}i} g^{\bar{j}k} \partial_t g_{k\bar{\ell}} = g^{\bar{\ell}i} g^{\bar{j}k} (R_{k\bar{\ell}} - \alpha_{k\bar{\ell}}) + g^{\bar{\ell}i} g^{\bar{j}k} g_{k\bar{\ell}},$$

and

$$\begin{aligned}\partial_t R_{i\bar{j}k\bar{\ell}} = {} & -\partial_i \partial_{\bar{j}} (\partial_t g_{k\bar{\ell}}) + \partial_t g^{\bar{q}p} \partial_i g_{k\bar{q}} \partial_{\bar{j}} g_{p\bar{\ell}} \\ & + g^{\bar{q}p} \partial_i (\partial_t g_{k\bar{q}}) \partial_{\bar{j}} g_{p\bar{\ell}} + g^{\bar{q}p} \partial_i g_{k\bar{q}} \partial_{\bar{j}} (\partial_t g_{p\bar{\ell}}),\end{aligned}$$

we obtain

$$\begin{aligned}\partial_t R_{i\bar{j}k\bar{\ell}} = {} & \partial_i \partial_{\bar{j}} (R_{k\bar{\ell}} - \alpha_{k\bar{\ell}} + g_{k\bar{\ell}}) + g^{\bar{s}p} g^{\bar{q}r} (R_{r\bar{s}} - \alpha_{r\bar{s}} + g_{r\bar{s}}) \Gamma^{a}_{ik} g_{a\bar{q}} \Gamma^{\bar{b}}_{\bar{j}\bar{\ell}} g_{\bar{b}p} \\ & - g^{\bar{q}p} \partial_i (R_{k\bar{q}} - \alpha_{k\bar{q}} + g_{k\bar{q}}) \partial_{\bar{j}} g_{p\bar{\ell}} - g^{\bar{q}p} \partial_{\bar{j}} (R_{p\bar{\ell}} - \alpha_{p\bar{\ell}} + g_{p\bar{\ell}}) \partial_i g_{k\bar{q}} \\ = {} & \partial_i \partial_{\bar{j}} (R_{k\bar{\ell}} - \alpha_{k\bar{\ell}}) + \Gamma^{p}_{ik} \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} (R_{p\bar{q}} - \alpha_{p\bar{q}}) \\ & - \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \partial_i (R_{k\bar{q}} - \alpha_{k\bar{q}}) - \Gamma^{p}_{ik} \partial_{\bar{j}} (R_{p\bar{\ell}} - \alpha_{p\bar{\ell}}) - R_{i\bar{j}k\bar{\ell}}.\end{aligned}$$

On the other hand, by the commutator formulas, we obtain

$$\begin{aligned}\nabla_i\nabla_{\bar j}\beta_{k\bar\ell} &= \partial_i\nabla_{\bar j}\beta_{k\bar\ell} - \Gamma^p_{ik}\nabla_{\bar j}\beta_{p\bar\ell}\\ &= \partial_i\left(\partial_{\bar j}\beta_{k\bar\ell} - \Gamma^{\bar q}_{\bar j\bar\ell}\beta_{k\bar q}\right) - \Gamma^p_{ik}\nabla_{\bar j}\beta_{p\bar\ell}\\ &= \partial_i\partial_{\bar j}\beta_{k\bar\ell} - \partial_i\Gamma^{\bar q}_{\bar j\bar\ell}\cdot\beta_{k\bar q} - \Gamma^{\bar q}_{\bar j\bar\ell}\partial_i\beta_{k\bar q} - \Gamma^p_{ik}\left(\partial_{\bar j}\beta_{p\bar\ell} - \Gamma^{\bar q}_{\bar j\bar\ell}\beta_{p\bar q}\right)\\ &= \partial_i\partial_{\bar j}\beta_{k\bar\ell} + \Gamma^p_{ik}\Gamma^{\bar q}_{\bar j\bar\ell}\beta_{p\bar q} - \Gamma^p_{ik}\partial_{\bar j}\beta_{p\bar\ell} - \Gamma^{\bar q}_{\bar j\bar\ell}\partial_i\beta_{k\bar q} + R_{i\bar j}{}^{\bar q}{}_{\bar\ell}\beta_{k\bar q}\end{aligned}$$

for any $(1,1)$-form $\beta_{k\bar\ell}$. Letting $\beta_{k\bar\ell} = R_{k\bar\ell} - \alpha_{k\bar\ell}$, it follows

$$\begin{aligned}\partial_t R_{i\bar j k\bar\ell} &= \nabla_i\nabla_{\bar j}(R_{k\bar\ell} - \alpha_{k\bar\ell}) - R_{i\bar j}{}^{\bar q}{}_{\bar\ell}(R_{k\bar q} - \alpha_{k\bar q}) - R_{i\bar j k\bar\ell}\\ &= \nabla_i\nabla_{\bar j}R_{k\bar\ell} - R_{i\bar j}{}^{\bar q}{}_{\bar\ell}R_{k\bar q} - R_{i\bar j k\bar\ell} - \nabla_i\nabla_{\bar j}\alpha_{k\bar\ell} + R_{i\bar j}{}^{\bar q}{}_{\bar\ell}\alpha_{k\bar q}.\end{aligned}$$

Using the formula in [3, 33],

$$\begin{aligned}\Delta_g R_{i\bar j k\bar\ell} &= \nabla_i\nabla_{\bar j}R_{k\bar\ell} - R_{i\bar j p\bar q}R^{\bar q p}{}_{k\bar\ell} + R_{i\bar p k\bar q}R^{\bar p}{}_{\bar j}{}^{\bar q}{}_{\bar\ell} - R_{i\bar p q\bar\ell}R^{\bar p}{}_{\bar j k}{}^{q} R_{k\bar q}R_{i\bar j}{}^{\bar q}{}_{\bar\ell}\\ &\quad + \frac{1}{2}\left(R_{i\bar p}R^{\bar p}{}_{\bar j k\bar\ell} + R_{p\bar j}R_i{}^p{}_{k\bar\ell} + R_{k\bar p}R_{i\bar j}{}^{\bar p}{}_{\bar\ell} + R_{p\bar\ell}R_{i\bar j k}{}^{p}\right).\end{aligned}$$

The proposition is thus proved. □

**Corollary 4.2.** *Along the flow (3.2), we have*

$$\partial_t R_{i\bar j} = \Delta_{g_t}R_{i\bar j} + R_{i\bar j k\bar\ell}R^{\bar\ell k} - R_{i\bar k}R^{\bar k}{}_{\bar j} - \nabla_i\nabla_{\bar j}\mathrm{tr}_{g_t}\alpha_t. \tag{4.2}$$

*Proof.* Taking the trace of (4.1), we have

$$\begin{aligned}\partial_t R_{i\bar j} &= R_{i\bar j k\bar\ell}\partial_t g^{\bar\ell k} + g^{\bar\ell k}\partial_t R_{i\bar j k\bar\ell}\\ &= R_{i\bar j k\bar\ell}g^{\bar q k}g^{\bar\ell p}(R_{p\bar q} - \alpha_{p\bar q} + g_{p\bar q}) + \Delta_{g_t}R_{i\bar j} + R_{i\bar p q\bar\ell}R^{\bar p}{}_{\bar j}{}^{\bar\ell q} - R_{i\bar p k\bar q}R^{\bar p}{}_{\bar j}{}^{\bar q k}\\ &\quad + R_{i\bar j p\bar q}R^{\bar q p} - \frac{1}{2}\left(R_{i\bar p}R^{\bar p}{}_{\bar j} + R_{p\bar j}R_i{}^p + R_{k\bar p}R_{i\bar j}{}^{\bar p k} + R_{p\bar\ell}R_{i\bar j}{}^{\bar\ell p}\right) - R_{i\bar j}\\ &\quad + R_{i\bar j}{}^{\bar q k}\alpha_{k\bar q} - \nabla_i\nabla_{\bar j}\mathrm{tr}_{g_t}\alpha_t\\ &= \Delta_{g_t}R_{i\bar j} + R_{i\bar j k\bar\ell}R^{\bar\ell k} - R_{i\bar k}R^{\bar k}{}_{\bar j} - \nabla_i\nabla_{\bar j}\mathrm{tr}_{g_t}\alpha_t.\end{aligned}$$

□

**Corollary 4.3.** *Along the flow (3.2), we have*

$$\partial_t R_{\omega_t} = \Delta_{g_t}R_{\omega_t} + |\rho(\omega_t)|^2_{\omega_t} + R_{\omega_t} - \Delta_{\omega_t}\mathrm{tr}_{\omega_t}\alpha_t - \langle\rho(\omega_t), \alpha_t\rangle_{\omega_t}. \tag{4.3}$$

*Proof.* Taking the trace of (4.2), we have

$$\begin{aligned}
\partial_t R_{g_t} &= -R_{i\bar{j}} g^{\bar{\ell} i} g^{\bar{j} k} \partial_t g_{k\bar{\ell}} \\
&\quad + g^{\bar{j} i} \left( \Delta_{g_t} R_{i\bar{j}} + R_{i\bar{j} k\bar{\ell}} R^{\bar{\ell} k} - R_{i\bar{k}} R^{\bar{k}}{}_{\bar{j}} - \nabla_i \nabla_{\bar{j}} \mathrm{tr}_{g_t} \alpha_t \right) \\
&= -R^{\bar{\ell} k} (-R_{k\bar{\ell}} - g_{k\bar{\ell}} + \alpha_{k\bar{\ell}}) \\
&\quad + \Delta_{g_t} R_{g_t} + R_{k\bar{\ell}} R^{\bar{\ell} k} - R_{i\bar{k}} R^{\bar{k} i} - \Delta_{g_t} \mathrm{tr}_{g_t} \alpha_t \\
&= R_{k\bar{\ell}} R^{\bar{\ell} k} + R_{g_t} - \alpha_{k\bar{\ell}} R^{\bar{\ell} k} + \Delta_{g_t} R_{g_t} - \Delta_{g_t} \mathrm{tr}_{g_t} \alpha_t \\
&= \Delta_{g_t} R_{\omega_t} + |\rho(\omega_t)|^2_{\omega_t} + R_{\omega_t} - \Delta_{\omega_t} \mathrm{tr}_{\omega_t} \alpha_t - \langle \rho(\omega_t), \alpha_t \rangle_{\omega_t}.
\end{aligned}$$

□

**Proposition 4.4.** *Along the flow (3.2), we have*

$$\partial_t \mathrm{tr}_{\omega_t} \alpha_t = \Delta_{\omega_t} \mathrm{tr}_{\omega_t} \alpha_t - |\alpha_t|^2_{\omega_t} + \langle \rho(\omega_t), \alpha_t \rangle_{\omega_t} + \mathrm{tr}_{\omega_t} \alpha_t. \tag{4.4}$$

*Proof.* Applying Propsition 2.3 and using the flow, we have

$$\begin{aligned}
\partial_t \mathrm{tr}_{\omega_t} \alpha_t &= -g^{\bar{\ell} i} g^{\bar{j} k} \partial_t g_{k\bar{\ell}} \cdot \alpha_{i\bar{j}} + g^{\bar{j} i} \partial_t \alpha_{i\bar{j}} \\
&= -g^{\bar{\ell} i} g^{\bar{j} k} (-R_{k\bar{\ell}} - g_{k\bar{\ell}} + \alpha_{k\bar{\ell}}) \alpha_{i\bar{j}} + \ g^{\bar{j} i} \partial_i \partial_{\bar{j}} \mathrm{tr}_{\omega_t} \alpha_t \\
&= \langle \rho(\omega_t), \alpha_t \rangle_{\omega_t} + \mathrm{tr}_{\omega_t} \alpha_t - |\alpha_t|^2_{\omega_t} + \Delta_{\omega_t} \mathrm{tr}_{\omega_t} \alpha_t.
\end{aligned}$$

□

**Proposition 4.5.** *Along the flow (3.2), we have*

$$\begin{aligned}
\partial_t |\alpha_t|^2_{\omega_t} &= \Delta_{\omega_t} |\alpha_t|^2_{\omega_t} - 2 |\nabla_{\omega_t} \alpha_t|^2_{\omega_t} + 2 |\alpha_t|^2_{\omega_t} \\
&\quad + 2 R_{i\bar{j} k\bar{\ell}} \alpha^{\bar{\ell} k} \alpha^{\bar{j} i} - 2 \alpha_{i\bar{j}} \alpha^{\bar{j}}{}_{\bar{\ell}} \alpha^{\bar{\ell} i}.
\end{aligned} \tag{4.5}$$

*Proof.* Applying Lemma 2.2, we obtain

$$\begin{aligned}
\partial_t |\alpha_t|^2_{\omega_t} &= 2 \partial_t g^{\bar{\ell} i} \cdot g^{\bar{j} k} \alpha_{i\bar{j}} \alpha_{k\bar{\ell}} + 2 g^{\bar{\ell} i} g^{\bar{j} k} \alpha_{k\bar{\ell}} \partial_t \alpha_{i\bar{j}} \\
&= 2 (R^{\bar{\ell} i} - \alpha^{\bar{\ell} i} + g^{\bar{\ell} i}) g^{\bar{j} k} \alpha_{i\bar{j}} \alpha_{k\bar{\ell}} \\
&\quad + 2 \alpha^{\bar{j} i} \left( \Delta_{\omega_t} \alpha_{i\bar{j}} + R_{k\bar{j} i\bar{\ell}} \alpha^{\bar{\ell} k} - \frac{1}{2} g^{\bar{\ell} k} (3 R_{k\bar{j}} \alpha_{i\bar{\ell}} + R_{i\bar{\ell}} \alpha_{k\bar{j}}) \right) \\
&= 2 \alpha^{\bar{j} i} \Delta_{\omega_t} \alpha_{i\bar{j}} + 2 |\alpha_t|^2_{\omega_t} - 2 \alpha^{\bar{\ell} i} \alpha_{i\bar{j}} \alpha^{\bar{j}}{}_{\bar{\ell}} \\
&\quad + 2 R_{k\bar{j} i\bar{\ell}} \alpha^{\bar{\ell} k} \alpha^{\bar{j} i} - 2 \alpha_{i\bar{j}} \alpha^{\bar{j}}{}_{\bar{\ell}} R^{\bar{\ell} i}.
\end{aligned}$$

Combining

$$\begin{aligned}\Delta_{\omega_t}\alpha_{i\bar{j}} &= \frac{1}{2}g^{\bar{\ell}k}\left(\nabla_k\nabla_{\bar{\ell}}\alpha_{i\bar{j}} + \nabla_{\bar{\ell}}\nabla_k\alpha_{i\bar{j}}\right)\\ &= \frac{1}{2}g^{k\bar{\ell}}\left(2\nabla_k\nabla_{\bar{\ell}}\alpha_{i\bar{j}} + R_{k\bar{\ell}i}{}^{p}\alpha_{p\bar{j}} + R_{k\bar{\ell}}{}^{\bar{q}}{}_{\bar{j}}\alpha_{i\bar{q}}\right)\\ &= g^{\bar{\ell}k}\nabla_k\nabla_{\bar{\ell}}\alpha_{i\bar{j}} + \frac{1}{2}R_{i\bar{p}}\alpha^{\bar{p}}{}_{\bar{j}} + \frac{1}{2}R_{q\bar{j}}\alpha_i{}^{q},\end{aligned}$$

where the second equality uses commutator formula, and

$$\Delta_{\omega_t}|\alpha_t|^2_{\omega_t} = 2\alpha^{\bar{j}i}g^{\bar{\ell}k}\nabla_k\nabla_{\bar{\ell}}\alpha_{i\bar{j}} + 2|\nabla_{\omega_t}\alpha_t|^2_{\omega_t},$$

the proposition is proved. □

### 4.2. Long-time existence with bounded curvatures in the general situation

For any two tensor fields $A$ and $B$ we denote by $A * B$ any linear combination of tensor products of tensors $A$ and $B$ formed by contractions on $A_{i_1\cdots i_k}$ and $B_{j_1\cdots j_l}$ using the metric $g$. For example,

$$R_{i\bar{p}q\bar{\ell}}R^{\bar{p}}{}_{\bar{j}k}{}^{q} = g^{-1} * g^{-1} * \mathrm{Rm}_g * \mathrm{Rm}_g.$$

Using the $*$-notion and Lemma 2.2, we have

$$\partial_t g_t = -\mathrm{Rc}_{g_t} - g_t + \alpha_t$$

and

$$\partial_t\alpha_t = \Delta_{\omega_t}\alpha_t + g_t^{-1} * g_t^{-1} * \mathrm{Rm}_{g_t} * \alpha_t - g_t^{-1} * \mathrm{Rc}_{g_t} * \alpha_t.$$

Furthermore, from Proposition 4.1 we arrive at

$$\begin{aligned}\partial_t\mathrm{Rm}_{g_t} &= \Delta_{g_t}\mathrm{Rm}_{g_t} - \mathrm{Rm}_{g_t} - \nabla^2_{g_t}\alpha_t\\ &\quad + g_t^{-1} * \left(g_t^{-1} * \mathrm{Rm}_{g_t} * \mathrm{Rm}_{g_t} + \mathrm{Rm}_{g_t} * \alpha_t\right).\end{aligned} \tag{4.6}$$

For any $t$-dependent tensor field $A(t)$, we have

$$\partial_t\nabla_{g_t}A(t) = \nabla_{g_t}\partial_t A(t) + (\partial_t\Gamma_{g_t}) * A(t). \tag{4.7}$$

On the other hand, the evolution equation of $\Gamma_{g_t}$ is given by

$$\partial_t\Gamma^k_{ij} = -g^{\bar{\ell}k}\nabla_i(R_{j\bar{\ell}} - \alpha_{j\bar{\ell}}), \tag{4.8}$$

from which we obtain

$$\partial_t \Gamma_{g_t} = g_t^{-1} * \nabla_{g_t}(\mathrm{Rc}_{g_t} - \alpha_t), \tag{4.9}$$

and thus

$$\partial_t \nabla_{g_t} A(t) = \nabla_{g_t} \partial_t A(t) + g_t^{-1} * \nabla_{g_t}(\mathrm{Rc}_{g_t} - \alpha_t) * A(t). \tag{4.10}$$

Another useful commutator formula is (see [3])

$$\begin{aligned} \Delta_g \nabla_g A = \nabla_g \Delta_g A &+ g^{-1} * g^{-1} * \mathrm{Rm}_g * \nabla_g A \\ &+ g^{-1} * g^{-1} * \nabla_g \mathrm{Rm}_g * A. \end{aligned} \tag{4.11}$$

We introduce a tensor field $S_{g(t)}$ defined by

$$S_{i\bar{j}k\bar{\ell}} := R_{i\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{k\bar{q}} - \nabla_i \nabla_{\bar{j}} \alpha_{k\bar{\ell}}. \tag{4.12}$$

This tensor field shares the same symmetric properties of $\mathrm{Rm}_{g(t)}$.

**Proposition 4.6.** *We have*

$$S_{i\bar{j}k\bar{\ell}} = S_{k\bar{j}i\bar{\ell}} = S_{i\bar{\ell}k\bar{j}} = S_{k\bar{\ell}i\bar{j}}, \quad \overline{S_{i\bar{j}k\bar{\ell}}} = S_{j\bar{i}\ell\bar{k}}. \tag{4.13}$$

*Proof.* By the symmetry of $R_{i\bar{j}k\bar{\ell}}$ and the identity $\nabla_{\bar{j}} \alpha_{k\bar{\ell}} = \partial_{\bar{j}} \alpha_{k\bar{\ell}} - \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \alpha_{k\bar{q}}$, we immediately get $S_{i\bar{j}k\bar{\ell}} = S_{i\bar{\ell}k\bar{j}}$. To prove $S_{i\bar{j}k\bar{\ell}} = S_{k\bar{j}i\bar{\ell}}$, we first compute

$$\begin{aligned} \nabla_k \nabla_{\bar{j}} \alpha_{i\bar{\ell}} - \nabla_i \nabla_{\bar{j}} \alpha_{k\bar{\ell}} &= \partial_k \nabla_{\bar{j}} \alpha_{i\bar{\ell}} - \Gamma^p_{ki} \nabla_{\bar{j}} \alpha_{p\bar{\ell}} - \partial_i \nabla_{\bar{j}} \alpha_{k\bar{\ell}} + \Gamma^p_{ik} \nabla_{\bar{j}} \alpha_{p\bar{\ell}} \\ &= \partial_k \left( \partial_{\bar{j}} \alpha_{i\bar{\ell}} - \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \alpha_{i\bar{q}} \right) - \partial_i \left( \partial_{\bar{j}} \alpha_{k\bar{\ell}} - \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \alpha_{k\bar{q}} \right) \\ &= \partial_i \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \cdot \alpha_{k\bar{q}} - \partial_k \Gamma^{\bar{q}}_{\bar{j}\bar{\ell}} \cdot \alpha_{i\bar{q}} \\ &= -R_{i\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{k\bar{q}} + R_{k\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{i\bar{q}}. \end{aligned}$$

Therefore, one obtains

$$R_{k\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{i\bar{q}} - \nabla_k \nabla_{\bar{j}} \alpha_{i\bar{\ell}} = R_{i\bar{j}}{}^{\bar{q}}{}_{\bar{\ell}} \alpha_{k\bar{q}} - \nabla_i \nabla_{\bar{j}} \alpha_{k\bar{\ell}},$$

i.e. $S_{k\bar{j}i\bar{\ell}} = S_{i\bar{j}k\bar{\ell}}$. The last symmetric identity follows from the above two identities. □

Consequently, the evolution (4.1) or (4.6) can be rewritten as the following nice form:

$$\partial_t \mathrm{Rm}_{g_t} = \Delta_{g_t} \mathrm{Rm}_{g_t} - \mathrm{Rm}_{g_t} + g_t^{-1} * g_t^{-1} * \mathrm{Rm}_{g_t} * \mathrm{Rm}_{g_t} + S_{g_t}. \tag{4.14}$$

In general, we have

**Proposition 4.7.** *Along the flow (3.2), for any nonnegative integer $k$, it follows*

$$\begin{aligned} \partial_t \nabla^k_{g_t} \mathrm{Rm}_{g_t} &= \Delta_{g_t} \nabla^k_{g_t} \mathrm{Rm}_{g_t} - \nabla^k_{g_t} \mathrm{Rm}_{g_t} \\ &\quad + \sum_{i+j=k} \nabla^i_{g_t} \mathrm{Rm}_{g_t} * \nabla^j_{g_t} \alpha_t * g_t^{-1} \\ &\quad + \sum_{i+j=k} \nabla^i_{g_t} \mathrm{Rm}_{g_t} * \nabla^j_{g_t} \mathrm{Rm}_{g_t} * g_t^{-1} * g_t^{-1} - \nabla^{k+2}_{g_t} \alpha_t. \end{aligned} \tag{4.15}$$

*Proof.* The case $k = 0$ was proved in (4.14). For convenience, we omit the subscripts $g_t$ and $t$ in the proof of the case $k > 0$. By (4.10), (4.11) and the inductive hypothesis, we have

$$\begin{aligned} \partial_t \nabla^{k+1} \mathrm{Rm} &= \nabla \partial_t \nabla^k \mathrm{Rm} + g^{-1} * \nabla^k \mathrm{Rm} * \nabla (g^{-1} * \mathrm{Rm} - \alpha) \\ &= \nabla \Bigg( \Delta \nabla^k \mathrm{Rm} - \nabla^k \mathrm{Rm} + \sum_{i+j=k} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * g^{-1} * g^{-1} \\ &\quad + \sum_{i+j=k} \nabla^i \mathrm{Rm} * \nabla^j \tilde{\alpha} * g^{-1} - \nabla^{k+2} \alpha \Bigg) \\ &\quad + g^{-1} * g^{-1} * \nabla \mathrm{Rm} * \nabla^k \mathrm{Rm} + g^{-1} * \nabla^k \mathrm{Rm} * \nabla \alpha \\ &= \Delta \nabla^{k+1} \mathrm{Rm} + g^{-1} * g^{-1} * \left( \mathrm{Rm} * \nabla^{k+1} \mathrm{Rm} + \nabla \mathrm{Rm} * \nabla^k \mathrm{Rm} \right) \\ &\quad - \nabla^{k+1} \mathrm{Rm} + \sum_{i+j=k+1} \nabla^i \mathrm{Rm} * \nabla^j \mathrm{Rm} * g^{-1} * g^{-1} \\ &\quad + \sum_{i+j=k+1} \nabla^i \mathrm{Rm} * \nabla^j \alpha * g^{-1} - \nabla^{k+3} \alpha + g^{-1} * \nabla^k \mathrm{Rm} * \nabla \alpha, \end{aligned}$$

yielding (4.15) for $k + 1$. □

**Proposition 4.8.** *Along the flow (3.2), for any nonnegative integer $k$, we have*

$$\partial_t \nabla^k_{g_t}\alpha_t = \Delta_{g_t}\nabla^k_{g_t}\alpha_t + \sum_{i+j=k}\nabla^i_{g_t}\mathrm{Rm}_{g_t} * \nabla^j_{g_t}\alpha_t * g_t^{-1} * g_t^{-1} + \sum_{i+j=k}\nabla^i_{g_t}\alpha_t * \nabla^j_{g_t}\alpha_t * g_t^{-1}. \tag{4.16}$$

*Proof.* Using again (4.10) and (4.11), we can prove (4.16) by induction on $k$. □

**Corollary 4.9.** *Along the flow (3.2), for any nonnegative integer $k$, we have*

$$\begin{aligned}\partial_t|\nabla^k_{g_t}\mathrm{Rm}_{g_t}|^2_{g_t} \leq{}& \Delta_{g_t}|\nabla^k_{g_t}\mathrm{Rm}_{g_t}|^2_{g_t} - 2|\nabla^{k+1}_{g_t}\mathrm{Rm}_{g_t}|^2_{g_t} + C_k|\nabla^k_{g_t}\mathrm{Rm}_{g_t}|^2_{g_t}\\ &+ \sum_{i+j=k} C_k|\nabla^i_{g_t}\mathrm{Rm}_{g_t}|_{g_t}|\nabla^j_{g_t}\mathrm{Rm}_{g_t}|_{g_t}|\nabla^k_{g_t}\mathrm{Rm}_{g_t}|_{g_t}\\ &+ \sum_{i+j=k} C_k|\nabla^i_{g_t}\mathrm{Rm}_{g_t}|_{g_t}|\nabla^j_{g_t}\alpha_t|_{g_t}|\nabla^k_{g_t}\mathrm{Rm}_{g_t}|_{g_t}\\ &+ |\nabla^{k+2}_{g_t}\alpha_t|^2_{g_t},\end{aligned} \tag{4.17}$$

$$\begin{aligned}\partial_t|\nabla^k_{g_t}\alpha_t|^2_{g_t} \leq{}& \Delta_{g_t}|\nabla^k_{g_t}\alpha_t|^2_{g_t} - 2|\nabla^{k+1}\alpha_t|^2_{g_t} + C_k|\nabla^k_{g_t}\alpha_t|^2_{g_t}\\ &+ \sum_{i+j=k} C_k|\nabla^i_{g_t}\mathrm{Rm}_{g_t}|_{g_t}|\nabla^j_{g_t}\alpha_t|_{g_t}|\nabla^k_{g_t}\alpha_t|_{g_t}\\ &+ \sum_{i+j=k} C_k|\nabla^i_{g_t}\alpha_t|_{g_t}|\nabla^j_{g_t}\alpha_t|_{g_t}|\nabla^k_{g_t}\alpha_t|_{g_t},\end{aligned} \tag{4.18}$$

*where $C_k$ is a positive constant depending only on $k$ and $n$.*

We now prove the higher order derivatives estimates for the flow (3.2), from which we can prove the long-time existence as long as the Riemann curvature tensor $\mathrm{Rm}_{g_t}$ and the $(1,1)$-form $\alpha_t$ are bounded.

**Proposition 4.10.** *Suppose that $(\omega_t, \alpha_t)_t$ is the solution to (3.2) on a closed Kähler manifold $(M, g)$ with a closed Hermitian $(1,1)$-form $\alpha$. Let $K$ be an arbitrary given positive constant. Then for each $a > 0$ and each integer $m \geq 1$ there exists a positive constant $\bar{C}_m$ depending only on $m, n, \max\{a, 1\}$,*

*$K$ such that if*

$$|\mathrm{Rm}_{g_t}|_{g_t} \leq K, \quad |\alpha_t|_{g_t} \leq K$$

*on $M$ for all $t \in [0, a/K]$, then*

$$\left|\nabla^{m-1}_{g_t}\mathrm{Rm}_{g_t}\right|_{g_t} + \left|\nabla^m_{g_t}\alpha_t\right|_{g_t} \leq \frac{\bar{C}_m}{t^{m/2}} \tag{4.19}$$

*on $M$ for all $t \in (0, a/K]$.*

*Proof.* The proof is standard as in the Ricci flow and we follow the idea used in [25] (see also [2, 3]). To simplify notions, we always omit the subscripts $g_t$ and $t$, and let $C, C', C_0, C_1, \ldots$ be positive constants depending only on $n, m, \max\{a, 1\}$, which may take different values at different places.

The basic idea to prove (4.19) is to find a suitable quantity $u$ so that along the flow (3.2) it is bounded from above. To motivate such an idea, we first consider the case $m = 1$. From Corollary 4.9, we have

$$\begin{aligned}
\partial_t|\mathrm{Rm}|^2 &\leq \Delta|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 + C_0K^2 + 2C_0K^3 + |\nabla^2\alpha|^2,\\
\partial_t|\nabla\mathrm{Rm}|^2 &\leq \Delta|\nabla\mathrm{Rm}|^2 - 2|\nabla^2\mathrm{Rm}|^2 + C_1|\nabla\mathrm{Rm}|^2\\
&\quad + 3C_1K|\nabla\mathrm{Rm}|^2 + C_1K|\nabla\mathrm{Rm}||\nabla\alpha| + |\nabla^3\alpha|^2,\\
\partial_t|\alpha|^2 &\leq \Delta|\alpha|^2 - 2|\nabla\alpha|^2 + C_0K^2 + 2C_0K^3,\\
\partial_t|\nabla\alpha|^2 &\leq \Delta|\nabla\alpha|^2 - 2|\nabla^2\alpha|^2 + C_1|\nabla\alpha|^2 + C_1K|\nabla\alpha|^2\\
&\quad + C_1K|\nabla\mathrm{Rm}||\nabla\alpha| + 2C_1K|\nabla\alpha|^2.
\end{aligned}$$

In order to control the bad terms $|\nabla^2\alpha|$ in the evolution equation of $|\mathrm{Rm}|^2$, we need the evolution equation of $|\nabla\alpha|^2$; similarly, to control the bad terms $|\nabla\alpha|, |\nabla\mathrm{Rm}|$ in the evolution equation of $|\nabla\alpha|^2$, we also need the evolution equation of $|\alpha|^2$. Therefore, we may consider the quantity

$$u := t(|\mathrm{Rm}|^2 + |\nabla\alpha|^2) + A|\alpha|^2. \tag{4.20}$$

Applying the above evolution inequalities in the evolution of $u$:

$$\begin{aligned}
\partial_t u &\leq \Delta u - 2t|\nabla\mathrm{Rm}|^2 + K^2 + C_0Ka + 2C_0K^2a + (C_0K^2 + 2C_0K^3)A\\
&\quad + (1 + C_1t + 3C_1Kt - 2A)|\nabla\alpha|^2 + C_1K\left(\sqrt{t}|\nabla\mathrm{Rm}|\right)\left(\sqrt{t}|\nabla\alpha|\right)\\
&\leq \Delta u + \left(1 + 3C_1a + \frac{C_1a}{K} + \frac{C_1^2a}{4}K - 2A\right)|\nabla\alpha|^2 + C_0aK\\
&\quad + (1 + 2C_0a)K^2 + K^2(C_0 + 2C_0K)A.
\end{aligned}$$

Taking $2A = 1 + 3C_1 a + \frac{C_1 a}{K} + \frac{C_1^2 a}{4} K$, we then obtain

$$\begin{aligned}\partial_t u &\le \Delta u + C_0 a K + (1 + 2C_0 a) K^2 \\ &\quad + \frac{C_0 K^2}{2}(1+2K)\left(1 + 3C_1 a + \frac{C_1 a}{K} + \frac{C_1^2 a}{4} K\right) \\ &\le \Delta u + C(K + K^2 + K^3 + K^4)\end{aligned}$$

for some positive constant $C = C(n, \max\{1, a\})$. Using $u(0) \le AK^2$ and the maximum principle, we arrive at

$$\begin{aligned}t(|\mathrm{Rm}|^2 + |\nabla\alpha|^2) &\le \frac{1}{2}\left(1 + 3C_1 a + \frac{C_1 a}{K} + \frac{C_1^2}{4} K\right) K^2 \\ &\quad + C(K^4 + K^3 + K^2 + K)t \\ &\le C'(K^3 + K^2 + K + 1)\end{aligned}$$

for some positive constant $C' = C'(n, \max\{1, a\})$. Consequently,

$$|\mathrm{Rm}| + |\nabla\alpha| \le \frac{\sqrt{2C'(K^3 + K^2 + K + 1)}}{t^{1/2}}.$$

In general, we consider the quantity

$$\begin{aligned}u &:= t^m(|\nabla^{m-1}\mathrm{Rm}|^2 + |\nabla^m \alpha|^2) \\ &\quad + \sum_{0 \le i \le m-1} A_i t^i (|\nabla^{i-1}\mathrm{Rm}|^2 + |\nabla^i \alpha|^2),\end{aligned} \tag{4.21}$$

where $m \ge 1$ and $|\nabla^{0-1}\mathrm{Rm}|^2 := 0$. Letting

$$\Phi_m := |\nabla^{m-1}\mathrm{Rm}|^2 + |\nabla^m \alpha|^2,$$

we have

$$\begin{aligned}\partial_t \Phi_m &\le \Delta \Phi_m - \Phi_{m+1} + \sum_{i+j=m} C_m' \Phi_{i+1}^{1/2} \Phi_j^{1/2} \Phi_m^{1/2} \\ &\quad + \sum_{i+j=m} C_m' \Phi_{i+1}^{1/2} \Phi_j^{1/2} \Phi_{m-1}^{1/2} + \sum_{i+j=m} C_m' \Phi_i^{1/2} \Phi_j^{1/2} \Phi_m^{1/2},\end{aligned}$$

where $C'_m$ are constants depending only on $C_m$ and $m$. In particular

$$\begin{aligned}
\partial_t\Phi_0 &\leq \Delta\Phi_0 - \Phi_1 + C'_0K^2\Phi_1^{1/2} + C'_0K^3 \\
&\leq \Delta\Phi_0 - \frac{1}{2}\Phi_1 + \frac{(C'_0K^2)^2}{2} + C'_0K^3, \\
\partial_t\Phi_1 &\leq \Delta\Phi_1 - \Phi_2 + 4C'_1K\Phi_1 + C'_1K\Phi_1^{1/2}\Phi_2^{1/2} + C'_1K^2\Phi_2^{1/2} \\
&\leq \Delta\Phi_1 - \frac{1}{2}\Phi_2 + (C'_1K)^2\Phi_1 + (C'_1K^2)^2, \\
\partial_t\Phi_2 &\leq \Delta\Phi_2 - \Phi_3 + 2C'_3\Phi_1^{1/2}\Phi_2 + 2KC'_3\Phi_2^{1/2}\Phi_3^{1/2} + C'_2K\Phi_1^{1/2}\Phi_3^{1/2} \\
&\quad + 3C'_2\Phi_1\Phi_2^{1/2} + 2KC'_2\Phi_2 \\
&\leq \Delta\Phi_2 - \frac{1}{2}\Phi_3 + [(2KC'_3)^2 + 2KC'_2]\Phi_2 + 2C'_3\Phi_1^{1/2}\Phi_2 \\
&\quad + 3C'_2\Phi_1\Phi_2^{1/2} + (C'_2K)^2\Phi_1.
\end{aligned}$$

Consequently, in the case $m = 2$,

$$\begin{aligned}
\partial_t u \leq \Delta u &+ \left[\frac{a(C'_2\overline{C}_1^2)^2}{4} + \frac{(C'_0K^2)^2A_0}{2} + C'_0K^3A_0 + \frac{A_1a(C'_1K^2)^2}{K}\right] \\
&+ \left[\frac{(C'_2K)^2a^2}{K^2} - \frac{A_0}{2} + \frac{A_1a(C'_1K)^2}{K} + A_1\right]\Phi_1 \\
&+ \left[\frac{4(C'_3a)^2}{K} + \frac{2C'_2a^2}{K} + \frac{2C'_3\overline{C}_1a^{3/2}}{K^{3/2}} + 1 + \frac{2a}{K} - \frac{a}{2K}A_1\right]\Phi_2.
\end{aligned}$$

Choosing sufficiently large $A_0$ and $A_1$, we can make the last two terms on the right-hand side of above inequality non-positive, and therefore, by the maximum principle, $u$ is bounded from above by some positive constant depending only on $n, \max\{1, a\}, K$. Thus we prove the estimate (4.19) for $m = 2$.

Using the same argument as in [2, 3, 25, 33], we can prove the full estimate (4.19). □

**Lemma 4.11.** *Suppose that $(\omega_t, \alpha_t)_{t\in[0,T)}$ for $T < \infty$ is the solution to the flow (3.2). If*

$$|\rho(\omega_t)|_{\omega_t} \leq K, \quad |\alpha_t|_{\omega_t} \leq K$$

*on $M$ for all $t \in [0, T)$, then*

$$e^{-C}\omega \leq \omega_t \leq e^C\omega$$

*on $M$ for all $t \in [0,T)$, where $C = T + 2Te^T K$. Moreover, the Kähler forms $\omega_t$ converge uniformly as $t \to T$ to a continuous fundamental form $\omega_T$ with $e^{-C}\omega \le \omega_T \le e^C \omega$.*

*Proof.* If $(\omega_t, \alpha_t)_t$ is the solution to the flow (3.2), then we can define a new family of Kähler metrics by setting

$$\tilde{g}_t := e^t g_t. \tag{4.22}$$

Hence $\tilde{g}_0 = g_0 = g$ and

$$\partial_t \tilde{g}_{i\bar{j}} = e^t(-R_{i\bar{j}} + \alpha_{i\bar{j}}).$$

By Hamilton's result [20] (or see [2]), it suffices to show

$$\int_0^T |\partial_t \tilde{g}_t|_{\tilde{g}_t}\, dt \le 2Te^T K. \tag{4.23}$$

Indeed, if (4.23) would be true, then Hamilton's result implies

$$e^{-2Te^T K}\omega \le \tilde{\omega}_t \le e^{2Te^T K}\omega.$$

Using $e^t\omega_t = \tilde{\omega}_t$, we immediately get the desired result. To prove (4.23), we integrate the inequality

$$|\partial_t \tilde{g}_t|_{\tilde{g}_t} \le e^t \left(|\rho(\omega_t)|_{\omega_t} + |\alpha_t|_{\omega_t}\right) \le 2Ke^T$$

from 0 to $T$ and then obtain the estimate. □

**Theorem 4.12.** *Let $[0,T)$ be the maximal time interval with $T < \infty$ such that $(\omega_t, \alpha_t)_{t\in[0,T)}$ is the solution to the flow (3.2). Then*

$$\limsup_{t\to T} \max_M \{|\mathrm{Rm}_{g_t}|_{g_t}, |\alpha_t|_{g_t}\} = \infty.$$

*Proof.* Otherwise, we can assume that $\max\{|\mathrm{Rm}_{g_t}|_{g_t}, |\alpha_t|_{g_t}\} \le K$ on $M$ for all $t \in [0,T)$, where $K$ is a positive constant. From Proposition 4.10 and Lemma 4.11, $\omega_t \to \omega_T$ as $t \to T$ in any $C^k$-norm. Hence $\omega_T$ is a smooth Kähler form. Similarly, it is not hard to see that $|\partial_t \alpha_t|_{\omega_t} \le C$ for some positive constant $C$ depending on $n, K, T$, and therefore, $e^{-TC}\alpha \le \alpha_t \le e^{TC}\alpha$. As a consequence, $\alpha_t$ converges uniformly as $t \to T$ to a continuous real $(1,1)$-form $\alpha_T$ satisfying $e^{-TC}\alpha \le \alpha_T \le e^{TC}\alpha$. Using again Theorem 4.10, it follows that $\alpha_t \to \alpha_T$ as $t \to T$ in any $C^k$-norm. Therefore $\alpha_T$ is a smooth real closed $(1,1)$-form. Starting from the pair $(\omega_T, \alpha_T)$, according to Theorem 3.1, we can extend the solution $(\omega_t, \alpha_t)_t$ to the time interval $[0, T+\epsilon]$ for some $\epsilon > 0$. This contradicts with the maximal time interval $[0, T)$. $\square$

### 4.3. Long-time existence with bounded curvatures under extra assumptions

In this subsection, we assume that $(M, \omega)$ is a closed Kähler manifold with a closed $(1,1)$-form $\alpha$ satisfying

$$[\alpha] \in \mathcal{K}'_M, \quad \omega \in -2\pi c_1(M) + [\alpha].$$

By Proposition 1.2 (see its proof in section 3.3), the flow equation (3.2) is equivalent to the following parabolic complex Monge-Ampère equation coupled with a heat equation

$$\partial_t \varphi_t = \log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n}{\Omega} - \varphi_t + f_t, \quad \partial_t f_t = \Delta_{\omega_t} f_t + \mathrm{tr}_{\omega_t}\alpha, \tag{4.24}$$

with $(\varphi_0, f_0) = (0, 0)$, where $\varphi_t, f_t$ are smooth functions on $M$ with desired normalization conditions such that $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$ and $\omega = \alpha + \sqrt{-1}\partial\bar{\partial}\log\Omega$ for some smooth volume form $\Omega$ on $M$.

**Lemma 4.13.** *Let $T < \infty$. If the Ricci curvature is uniformly bounded, i.e. $|\rho(\omega_t)|_{\omega_t} \le C_1$ on $M \times [0, T)$ for some positive constant $C_1$, then there exists some positive uniform constant $C_2$ depending on $\sup_M \mathrm{tr}_\omega \alpha$, $n$, $C_1$, and $T$, such that*

$$\mathrm{tr}_{\omega_t}\alpha_t \le C_2 \tag{4.25}$$

*on $M \times [0, T)$.*

*Proof.* From (4.4), we have

$$\partial_t \left(e^{-t}\mathrm{tr}_{\omega_t}\alpha_t\right) = \Delta_{\omega_t}\left(e^{-t}\mathrm{tr}_{\omega_t}\alpha_t\right) - e^{-t}\left(|\alpha_t|^2_{\omega_t} + \langle \rho(\omega_t), \alpha_t\rangle_{\omega_t}\right).$$

Then, by applying the Cauchy-Schwarz inequality,

$$\partial_t \left(e^{-t}\mathrm{tr}_{\omega_t}\alpha_t\right) \le \Delta_{\omega_t}\left(e^{-t}\mathrm{tr}_{\omega_t}\alpha_t\right) - \frac{1}{2}e^{-t}\left(|\alpha_t|^2_{\omega_t} - |\rho(\omega_t)|^2_{\omega_t}\right).$$

Suppose $e^{-t}\mathrm{tr}_{\omega_t}\alpha_t$ achieves its maximum at a point $(x_0, t_0) \in M \times [0, T)$. If $t_0 = 0$, then $e^{-t}\mathrm{tr}_{\omega_t}\alpha_t \leq \mathrm{tr}_\omega\alpha(x_0) \leq \sup_M \mathrm{tr}_\omega\alpha$. If $t_0 > 0$, then by the maximum principle we have

$$|\alpha_t|^2_{\omega_t} \leq |\rho(\omega_t)|^2_{\omega_t} \quad \text{at } (x_0, t_0),$$

and hence $|\alpha_t|^2_{\omega_t} \leq C_1^2$ at $(x_0, t_0)$. The inequality $\mathrm{tr}_{\omega_t}\alpha_t \leq \sqrt{n}|\alpha_t|_{\omega_t}$ implies $\mathrm{tr}_{\omega_t}\alpha_t \leq \sqrt{n}C_1$ at $(x_0, t_0)$. Consequently, $\mathrm{tr}_{\omega_t}\alpha_t \leq \sqrt{n}C_1e^T$ on $M \times [0, T)$. □

As a consequence of Lemma 4.13, we can get the uniform $C^0$ bound on $\varphi_t$ provided that $\alpha$ is nonnegative.

**Proposition 4.14.** *Suppose $T < \infty$ and $\alpha$ is nonnegative. If the Ricci curvature is uniformly bounded, i.e., $|\rho(\omega_t)|_{\omega_t} \leq C_1$ on $M \times [0, T)$ for some positive uniform constant $C_1$, then there exist positive uniform constants $C_3, C_4, C_5$ depending only on $\omega, \Omega, \alpha, n, C_1, T$ such that*

$$|f_t| \leq C_3, \quad \dot{\varphi}_t \leq C_4, \quad |\varphi_t| \leq C_5, \tag{4.26}$$

*on $M \times [0, T)$.*

*Proof.* Since $\alpha$ is nonnegative, we have $\mathrm{tr}_{\omega_t}\alpha \geq 0$ along the flow (4.24) and then $\partial_t f_t \geq \Delta_{\omega_t} f_t$. By the maximum principle, we must have $f_t \geq 0$ on $M \times [0, T)$. On the other hand, applying Lemma 4.13 to flow (4.24), we have

$$\partial_t f_t \leq C_2$$

on $M \times [0, T)$. Integrating with respect to $t$, we obtain $f_t \leq C_2 t \leq C_2 T$ on $M \times [0, T)$. Thus we prove $|f_t| \leq C_3$ on $M \times [0, T)$ for some positive uniform constant $C_3$.

Differentiation of (4.24) with respect to $t$ yields

$$\partial_t\dot{\varphi}_t = \mathrm{tr}_{\omega_t}(\sqrt{-1}\partial\bar{\partial}\dot{\varphi}_t) - \dot{\varphi}_t + \dot{f}_t = \Delta_{\omega_t}\dot{\varphi}_t - \dot{\varphi}_t + \dot{f}_t.$$

By Lemma 4.13, $\dot{f}_t = \mathrm{tr}_{\omega_t}\alpha_t \leq C_2$, and therefore, it follows from the maximum principle that,

$$\dot{\varphi}_t \leq C_2.$$

Moreover, it follows from the parabolic maximum principle for the first equation in Proposition 1.2 that

$$|\varphi_t| \leq \max_M \left|\log\frac{\omega^n}{\Omega}\right| + \max_{M\times[0,T)} |f_t| =: C_5. \qquad \square$$

**Proposition 4.15.** *Suppose $T<\infty$ and $\alpha$ is nonnegative. If $|\mathrm{Rm}_{\omega_t}|_{\omega_t}\le C_0$ on $M\times[0,T)$ for some positive uniform constant $C_0$, then there exists positive uniform constants $C_6, C_7$ such that*

$$\frac{1}{C_6}\omega \le \omega_t \le C_6\omega, \quad |\alpha_t|^2_{\omega_t}\le C_7 \tag{4.27}$$

*on $M\times[0,T)$.*

*Proof.* Since $\rho(\omega_t)$ is uniformly bounded along the flow by our assumption, it follows from Proposition 4.14 that $\varphi_t$ is uniformly bounded. Therefore, by the compactness theorem in [7], $\omega_t$ and $\omega$ are uniformly equivalent for any $t\in[0,T)$. Namely, there exists a positive constant $C_6$, such that

$$\frac{1}{C_6}\omega \le \omega_t \le C_6\omega.$$

To simplify the notion, we use $\hat{\omega}$ (or $\hat{g}$, respectively) to denote $\omega$ (or $g$, respectively), use $\tilde{\omega}$ (or $\tilde{g}$) to denote $\omega_t$ (or $g_t$) and use $\tilde{\alpha}$ to denote $\alpha_t$. Then

$$\begin{aligned}\partial_t|\alpha_t|^2_\omega &\equiv \partial_t|\tilde{\alpha}|^2_{\hat{\omega}} = \partial_t\left(\hat{g}^{\bar{\ell}j}\hat{g}^{\bar{j}k}\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\right)\\ &= 2\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{k\bar{\ell}}\partial_t\tilde{\alpha}_{i\bar{j}} = 2\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{k\bar{\ell}}\overline{\Box}_{\tilde{g}}\tilde{\alpha}_{i\bar{j}}\\ &= 2\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{k\bar{\ell}}\bigg(\Delta_{\tilde{\omega}}\alpha_{i\bar{j}} + \tilde{R}_{i\bar{j}r\bar{s}}\tilde{g}^{\bar{s}p}\tilde{g}^{\bar{q}r}\tilde{\alpha}_{p\bar{q}} - \frac{3}{2}\tilde{g}^{\bar{q}p}\tilde{R}_{p\bar{j}}\tilde{\alpha}_{i\bar{q}} - \frac{1}{2}\tilde{g}^{\bar{q}p}\tilde{R}_{i\bar{q}}\tilde{\alpha}_{p\bar{j}}\bigg).\end{aligned}$$

From the identity

$$\Delta_{\tilde{\omega}}\tilde{\alpha}_{i\bar{j}} = \tilde{g}^{\bar{q}p}\nabla_p\nabla_{\bar{q}}\tilde{\alpha}_{i\bar{j}} + \frac{1}{2}\tilde{R}_{i\bar{p}}\tilde{g}^{\bar{p}q}\tilde{\alpha}_{q\bar{j}} + \frac{1}{2}\tilde{R}_{q\bar{j}}\tilde{g}^{\bar{p}q}\tilde{\alpha}_{i\bar{p}},$$

we arrive at

$$\partial_t|\alpha_t|^2_\omega = 2\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{k\bar{\ell}}\left(\tilde{g}^{\bar{q}p}\nabla_p\nabla_{\bar{q}}\tilde{\alpha}_{i\bar{j}} + \tilde{R}_{i\bar{j}r\bar{s}}\tilde{g}^{\bar{s}p}\tilde{g}^{\bar{q}r}\tilde{\alpha}_{p\bar{q}} - \tilde{g}^{\bar{r}s}\tilde{R}_{s\bar{j}}\tilde{\alpha}_{i\bar{r}}\right).$$

Choose normal coordinates on $M$ for $\tilde{\omega}=\omega_t$ such that

$$\tilde{g}_{i\bar{j}} = \delta_{ij}, \quad \partial_k\tilde{g}_{i\bar{j}} = 0, \quad \hat{g}_{i\bar{j}} = \lambda_i\delta_{ij}.$$

In particular, $\tilde{R}_{i\bar{j}k\bar{\ell}} = -\partial_i\partial_{\bar{j}}\tilde{g}_{k\bar{\ell}} = -\tilde{g}_{p\bar{\ell}}\partial_{\bar{j}}\tilde{\Gamma}^p_{ik}$. Consequently,

$$\nabla_k\nabla_{\bar{\ell}}\tilde{\alpha}_{i\bar{j}} = \partial_k\nabla_{\bar{\ell}}\tilde{\alpha}_{i\bar{j}} = \partial_k\left(\partial_{\bar{\ell}}\tilde{\alpha}_{i\bar{j}} - \overline{\tilde{\Gamma}^q_{\ell j}}\tilde{\alpha}_{i\bar{q}}\right) = \partial_k\partial_{\bar{\ell}}\tilde{\alpha}_{i\bar{j}} + \tilde{g}^{\bar{q}p}\tilde{R}_{k\bar{\ell}p\bar{j}}\tilde{\alpha}_{i\bar{q}}$$

and

$$\begin{aligned}\partial_t|\alpha_t|^2_\omega &= 2\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{k\bar{\ell}}\left(\tilde{g}^{\bar{q}p}\partial_p\partial_{\bar{q}}\tilde{\alpha}_{i\bar{j}} + \tilde{R}_{i\bar{j}r\bar{s}}\tilde{g}^{\bar{s}p}\tilde{g}^{\bar{q}r}\tilde{\alpha}_{p\bar{q}}\right)\\ &= \frac{2}{\lambda_i\lambda_j}\delta_{i\ell}\delta_{jk}\tilde{\alpha}_{k\bar{\ell}}\left(\delta_{pq}\partial_p\partial_{\bar{q}}\tilde{\alpha}_{i\bar{j}} + \tilde{R}_{i\bar{j}r\bar{s}}\delta_{sp}\delta_{qr}\tilde{\alpha}_{p\bar{q}}\right)\\ &= \sum_{1\le i,j,p\le n}\left(\frac{2\alpha_{j\bar{i}}}{\lambda_i\lambda_j}\partial_p\partial_{\bar{p}}\tilde{\alpha}_{i\bar{j}} + \frac{2\tilde{\alpha}_{j\bar{i}}\tilde{\alpha}_{p\bar{q}}}{\lambda_i\lambda_j}\tilde{R}_{i\bar{j}q\bar{p}}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta_{\omega_t}|\alpha_t|^2_\omega &= \tilde{g}^{\bar{q}p}\partial_p\partial_{\bar{q}}\left(\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\right)\\ &= 2\tilde{g}^{\bar{q}p}\partial_p\left(\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\hat{g}^{\bar{j}k}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i} + \tilde{\alpha}_{k\bar{\ell}}\hat{g}^{\bar{\ell}i}\hat{g}^{\bar{j}k}\partial_{\bar{q}}\tilde{\alpha}_{i\bar{j}}\right)\\ &= I_1 + I_2,\end{aligned}$$

where $I_1$ and $I_2$ denote the first and second term on the right-hand side. For $I_1$, using the normal coordinates, we have

$$\begin{aligned}I_1 &= 2\tilde{g}^{\bar{q}p}\bigg(\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\hat{g}^{\bar{j}k}\partial_p\partial_{\bar{q}}\hat{g}^{\bar{\ell}i} + \tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\hat{g}^{\bar{j}k}\\ &\qquad + \tilde{\alpha}_{k\bar{\ell}}\hat{g}^{\bar{j}k}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\tilde{\alpha}_{i\bar{j}} + \tilde{\alpha}_{i\bar{j}}\hat{g}^{\bar{j}k}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\tilde{\alpha}_{k\bar{\ell}}\bigg)\\ &= 2\delta_{pq}\bigg(\frac{\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}}{\lambda_k}\delta_{jk}\partial_p\partial_{\bar{q}}\hat{g}^{\bar{\ell}i} + \tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\hat{g}^{\bar{j}k}\\ &\qquad + \frac{\tilde{\alpha}_{k\bar{\ell}}\delta_{jk}}{\lambda_k}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\tilde{\alpha}_{i\bar{j}} + \frac{\tilde{\alpha}_{i\bar{j}}\delta_{jk}}{\lambda_k}\partial_{\bar{q}}\hat{g}^{\bar{\ell}i}\partial_p\tilde{\alpha}_{k\bar{\ell}}\bigg)\\ &= \sum_{1\le i,j,k,\ell\le n}\bigg(\frac{2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{j\bar{\ell}}}{\lambda_j}\partial_k\partial_{\bar{k}}\hat{g}^{\bar{\ell}i} + \sum_{1\le p\le n}2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\partial_p\hat{g}^{\bar{j}k}\partial_{\bar{p}}\hat{g}^{\bar{\ell}i}\\ &\qquad + \frac{2\tilde{\alpha}_{j\bar{\ell}}}{\lambda_j}\partial_k\tilde{\alpha}_{i\bar{j}}\partial_{\bar{k}}\hat{g}^{\bar{\ell}i} + \frac{2\tilde{\alpha}_{\ell\bar{j}}}{\lambda_j}\partial_k\tilde{\alpha}_{j\bar{i}}\partial_{\bar{k}}\hat{g}^{\bar{i}\ell}\bigg).\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}I_2 &= \sum_{1\le i,j,p\le n}\frac{2\tilde{\alpha}_{j\bar{i}}}{\lambda_i\lambda_j}\partial_p\partial_{\bar{p}}\tilde{\alpha}_{i\bar{j}} + \sum_{1\le i,j,p\le n}\frac{2}{\lambda_i\lambda_j}\partial_p\tilde{\alpha}_{j\bar{i}}\partial_{\bar{p}}\tilde{\alpha}_{i\bar{j}}\\ &\quad + \sum_{1\le i,j,k,\ell\le n}\left(\frac{2\tilde{\alpha}_{j\bar{\ell}}}{\lambda_j}\partial_{\bar{k}}\tilde{\alpha}_{i\bar{j}}\partial_k\hat{g}^{\bar{\ell}i} + \frac{2\tilde{\alpha}_{\ell\bar{j}}}{\lambda_j}\partial_{\bar{k}}\tilde{\alpha}_{j\bar{i}}\partial_k\hat{g}^{\bar{i}\ell}\right).\end{aligned}$$

From $I_1, I_2$, we finally obtain

$$\begin{aligned}\partial_t|\alpha_t|^2_\omega = \Delta_{\omega_t}|\alpha_t|^2_\omega &+ \sum_{1\leq i,j,p,q\leq n}\frac{2\tilde{\alpha}_{j\bar{i}}\tilde{\alpha}_{p\bar{q}}}{\lambda_i\lambda_j}\tilde{R}_{i\bar{j}q\bar{p}} - \sum_{1\leq i,j,p\leq n}\frac{2}{\lambda_i\lambda_j}\left|\partial_p\tilde{\alpha}_{j\bar{i}}\right|^2\\ &- \sum_{1\leq i,j,k,\ell\leq n}\frac{2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{j\bar{\ell}}}{\lambda_j}\partial_k\partial_{\bar{k}}\hat{g}^{\bar{\ell}i} - \sum_{1\leq i,j,k,\ell,p\leq n}2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\partial_p\hat{g}^{\bar{j}k}\partial_{\bar{p}}\hat{g}^{\bar{\ell}i}\\ &+ \sum_{1\leq i,j,k,\ell\leq n}\frac{4}{\lambda_i}\Re\left(\tilde{\alpha}_{j\bar{\ell}}\partial_k\tilde{\alpha}_{i\bar{j}}\partial_{\bar{k}}\hat{g}^{\bar{\ell}i}\right)\\ &+ \sum_{1\leq i,j,k,\ell\leq n}\frac{4}{\lambda_j}\Re\left(\tilde{\alpha}_{\ell\bar{j}}\partial_k\tilde{\alpha}_{j\bar{i}}\partial_{\bar{k}}\hat{g}^{\bar{i}\ell}\right),\end{aligned}\tag{4.28}$$

where $\Re(Z)$ means the real part of $Z$. Since $\omega$ is a Kähler form on the compact manifold $M$, we can find two positive uniform constants $\lambda_{\min}$ and $\lambda_{\max}$ such that $\lambda_{\min}\leq\lambda_i,\ldots,\lambda_n\leq\lambda_{\max}$ and hence

$$\frac{1}{\lambda^2_{\max}}\sum_{1\leq i,j\leq n}|\tilde{\alpha}_{i\bar{j}}|^2\leq|\alpha_t|^2_\omega\leq\frac{1}{\lambda^2_{\min}}\sum_{1\leq i,j\leq n}|\tilde{\alpha}_{i\bar{j}}|^2.\tag{4.29}$$

We denote by $\mathcal{P}\lesssim\mathcal{Q}$ if $\mathcal{P}\leq C\mathcal{Q}$ for some positive uniform constant $C$. Then

$$\sum_{1\leq i,j\leq n}|\tilde{\alpha}_{i\bar{j}}|^2\lesssim|\alpha_t|^2_\omega.$$

Since the curvature is uniformly bounded along the flow, it follows that

$$\begin{aligned}\sum_{i,j,p,q}\frac{2\tilde{\alpha}_{j\bar{i}}\tilde{\alpha}_{p\bar{q}}}{\lambda_i\lambda_j}\tilde{R}_{i\bar{j}q\bar{p}} &\lesssim \sum_{i,j,p,q}2|\tilde{\alpha}_{j\bar{i}}\tilde{\alpha}_{p\bar{q}}|\leq\sum_{i,j,p,q}\left(|\tilde{\alpha}_{j\bar{i}}|^2+|\tilde{\alpha}_{p\bar{q}}|^2\right)\\ &\leq n^2\sum_{i,j}|\tilde{\alpha}_{j\bar{i}}|^2+n^2\sum_{p,q}|\tilde{\alpha}_{p\bar{q}}|^2\lesssim|\alpha_t|^2_\omega,\\ \sum_{i,j,k,\ell}\frac{2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{j\bar{\ell}}}{\lambda_j}\partial_k\partial_{\bar{k}}\hat{g}^{\bar{\ell}i} &\lesssim\sum_{i,j,k,\ell}2|\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{j\bar{\ell}}|\leq\sum_{i,j,k,\ell}\left(|\tilde{\alpha}_{i\bar{j}}|^2+|\tilde{\alpha}_{j\bar{\ell}}|^2\right)\\ &\leq n^2\sum_{i,j}|\tilde{\alpha}_{i\bar{j}}|^2+n^2\sum_{j,\ell}|\tilde{\alpha}_{j\bar{\ell}}|^2\lesssim|\alpha_t|^2_\omega,\\ \sum_{i,j,k,\ell,p}2\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}\partial_p\hat{g}^{\bar{j}k}\partial_{\bar{p}}\hat{g}^{\bar{\ell}i} &\lesssim\sum_{i,j,k,\ell,p}2|\tilde{\alpha}_{i\bar{j}}\tilde{\alpha}_{k\bar{\ell}}|\leq n\sum_{i,j,k,\ell}\left(|\tilde{\alpha}_{i\bar{j}}|^2+|\tilde{\alpha}_{k\bar{\ell}}|^2\right)\\ &\leq n^2\sum_{i,j}|\tilde{\alpha}_{i\bar{j}}|^2+n^2\sum_{j,\ell}|\tilde{\alpha}_{j\bar{\ell}}|^2\lesssim|\alpha_t|^2_\omega,\end{aligned}$$

where all indices are taken in the set $\{1,\dots,n\}$. By the Cauchy-Schwarz inequality, we have

$$4\sum_{i,j,k,\ell}\frac{1}{\lambda_i}\Re\left(\tilde{\alpha}_{j\bar{\ell}}\partial_k\tilde{\alpha}_{i\bar{j}}\partial_{\bar{k}}\hat{g}^{\bar{\ell}i}\right)\leq 4\left|\sum_{i,j,k}\left(\frac{1}{\sqrt{\lambda_i\lambda_j}}\partial_k\tilde{\alpha}_{i\bar{j}}\right)\left(\sqrt{\frac{\lambda_j}{\lambda_i}}\sum_{\ell}\tilde{\alpha}_{j\bar{\ell}}\partial_{\bar{k}}\hat{g}^{\bar{\ell}i}\right)\right|$$
$$\leq\sum_{i,j,k}\frac{1}{\lambda_i\lambda_j}\left|\partial_k\tilde{\alpha}_{i\bar{j}}\right|^2+4\sum_{i,j,k}\frac{\lambda_j}{\lambda_i}\left|\sum_{\ell}\tilde{\alpha}_{j\bar{\ell}}\partial_{\bar{k}}\hat{g}^{\bar{\ell}i}\right|^2;$$

by the Cauchy-Schwarz inequality again, the second term on the right-hand side is bounded from above by

$$4n\frac{\lambda_{\max}}{\lambda_{\min}}\left(\sum_{i,j,k}\left|\partial_{\bar{k}}\hat{g}^{\bar{j}i}\right|^2\right)\left(\sum_{j,\ell}|\tilde{\alpha}_{j\bar{\ell}}|^2\right)\lesssim|\alpha_t|^2_{\omega}$$

according to (4.29). Similarly, we can show that

$$4\sum_{i,j,k,\ell}\frac{1}{\lambda_i}\Re\left(\tilde{\alpha}_{\ell\bar{j}}\partial_k\tilde{\alpha}_{j\bar{i}}\partial_{\bar{k}}\hat{g}^{i\bar{\ell}}\right)\leq\sum_{i,j,k}\frac{1}{\lambda_i\lambda_j}|\partial_k\tilde{\alpha}_{j\bar{i}}|^2+4\sum_{i,j,k}\frac{\lambda_j}{\lambda_i}\left|\sum_{\ell}\tilde{\alpha}_{\ell\bar{j}}\partial_{\bar{k}}\hat{g}^{\bar{i}\ell}\right|^2$$

where the second term on the right-hand side is still $\lesssim|\alpha_t|^2_{\omega}$. Plugging those estimates into (4.28), we arrive at

$$(\partial_t-\Delta_{\omega_t})\,|\alpha_t|^2_{\omega}\lesssim|\alpha_t|^2_{\omega}.$$

Hence, we can find a positive uniform constant $C'$ depending on $\omega,m$ such that

$$\partial_t|\alpha_t|^2_{\omega}\leq\Delta_{\omega_t}|\alpha_t|^2_{\omega}+C'|\alpha_t|^2_{\omega}.$$

According to the maximum principle, we immediately obtain

$$|\alpha_t|^2_{\omega}\leq e^{C't}\sup_M|\alpha|^2_{\omega}\leq e^{C'T}\sup_M|\alpha|^2_{\omega}.$$

Thus $|\alpha_t|^2_{\omega}$ is uniformly bounded from above, and, by the fact that $\omega_t$ is uniformly equivalent to $\omega$, we have $|\alpha_t|^2_{\omega_t}\leq C_7$ for some positive uniform constant $C_7$. □

Combining Theorem 4.12 with Proposition 4.15, the main theorem follows from the similar argument as in the proof of Theorem 4.12.

**Theorem 4.16.** *Let $(M,\omega)$ be a compact Kähler manifold and $\alpha$ be a closed nonnegative $(1,1)$-form satisfying*

$$[\alpha] \in \mathcal{K}'_M, \quad \omega \in -2\pi c_1(M) + [\alpha].$$

*Let $[0,T)$ with $T<\infty$ be the maximal time interval such that $(\omega_t,\alpha_t)_{t\in[0,T)}$ is the solution to the flow (1.1) with the initial condition $(\omega,\alpha)$. Then*

$$\limsup_{t\to T} \max_M |\mathrm{Rm}_{\omega_t}|_{\omega_t} = \infty.$$

By using the standard blow-up argument as in [32], we have following corollary:

**Corollary 4.17.** *Assume that $\alpha$ is a closed nonnegative $(1,1)$-form such that*

$$\omega \in -2\pi c_1(M) + [\alpha] > 0.$$

*Let $(\omega_t,\alpha_t)$ be the solution to the flow (1.1) for $t\in[0,T)$ with the initial condition $(\omega,\alpha)$. Suppose that the Ricci curvature of $\omega_t$ and $|\alpha_t|_{\omega_t}$ are uniformly bounded on $[0,T)$. Then the solution $(\omega_t,\alpha_t)$ can be extended past time $T$.*

*Proof.* We only sketch the proof by pointing out the difference from [32]. Let $s=e^t-1, \tilde{\omega}_s = e^t\omega_t$ and $\tilde{\alpha}_s = \alpha_t$. Then the equation (1.1) can be rewritten as

$$\partial_s\tilde{\omega}_s = -\mathrm{Ric}(\tilde{\omega}_s) + \tilde{\alpha}_s, \quad \partial_s\tilde{\alpha}_s = \overline{\Box}_{\tilde{\omega}_s}\tilde{\alpha}_s, \quad (\tilde{\omega}_0,\tilde{\alpha}_0) = (\omega,\alpha), \tag{4.30}$$

Suppose on the contrary that $(\omega_t,\alpha_t)$ cannot be extended past $T$. Then $(\tilde{\omega}_s,\tilde{\alpha}_s)$ cannot be extended past $e^T-1$. By Theorem 4.16, there exists a sequence of points and times $(x_i,s_i)$ with $s_i\to e^T-1$ such that

$$K_i = |\mathrm{Rm}_{\tilde{\omega}}|_{\tilde{\omega}}(x_i,s_i) = \sup_{M\times[0,s_i]} |\mathrm{Rm}_{\tilde{\omega}}|_{\tilde{\omega}}(x,s) \to \infty.$$

The pointed rescaled solutions $(M,\omega_i(s),\alpha_i(s),x_i)$ for $s\in[-s_iK_i,0]$ are defined as

$$\omega_i(s) = K_i\tilde{\omega}(s_i + K_i^{-1}s), \quad \alpha_i(s) = \tilde{\alpha}(s_i+K_i^{-1}s).$$

Obviously, $|\mathrm{Rm}_{\omega_i}|\le 1$; $|\mathrm{Rm}_{\omega_i}|(x_i,0)=1$; $(\omega_i,\alpha_i)$ satisfies $\partial_s\omega_i(s) = -\mathrm{Ric}(\omega_i(s)) + \alpha_i(s)$ and $\partial_s\alpha_i(s) = \overline{\Box}_{\omega_i(s)}\tilde{\alpha}_i(s)$. Since $|\mathrm{Ric}(\omega_t)|_{\omega_t}$ and $|\alpha_t|_{\omega_t}$ are uniformly bounded, then $\tilde{\omega}_s$ is uniformly continuous and $|\mathrm{Ric}(\tilde{\omega}_s)|_{\tilde{\omega}_s}$ is also uniformly bounded. By the standard argument as in Ricci flow [32], the injective radius $\mathrm{inj}_{\omega_i(0)}(x_i)$ has a positive lower bound independent of $i$. It follows from the Cheeger-Gromov compactness theorem and Hamilton compactness theorem [21] that $(M, \omega_i, x_i)$ subconverges to a complete pointed solution $(M, \omega_\infty, x_\infty)$ with $|\mathrm{Rm}_{\omega_\infty}|_{\omega_\infty}(x_\infty, 0) = 1$ by applying Proposition 4.10 to $(\omega_i, \alpha_i)$. Moreover, the Ricci curvature tensor $\mathrm{Ric}_{\omega_\infty} \equiv 0$. Again by the standard argument as in Ricci flow [32], the metric $\omega_\infty(0)$ has Euclidean volume growth. Therefore, Bishop-Gromov volume comparison theorem implies that $\omega_\infty(0)$ is Euclidean. This contradicts to $|\mathrm{Rm}_{\omega_\infty}|_{\omega_\infty}(x_\infty, 0) = 1$. □

## References


[1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math., (2) **102** (1978), no. 1, 63–95.

[2] B. Chow, P. Lu, and L. Ni, Hamilton's Ricci Flow, Graduate Studies in Mathematics **77**, American Mathematical Society, Providence, RI; Science Press, New York, (2006), xxxv+608 pp.

[3] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects, Mathematical Surveys and Monographs **135**, American Mathematical Society, Providence, RI, (2007), xxiv+536 pp.

[4] E. Calabi, *Extremal Kähler metrics*, in: Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, NJ, (1982).

[5] E. Calabi, *Extremal Kähler metrics. II*, in: Differential Geometry and Complex Analysis, pp. 95–114, Springer, Berlin, (1985).

[6] H. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372.

[7] X. Chen and W. He, *On the Calabi flow*, Amer. J. Math. **130** (2008), no. 2, 539–570.

[8] X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[9] X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds, II: limits with cone angle less than* $2\pi$, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[10] X. Chen, S. Donaldson, and S. Sun, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[11] X. Chen and S. Sun, *Calabi flow, Geodesic rays, and uniqueness of constant scalar curvature Kähler metrics*, Ann. of Math., (2) **180** (2014), no. 2, 407–454.

[12] X. Chen, S. Sun, and B. Wang, *Kähler-Ricci flow, Kähler-Einstein metric, and K-stability*, Geom. Topol. **22** (2018), no. 6, 3145–3173.

[13] X. Chen and G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs*, Publ. Math. Inst. Hautes Etudes Sci. **107** (2008), 1–107.

[14] X. Chen and K. Zheng, *Pseudo-Calabi flow*, J. Reine Angew. Math. **674** (2013), 195–251.

[15] S. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), no. 3, 479–522.

[16] S. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349.

[17] S. Donaldson, *Constant scalar curvature metrics on toric surfaces*, Geom. Funct. Anal. **19** (2009), no. 1, 83–136.

[18] J. Fine, *Calabi flow and projective embeddings. With an appendix by Kefeng Liu and Xiaonan Ma*, J. Differential Geom. **84** (2010), no. 3, 489–523.

[19] R. Feng and H. Huang, *The global existence and convergence of the Calabi flow on $\mathbb{C}^n/\mathbb{Z}^n + i\mathbb{Z}^n$*, J. Funct. Anal. **263** (2012), no. 4, 1129–1146.

[20] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[21] R. S. Hamilton, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. **117** (1995), 545–572.

[22] W. He, *On the convergence of the Calabi flow*, Proc. Amer. Math. Soc. **143** (2015), no. 3, 1273–1281.

[23] H. Huang, *Toric surfaces, K-stability and Calabi flow*, Math. Z. **276** (2014), no. 3-4, 953–968.

[24] K. Kodaira, Complex Manifolds and Deformation of Complex Structures [translated from the 1981 Japanese original by Kazuo Akao], reprint of the 1986 English edition, Classics in Mathematics, Springer-Verlag, Berlin, (2005), x+465 pp.

[25] Y. Li, *Generalized Ricci flow I: higher derivatives estimates for compact manifolds*, Anal. PDE. **5** (2012), no. 4, 747–775.

[26] H. Li, B. Wang, and K. Zheng, *Regularity scales and convergence of the Calabi flow*, J. Geom. Anal. **28** (2018), no. 3, 2050–2101.

[27] K. Liu and X. Yang, *Geometry of Hermitian manifolds*, Internat. J. Math. **23** (2012), no. 6, 1250055, 40 pp.

[28] D. H. Phong, *Parabolic flows in complex geometry*, Acta Math. Vietnam **39** (2014), no. 1, 35–54.

[29] D. H. Phong and J. Sturm, *On stability and the convergence of the Kähler-Ricci flow*, J. Differential Geom. **72** (2006), no. 1, 149–168.

[30] D. H. Phong and J. Sturm, *Lectures on stability and constant scalar curvature*, Handbook of Geometric Analysis, no. 3, pp. 357–436, Adv. Lect. Math. (ALM) **14**, Int. Press, Somerville, MA, (2010).

[31] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, *The Kähler-Ricci flow and the $\bar{\partial}$ operator on vector fields*, J. Differential Geom. **81** (2009), no. 3, 631–647.

[32] N. Sesum, *Curvature tensor under the Ricci flow*, Amer. J. Math. **127** (2005), no. 6, 1315–1324.

[33] J. Song and B. Weinkove, *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. **61** (2008), no. 2, 210–229.

[34] J. Song and B. Weinkove, An Introduction to the Kähler-Ricci Flow, pp. 89–188, Lecture Notes in Math. **2086**, Springer, Cham, (2013).

[35] J. Streets, *Long time existence of minimizing movement solutions of Calabi flow*, Adv. Math. **259** (2014), 688–729.

[36] G. Székelyhidi, An Introduction to Extremal Kähler Metrics, Graduate Studies in Mathematics **152**, American Mathematical Society, Providence, RI, (2014), xvi+192 pp. ISBN 978-1-4704-1047-6.

[37] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37.

[38] G. Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156.

[39] V. Tosatti and B. Weinkove, *The Calabi flow with small initial energy*, Math. Res. Lett. **14** (2007), no. 6, 1033–1039.

[40] V. Tosatti and B. Weinkove, *On the evolution of a Hermitian metric by its Chern-Ricci form*, J. Differential Geom. **99** (2015), no. 1, 125–163.

[41] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[42] S.-T. Yau, *Open problems in geometry*, in: Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990), pp. 1–28, Proc. Sympos. Pure. Math. **54**, Part 1, Amer. Math. Soc., Providence, RI, (1993).

[43] Z. Zhang, *Scalar curvature behavior for finite-time singularity of Kähler-Ricci flow*, Michigan Math. J. **59** (2010), no. 2, 419–433.

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