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# Geometric quantities arising from bubbling analysis of mean field equations

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Let $E = \mathbb{C}/\Lambda$ be a flat torus and $G$ be its Green function with singularity at 0. Consider the multiple Green function $G_n$ on $E^n$:

$$G_n(z_1, \ldots, z_n) := \sum_{i<j} G(z_i - z_j) - n \sum_{i=1}^{n} G(z_i).$$

A critical point $a = (a_1, \ldots, a_n)$ of $G_n$ is called *trivial* if $\{a_1, \ldots, a_n\} = \{-a_1, \ldots, -a_n\}$. For such a point $a$, two geometric quantities $D(a)$ and $H(a)$ arising from bubbling analysis of mean field equations are introduced. $D(a)$ is a global quantity measuring asymptotic expansion and $H(a)$ is the Hessian of $G_n$ at $a$. By way of geometry of Lamé curves developed in [3], we derive precise formulas to relate these two quantities.

## 1. Introduction

Let $E = E_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in \mathbb{H} = \{\tau \mid \operatorname{Im} \tau > 0\}$. We use the convention $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1 + \tau$. Consider the following mean field equation with singular strength $\rho > 0$:

$$\triangle u + e^u = \rho\, \delta_0 \quad \text{in } E, \tag{1.1}$$

where $\delta_0$ is the Dirac measure at 0. Solutions to this simple looking equation (1.1) possess a rich structure from either the point of view of partial differential equations or of integrable systems. See [3, 4, 6].

Not surprisingly, (1.1) is related to various research areas. In conformal geometry, a solution $u(x)$ to (1.1) leads to a metric $ds^2 = \frac{1}{2} e^u\,(dx^2 + dy^2)$ with constant Gaussian curvature +1 acquiring a conic singularity at 0. It also appears in statistical physics as the equation for the *mean field limit* of the Euler flow in Onsager's vortex model, hence its name. In the physical model of superconductivity, (1.1) is one of limiting equations of the well-known Chern–Simons–Higgs equation as the coupling parameter tends to 0.

We refer the interested readers to [2, 5, 7, 8, 11, 12] and references therein for recent development on this equation.

One important feature of (1.1) is the so-called *bubbling phenomena.* Let $u_k$ be a sequence of solutions to (1.1) with $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$, and $\max_E u_k(z) \to +\infty$ as $k \to +\infty$. Then it was proved in [5] that $u_k$ has exactly $n$ blowup points $\{a_1, \ldots, a_n\}$ in $E$ and $a_i \neq 0$ for all $i$. The well-known *Pohozaev identity* says that the position of these blowup points are determined by the following system of equations:

$$n\nabla G(a_i) = \sum_{j\neq i}^{n} \nabla G(a_i - a_j), \qquad 1 \leq i \leq n. \tag{1.2}$$

Here $G(z, w) = G(z - w)$ is the Green function on $E$ defined by

$$\begin{cases} -\triangle G = \delta_0 - \dfrac{1}{|E|} & \text{on } E, \\ \int_E G = 0, \end{cases} \tag{1.3}$$

and $|E|$ is the area of $E$.

If $\rho_k = 8\pi n$ for all $k$, then $\{u_k\}$ consists of *type II solutions* with explicit blowup behavior (cf. [3]). On the other hand, we have

**Theorem A.** [3, 4] *Let $u_k$ be a sequence of bubbling solutions of equation (1.1) with $\rho = \rho_k \to 8\pi n$, $n \in \mathbb{N}$. If $\rho_k \neq 8\pi n$ for large $k$, then*

(1) *The blowup set $a = \{a_1, \ldots, a_n\}$ satisfies*

$$\{a_1, \ldots, a_n\} = \{-a_1, \ldots, -a_n\} \quad \text{in } E.$$

(2) *Let $\lambda_k := \max_E u_k(z)$, then there is a constant $D(a)$ such that*

$$\rho_k - 8\pi n = (D(a) + o(1))e^{-\lambda_k}. \tag{1.4}$$

From (1.4), the quantity $D(a)$ plays a fundamental role in controling the sign of $\rho_k - 8\pi n$. Thus it provides one of the key geometric messages for bubbling solutions $u_k$. The question is how to compute $D(a)$?

There exists a complicate expression for $D(a)$ which we will recall in (1.7) below. Define the regular part $\tilde{G}(z, w)$ of $G(z, w)$ by

$$\tilde{G}(z, w) := G(z, w) + \frac{1}{2\pi} \log |z - w|.$$

Given a blowup set $a = \{a_1, \dots, a_n\}$ as in Theorem A (1), we set

$$f_{a_i}(z) = 8\pi\Bigg(\tilde{G}(z, a_i) - \tilde{G}(a_i, a_i) + \sum_{j\neq i}(G(z, a_j) - G(a_i, a_j)) - n(G(z) - G(a_i))\Bigg), \tag{1.5}$$

$$\mu_i := \exp\Bigg(8\pi(\tilde{G}(a_i, a_i) + \sum_{j\neq i} G(a_i, a_j) - nG(a_i))\Bigg). \tag{1.6}$$

Then $D(a)$ can be calculated by

$$D(a) = \lim_{r\to 0}\sum_{i=1}^{n}\mu_i\left(\int_{\Omega_i\setminus B_r(a_i)}\frac{e^{f_{a_i}(z)}-1}{|z-a_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i}\frac{1}{|z-a_i|^4}\right), \tag{1.7}$$

where $\Omega_i$ is any open neighborhood of $a_i$ in $E$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n}\bar{\Omega}_i = E$. The limit exists since $f_{a_i}(z) = O(|z - a_i|^3)$ plus a quadratic harmonic function for all $i$. For a proof, see [11].

Consider the divisor (complete diagonal) in $(E^\times)^n$:

$$\Delta_n = \{(z_1, \dots, z_n) \in (E^\times)^n \mid z_i = z_j \text{ for some } i \neq j\}$$

and define the *multiple Green function* $G_n(z) = G_n(z; \tau)$ on $(E^\times)^n \setminus \Delta_n$ by

$$G_n(z) := \sum_{i<j} G(z_i - z_j) - n\sum_{i=1}^{n} G(z_i). \tag{1.8}$$

Notice that $G_n$ is invariant under the permutation group $S_n$. It is clear that the system (1.2) gives the critical point equations of $G_n$. A critical point $a$ is called *trivial* if $\{a_1, \dots, a_n\} = \{-a_1, \dots, -a_n\}$ in $E$. Theorem A (1) says that the blowup set of a sequence of bubbling solutions $u_k$ of (1.1) with $\rho_k \neq 8\pi n$ for large $k$ is a trivial critical point of $G_n$.

To proceed, it is crucial and natural to ask when is a trivial critical point a degenerate critical point? To answer this question, we need to study the Hessian $H(a)$ at a trivial critical point $a$:

$$H(a) := \det D^2 G_n(a). \tag{1.9}$$

The quantity $H(a)$ can be used to determine the local maximum points of $u_k$ near $a_i$, $1 \leq i \leq n$, and to provide other useful geometric information for the bubbling solutions $u_k$ (cf. [4]).

There are many potential applications of these two quantities. For example, $H(a)$ and $D(a)$ together imply *local uniqueness* of bubbling solutions, as described in the following theorem:

**Theorem B.** *Let $u_k(z)$ and $\tilde{u}_k(z)$ be two sequences of solutions to equation (1.1) with the same parameter $\rho_k \to 8\pi n$ and $\rho_k \neq 8\pi n$ for large $k$. If they have the same blowup set $a = \{a_1, \ldots, a_n\}$ and both $H(a)$ and $D(a)$ do not vanish, then $u_k(z) = \tilde{u}_k(z)$ for large $k$.*

The proof of Theorem B will be given in a forthcoming paper by the first author. It is unexpected since after some suitable scaling at each blowup point $a_i$, the solution $u_k(z)$ (resp. $\tilde{u}_k(z)$) converge to a solution of equation

$$\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^w < \infty,$$

and it is easy to see that the linearized operator $\Delta + e^w$ has non-trivial kernel. To prove the uniqueness, we have to overcome the difficulty caused by the degeneracy of the operator $\Delta + e^w$.

Surprisingly, these two quantities $D(a)$ and $H(a)$ are related to each other as shown by the main result of this paper:

**Theorem 1.1 (=Theorem 4.1).** *For fixed $n \in \mathbb{N}$ and any trivial critical point $a$ of $G_n(z)$, there exists $c_a \geq 0$ such that*

$$H(a) = (-1)^n c_a D(a). \tag{1.10}$$

*Moreover, $c_a > 0$ if and only if $B_a := (2n-1)\sum_{i=1}^n \wp(a_i)$ is not a multiple root of the Lamé polynomial $\ell_n(B)$.*

Here is an outline of the proof, together with a brief description on the content of each section:

The mean field equation (1.1) is closely related to the *Lamé equation* $y'' = (n(n+1)\wp + B)y$. To prove (1.10), a key step is to express $D(a)$ in terms of quantities at a branch point of the hyperelliptic curve $Y_n \to \mathbb{C}$ associated to the Lamé equation. This *Lamé curve* $Y_n$ can be represented by $C^2 = \ell_n(B)$ where the *Lamé polynomial* $\ell_n(B)$ has no multiple roots except for *finitely many* isomorphic classes of tori. This theory is well developed in [3] and the results we need will be reviewed in §2 (cf. Theorem 2.4).

In §3 we study the quantity $D(a)$ in details and derive the above mentioned expression of $D(a)$ in Theorem 3.4. In fact, the Lamé curve encodes the $n-1$ algebraic constraints of the system (1.2), with the remaining analytic constraint being $\sum_{i=1}^n \nabla G(a_i) = 0$. It is thus natural to study the map $a \mapsto \phi(a) := -4\pi \sum_{i=1}^{n} \nabla G(a_i)$ for $a \in Y_n$. It turns out that $D(a)$ is expressible in terms of the Jacobian of $\phi$ (Corollary 3.6).

The proof of Theorem 1.1 is completed in §4 by a process called *analytic adjunction*. The idea is simple: The quantity $H(a)$ is a (real) $2n$-dimensional Hessian on $E^n/S_n$ while $D(a)$ can be regarded as a two dimensional Hessian on $Y_n \subset E^n/S_n$. To relate $H(a)$ with $D(a)$ it amounts to reducing the determinant by substituting the $n-1$ (complex) algebraic equations defining $Y_n$ into it. We end this paper by investigating the case $n = 2$ in Example 4.2 where the value of $c_a$ (given in (4.9)) is written in more explicit terms.

## 2. Lamé equations and Lamé curves [3, 10]

Let $\wp(z) = \wp(z;\tau)$ be the Weierstrass elliptic function with periods $\Lambda_\tau$:

$$\wp(z;\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

which satisfies the well known cubic equation

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau),$$

where

$$g_2(\tau) = 60 \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^4}, \qquad g_4(\tau) = 140 \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \frac{1}{\omega^6}$$

are the weight 4 and weight 6 Eisenstein series respectively.

Let $\zeta(z) = \zeta(z;\tau) := -\int^z \wp(\xi;\tau)\, d\xi$ be the Weierstrass zeta function with quasi-periods $\eta_1(\tau)$ and $\eta_2(\tau)$:

$$\eta_i(\tau) := \zeta(z + \omega_i;\tau) - \zeta(z;\tau), \quad i = 1, 2,$$

and $\sigma(z) = \sigma(z;\tau)$ be the Weierstrass sigma function defined by $\sigma(z) = \exp \int^z \zeta(\xi)\, d\xi$. $\sigma(z)$ is an odd entire function with simple zeros at $\Lambda_\tau$.

The Green function on $E$ (defined in (1.3)) can be expressed in terms of elliptic functions. In [8], we proved that

$$-4\pi \frac{\partial G}{\partial z}(z) = \zeta(z) - r\eta_1 - s\eta_2 = \zeta(z) - z\eta_1 + 2\pi i s, \tag{2.1}$$

where $z = r + s\tau$ with $r, s \in \mathbb{R}$. Using (2.1), equations (1.2) can be translated into the following equivalent system: Consider $a = (a_1, \ldots, a_n) \in E^n$, subject to the constraint $a \in (E^\times)^n \setminus \Delta_n$, that is

$$a_i \neq 0, \quad a_i \neq a_j \text{ for } i \neq j. \tag{2.2}$$

Then

$$\sum_{j \neq i} (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = 0, \qquad 1 \le i \le n \tag{2.3}$$

(there are only $n-1$ independent equations), and

$$\sum_{i=1}^{n} \nabla G(a_i) = 0. \tag{2.4}$$

We will use (2.2)–(2.4) to connect a critical point of $G_n$ defined in (1.8) with the classical Lamé equation. For the reader's convenience, we review some basics on it and refer the readers to [3, 13, 14] for further details.

Recall the Lamé equation

$$\mathcal{L}_{n,B}: \quad y''(z) = (n(n+1)\wp(z) + B)y(z), \tag{2.5}$$

where $n \in \mathbb{R}_{\ge -1/2}$ and $B \in \mathbb{C}$ are its *index* and *accessory parameter* respectively. In general, a solution $y(z)$ is a multi-valued meromorphic function on $\mathbb{C}$ with branch points at $\Lambda$. Any lattice point is a regular singular point with local exponents $-n$ and $n+1$. In this paper we consider only $n \in \mathbb{N}$.

For $a = (a_1, \ldots, a_n)$, we consider the *Hermite–Halphen ansatz*:

$$y_a(z) := e^{z \sum_{i=1}^{n} \zeta(a_i)} \frac{\prod_{i=1}^{n} \sigma(z - a_i)}{\sigma(z)^n}. \tag{2.6}$$

**Theorem 2.1 ([3, 14]).** *Suppose that $a = (a_1, \ldots, a_n) \in (E^\times)^n \setminus \Delta_n$. Then $y_a(z)$ is a solution to $\mathcal{L}_{n,B}$ for some $B$ if and only if $a$ satisfies* (2.3) *and*

$$B = B_a := (2n-1) \sum_{i=1}^{n} \wp(a_i). \tag{2.7}$$

Note that if $a = (a_1, \ldots, a_n) \in (E^\times)^n \setminus \Delta_n$ satisfies (2.3), then so does $-a = (-a_1, \ldots, -a_n)$, and then $y_{-a}(z)$ is also a solution of the same Lamé equation because $B_a = B_{-a}$. Clearly $y_a(z)$ and $y_{-a}(z)$ are linearly independent if and only if $\{a_1, \ldots, a_n\} \neq \{-a_1, \ldots, -a_n\}$ in $E$. Furthermore, the condition actually implies that

$$\{a_1, \ldots, a_n\} \cap \{-a_1, \ldots, -a_n\} = \emptyset \tag{2.8}$$

because $y_a(z)$ and $y_{-a}(z)$ can not have common zeros. For otherwise the Wronskian of $(y_a(z), y_{-a}(z))$ would be identically zero, which forces that $y_a(z)$, $y_{-a}(z)$ are linearly dependent.

**Definition 2.2.** *Suppose that $a = (a_1, \ldots, a_n) \in (E^\times)^n \setminus \Delta_n$ satisfies (2.3). Then $a$ is called a branch point if $\{a_1, \ldots, a_n\} = \{-a_1, \ldots, -a_n\}$ in $E$.*

Note that if $a$ is *not* a branch point, then $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$. By the addition formula

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)},$$

the system (2.3) is equivalent to

$$\sum_{j \neq i} \frac{\wp'(a_i) + \wp'(a_j)}{\wp(a_i) - \wp(a_j)} = 0, \qquad 1 \leq i \leq n. \tag{2.9}$$

The following non-obvious equivalence is crucial for our purpose:

**Proposition 2.3.** *[3, Proposition 5.8.3] Suppose that $a = (a_1, \ldots, a_n) \in (E^\times)^n$ satisfies $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$. Then (2.9) is equivalent to*

$$\sum_{i=1}^{n} \wp'(a_i)\wp(a_i)^l = 0, \qquad 0 \leq l \leq n-2. \tag{2.10}$$

Let $a \in (E^\times)^n \setminus \Delta_n$ satisfy (2.3) and suppose that it is not a branch point. Then (2.10) implies that

$$g_a(z) := \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{C(a)}{\prod_{i=1}^{n}(\wp(z) - \wp(a_i))} \tag{2.11}$$

for a constant $C(a) \neq 0$. Equivalently,

$$C(a) = \sum_{i=1}^{n} \wp'(a_i) \prod_{j \neq i} (\wp(z) - \wp(a_j)). \tag{2.12}$$

There are various ways to represent $C(a)$ by plugging in different values of $z$ in (2.12). For example, for $z = a_i$ we get

$$C(a) = \wp'(a_i) \prod_{j \neq i} (\wp(a_i) - \wp(a_j)) \tag{2.13}$$

which is independent of the choices of $i$. Notice that if $a$ is a branch point then $g_a(z) \equiv 0$ and so $C(a) = 0$.

Then we have the following important result:

**Theorem 2.4.** *[3] There exists a polynomial* $\ell_n(B) = \ell_n(B; g_2, g_3) \in \mathbb{Q}[g_2, g_3][B]$ *of degree* $2n + 1$ *in* $B$ *such that if* $a \in (E^\times)^n \setminus \Delta_n$ *satisfies* (2.3), *then* $C^2 = \ell_n(B)$, *where* $C = C(a)$ *and* $B = B_a$ *are given in* (2.13) *and* (2.7) *respectively.*

This polynomial $\ell_n(B)$ is called the *Lamé polynomial* in the literature.

Let $Y_n = Y_n(\tau) \subset \operatorname{Sym}^n E = E^n / S_n$ be the set of $a = \{a_1, \ldots, a_n\}$ which satisfies (2.2) and (2.3). Clearly $-a := \{-a_1, \ldots, -a_n\} \in Y_n$ if $a \in Y_n$, and $a \in Y_n$ is a *branch point* if $a = -a$ in $E$. Then the map $B : Y_n \to \mathbb{C}$ in (2.7) is a ramified covering of degree 2, and Theorem 2.4 implies that

$$Y_n \cong \left\{ (B, C) \mid C^2 = \ell_n(B) \right\},$$

(cf. [3, Theorem 7.4]). Therefore, $Y_n$ is a hyperelliptic curve, known as the *Lamé curve*. Furthermore, $Y_n$ *is singular at a trivial critical point* $a$ *if and only if* $B_a$ *is a multiple zero of* $\ell_n(B)$. For later usage, we denote

$$X_n := \{a \in Y_n \mid a \text{ is not a branch point}\} \subset Y_n.$$

Since $a$ is a branch point of $Y_n$ if and only if it is a trivial critical point of $G_n$. From now on we will switch these two notions freely.

There are several ways to compute the Lamé polynomial $\ell_n(B)$. A recursive construction can be found in [3, Theorem 7.4].

**Example 2.5.** [1, 3] $\ell_n(B)$ for $n = 1, 2$. Denote $e_k = \wp(\frac{\omega_k}{2})$ for $k = 1, 2, 3$.

(1) $n = 1$, $\bar{X}_1 \cong E$, $C^2 = \ell_1(B) = 4B^3 - g_2 B - g_3 = 4 \prod_{i=1}^3 (B - e_i)$.

(2) $n = 2$ (notice that $e_1 + e_2 + e_3 = 0$),

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81} B^5 - \tfrac{7}{27} g_2 B^3 + \tfrac{1}{3} g_3 B^2 + \tfrac{1}{3} g_2^2 B - g_2 g_3 \\ &= \frac{2^2}{3^4} (B^2 - 3g_2) \prod_{i=1}^3 (B + 3e_i). \end{aligned}$$

Consequently, $\ell_2(B;\tau)$ has multiple zeros if and only if $g_2(\tau)=0$, that is $\tau$ is equivalent to $e^{\pi i/3}$ under the $\mathrm{SL}(2,\mathbb{Z})$ action.

If $a=\{a_1,a_2\}$ is a branch point of $Y_2$, then $\{a_1,a_2\}=\{-a_1,-a_2\}$ in $E$ implies that either (1) $a=\{\frac{1}{2}\omega_i,\frac{1}{2}\omega_j\}$ with $\{i,j,k\}=\{1,2,3\}$, which corresponds to $B_a=3(e_i+e_j)=-3e_k$, or (2) $a_1=-a_2\neq\frac{\omega_k}{2}$. Then $\pm\sqrt{3g_2}=B_a=6\wp(a_1)$, i.e. $\wp(a_1)=\pm\sqrt{g_2/12}$. We conclude that the branch points of $Y_2$ are given by $\{(\frac{1}{2}\omega_i,\frac{1}{2}\omega_j)\mid i\neq j\}$ and $\{(q_\pm,-q_\pm)\mid\wp(q_\pm)=\pm\sqrt{g_2/12}\}$.

From Example 2.5 we see that the singularity of $Y_2(\tau)$ is no worse then a double point for any $\tau\in\mathbb{H}$. It is an old conjecture that this property holds true for all $n\in\mathbb{N}$.

## 3. The invariant $D(a)$ and its geometric meaning

The purpose of this section is to generalize the invariant $D(a)$ studied in [9] for $\rho=8\pi$, where $a$ is a half-period point, to the general case $\rho=8\pi n$ for all $n\in\mathbb{N}$. $D(a)$ is fundamental in analyzing the bubbling behavior of a sequence $u_k$ with $\rho_k\to 8\pi n$. By Theorem A, the bubbling loci $a=\{a_1,\ldots,a_n\}$ must be a branch point of $Y_n$ if $\rho_k\neq 8\pi n$ for $k$ large. Thus it is essential to study the geometric meaning of $D(a)$ at those $2n+1$ branch points as in the case $n=1$ in [9, Theorem 0.4].

For $a=(a_1,\ldots,a_n)\in(E^\times)^n\setminus\Delta_n$ a trivial critical point, we recall (1.7):

$$D(a):=\lim_{r\to 0}\sum_{i=1}^n\mu_i\left(\int_{\Omega_i\setminus B_r(a_i)}\frac{e^{f_{a_i}(z)}-1}{|z-a_i|^4}-\int_{\mathbb{R}^2\setminus\Omega_i}\frac{1}{|z-a_i|^4}\right),\tag{3.1}$$

where $f_{a_i}(z)$, $\mu_i$ are defined in (1.5) and (1.6) respectively. Notice that the sum in the RHS of (3.1) can be written as

$$\sum_{i=1}^n\left(\int_{\Omega_i\setminus B_r(a_i)}\frac{\mu_i e^{f_{a_i}(z)}}{|z-a_i|^4}-\int_{\mathbb{R}^2\setminus B_r(a_i)}\frac{\mu_i}{|z-a_i|^4}\right),$$

where

$$K(z):=\frac{\mu_i e^{f_{a_i}(z)}}{|z-a_i|^4}=\exp\left(8\pi\sum_{j=1}^n G(z,a_j)-8\pi nG(z)\right)\tag{3.2}$$

is independent of $i$. Hence (3.1) of is independent of the choices of $\Omega_i$'s.

From now on, we use notation $p=\{p_1,\ldots,p_n\}$ instead of $a=\{a_1,\ldots,a_n\}$ to denote branch points. Assume that $p=\{p_1,\ldots,p_n\}\in Y_n\setminus X_n$ is a branch point. Then $\{p_1, \ldots, p_n\} = \{-p_1, \ldots, -p_n\}$ and

$$\begin{aligned} K(z) &= \exp 4\pi \left( \sum_{j=1}^{n} \big(G(z, p_j) + G(z, -p_j) - 2G(z)\big) \right) \\ &= e^c \prod_{i=1}^{n} |\wp(z) - \wp(p_i)|^{-2} \end{aligned} \tag{3.3}$$

for some constant $c \in \mathbb{R}$. The last equality follows by the comparison of singularities. We remark here that, in comparison with [9, §2], for non-half period points the simultaneous appearance of $\pm p_i$ is essential to arrive at the above simple looking closed form.

For convenience, we define $\Lambda_2 = \{\, i \mid p_i \in E[2]\,\}$, the two-torsion part, and for $i \notin \Lambda_2$ we define $i^* \notin \Lambda_2$ to be the index so that $p_{i^*} = -p_i$.

Choose a sequence $a^k \in X_n$ with $\lim_{k\to\infty} a^k = p$. For ease of notations we drop the index $k$ and simply denote $a = (a_1, \ldots, a_n) \to (p_1, \ldots, p_n)$.

In §2 we show that $a \in X_n$ is equivalent to the following equation:

$$g_a(z) := \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{C(a)}{\prod_{i=1}^{n}(\wp(z) - \wp(a_i))} \tag{3.4}$$

(so that $\operatorname{ord}_{z=0} g_a(z) = 2n$) for a constant $C(a) \neq 0$ given by

$$C(a) = \wp'(a_i) \prod_{j\neq i} (\wp(a_i) - \wp(a_j)), \quad \text{for any } i = 1, \ldots, n.$$

For $a \in Y_n$, $C(a) = 0$ if and only if $a$ is a branch point. It is easy to describe the behavior of the limit $C(a) \to C(p) = 0$ as $a \to p$:

**Lemma 3.1.** *Let $p \in Y_n \setminus X_n$ and $a \in X_n$ near $p$. If $i \in \Lambda_2$ then*

$$C(a) = \wp''(p_i) \prod_{j\neq i} (\wp(p_i) - \wp(p_j))(a_i - p_i) + o(|a_i - p_i|), \tag{3.5}$$

*and if $i \notin \Lambda_2$ then*

$$C(a) = \wp'(p_i)^2 \prod_{j\neq i,\, i^*} (\wp(p_i) - \wp(p_j))(a_i + a_{i^*}) + o(|a_i + a_{i^*}|). \tag{3.6}$$

**Lemma 3.2.** *For $p \in Y_n$ being a branch point, the residue for*

$$P_p(z) := \prod_{i=1}^{n} (\wp(z) - \wp(p_i))^{-1}$$

*at $p_i$ is zero for all $i = 1, \ldots, n$.*

*Proof.* Choose $a \in X_n$ with $a \to p$ as above. We compute from (3.4) that

$$P_a(z) = \frac{g_a(z)}{C(a)} = \frac{1}{C(a)} \sum_{i \in \Lambda_2} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} + \frac{1}{2C(a)} \sum_{i \notin \Lambda_2} \left( \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} + \frac{\wp'(a_{i^*})}{\wp(z) - \wp(a_{i^*})} \right).$$

By Lemma 3.1, the first sum has limit

$$\sum_{i \in \Lambda_2} \frac{\prod_{j \neq i} (\wp(p_i) - \wp(p_j))^{-1}}{\wp(z) - \wp(p_i)}$$

when $a \to p$, which obviously has zero residue at $p_i$ because $i \in \Lambda_2$ means $p_i = \frac{1}{2}\omega_k$ in $E$ for some $k \in \{1, 2, 3\}$.

For the second sum, we rewrite each $i$-th summand as

$$\frac{1}{2C} \frac{\wp'(a_i) - \wp'(-a_{i^*})}{\wp(z) - \wp(a_i)} - \frac{\wp'(a_{i^*})}{2C} \frac{\wp(a_i) - \wp(a_{i^*})}{(\wp(z) - \wp(a_i))(\wp(z) - \wp(-a_{i^*}))},$$

which has limit

$$\frac{1}{2} \left( \frac{\wp''(p_i)}{\wp'(p_i)^2} \frac{1}{\wp(z) - \wp(p_i)} + \frac{1}{(\wp(z) - \wp(p_i))^2} \right) \prod_{j \neq i,\, i^*} (\wp(p_i) - \wp(p_j))^{-1}.$$

A direct Taylor expansion shows that the residues of both terms at $p_i$ ($i \notin \Lambda_2$) cancel out with each other. This proves the lemma. □

By Lemma 3.2, we may rewrite

$$P_p(z) = \prod_{i=1}^{n} (\wp(z) - \wp(p_i))^{-1} = \sum_{j=1}^{n} c_j \wp(z - p_j) + c_0. \tag{3.7}$$

Since the vanishing order of the LHS at $z = 0$ is $2n$, the coefficients must satisfy the constraints

$$\sum_{j=1}^{n} c_j \wp(p_j) + c_0 = 0, \tag{3.8}$$
$$\sum_{j=1}^{n} c_j \wp^{(k)}(-p_j) = 0, \quad \text{for } k = 1, \ldots, 2n-1.$$

Also, it is easy to see from (3.7) that for $i \in \Lambda_2$,

$$c_i = 2\wp''(p_i)^{-1} \prod_{j \neq i} (\wp(p_i) - \wp(p_j))^{-1}, \tag{3.9}$$

and if $i \notin \Lambda_2$ then

$$c_i = \wp'(p_i)^{-2} \prod_{j \neq i, i^*} (\wp(p_i) - \wp(p_j))^{-1}. \tag{3.10}$$

In particular $c_{i^*} = c_i$.

This vector $\vec{c} = (c_1, \ldots, c_n)$ indeed has important geometric meaning:

**Lemma 3.3.** *By considering $C$ as the local holomorphic coordinate for (a branch of) the hyperelliptic curve $Y_n \ni a(C)$ near a branch point $p$, then we have $a'(0) = \vec{c}/2$. Moreover,*

$$\frac{\partial a_j}{\partial C}(0) = \frac{c_j}{2} \notin \{0, \infty\}$$

*for $j = 1, \ldots, n$.*

*Proof.* We first show that if $i \notin \Lambda_2$ then

$$\frac{\partial a_i}{\partial C}(0) = \frac{\partial a_{i^*}}{\partial C}(0). \tag{3.11}$$

Suppose that $a(C) = (a_i(C))$ represents the point $(B, C) \in Y_n$ close to $p$, where $B = (2n-1)\sum_{i=1}^{n} \wp(a_i(C))$. Then $\tilde{a}(C) = (a_i(-C))$ represent the other point $(B, -C)$ with the same $B$. That is, $B = (2n-1)\sum_{i=1}^{n} \wp(a_i(-C))$

too. By the hyperelliptic structure on $Y_n$, we conclude that

$$\{a_1(-C), \ldots, a_n(-C)\} = \{-a_1(C), \ldots, -a_n(C)\}.$$

If $i \notin \Lambda_2$, then we must have $a_i(-C) = -a_{i^*}(C)$ and $a_{i^*}(-C) = -a_i(C)$. Therefore, $a_i(-C) + a_{i^*}(-C) = -(a_i(C) + a_{i^*}(C))$ and

$$a_i(-C) - a_{i^*}(-C) = a_i(C) - a_{i^*}(C).$$

That is, $a_i(C) - a_{i^*}(C)$ is even in $C$, which implies (3.11).

The lemma now follows from (3.5)-(3.6) in Lemma 3.1. For example, if $i \in \Lambda_2$, then (3.5) implies $\lim_{C\to 0} \frac{a_i(C)-p_i}{C} = \frac{c_i}{2}$. If $i \notin \Lambda_2$, then (3.11) and (3.6) imply

$$2\frac{\partial a_i}{\partial C}(0) = \frac{\partial a_i}{\partial C}(0) + \frac{\partial a_{i^*}}{\partial C}(0) = \lim_{C\to 0} \frac{a_i(C) + a_{i^*}(C)}{C} = c_i.$$

Notice that the property $c_j \neq 0, \infty$ for all $j$ is clear from the expressions in (3.9) and (3.10) since (i) $p_i \notin \Lambda$ for all $i$ and $\wp(p_i) \neq \wp(p_j)$ for all $i \neq j$, and (ii) $\wp''(p_i) \neq 0$ for $i \in \Lambda_2$ and $\wp'(p_i) \neq 0$ for $i \notin \Lambda_2$. $\square$

Using the tangent vector $\vec{c}$, we may derive a simple formula for $D(p)$.

**Theorem 3.4.** *Let $p \in Y_n \setminus X_n$ be a branch point of the hyperelliptic curve $Y_n$ defined by $C^2 = \ell_n(B)$. Consider the local parameter $C$ near $p$ and let $\vec{c} = 2a'(0) = 2\partial a/\partial C|_{C=0}$. Denote also by $s = \sum_{j=1}^n c_j$ and $c_0 = -\sum_{j=1}^n c_j \wp(p_j)$. Then*

$$D(p) = \operatorname{Im}\tau \cdot e^c \left( |c_0 - s\eta_1|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Re} \bar{s}(c_0 - s\eta_1) \right) \tag{3.12}$$
$$= \operatorname{Im}\tau \cdot e^c |s|^2 \left( \left| \frac{c_0}{s} - \eta_1 \right|^2 + \frac{2\pi}{\operatorname{Im}\tau} \operatorname{Re} \left( \frac{c_0}{s} - \eta_1 \right) \right).$$

*Proof.* By Lemma 3.3, $\vec{c}$ coincides with the vector formed by the coefficients $c_1, \ldots, c_n$ appeared in the expansion formula of $P_p(z)$ in (3.7).

Let $T \subset \mathbb{R}^2$ be a fundamental domain of $E_\tau$ with $p \cap \partial T = \emptyset$. Then

$$D(p) = \lim_{r\to 0} \left( e^c \int_{T \setminus \cup_i B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^n \int_{\mathbb{R}^2 \setminus B_r(p_i)} \frac{\mu_i}{|z - p_i|^4} \right) \tag{3.13}$$
$$= \lim_{r\to 0} \left( e^c \int_{T \setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^n \frac{\pi \mu_i}{r^2} \right).$$

Consider an anti-derivative of $P_p(z)$:

$$L_p(z) := \int_0^z P_p(w)\,dw = -\sum_{j=1}^n c_j \zeta(z - p_j) + c_0 z. \tag{3.14}$$

For $i = 1, 2$, we define the “quasi-periods” $\chi_i$ by

$$\chi_i = L_p(z + \omega_i) - L_p(z) = c_0\omega_i - s\eta_i. \tag{3.15}$$

To compute $D(p)$, we note from (3.7) that

$$P_p(z) = \frac{c_i}{(z - p_i)^2} + O(1)$$

and from (3.2), the definition (1.6) of $\mu_i$ that

$$K(z) = \frac{\mu_i}{|z - p_i|^4} + O(|z - p_i|^{-2}).$$

Inserting these into (3.3) leads to

$$\mu_i = e^c |c_i|^2, \quad \forall 1 \le i \le n. \tag{3.16}$$

Now we denote

$$L_p(z) = u + \sqrt{-1}v, \quad z = x + \sqrt{-1}y.$$

Then $P_p(z) = L_p'(z) = u_x - iu_y$, i.e.

$$|P_p(z)|^2 = u_x^2 + u_y^2 = (uu_x)_x + (uu_y)_y \text{ for } z \text{ outside } \{p_1, \cdots, p_n\},$$

so

$$\begin{aligned}\int_{T \setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2 &= \int_{\partial T} \big(uu_x dy - uu_y dx\big) \\ &\quad - \sum_{i=1}^n \int_{|z - p_i| = r} \big(uu_x dy - uu_y dx\big).\end{aligned}$$

Applying (3.15) we obtain

$$\begin{aligned}\int_{\partial T} \big(uu_x dy - uu_y dx\big) &= \int_{\partial T} u dv = -\frac{1}{2}\operatorname{Im}\int_{\partial T} L_p d\bar{L}_p \\ &= \frac{1}{2}\operatorname{Im}(\bar{\chi}_1\chi_2 - \chi_1\bar{\chi}_2).\end{aligned} \tag{3.17}$$

Since near $p_i$,

$$u + \sqrt{-1}v = L_p(z) = -\frac{c_i}{z - p_i} + f(z),$$

where $f(z)$ is holomorphic in a neighborhood of $p_i$, it is easy to prove that

$$-\int_{|z-p_i|=r} \big(uu_x dy - uu_y dx\big) = -\int_{|z-p_i|=r} u dv = \frac{\pi |c_i|^2}{r^2} + O(r).$$

Therefore, we conclude from (3.16) and (3.17) that

$$\begin{aligned} D(p) &= \lim_{r\to 0} \left( e^c \int_{T\setminus \cup_{i=1}^n B_r(p_i)} |P_p(z)|^2 - \sum_{i=1}^n \frac{\pi \mu_i}{r^2} \right) \\ &= \frac{e^c}{2} \operatorname{Im}(\bar{\chi}_1 \chi_2 - \chi_1 \bar{\chi}_2). \end{aligned} \tag{3.18}$$

By direct substitution, we compute

$$\begin{aligned} \bar{\chi}_1 \chi_2 - \chi_1 \bar{\chi}_2 = |c_0|^2 (\bar{\omega}_1 \omega_2 - \omega_1 \bar{\omega}_2) + |s|^2 (\bar{\eta}_1 \eta_2 - \eta_1 \bar{\eta}_2) \\ + \bar{c}_0 s (\eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1) - c_0 \bar{s} (\bar{\eta}_1 \omega_2 - \bar{\eta}_2 \omega_1). \end{aligned}$$

Now we plug in $\omega_1 = 1$, $\omega_2 = \tau = a + bi$, and use the Legendre relation $\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$. Then

$$\begin{aligned} \bar{\omega}_1 \omega_2 - \omega_1 \bar{\omega}_2 &= 2ib, \\ \bar{\eta}_1 \eta_2 - \eta_1 \bar{\eta}_2 &= 2i(b|\eta_1|^2 - \pi(\eta_1 + \bar{\eta}_1)), \\ \eta_1 \bar{\omega}_2 - \eta_2 \bar{\omega}_1 &= 2i(\pi - \eta_1 b). \end{aligned}$$

Hence

$$\begin{aligned} D(p) &= e^c \left( b(|c_0|^2 + |s\eta_1|^2) + 2\operatorname{Re}\big(c_0 \bar{s}(\pi - \bar{\eta}_1 b) - |s|^2 \pi \eta_1\big)\right) \\ &= be^c \left( |c_0 - s\eta_1|^2 + \frac{2\pi}{b} \operatorname{Re} \bar{s}(c_0 - s\eta_1) \right). \end{aligned}$$

This proves the theorem. □

In fact there is a simple geometric interpretation of the expression appeared in the RHS of (3.12).

**Proposition 3.5.** *Consider the vector-valued map* $(E^\times)^n \to \mathbb{R}^2$ *defined by*

$$a \mapsto \phi(a) := -4\pi \sum_{i=1}^{n} \nabla G(a_i).$$

*Let* $C = u + iv \mapsto a(C) \in E^n$ *be a local holomorphic parametrization of a Riemann surface* $V \subset E^n$. *Then the Jacobian* $J(\phi \circ a)(u,v)$ *is given by*

$$\det\left(\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}\right) = -\left(|c_0 - s\eta_1|^2 + \frac{2\pi}{\operatorname{Im}\tau}\operatorname{Re}\bar{s}(c_0 - s\eta_1)\right), \tag{3.19}$$

*where* $\vec{c} = (c_i) := 2a'(C)$, $s := \sum_{i=1}^n c_i$, *and* $c_0 := -\sum_{i=1}^n c_i \wp(a_i)$.

*Proof.* Denote $a_j = x_j + \sqrt{-1}y_i$, $b = \operatorname{Im}\tau$ and $\phi = (\phi_1, \phi_2)^T$. By (2.1), we have

$$\begin{aligned} \phi_1 &= 2\operatorname{Re}\left(\sum_i \zeta(a_i) - \eta_1 a_i\right), \\ \phi_2 &= -2\operatorname{Im}\left(\sum_i \zeta(a_i) - \eta_1 a_i\right) - \frac{4\pi}{b}\sum y_i. \end{aligned} \tag{3.20}$$

The chain rule shows that

$$\begin{aligned} \partial_u \phi_1 &= -2\operatorname{Re}\left[\sum_i (\wp(a_i) + \eta_1)\frac{\partial a_i}{\partial C}\right] = \operatorname{Re}(c_0 - s\eta_1), \\ \partial_v \phi_1 &= -2\operatorname{Re}\left[\sum_i (\wp(a_i) + \eta_1)\frac{\partial a_i}{\partial C}\sqrt{-1}\right] = -\operatorname{Im}(c_0 - s\eta_1), \\ \partial_u \phi_2 &= 2\operatorname{Im}\left[\sum_i (\wp(a_i) + \eta_1)\frac{\partial a_i}{\partial C}\right] - \frac{4\pi}{b}\sum_i \frac{\partial y_i}{\partial u} \\ &= -\operatorname{Im}(c_0 - s\eta_1) - \frac{2\pi}{b}\operatorname{Im} s, \\ \partial_v \phi_2 &= 2\operatorname{Im}\left[\sum_i (\wp(a_i) + \eta_1)\frac{\partial a_i}{\partial C}\sqrt{-1}\right] - \frac{4\pi}{b}\sum_i \frac{\partial y_i}{\partial v} \\ &= -\operatorname{Re}(c_0 - s\eta_1) - \frac{2\pi}{b}\operatorname{Re} s. \end{aligned}$$

Hence the Jacobian is given by

$$\begin{aligned}&-|c_0 - s\eta_1|^2 - \frac{2\pi}{b}(\mathrm{Re}\,(c_0 - s\eta_1)\,\mathrm{Re}\,s + \mathrm{Im}\,(c_0 - s\eta_1)\,\mathrm{Im}\,s)\\ &= -\left(|c_0 - s\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}\,\bar{s}(c_0 - s\eta_1)\right)\end{aligned}$$

as expected. □

**Corollary 3.6.** *For $p \in Y_n \setminus X_n$ with local coordinate $C$, we have*

$$D(p) = -\mathrm{Im}\tau\, e^c J(\phi \circ a)(0) \tag{3.21}$$

*for some constant $c$.*

*Proof.* This follows from Theorem 3.4 and Proposition 3.5. □

Corollary 3.6 will play important role in our subsequent degeneration analysis of these branch points $p \in Y_n \setminus X_n$. One may also interpret the above proof of it as a stationary phase integral calculation.

**Example 3.7.** For $n = 1$, $c_0 = -c_1\wp(p_1) = -c_1 e_i$ if $p_1 = \frac{1}{2}\omega_i$, and $s = c_1$. The formula reduces to the one for $\det D^2G(p)$ first studied in [8]:

$$|e_i + \eta_1|^2 - \frac{2\pi}{\mathrm{Im}\tau}\mathrm{Re}(e_i + \eta_1).$$

## 4. Proof of Theorem 1.1: Analytic adjunction

It is elementary to see that for $\chi_1 = a_1 + b_1 i$ and $\chi_2 = a_2 + b_2 i$,

$$\bar{\chi}_1\chi_2 - \chi_1\bar{\chi}_2 = 2i(a_1b_2 - a_2b_1).$$

Hence the formula in (3.18) says that $D(p)$ is exactly $e^c$ times the signed area spanned by $\chi_1$ and $\chi_2$ in $\mathbb{R}^2$. Indeed, $\chi_1 = c_0 - s\eta_1 = -\sum_{j=1}^n c_j(\wp(p_j) + \eta_1)$. So we may rewrite (3.12) as

$$D(p) = -\mathrm{Im}\tau\, e^c|s|^2 \begin{vmatrix} -\mathrm{Re}\,\chi_1 s^{-1} & +\mathrm{Im}\,\chi_1 s^{-1} \\ +\mathrm{Im}\,\chi_1 s^{-1} & \mathrm{Re}\,\chi_1 s^{-1} + \dfrac{2\pi}{\mathrm{Im}\tau} \end{vmatrix}. \tag{4.1}$$

Formula (4.1) suggests the possibility for interpreting $D(p)$ in terms of the determinant of the Hessian of some "Green function" for general $n \in \mathbb{N}$. To find such a Green function on $\bar{X}_n$ will require the search for a suitable conformal metric on it. Alternatively we consider the multiple Green function $G_n$ defined in (1.8):

$$G_n(z_1, \ldots, z_n) := \sum_{i<j} G(z_i - z_j) - n \sum_{i=1}^{n} G(z_i)$$

for $z = (z_1, \ldots, z_n) \in (E^\times)^n \setminus \Delta_n$. Then $G_n$ is a Green function on $E^n$ with divisor $D_n$ where $(E^\times)^n \setminus \Delta_n = E^n \setminus D_n$. Recall that *$p$ is a branch point of $Y_n$ if and only if it is a trivial critical point of $G_n$.*

**Theorem 4.1 (Analytic adjunction formula).** *For any fixed $n \in \mathbb{N}$ and any branch point $p = (p_1, \ldots, p_n) \in Y_n$, there is a constant $c_p \geq 0$ such that*

$$\det D^2 G_n(p) = (-1)^n c_p D(p).$$

*Moreover, $c_p = 0$ precisely when the associated hyperelliptic curve $Y_n(\tau)$ for $E = E_\tau$ is singular at $p$. There are only finitely many such tori $E_\tau$ for each $n$.*

For $n = 1$, this is [9, Theorem 0.4]. For $n = 2$, a direct check based on Theorem 3.4 is still possible (c.f. Example 4.2). For $n \geq 3$ the $D^2 G_n$ is a $2n \times 2n$ matrix and it is cumbersome to compute $\det D^2 G_n(p)$ directly. The proof of Theorem 4.1 given below is based on Corollary 3.6.

*Proof.* It was proved in [3, §5.3] (recalled in (2.3)–(2.4)) that the system of equations (1.2) given by $-2\pi \nabla G_n(a) = 0$ is equivalent to holomorphic equations $g^1(a) = \cdots = g^{n-1}(a) = 0$ with

$$g^i(a) = \sum_{j \neq i}^{n} (\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)), \quad 1 \leq i \leq n-1, \tag{4.2}$$

which defines $Y_n$, and the non-holomorphic equation $g^n(a) = 0$ with

$$g^n(a) = \tfrac{1}{2}\phi(a) = -2\pi \sum_{i=1}^{n} \nabla G(a_i). \tag{4.3}$$

By (2.1), we easily obtain for $1 \leq i \leq n-1$,

$$g^i(a) = -2\pi \left( \sum_{j \neq i} 2G_z(a_i - a_j) - 2nG_z(a_i) + \sum_{j=1}^{n} 2G_z(a_j) \right). \tag{4.4}$$

For any $i$, we have

$$\nabla_i G_n(a) = \sum_{j\neq i} \nabla G(a_i - a_j) - n\nabla G(a_i).$$

By taking into account that $\nabla G \mapsto 2G_z$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $g^n = \frac{2\pi}{n}\sum_{i=1}^n \nabla_i G_n$, the equivalence between the map $a \mapsto g(a) := (g^1(a), \dots, g^n(a))^T$ and $-2\pi\nabla G_n$ is induced by a real $2n \times 2n$ matrix $A$ given by

$$A = \begin{bmatrix} 1 & & & & & 1 & \\ & 1 & & & & & -1 \\ & & \ddots & & & \vdots & \vdots \\ & & & 1 & & 1 & \\ & & & & 1 & & -1 \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ \frac{-1}{n} & & \cdots & \frac{-1}{n} & & \frac{-1}{n} & & \\ & \frac{-1}{n} & \cdots & & \frac{-1}{n} & & \frac{-1}{n} \end{bmatrix}.$$

In other words, by considering $g^k = (\mathrm{Re}g^k, \mathrm{Im}g^k)^T$ for $1 \le k \le n-1$ and $G_z = (\mathrm{Re}G_z, \mathrm{Im}G_z)^T$, we have $2G_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\nabla G$. Inserting this into (4.4), it is easy to obtain $g(a) = -2\pi A\nabla G_n(a)$. Consequently,

$$J(g)(a) = J(-2\pi A\nabla G_n)(a) = \frac{(-1)^{n-1}}{n^2}(2\pi)^{2n} \det D^2 G_n(a), \tag{4.5}$$

so it suffices to compute (the real Jacobian) $J(g)$.

Let $p \in Y_n \setminus X_n$ and consider a holomorphic parametrization $C \mapsto a(C)$ of a branch of $Y_n$ near $p$, where $p$ corresponds to $C = 0$. Notice that if $p$ is not a singular point of $Y_n$, i.e. $B_p$ is a simple zero of $\ell_n(B) = 0$, then there is only one branch of $Y_n$ near $p$ and the map $C \to a(C)$ is unique.

We denote

$$C = u + \sqrt{-1}v, \quad a_k = x^k + \sqrt{-1}y^k, \quad g^k = U^k + \sqrt{-1}V^k, \quad 1 \le k \le n.$$

Along $Y_n$ we have by chain rule (denote $g^i_j = \partial g^i/\partial a_j$)

$$0 = \frac{\partial g^i}{\partial C} = \sum_{j=1}^n g^i_j \frac{\partial a_j}{\partial C}, \qquad 1 \le i \le n-1. \tag{4.6}$$

If $g^n$ is also holomorphic, then (4.6) can be used to evaluate the "complex determinant" $\det D^{\mathbb{C}} g = \det(g^i_j)$ by elementary column operations. For example, if $\partial a_k/\partial C \neq 0$ then we may eliminate all the entries of the $k$-th column except the last ($n$-th) one. The case $k = n$ reads as:

$$\det D^{\mathbb{C}} g = \det(g^i_j)_{i,j=1}^{n-1} \times \frac{\partial g^n}{\partial C} \times \left(\frac{\partial a_n}{\partial C}\right)^{-1}. \tag{4.7}$$

In the current case $g^n = \frac{1}{2}\phi$ is not holomorphic (see (3.20) for the additional linear term $-2\pi \sum_k y^k/b$ for $V^n = \frac{1}{2}\phi_2$). The same argument via implicit functions still applies if we work with the real components $U^k, V^k$ and real variables $x^k, y^k$ and $u, v$ instead.

More precisely, (4.6) takes the real form: For $1 \leq i \leq n-1$,

$$0 = \begin{bmatrix} U^i_u & U^i_v \\ V^i_u & V^i_v \end{bmatrix} = \sum_{k=1}^n \begin{bmatrix} U^i_{x^k} & U^i_{y^k} \\ V^i_{x^k} & V^i_{y^k} \end{bmatrix} \begin{bmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{bmatrix}. \tag{4.8}$$

The two rows are equivalent by the Cauchy–Riemann equation.

The elementary column operation on the $2n \times 2n$ real jacobian matrix $Dg$ is now replaced by the right multiplication with the matrix

$$R_n := \begin{bmatrix} 1 & & & x^1_u & x^1_v \\ & 1 & & y^1_u & y^1_v \\ & & \ddots & \vdots & \vdots \\ & & & x^n_u & x^n_v \\ & & & y^n_u & y^n_v \end{bmatrix}.$$

In fact we may do so for any $(2k-1, 2k)$-th pair of columns—since

$$\begin{vmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{vmatrix} = |a_k'(0)|^2 \neq 0, \infty$$

by Lemma 3.3, and get a similar matrix $R_k$. We take $R = R_n$ below.

Denote by $D'g$ the principal $2(n-1) \times 2(n-1)$ sub-matrix of $Dg$. Notice from (4.3) that

$$\tfrac{1}{2} D(\phi \circ a) = \begin{bmatrix} U^n_u & U^n_v \\ V^n_u & V^n_v \end{bmatrix} = \sum_{k=1}^n \begin{bmatrix} U^n_{x^k} & U^n_{y^k} \\ V^n_{x^k} & V^n_{y^k} \end{bmatrix} \begin{bmatrix} x^k_u & x^k_v \\ y^k_u & y^k_v \end{bmatrix},$$

which is precisely the right bottom $2 \times 2$ sub-matrix of $(Dg)R$. Hence it follows from (4.8) that

$$(Dg)R = \begin{bmatrix} D'g & 0 \\ * & \frac{1}{2} D(\phi \circ a) \end{bmatrix},$$

which can be used to calculate the determinant:

$$\det Dg \det R = \det((Dg)R) = \det D'g \det \tfrac{1}{2}D(\phi \circ a).$$

By (4.5) and Corollary 3.6, we get

$$\det D^2 G_n(p) = \frac{(-1)^n n^2 e^{-c}}{4\mathrm{Im}\,\tau\,(2\pi)^{2n}} \frac{|\det D'^{\mathbb{C}} g(p)|^2}{|a_n'(0)|^2} D(p),$$

where $D'^{\mathbb{C}}g$ is the principal $(n-1)\times(n-1)$ sub-matrix of $D^{\mathbb{C}}g$. Here we used $\det D'g(p) = |\det D'^{\mathbb{C}}g(p)|^2$ because $g^k$ is holomorphic for any $1 \le k \le n-1$. Thus

$$c_p = \frac{n^2 e^{-c}}{4\mathrm{Im}\,\tau\,(2\pi)^{2n}} \frac{|\det D'^{\mathbb{C}} g(p)|^2}{|a_n'(0)|^2} \ge 0. \tag{4.9}$$

To complete the proof of Theorem 4.1, we recall the standard Jacobian criterion for smoothness of the point $p \in Y_n$. Since $g^1 = 0, \ldots, g^{n-1} = 0$ are the defining equations for $Y_n$, $p \in Y_n$ is a non-singular point if and only if there is some $(n-1)\times(n-1)$ minor of the $(n-1)\times n$ matrix $D^{\mathbb{C}}\tilde{g}(p)$ which does not vanish at $p$, where $\tilde{g} := (g^1, \ldots, g^{n-1})^T$. Notice that (4.9) is valid for all choices of those minors (with $a_n'(0)$ being replaced by $a_k'(0)$), thus $p \in Y_n$ is non-singular is indeed equivalent to $\det D'^{\mathbb{C}}g(p) \neq 0$ (which actually implies that any $(n-1)\times(n-1)$ minor does not vanish at $p$). Since $p \in Y_n \setminus X_n$ is a branch point and $Y_n$ is defined by the hyperelliptic equation $C^2 = \ell_n(B)$, this is precisely the case when $B_p$ is a simple zero of $\ell_n(B) = 0$. The proof is complete. □

**Example 4.2 (The case $n = 2$).** For any flat torus $E_\tau$ and $p \in Y_2(\tau) \setminus X_2(\tau)$, we compute directly the constant $c_p = c_p(\tau) \ge 0$ such that

$$\det D^2 G_2(p) = c_p D(p).$$

To serve as a consistency check with (4.9) we will not follow the procedure used in the proof of Theorem 4.1. Instead we will compute $\det D^2 G_2(p)$ directly. It will be clear that $c_p(\tau) > 0$ if $\tau \not\equiv e^{\pi/3}$ under the $\mathrm{SL}(2,\mathbb{Z})$ action.

By Example 2.5 (2), we see that the five branch points in $Y_n$ are given by $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) \mid i \neq j\}$ and $\{(q_\pm, -q_\pm) \mid \wp(q_\pm) = \pm\sqrt{g_2/12}\}$. Note that $\wp^2(q) = \frac{1}{12}g_2$ if and only if $\wp''(q) = 0$. The only case that these five points reduce to four points is when $g_2 = 0$. This happens precisely when $\tau \equiv e^{\pi/3}$ and then $\bar{Y}_n$ becomes a singular (nodal) hyperelliptic curve.

To compute the Hessian of $G_2$, we recall the formulae [9, (2.4) and (2.5)] for the second partial derivatives of $G$. Namely,

$$\det D^2G = \frac{-1}{4\pi^2}\left(|(\log\vartheta)_{zz}|^2 + \frac{2\pi}{b}\mathrm{Re}\,(\log\vartheta)_{zz}\right), \tag{4.10}$$

where $b = \mathrm{Im}\,\tau$ and in terms of the Weierstrass theory

$$(\log\vartheta)_{zz}(\tfrac{1}{2}\omega_i) = -\wp(\tfrac{1}{2}\omega_i) - \eta_1 = -(e_i + \eta_1), \tag{4.11}$$

where $(\log\vartheta)_z(z) = \zeta(z) - \eta_1 z$ is used.

First we compute $D^2G(p)$ for $p = (\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$. Denote by $\frac{1}{2}\omega_k$ the third remaining half period point. Notice that $\wp(\frac{1}{2}\omega_i - \frac{1}{2}\omega_j) = \wp(\frac{1}{2}\omega_k) = e_k$. For simplicity we write

$$w_k := (\log\vartheta)_{zz}(\tfrac{1}{2}\omega_k) = -(e_k + \eta_1) = u_k + v_k i,$$

and similarly for the indices $i, j$. Then we have

$$D^2G_2(p) = \frac{1}{2\pi}\begin{pmatrix} -u_k + 2u_i & v_k - 2v_i & u_k & -v_k \\ v_k - 2v_i & u_k - 2u_i - \frac{2\pi}{b} & -v_k & -u_k - \frac{2\pi}{b} \\ u_k & -v_k & -u_k + 2u_j & v_k - 2v_j \\ -v_k & -u_k - \frac{2\pi}{b} & v_k - 2v_j & u_k - 2u_j - \frac{2\pi}{b} \end{pmatrix}.$$

A lengthy yet straightforward calculation shows that

$$\begin{aligned} &\det D^2G_2(p) = \\ &\quad \frac{4}{(2\pi)^4}\left(|2e_ie_j + e_k^2 - 3e_k\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}\left(3\bar{e}_k(2e_ie_j + e_k^2 - 3e_k\eta_1)\right)\right). \end{aligned} \tag{4.12}$$

The details will be omitted here. We only note that when $\tau \in i\mathbb{R}$, all $e_l$'s and $\eta_1$ are real numbers. Thus all the imaginary parts vanish: $v_1 = v_2 = v_3 = 0$. In this case (4.12) can be verified easily.

By (3.9) and the fact that $\wp''(\frac{1}{2}\omega_i) = 2(e_i - e_j)(e_i - e_k)$, we compute

$$\begin{aligned} c_1 &= 2\wp''(\tfrac{1}{2}\omega_i)^{-1}(e_i - e_j)^{-1} = (e_i - e_j)^{-2}(e_i - e_k)^{-1}, \\ c_2 &= (e_j - e_i)^{-2}(e_j - e_k)^{-1}, \\ s &= c_1 + c_2 = -3e_k(e_i - e_j)^{-2}(e_i - e_k)^{-1}(e_j - e_k)^{-1}, \\ c_0 &= -(c_1e_i + c_2e_j) = -(2e_ie_j + e_k^2)(e_i - e_j)^{-2}(e_i - e_k)^{-1}(e_j - e_k)^{-1}. \end{aligned}$$

By Theorem 3.4, we get

$$D(p) = c(p)\left(|2e_ie_j + e_k^2 - 3e_k\eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}\left(3\bar{e}_k(2e_ie_j + e_k^2 - 3e_k\eta_1)\right)\right)$$

with $c(p) = be^c|e_i - e_j|^{-2}|e_i - e_k|^{-1}|e_j - e_k|^{-1}$. Thus

$$\det D^2G_2(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j) = c_p D(\tfrac{1}{2}\omega_i, \tfrac{1}{2}\omega_j)$$

with

$$c_p = (4\pi^4 c(p))^{-1} = e^{-c}|e_i - e_j||e_i - e_k||e_j - e_k|/(4b\pi^4) > 0. \tag{4.13}$$

Next we consider $p = (q, -q)$ with $q \in \{q_+, q_-\}$. Let $\mu = \wp(q)$. Since $\wp''(q) = 0$, we have also $\wp(2q) = -2\wp(q) = -2\mu$ by the addition (duplication) formula. Denote by $\mu = u + iv$ and $\eta_1 = s + it$. Then we have

$$D^2G_2(p) = \frac{1}{2\pi}\begin{pmatrix} -4u - s & 4v + t & 2u - s & -2v + t \\ 4v + t & 4u + s - \frac{2\pi}{b} & -2v + t & -2u + s - \frac{2\pi}{b} \\ 2u - s & -2v + t & -4u - s & 4v + t \\ -2v + t & -2u + s - \frac{2\pi}{b} & 4v + t & 4u + s - \frac{2\pi}{b} \end{pmatrix}.$$

A straightforward calculation easier than the previous case shows that the determinant is given by

$$\begin{aligned} \det D^2G_2(p) &= \frac{144}{(2\pi)^4}(u^2 + v^2)\left((u + s)^2 + (v + t)^2 - \frac{2\pi}{b}(u + s)\right) \\ &= \frac{9}{\pi^4}|\wp(q)|^2\left(|\wp(q) + \eta_1|^2 - \frac{2\pi}{b}\mathrm{Re}(\wp(q) + \eta_1)\right). \end{aligned} \tag{4.14}$$

By (3.10), we compute easily that $c_1 = c_2 = \wp'(q)^{-2}$, $c_0 = -2\wp(q)\wp'(q)^{-2}$, and $s = c_1 + c_2 = 2\wp'(q)^{-2}$. Hence by Theorem 3.4

$$\begin{aligned} D(p) &= 4be^c|\wp'(q)|^{-4}\left(|-\wp(q) - \eta_1|^2 + \frac{2\pi}{b}\mathrm{Re}(-\wp(q) - \eta_1)\right) \\ &= c_p^{-1}\det D^2G_2(p), \end{aligned}$$

where

$$c_p = \frac{9e^{-c}}{4b\pi^4}|\wp(q)|^2|\wp'(q)|^4 \geq 0. \tag{4.15}$$

Since $\wp'(q) \neq 0$, $c_p > 0$ unless $\wp(q) = \pm\sqrt{g_2/12} = 0$. This is the case precisely when $\tau$ is equivalent to $e^{\pi i/3}$. We leave the simple consistency check of (4.13) and (4.15) with the general formula (4.9) to the readers.

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