Communications in
Analysis and Geometry
Volume 28, Number 6, 1337–1406, 2020

On the Morse index of Willmore spheres in S^3

Alexis Michelat

We obtain an upper bound for the Morse index of Willmore spheres $\Sigma \subset S^3$ coming from an immersion of S^2. The quantization of Willmore energy, which is a consequence of the classification of Willmore spheres in S^3 by Robert Bryant, shows that there exists an integer m such that $\mathscr{W}(\Sigma) = 4\pi m$. We show that the Morse index $\mathrm{Ind}_{\mathscr{W}}(\Sigma)$ of the Willmore sphere Σ satisfies the inequality $\mathrm{Ind}_{\mathscr{W}}(\Sigma) \leq m$.

1. Introduction

1.1. Definitions and statement of the main results

The problem of estimating the index of a minimal surface is rich, and has strong connections with complex analysis and algebraic geometry. In this paper the goal is to study the index of Willmore spheres from the 2-sphere S^2 into the 3-sphere S^3. We first recall a few definitions. Let (N^n, h) be a smooth Riemannian manifold, and Σ be a connected (possibly non-compact) Riemann surface. For every smooth immersion $\vec{\Psi} : \Sigma \to N^n$, we define the

Willmore energy of Ψ by

$$W_{N^n}(\vec{\Psi}) = \int_\Sigma |\vec{H}_g|^2 \, d\mathrm{vol}_g, \tag{1.1}$$

where $g = \vec{\Psi}^* h$ is the induced metric on Σ, and $\vec{H}_g$ is the mean curvature tensor of $\vec{\Psi}(\Sigma)$. Then we can define a conformal Willmore functional by

$$\mathscr{W}_{N^n}(\vec{\Psi}) = \int_\Sigma \left(|\vec{H}_g|^2 + K_h\right) \, d\mathrm{vol}_g \tag{1.2}$$

if $K_h = K_{N^n}(\vec{\Psi}_* T\Sigma)$ is the sectional curvature of the 2-plan $\vec{\Psi}_* T\Sigma \subset TN^n$. It is conformally invariant in the sense that for every conformal diffeomorphism $\varphi : (N^n, h) \to (\widetilde{N}^n, \widetilde{h})$ and for every immersion $\vec{\Psi} : \Sigma \to N^n$ of a closed surface Σ, we have

$$\mathscr{W}_{N^n}(\vec{\Psi}) = \mathscr{W}_{\widetilde{N}^n}(\varphi \circ \vec{\Psi}).$$

In this general setting, this property is a theorem of Bang-Yen Chen (see [4], [5]). In the case where $N^n = S^n$ and h is the standard metric, we simply have

$$\mathscr{W}_{S^n}(\vec{\Psi}) = \int_\Sigma \left(1 + |\vec{H}_g|^2\right) d\mathrm{vol}_g.$$

Furthermore, if $p \in S^n$ and $\pi : S^n \setminus \{p\} \to \mathbb{R}^n$ is a stereographic projection from p such that $p \notin \vec{\Phi}(\Sigma) \subset S^n$, as π is conformal when $\mathbb{R}^n$ is equipped with its flat metric, we have

$$\mathscr{W}_{S^n}(\vec{\Psi}) = \mathscr{W}_{\mathbb{R}^n}(\pi \circ \vec{\Psi}).$$

In $\mathbb{R}^n$, we obviously have

$$W_{\mathbb{R}^n}(\vec{\Psi}) = \int_\Sigma |\vec{H}_g|^2 \, d\mathrm{vol}_g.$$

We also notice that for every conformal transformation $\varphi : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ such that $\varphi^{-1}(\infty) \cap \vec{\Psi}(\Sigma) = \varnothing$, we have

$$W_{\mathbb{R}^n}(\varphi \circ \vec{\Psi}) = W_{\mathbb{R}^n}(\vec{\Psi}).$$

However, this identity is no longer true in general if $\varphi^{-1}(\infty) \cap \vec{\Psi}(\Sigma) \neq \varnothing$. Inversions of minimal surfaces are counter-examples, as we shall see later. In the special case of $\mathbb{R}^n$, we define another globally conformal invariant, where we do not take the extrinsic curvature K_h (as in (1.1)) which vanishes identically, but rather the Gauss curvature K_g of $\Psi(\Sigma) \subset \mathbb{R}^n$. It is defined by

$$\mathscr{W}_{\mathbb{R}^n}(\vec{\Psi}) = \int_\Sigma \left(|\vec{H}_g|^2 - K_g\right) d\mathrm{vol}_g.$$

Notice in particular that it differs with $\mathscr{W}_{\mathbb{R}^n}$ as defined in (1.2). As the 2-form $(|\vec{H}_g|^2 - K_g)d\mathrm{vol}_g$ is invariant under any conformal transformation in $\mathbb{R}^n$ (see [2], [32]), the functional $\mathscr{W}_{\mathbb{R}^n}$ is *a fortiori* a conformal invariant. If Σ is closed, then by the Gauss-Bonnet theorem, we have

$$\int_\Sigma K_g d\mathrm{vol}_g = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Therefore, the two functionals $W_{\mathbb{R}^n}$ and $\mathscr{W}_{\mathbb{R}^n}$ only differ by a constant when we consider immersions of a closed (compact connected) Riemann surface Σ. We say that an immersion $\vec{\Psi} : \Sigma \to \mathbb{R}^n$ is a Willmore immersion if it is a critical point of $W_{\mathbb{R}^n}$. We will always assume that Willmore immersions do not have branched points when they are defined on a closed Riemann surface Σ. For a smooth immersion and in codimension 1, this is equivalent to

$$\Delta_g H_g + 2H_g(H_g^2 - K_g) = 0.$$

Here, H_g is the mean curvature *function* of $\vec{\Psi}(\Sigma) \subset \mathbb{R}^3$ (see [24] for a weak formulation of this formula and its consequences). This equation goes back to the works of Blaschke ([2]) and Thomsen ([29]). We remark that for a minimal surface $\vec{\Phi} : \Sigma \to \mathbb{R}^n$, we have $\vec{H}_g = 0$, so $\vec{\Phi}$ is an absolute minimiser of the Willmore energy, and by conformal invariance, we deduce that its inversion centred at a point *outside* of $\vec{\Phi}(\Sigma)$ is a compact Willmore surfaces. Likewise, its inverse stereographic projection in S^3 is a Willmore surface. These are the simplest cases of Willmore surface.

In the special case that $\Sigma = S^2$, we call a Willmore immersion a Willmore sphere. To define the (Morse) index for critical points of the Willmore energy, we first need the following definition.

Definition. Let Σ be a connected Riemann surface, and (N^n, h) be a Riemannian manifold that we suppose isometrically embedded in some Euclidean space $\mathbb{R}^q$. The set of weak immersions from Σ into N^n is as follows

$$\begin{aligned} \mathrm{W}^{2,2}_\iota(\Sigma, N^n) = {} & \mathrm{W}^{2,2}(\Sigma, \mathbb{R}^q) \\ & \cap \left\{\vec{\Psi} : \vec{\Phi}(x) \in N^n \text{ and } d\vec{\Psi}(x) \text{ is injective for a.e. } x \in \Sigma\right\}, \end{aligned}$$

Then for all $\vec{\Psi} \in \mathrm{W}^{2,2}_{\iota}(\Sigma, N^n)$, we define

$$\mathrm{W}^{2,2}_{\vec{\Psi}}(\Sigma, TN^n) = \mathrm{W}^{2,2}(\Sigma, TN^n) \cap \left\{\vec{w} : \vec{w}(x) \in T_{\vec{\Psi}(x)}N^n \text{ for a.e. } x \in \Sigma\right\}.$$

The index of a critical point $\vec{\Psi} \in \mathrm{W}^{2,2}_{\iota}(\Sigma, N^n) \cap \mathrm{W}^{1,\infty}(\Sigma, N^n)$ of the Willmore functional W, denoted by $\mathrm{Ind}_W(\vec{\Psi})$, is defined as the dimension of the maximal subspace of $\mathrm{W}^{2,2}_{\vec{\Psi}}(\Sigma, TN^n) \cap \mathrm{W}^{1,\infty}(\Sigma, TN^n)$ where the second derivative $D^2W(\vec{\Psi})$ is negative definite.

Our main result is the following.

Theorem 1.1. *Let $\vec{\Psi} : S^2 \to S^3$ be a Willmore sphere, and $m \in \mathbb{N}$ be the integer defined by*

$$m = \frac{1}{4\pi}\mathscr{W}(\vec{\Psi}) = \frac{1}{4\pi}\int_{S^2}(1 + H_g^2)\, d\mathrm{vol}_g.$$

Then we have

$$\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi}) \leq m. \tag{1.3}$$

These Willmore spheres are always assumed to be globally defined, *i.e.* they do not admit branched points. This implies that they are smooth (see [25], [27]). This hypothesis is made in order to apply Bryant's theorem ([3]).

One may conjecture that the index of a Willmore sphere, seen as a function depending on the number of ends m of the dual minimal surfaces given by Bryant's theorem, is bounded by two affine functions of m with positive slopes. One reason this might hold comes from minimal surfaces theory, where this property is exactly verified. (see [10], [30], [8], [7]). However, we do not obtain any lower bound in this work. Furthermore, we think that it is possible to improve the bound $\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi}) \leq m$ (if $\mathscr{W}_{S^3}(\vec{\Psi}) = 4\pi m$) to the bound $\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi}) \leq m - 3$. The reason is that the first non-trivial Willmore sphere has energy 16π, so for $m = 4$ there should be only one direction to decrease the energy in the class of *non-branched* Willmore spheres. The non-existence of Willmore spheres with energy 8π and 12π is a direct consequence to the non-existence of minimal surfaces with genus 0, embedded planar ends and total curvature -4π and -8π (see [17], [18]).

Another reason why this Willmore surface of energy 16π should have index less or equal than 1 is the following: if, as conjectured by Kusner (see [14]), the Bryant Willmore surface with energy 16π realises the *min-max sphere eversion* (see also open problems 4 and 7 in [26]), its Morse index should be bounded by the number of parameters of the min-max (see for example

[19] for such theorem in the case of minimal surfaces). As such a critical point would correspond to a path of immersions between Euclidean spheres of opposite orientations (that is, a 1-dimensional min-max), its Morse index should be less or equal than 1.

Acknowledgements. It is a pleasure to thank my advisor Tristan Rivière for useful discussions and motivation about this problem. The author is grateful to the referees for their very careful reading of the preliminary versions of the present paper and for their precise questions which helped improve significantly the quality of the manuscript.

1.2. Organisation of the paper

In Section 2, we will derive the first and the second variation of the Willmore functional in a general setting, and in Section 3, we will derive the first and the second variation of the Gauss curvature. As Sections 2 and 3 are rather computational, they are self-explanatory, we will describe in more details the content of Section 4, which contains the formulae for the index and the proof of Theorem 1.1.

We first recall one of the main result of Bryant's classification theory.

Theorem 1.2 (Bryant, [3]). *Let $\vec{\Psi}: S^2 \to S^3$ be a Willmore sphere. Either $\vec{\Psi}$ is completely umbilic and $\vec{\Psi}(S^2) \subset S^3$ is a round sphere or there exists a point $p \in \vec{\Psi}(S^2) \subset S^3$ and a stereographic projection $\pi : S^3 \setminus \{p\} \to \mathbb{R}^3$ such that $\vec{\Psi}^{-1}(\{p\})$ is discrete and*

$$\pi \circ \vec{\Psi} : S^2 \setminus \vec{\Psi}^{-1}(\{p\}) \to \mathbb{R}^3$$

is a complete minimal sphere in $\mathbb{R}^3$ with finite total curvature and embedded planar ends.

By the conformal invariance of $\mathscr{W}$, the index of a Willmore sphere $\vec{\Psi}: S^2 \to S^3$ is equal to the index of $\pi \circ \vec{\Psi} : S^2 \to \mathbb{R}^3$, where π is a stereographic projection whose domain includes the image of $\vec{\Psi}$.

Therefore, we fix some Willmore sphere $\vec{\Psi} : S^2 \to \mathbb{R}^3$. Up to translation, by the theorem of Robert Bryant ([3]), the image $\vec{\Phi} = i \circ \vec{\Psi}$ is a branched minimal surface with finite total curvature and planar ends (we refer to Section 4 for the definitions), where $i : \mathbb{R}^3 \cup \{\infty\} \to \mathbb{R}^3 \cup \{\infty\}$ is the inversion centred at 0. Let $\vec{n}_{\vec{\Psi}}$ (resp. $\vec{n}_{\vec{\Phi}}$) be the unit normal of $\vec{\Psi}$ (resp. of $\vec{\Phi}$). We deduce that if $\vec{v}$ is a normal variation of $\vec{\Psi}$, that is, $\vec{v} = v\,\vec{n}_{\vec{\Psi}}$ for some function v, then

$$D^2W(\vec{\Psi})[\vec{v},\vec{v}] = D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}] = D^2\mathscr{W}(\vec{\Phi})[\vec{w},\vec{w}]$$

where $\vec{w} = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}$. So the index of $\vec{\Psi}$ is equal to the index of $\vec{\Phi}$ for normal variations of the form $\vec{w} = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}$. The later can be explicitly computed, as for a minimal surface the index quadratic form simplifies significantly. Indeed, we have the following intermediate result, which is the main object of Section 3.

Proposition 1.3. *Let Σ be a connected Riemann surface and $\vec{\Phi}:\Sigma\to\mathbb{R}^3$ be a complete minimal immersion. Then for every normal variation $\vec{w} = w\,\vec{n}_{\vec{\Phi}}$, we have*

$$D^2\mathscr{W}(\vec{\Phi})[\vec{w},\vec{w}] = \int_\Sigma \left\{\frac{1}{2}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - d\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right)\right\}, \tag{1.4}$$

where $\star$ is the Hodge star operator.

In particular, we will see that the index of $\vec{\Psi}$ for W (or $\mathscr{W}$) is equal to the index of $\vec{\Phi}$ for $\mathscr{W}$ for the special class of variations $\vec{w} = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}$. The residue term coming from the exact form on the right-hand side of (1.4) will actually give all the negative directions, and it can be computed explicitly thanks to the Weierstrass parametrisation and the planarity of the ends of the minimal surface $\vec{\Phi}(\Sigma)\subset\mathbb{R}^3$.

Theorem 1.4. *Let Σ be a closed Riemann surface and $\vec{\Phi}:\Sigma\setminus\{p_1,\dots,p_m\}\to\mathbb{R}^3$ be a complete minimal immersion with m embedded planar ends such that $\vec{\Psi} = i\circ\vec{\Phi}:\Sigma\to\mathbb{R}^3$ is a non-branched Willmore immersion. For all $1\leq j\leq m$, we fix some small enough disjoint charts (U_j,u_j), where u_j is a complex coordinate $u_j:U_j\to D^2\subset\mathbb{C}$ such that $u_j(p_j)=0$. For all normal variations $\vec{v} = v\,\vec{n}_{\vec{\Psi}}$ of $\vec{\Psi}$, we have*

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}] = \lim_{R\to 0}\left\{\frac{1}{2}\int_{\Sigma_R}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - 4\pi\sum_{j=1}^m \frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}\,v^2(p_j)\right\}. \tag{1.5}$$

Here, $w = |\vec{\Phi}|^2 v$, $g = \vec{\Phi}^* g_{\mathbb{R}^3}$, *the set* Σ_R *is defined by*

$$\Sigma_R = \Sigma \setminus \bigcup_{j=1}^{m} D^2_\Sigma(p_j, R)$$

where for all $1 \leq j \leq n$, *we have* $D^2_\Sigma(p_j, R) = u_j^{-1}(D^2_{\mathbb{C}}(0,R))$ *(for* $0 < R < 1$*), and finally,* $\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)$ *is a positive number given by Definition 4.6 (it is independent of* u_j, but not of U_j*). In particular, the limit on the right-hand side of* (4.31) *exists and is a well-defined real number for every normal variation*

$$\vec{v} \in W^{2,2}_{\vec{\Psi}}(\Sigma, \mathbb{R}^3) \cap W^{1,\infty}_{\vec{\Psi}}(\Sigma, \mathbb{R}^3).$$

The residues $\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)$ of the minimal immersion $\vec{\Phi}$ with embedded planar ends at the points p_j $(1 \leq j \leq m)$ are defined in Definition 4.6. The quadratic form $D^2\mathscr{W}(\vec{\Psi})$ can be seen as a special case of a family of Schrödinger operators associated to meromorphic functions. This demonstrates the strong analogy to the index theory for minimal surfaces (see Section 4.5 for the general discussion and the papers of Shiu-Yuen Cheng and Johan Tysk [6], Sebastián Montiel and Antonio Ros [20] for the links between Schrödinger operators and the index of minimal surfaces).

2. First and second variation of the Willmore functional

2.1. Definitions and notations

Let (N^n, h) be a smooth Riemannian manifold, ∇ be its Levi-Civita connection, R be its Riemann tensor curvature, and $M^m \subset N^n$ be an isometrically embedded sub-manifold of N^n. The induced Levi-Civita connection on M^n by the injection $\iota : M^m \to N^n$ will be denoted by $\overline{\nabla} = \iota^*\nabla$ and $\overline{R}$ will denote the Riemann curvature tensor of $\overline{\nabla}$. If $\pi_M : TN \to TM$ is the orthogonal projection, then $\overline{\nabla}$ is characterised by the condition

$$\overline{\nabla} X = (\nabla X)^\top = \pi_M(\nabla X)$$

for every vector field $X \in \Gamma(TN)$. As a consequence, we have for every vector fields $X, Y, Z \in \Gamma(TM)$ the identity

$$\langle \nabla_X Y, Z \rangle = \left\langle \overline{\nabla}_X Y, Z \right\rangle.$$

Let $\vec{\mathbb{I}}$ be the second fundamental form of $M^m \subset N^n$, the symmetric two-tensor $\vec{\mathbb{I}} \in \Gamma((T^*M)^{\otimes 2} \otimes (TM)^\perp)$, characterised by the following condition : for all vector fields $X, Y \in \Gamma(TM)$, we have

$$\vec{\mathbb{I}}(X,Y) = (\nabla_X Y)^\perp = \pi_M^\perp(\nabla_X Y)$$

As ∇ is torsion-free, we have

$$\vec{\mathbb{I}}(X,Y) = (\nabla_X Y)^\perp = (\nabla_Y X - [X,Y])^\perp = (\nabla_Y X)^\perp = \vec{\mathbb{I}}(Y,X),$$

so $\vec{\mathbb{I}}$ is symmetric. The connection ∇ also induces a connection $\nabla^\perp$, which we call the normal connection, such that on the total bundle $\iota^* TN^n$, we have the decomposition $\nabla = \overline{\nabla} + \nabla^\perp$. Furthermore, $\nabla^\perp$ induces a connection on the bundle $(T^*M)^{\otimes 2} \otimes (TM)^\perp$, still denoted by $\nabla^\perp$, such that for all vector fields $X, Y, Z \in \Gamma(TM)$, there holds

$$(\nabla^\perp_X \vec{\mathbb{I}})(Y,Z) = \nabla^\perp_X\left(\vec{\mathbb{I}}(Y,Z)\right) - \vec{\mathbb{I}}(\overline{\nabla}_X Y, Z) - \vec{\mathbb{I}}(Y, \overline{\nabla}_X Z).$$

The main symmetries of the second fundamental form are gathered in the following theorem (see [23]), which we explicitly recall for the convenience of the reader and to fix notations.

Theorem. (i) *(**Gauss formula**) For all $X, Y \in \Gamma(TM)$,*

$$\nabla_X Y = \overline{\nabla}_X Y + \vec{\mathbb{I}}(X,Y)$$

(ii) *(**Gauss equation**) For all $X, Y, Z, W \in \Gamma(TM)$,*

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \left\langle \vec{\mathbb{I}}(Y,Z), \vec{\mathbb{I}}(X,W) \right\rangle - \left\langle \vec{\mathbb{I}}(X,Z), \vec{\mathbb{I}}(Y,W) \right\rangle.$$

(iii) *(**Codazzi-Mainardi identity**) For all $X, Y, Z \in \Gamma(TM)$, we have*

$$(\nabla^\perp_X \vec{\mathbb{I}})(Y,Z) = (\nabla^\perp_Y \vec{\mathbb{I}})(X,Z) + (R(X,Y)Z)^\perp.$$

Here and subsequently, assume N to be 3-dimensional. Let Σ be a connected Riemann surface, and $\vec{\Phi} \in \mathrm{W}^{2,2}_\iota(\Sigma, N^3)$ be a smooth immersion. We restrict ourselves in the following computations to dimension 3 only to simplify the presentation, as we will deal with a local normal unit vector-field inducing locally the second fundamental form, whereas for a n-manifold, we need to deal with a $(n-2)$-vector field, adding sums only in computations, and not in final formulae. Let $g = \vec{\Phi}^* h$ be the induced metric on Σ, $(g_{i,j})_{1 \leq i,j \leq 2}$

its local components and $(g^{i,j})_{1\leq i,j\leq 2}$ be the components of the inverse of g. We define the mean-curvature tensor field $\vec{H}_g$ of the immersion $\vec{\Phi}:\Sigma\to N^3$ by

$$\vec{H}_g=\frac{1}{2}\sum_{i,j=1}^{2}g^{i,j}\vec{\mathbb{I}}_{i,j}$$

where for $1\leq i,j\leq 2$, we define $\vec{\mathbb{I}}_{i,j}=\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)=(\nabla_{\vec{e}_i}\vec{e}_j)^{\perp}$ if ∇ is the Levi-Civita connection on (N^3,h), $\vec{e}_k=\partial_{x_k}\vec{\Phi}$ for $k=1,2$, and $\perp\colon TN^3\to\vec{\Phi}_*(T\Sigma)^{\perp}$ is the orthogonal projection.

2.2. First variation of W

We define for $i=1,2$, $\vec{e}_i=\partial_{x_i}\vec{\Phi}$, and we use a conformal local chart where

$$\langle\vec{e}_i,\vec{e}_j\rangle=e^{2\lambda}\delta_{i,j}.$$

In particular, we have

$$e^{2\lambda}=|\partial_{x_1}\vec{\Phi}|^2=|\partial_{x_2}\vec{\Phi}|^2=\frac{1}{2}|\nabla\vec{\Phi}|^2.$$

In the following Sections, we will compute the first and the second variation of a weak immersion $\vec{\Phi}\in \mathrm{W}_{\iota}^{2,2}(\Sigma,N^n)\cap W^{1,\infty}(\Sigma,N^3)$. Notice that can always, by a standard approximation argument, suppose that the immersion is smooth.

Definition 2.1. Let Σ be a closed Riemann surface and $\vec{\Phi}\in C^\infty\cap W_{\iota}^{2,2}(\Sigma,N^3)$ be a smooth immersion. An admissible variation of $\vec{\Phi}$ is a C^2 function

$$\{\vec{\Phi}_t\}_{t\in I}\in C^2\left(I,C^\infty\cap W_{\iota}^{2,2}(\Sigma,N^3)\right),$$

such that $\vec{\Phi}_0=\vec{\Phi}$. Here, I is an open interval of $\mathbb{R}$ containing 0, and we define by abuse of notation the variation of $\{\vec{\Phi}_t\}_{t\in I}$ by

$$\vec{w}=\left(\frac{d}{dt}\vec{\Phi}_t\right)_{|t=0}.$$

Furthermore, for all $t\in I$, we set

$$\vec{w}_t=\frac{d}{dt}\vec{\Phi}_t,\quad e^{2\lambda(t)}=\frac{|\nabla\vec{\Phi}_t|^2}{2},\quad \vec{e}_i(t)=\partial_{x_i}\vec{\Phi}_t,\quad i=1,2.$$

In particular, in this definition, we have

$$\vec{w}_0 = \vec{w}.$$

2.2.1. First variation of the metric. Here we compute the first variation of the metric, of its inverse and of the induced volume form.

Lemma 2.2. *Under the preceding hypotheses, we have for $1 \leq i, j \leq 2$, and for all $t \in I$,*

$$\frac{d}{dt} g_{i,j}(t) = \left\langle \nabla_{\vec{e}_i(t)} \vec{w}_t, \vec{e}_j(t) \right\rangle + \left\langle \nabla_{\vec{e}_j(t)} \vec{w}_t, \vec{e}_i(t) \right\rangle, \tag{2.1}$$

$$\frac{d}{dt} g^{i,j}(t) = -e^{-4\lambda(t)} \left(\left\langle \nabla_{\vec{e}_i(t)} \vec{w}_t, \vec{e}_j(t) \right\rangle + \left\langle \nabla_{\vec{e}_j(t)} \vec{w}_t, \vec{e}_i(t) \right\rangle \right), \tag{2.2}$$

$$\frac{d}{dt} d\mathrm{vol}_{g(t)} = \left\langle d\vec{\Phi}_t, d\vec{w}_t \right\rangle_{g(t)} d\mathrm{vol}_{g(t)}. \tag{2.3}$$

Here, for all $t \in I$, the metric $g(t) = \vec{\Phi}_t^{\ast} h$ is the pull-back of the metric of (N^3, h) by $\vec{\Phi}_t$.

Remark 2.3. *Notice that in this Section, we only need the value of the derivative at $t = 0$ for the first variation, where $g_{i,j} = \delta_{i,j}$. We shall however need the value for all $t \in I$ in the next Section where we derive the second derivative of the Willmore energy.*

Proof. To simplify the notations, we will drop most of the t indices in the proof. Fix $1 \leq i, j \leq 2$. Locally, we have $g_{i,j} = \left\langle \partial_{x_i} \vec{\Phi}_t, \partial_{x_j} \vec{\Phi}_t \right\rangle$. Therefore, the compatibility of the Levi-Civita connection ∇ with the metric gives

$$\begin{aligned}\frac{d}{dt} \left\langle \partial_{x_i} \vec{\Phi}_t, \partial_{x_j} \vec{\Phi}_t \right\rangle &= \left\langle \nabla_{\frac{d}{dt}} \partial_{x_i} \vec{\Phi}, \partial_{x_j} \vec{\Phi}_t \right\rangle + \left\langle \nabla_{\frac{d}{dt}} \partial_{x_j} \vec{\Phi}_t, \partial_{x_i} \vec{\Phi}_t \right\rangle \\ &= \langle \nabla_{\vec{e}_i} \vec{w}, \vec{e}_j \rangle + \langle \nabla_{\vec{e}_j} \vec{w}, \vec{e}_i \rangle.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\frac{d}{dt} \det g(t) &= \frac{d}{dt} \left(g_{1,1} g_{2,2} - g_{1,2}^2 \right) \\ &= 2\big(\langle \nabla_{\vec{e}_1} \vec{w}, \vec{e}_1 \rangle g_{2,2} + \langle \nabla_{\vec{e}_2} \vec{w}, \vec{e}_2 \rangle g_{1,1} \\ &\qquad - (\langle \nabla_{\vec{e}_1} \vec{w}, \vec{e}_2 \rangle + \langle \nabla_{\vec{e}_2} \vec{w}, \vec{e}_1 \rangle) g_{1,2} \big)\end{aligned}$$

which specializes for $t=0$ to

$$\frac{d}{dt}\det g_{t_{|t=0}} = 2e^{2\lambda}\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle + \langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\right) = 2e^{4\lambda}\left\langle d\vec{\Phi}, d\vec{w}\right\rangle_g.$$

This implies as $d\mathrm{vol}_{g_t} = \sqrt{\det g_t}\,dx_1\wedge dx_2$ locally that

$$\frac{d}{dt}\mathrm{vol}_{g_{t_{|t=0}}} = \left\langle d\vec{\Phi}, d\vec{w}\right\rangle_g d\mathrm{vol}_g.$$

Now, the explicit formula

$$g^{i,j} = (-1)^{i+j}\frac{g_{i+1,j+1}}{\det g}$$

gives

$$\frac{d}{dt}g^{i,j} = (-1)^{i+j}\Bigg(e^{-4\lambda}\left(\langle\nabla_{\vec{e}_{i+1}}\vec{w},\vec{e}_{j+1}\rangle + \langle\nabla_{\vec{e}_{j+1}}\vec{w},\vec{e}_{i+1}\rangle\right) - \delta_{i,j}e^{2\lambda}\frac{2e^{2\lambda}(\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_i\rangle + \langle\nabla_{\vec{e}_{i+1}}\vec{w},\vec{e}_{i+1}\rangle)}{e^{8\lambda}}\Bigg). \tag{2.4}$$

The formula (4.45) specialises for $i=j$ to

$$\frac{d}{dt}g^{i,i} = -2e^{-4\lambda}\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_i\rangle$$

while for $(i,j)=(1,2)$, we obtain

$$\frac{d}{dt}g_{1,2} = -e^{-4\lambda}\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle + \langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right).$$

Finally, we deduce that

$$\frac{d}{dt}g^{i,j} = -e^{-4\lambda}\left(\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle\right),$$

which concludes the proof of the lemma. □

2.2.2. First variation of the second fundamental form. We notice that even if we have chosen a local chart where $\vec{\Phi}$ is conformal, for $t\neq 0$, $\vec{\Phi}_t$ is not conformal in general, and as we aim at computing second derivative, we must keep track of the exact quantities depending on t. Therefore, we introduce the following definition.

Definition 2.4. For all $t \in I$, and $i = 1, 2$, we define the two quantities $e^{2\lambda(t)}$ and $e^{2\lambda_i(t)}$ by

$$e^{2\lambda(t)} = |\partial_{x_1}\vec{\Phi}_t \wedge \partial_{x_2}\vec{\Phi}_t|, \quad e^{2\lambda_i(t)} = g_{i,i}(t), \quad \mu(t) = g_{1,2}(t) = \left\langle \partial_{x_1}\vec{\Phi}_t, \partial_{x_2}\vec{\Phi}_t \right\rangle$$

Lemma 2.5. *Let $\vec{v}_t \in \Gamma(TN^3)$ be a time-dependant smooth vector field on N^3, and let us denote $\pi_{\vec{\Phi}_t}$ the orthogonal projection $TN^3 \to \vec{\Phi}_{t*}(T\Sigma)$. Then we have*

$$\pi_{\vec{\Phi}_t}(\vec{v}_t) = g_t^{-1} \begin{pmatrix} \left\langle \vec{v}_t, \partial_{x_1}\vec{\Phi}_t \right\rangle \\ \left\langle \vec{v}_t, \partial_{x_2}\vec{\Phi}_t \right\rangle \end{pmatrix} (\partial_{x_1}\vec{\Phi}_t,\, \partial_{x_2}\vec{\Phi}_t)$$

where g_t^{-1} is the inverse of the metric $g_t = \vec{\Phi}_t^* h$, viewed as a squared 2-matrix.

Lemma 2.6. *For all $1 \leq i, j \leq 2$, we have*

$$\begin{aligned} \nabla^\perp_{\frac{d}{dt}} \vec{\mathbb{I}}_{i,j} &= \left((\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top})\vec{w} + R(\vec{w}, \vec{e}_i)\vec{e}_j \right)^\perp \\ &= \nabla^\perp_{\vec{e}_i}\nabla^\perp_{\vec{e}_j}\vec{w} - \sum_{k=1}^2 \Gamma^k_{i,j}\nabla^\perp_{\vec{e}_k}\vec{w} + \sum_{k,l=1}^2 g^{k,l}\left\langle \nabla_{\vec{e}_j}\vec{w}, \vec{e}_k \right\rangle \vec{\mathbb{I}}_{i,l} + (R(\vec{w}, \vec{e}_i)\vec{e}_j)^\perp \end{aligned} \tag{2.5}$$

if $\Gamma^k_{i,j}$ are the Christoffel symbols of the metric $g_t = \vec{\Phi}_t^* h$.

Proof. As $\vec{n}$ is a unit vector, we have $\nabla^\perp_{\frac{d}{dt}}\vec{n}_t = 0$, as

$$\left\langle \nabla_{\frac{d}{dt}}\vec{n}_t, \vec{n}_t \right\rangle = \left\langle \nabla^\perp_{\frac{d}{dt}}\vec{n}_t, \vec{n}_t \right\rangle = 0,$$

and furthermore, by lemma (2.5),

$$\begin{aligned} \nabla_{\frac{d}{dt}}\vec{n}_t &= -e^{-4\lambda(t)}\left(e^{2\lambda_2(t)}\left\langle \vec{n}_t, \nabla_{\partial_{x_1}\vec{\Phi}_t}\vec{w}_t \right\rangle - \mu(t)\left\langle \vec{n}_t, \nabla_{\partial_{x_2}\vec{\Phi}_t}\vec{w}_t \right\rangle \right)\partial_{x_1}\vec{\Phi}_t \\ &\quad - e^{-4\lambda(t)}\left(-\mu(t)\left\langle \vec{n}_t, \nabla_{\partial_{x_1}\vec{\Phi}_t}\vec{w}_t \right\rangle + e^{2\lambda_1(t)}\left\langle \vec{n}_t, \nabla_{\partial_{x_2}\vec{\Phi}_t}\vec{w}_t \right\rangle \right)\partial_{x_2}\vec{\Phi}_t \\ &= -e^{-4\lambda}\left(e^{2\lambda_2}\left\langle \vec{n}, \nabla_{\vec{e}_1}\vec{w} \right\rangle - \mu\left\langle \vec{n}, \nabla_{\vec{e}_2}\vec{w} \right\rangle \right)\vec{e}_1 \\ &\quad - e^{-4\lambda}\left(-\mu\left\langle \vec{n}, \nabla_{\vec{e}_1}\vec{w} \right\rangle + e^{2\lambda_1}\left\langle \vec{n}, \nabla_{\vec{e}_2}\vec{w} \right\rangle \right)\vec{e}_2 \end{aligned}$$

where $\vec{w}_t = \nabla_{\frac{d}{dt}}\vec{\Phi}_t$, dropping the t index on the last line. And, if $\vec{v}_t$ is a smooth vector-field on N^3 depending of t, we have

$$\begin{aligned}\left\langle \vec{v}_t, \nabla_{\frac{d}{dt}}\vec{n}_t\right\rangle &= \Big\langle \vec{v}_t, -e^{-4\lambda}\left(e^{2\lambda_2}\langle \vec{n}, \nabla_{\vec{e}_1}\vec{w}\rangle - \mu\langle \vec{n}, \nabla_{\vec{e}_2}\vec{w}\rangle\right)\vec{e}_1\\ &\qquad - e^{-4\lambda}\left(-\mu\langle \vec{n}, \nabla_{\vec{e}_1}\vec{w}\rangle + e^{2\lambda_1}\langle \vec{n}, \nabla_{\vec{e}_2}\vec{w}\rangle\right)\vec{e}_2\Big\rangle\\ &= -e^{-4\lambda}\left(e^{2\lambda_2}\langle \vec{v}_t, \vec{e}_1\rangle - \mu(t)\langle \vec{v}_t, \vec{e}_2\rangle\right)\langle \vec{n}, \nabla_{\vec{e}_1}\vec{w}\rangle\\ &\quad - e^{-4\lambda}\left(e^{-2\lambda_1}\langle \vec{v}_t, \vec{e}_2\rangle - \mu\langle \vec{v}_t, \vec{e}_1\rangle\right)\langle \vec{n}, \nabla_{\vec{e}_2}\vec{w}\rangle\\ &= -\left\langle \vec{n}_t, \nabla_{\pi_{\vec{\Phi}}(\vec{v}_t)}\vec{w}_t\right\rangle = -\left\langle \vec{n}_t, \nabla_{(\vec{v}_t)^\top}\vec{w}_t\right\rangle\end{aligned}$$

so we get

$$\nabla^\perp_{\frac{d}{dt}}(\vec{v}_t)^\perp = \nabla^\perp_{\frac{d}{dt}}\vec{v}_t - \nabla^\perp_{\vec{v}_t^\top}\vec{w}_t. \tag{2.6}$$

Therefore, we deduce that

$$\nabla^\perp_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j} = \left\langle \nabla_{\frac{d}{dt}}\nabla_{\vec{e}_i}\vec{e}_j, \vec{n}\right\rangle\vec{n} - \left\langle \vec{n}, \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top}\vec{w}\right\rangle\vec{n}$$

and

$$\begin{aligned}\nabla_{\frac{d}{dt}}\nabla_{\vec{e}_i}\vec{e}_j &= \nabla_{\vec{e}_i}\nabla_{\frac{d}{dt}}\vec{e}_j + R(\vec{w}, \vec{e}_i)\vec{e}_j\\ &= \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w} + R(\vec{w}, \vec{e}_i)\vec{e}_j.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\nabla^\perp_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j} &= \left\langle \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w} + R(\vec{w}, \vec{e}_i)\vec{e}_j - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top}\vec{w}, \vec{n}\right\rangle\vec{n}\\ &= \left\langle (\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top})\vec{w} + R(\vec{w}, \vec{e}_i)\vec{e}_j, \vec{n}\right\rangle\vec{n}\\ &= \left((\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top})\vec{w} + R(\vec{w}, \vec{e}_i)\vec{e}_j\right)^\perp\end{aligned}$$

which concludes the proof of the lemma. □

2.2.3. First variation of the mean curvature. We deduce from (2.2) and (2.6) that, making the shift of notation $\vec{e}_i = e^{-\lambda}\partial_{x_i}\vec{\Phi}$ (therefore $(\vec{e}_1, \vec{e}_2)$

is an orthonormal frame for $\vec{\Phi}$)

$$
\begin{aligned}
\nabla^{\perp}_{\frac{d}{dt}} H_{g_t} &= \frac{1}{2}\sum_{i,j=1}^{2}\left(\frac{d}{dt}g^{i,j}\right)\vec{\mathbb{I}}_{i,j} + g^{i,j}\left(\nabla^{\perp}_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j}\right) \qquad (2.7)\\
&= \frac{1}{2}\sum_{i,j=1}^{2}\Big(-\left(\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle\nabla_{\vec{e}_j}\vec{w},\vec{e}_i\rangle\right)\vec{\mathbb{I}}_{i,j}\Big)\\
&\quad + \frac{1}{2}e^{-2\lambda}\sum_{k=1}^{2}\left(\left(\nabla_{\vec{e}_k}\nabla_{\vec{e}_k} - \nabla_{(\nabla_{\vec{e}_k}\vec{e}_k)^{\top}}\right)\vec{w} + R(\vec{w},\vec{e}_k)\vec{e}_k\right)^{\perp}\\
&= \frac{1}{2}\left(\Delta_g^{\vec{n}}\vec{w} + \mathscr{R}_1^{\vec{n}}(\vec{w}) - \sum_{i,j=1}^{2}\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle\,\vec{\mathbb{I}}_{i,j}\right),
\end{aligned}
$$

if $\Delta_g^{\vec{n}}$ is the normal Laplacian, defined by

$$
\Delta_g^{\vec{n}}\vec{w} = \sum_{k=1}^{2}\left(\nabla^{\perp}_{\vec{e}_k}\nabla^{\perp}_{\vec{e}_k} - \nabla^{\perp}_{\left(\nabla_{\vec{e}_k}\vec{e}_k\right)^{\top}}\right)\vec{w},
$$

and

$$
\mathscr{R}_1^{\vec{n}}(\vec{w}) = \left(\sum_{k=1}^{2}R(\vec{w},\vec{e}_k)\vec{e}_k\right)
$$

is a curvature operator. Indeed, we define the second order differential operator $\nabla^2_{X,Y}$ (see [9], 5.4.12 for example),acting on Section of the total pull-back bundle $\vec{\Phi}^*TN^3$, such that

$$
\nabla^2_{X,Y} = \nabla^{\perp}_X\nabla^{\perp}_Y - \nabla^{\perp}_{(\nabla_X Y)^{\top}}
$$

Then, we define the normal Laplacian $\Delta_g^{\vec{n}}$ as

$$
\Delta_g^{\vec{n}} = \operatorname{Tr}\overline{\nabla}^2 = \sum_{i=1}^{2}\overline{\nabla}^2_{\vec{e}_i,\vec{e}_i},
$$

if $(\vec{e}_1, \vec{e}_2)$ is a local orthonormal frame. In particular, we have for any $\vec{w}$ as before

$$
\begin{aligned}
(2.8)\qquad \left(\left(\nabla_X\nabla_Y-\nabla_{(\nabla_X Y)^\top}\right)\vec{w}\right)^\perp &= \nabla^2_{X,Y}\vec{w}+\nabla^\perp_X\nabla^\top_Y\vec{w}\\
&= \nabla^2_{X,Y}\vec{w}+\nabla^\perp_X\left(\sum_{j=1}^2\langle\nabla_Y\vec{w},\vec{e}_j\rangle\,\vec{e}_j\right)\\
&= \nabla^2_{X,Y}\vec{w}+\sum_{i=1}^2\langle\nabla_Y\vec{w},\vec{e}_j\rangle\,\vec{\mathbb{I}}(X,\vec{e}_j).
\end{aligned}
$$

Then (2.8) implies that

$$
\begin{aligned}
\sum_{i=1}^2\left(\left(\nabla_{\vec{e}_i}\nabla_{\vec{e}_i}-\nabla_{(\nabla_{\vec{e}_i}\vec{e}_i)^\top}\right)\vec{w}\right)^\perp &= \sum_{i=1}^2\nabla^2_{\vec{e}_i,\vec{e}_i}\vec{w}+\sum_{i,j=1}^2\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle\,\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&= \Delta_g^{\vec{n}}\vec{w}+\sum_{i,j=1}^2\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle\,\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)
\end{aligned}
$$

Therefore, we deduce that

$$
\begin{aligned}
DW(\vec{\Phi})\cdot\vec{w} &= \left(\frac{d}{dt}\int_\Sigma|\vec{H}_{g_t}|^2 d\mathrm{vol}_{g_t}\right)_{|t=0}\\
&= \left(\int_\Sigma 2\left\langle\nabla_{\frac{d}{dt}}\vec{H}_{g_t},\vec{H}_{g_t}\right\rangle d\mathrm{vol}_{g_t}+\int_\Sigma|\vec{H}_{g_t}|^2\left(\frac{d}{dt}d\mathrm{vol}_{g_t}\right)\right)_{|t=0}\\
&= \int_\Sigma\left\langle\Delta_g^{\vec{n}}\vec{w}+\mathscr{R}_1^{\vec{n}}(\vec{w})-\sum_{i,j=1}^2\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle\,\vec{\mathbb{I}}_{i,j},\vec{H}_g\right\rangle d\mathrm{vol}_g\\
&\quad+\int_\Sigma|\vec{H}_g|^2\left\langle d\vec{\Phi},d\vec{w}\right\rangle_g d\mathrm{vol}_g.
\end{aligned}
$$

We notice that this formula holds for the minimal regularity assumption *i.e.* for $\vec{\Phi}\in \mathrm{W}^{2,2}_\iota(\Sigma,N^n)\cap W^{1,\infty}(\Sigma,N^3)$. Furthermore, as mentioned in the introduction, it does not depend on the dimension of N, and is actually valid in any dimension. Indeed, in a Riemannian manifold (N^n,h) one simply needs to replace $\vec{n}$ with a $(n-2)$-vector inducing the second fundamental form, still denoted by $\vec{n}$. Then locally, $\vec{n}=\vec{n}_1\wedge\cdots\vec{n}_{n-2}$ where $(\vec{n}_1,\dots,\vec{n}_{n-2})$ is an orthonormal basis of the normal bundle of $\vec{\Phi}_t$. Extending by parallel transport the $\vec{n}_j$ $(1\leq j\leq n-2)$ such that $\nabla^\perp_{\frac{d}{dt}}\vec{n}_j=0$, the formula (2.6) is still correct and we get immediately the result.

If $\vec{\Phi}$ is smooth, and $\vec{w}$ is a normal variation, $\langle \nabla_{\vec{e}_i}\vec{w}, \vec{e}_j\rangle = -\left\langle \vec{w}, \vec{\mathbb{I}}_{i,j}\right\rangle$, so we get

$$\begin{aligned}\frac{d}{dt}\int_\Sigma H_g^2 d\mathrm{vol}_g &= \int_\Sigma \left\langle \Delta_g^{\vec{n}}\vec{w} + \mathscr{R}^\perp(\vec{w}) + \sum_{i,j=1}^2 \left\langle \vec{w}, \vec{\mathbb{I}}_{i,j}\right\rangle \vec{\mathbb{I}}_{i,j}, \vec{H}_g\right\rangle d\mathrm{vol}_g \\ &\quad - 2\int_\Sigma |\vec{H}_g|^2 \left\langle \vec{H}_g, \vec{w}\right\rangle d\mathrm{vol}_g \\ &= \int_\Sigma \left\langle \Delta_g^{\vec{n}}\vec{H}_g - 2|\vec{H}_g|^2\vec{H}_g + \mathscr{A}(\vec{H}_g) + \mathscr{R}_1^{\vec{n}}(\vec{H}_g), \vec{w}\right\rangle d\mathrm{vol}_g.\end{aligned}$$

This gives the classical Willmore equation (see the paper of Joel Weiner [31], and notice the different conventions we use here)

$$\Delta_g^{\vec{n}}\vec{H}_g - 2|\vec{H}_g|^2\vec{H}_g + \mathscr{A}(\vec{H}_g) + \mathscr{R}_1^{\vec{n}}(\vec{H}_g) = 0. \tag{2.9}$$

which is valid in an arbitrary Riemannian manifold (N^n, h) such that $\vec{\Phi} : \Sigma \to N^n$ is an immersion. Here $\mathscr{A}$ is the Simons' operation, defined by

$$\mathscr{A}(\vec{H}_g) = \sum_{i,j=1}^2 \left\langle \vec{H}_g, \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j)\right\rangle \vec{\mathbb{I}}(\vec{e}, \vec{e}_j)$$

if $(\vec{e}_1, \vec{e}_2)$ is a orthonormal frame on $\vec{\Phi}_*(T\Sigma)$ (we recall the shift of notation $\vec{e}_i = e^{-\lambda}\partial_{x_i}\vec{\Phi}$). In dimension 3, the equation can sometimes be written in a simpler way if $\vec{\Phi}(\Sigma)$ has a trivial normal bundle. Indeed, in this case, we can define up to the sign of a normal vector-field, the scalar mean curvature H_g, defined by $\vec{H}_g = H_g\vec{n}$. This easily gives

$$\Delta_g^{\vec{n}}\vec{H}_g = (\Delta_g H_g)\vec{n}.$$

As

$$\mathscr{A}(\vec{H}_g) = |\vec{\mathbb{I}}_g|^2 H_g\vec{n}$$

we finally obtain

$$\begin{aligned}&\Delta_g^{\vec{n}}\vec{H}_g - 2|\vec{H}_g|^2\vec{H}_g + \mathscr{A}(\vec{H}_g) + \mathscr{R}^\perp(\vec{H}_g) \\ &= \left(\Delta_g H_g - 2H_g^3 + |\vec{\mathbb{I}}_g|^2 H_g + \mathrm{Ric}(\vec{n},\vec{n})H_g\right)\vec{n} \\ &= \left(\Delta_g H_g - 2H_g^3 + (4H_g^2 - 2K_g)H_g + \mathrm{Ric}(\vec{n},\vec{n})H_g\right)\vec{n} \\ &= \left(\Delta_g H_g + 2H_g(H_g^2 - K_g) + \mathrm{Ric}(\vec{n},\vec{n})H_g\right)\vec{n}\end{aligned}$$

which is equivalent to

$$\Delta_g H_g + 2H_g(H_g^2 - K_g) + \mathrm{Ric}(\vec{n},\vec{n})H_g = 0 \tag{2.10}$$

if Ric is the Ricci curvature of (N^3, h).

2.3. Second variation of W

Let $\vec{\Phi} : \Sigma \to N^3$ be a smooth critical point of W. Then the second variation of W is well-defined, and does not depend on the variation $\vec{w}$ such that

$$\vec{w} = \left(\frac{d}{dt}\vec{\Phi}_t\right)_{|t=0}.$$

Therefore, we choose a variation $\vec{\Phi}_t$ such that

$$\nabla_{\frac{d}{dt}}\frac{d}{dt}\vec{\Phi}_t = 0.$$

and we abbreviate this expression by abuse of notation as $\nabla_{\vec{w}}\vec{w} = 0$. We will omit the proof of the following lemmas, as they are analogous to the previous ones.

2.3.1. Second variation of the metric. We split the preliminary computation into two lemmas.

Lemma 2.7. *Let $1 \leq i,j \leq 2$ be fixed indices. We have*

$$\left(\frac{d^2}{dt^2}g_{i,j}\right)_{|t=0} = 2\left\langle \nabla_{\vec{e}_i}\vec{w}, \nabla_{\vec{e}_j}\vec{w}\right\rangle - 2\left\langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_j\right\rangle, \tag{2.11}$$

$$\begin{aligned}\left(\frac{d^2}{dt^2}g^{i,j}\right)_{|t=0} &= 2e^{-2\lambda}\left(-\left\langle \nabla_{\vec{e}_i}\vec{w}, \nabla_{\vec{e}_j}\vec{w}\right\rangle_g + \left\langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_j\right\rangle_g\right)\\ &\quad + 4e^{-2\lambda}\left(\left\langle \nabla_{\vec{e}_i}\vec{w},\vec{e}_j\right\rangle_g + \left\langle \nabla_{\vec{e}_j}\vec{w},\vec{e}_i\right\rangle_g\right)\\ &\quad\quad \times\left(\left\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\right\rangle_g + \left\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\right\rangle_g\right)\\ &\quad - 2\delta_{i,j}e^{-2\lambda}\Bigg(4e^{-2\lambda}\left\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\right\rangle\left\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\right\rangle_g\\ &\quad\quad - \left(\left\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\right\rangle_g + \left\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\right\rangle_g\right)^2\Bigg),\end{aligned} \tag{2.12}$$

$$
\begin{aligned}
(2.13)\qquad \left(\frac{d^2}{dt^2}d\mathrm{vol}_{g_t}\right)_{|t=0} = \Big(&|d\vec{w}|_g^2 - \mathscr{R}_2(\vec{w},\vec{w})\\
&-2\left\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\right\rangle_g^2 - 2\left\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\right\rangle_g^2\\
&-\left(\left\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_2\right\rangle_g + \left\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_1\right\rangle_g\right)^2\Big)d\mathrm{vol}_g.
\end{aligned}
$$

where

$$
\mathscr{R}_2(\vec{w},\vec{w}) = e^{-2\lambda}\sum_{i=1}^{2}\left\langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_i\right\rangle.
$$

2.3.2. Second variation of the second fundamental form.

Lemma 2.8. *We have for* $1\leq i,j\leq 2$

$$
\begin{aligned}
\nabla^{\perp}_{\frac{d}{dt}}\nabla^{\perp}_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j} = \Big(&R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_j)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_j)\vec{w}\\
&+R(\vec{w},\vec{\mathbb{I}}_{i,j})\vec{w} + R(\vec{w},\vec{e}_j)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_j\\
&+R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_j + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w}\\
&-\sum_{k=1}^{2}\left\langle(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j,\vec{e}_k\right\rangle\nabla_{\vec{e}_k}\vec{w}\Big)^{\perp}.
\end{aligned}
$$

Proof. Thanks to (2.6), we have in the preceding notations of the proof of Lemma 2.6 (φ_t is a smooth vector-field on N^3 which is C^1 regular with respect to time)

$$
\nabla^{\perp}_{\frac{d}{dt}}(\vec{v}_t)^{\perp} = \nabla^{\perp}_{\frac{d}{dt}}\vec{v}_t - \nabla^{\perp}_{\vec{v}_t^{\top}}\vec{w}_t, \tag{2.14}
$$

and

$$
\nabla^{\perp}_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j} = \left((\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j\right)^{\perp}.
$$

Therefore, we first have

$$
\begin{aligned}
(2.15)\qquad \nabla^{\perp}_{\frac{d}{dt}}\left(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w}\right)^{\perp} &= \left(\nabla_{\frac{d}{dt}}\nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w}\right)^{\perp} - \nabla^{\perp}_{(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w})^{\top}}\vec{w}\\
&= \left(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}\nabla_{\vec{w}}\nabla_{\vec{e}_j}\vec{w}\right)^{\perp} - \sum_{k=1}^{2}\left\langle\nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w},\vec{e}_k\right\rangle\nabla^{\perp}_{\vec{e}_k}\vec{w}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}\Big(R(\vec{w},\vec{e}_j)\vec{w} + \nabla_{\vec{e}_j}\nabla_{\frac{d}{dt}}\vec{w}\Big)\Big)^{\perp} \\
&\quad - \sum_{k=1}^{2}\left\langle \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w}, \vec{e}_k\right\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w} \\
&= \left(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}\left(R(\vec{w},\vec{e}_j)\vec{w}\right)\right)^{\perp} - \sum_{k=1}^{2}\left\langle \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w}, \vec{e}_k\right\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w} \\
&= \big(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_j)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_j)\vec{w} \\
&\qquad + R(\vec{w},\nabla_{\vec{e}_i}\vec{e}_j)\vec{w} + R(\vec{w},\vec{e}_j)\nabla_{\vec{e}_i}\vec{w}\big)^{\perp} \\
&\quad - \sum_{k=1}^{2}\left\langle \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{w}, \vec{e}_k\right\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w}.
\end{aligned}
$$

Then, we compute

$$
\begin{aligned}
\nabla_{\frac{d}{dt}}\Big(\nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}}\vec{w}\Big)^{\perp} &= \Big(\nabla_{\frac{d}{dt}}\nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}}\vec{w}\Big)^{\perp} - \nabla^{\perp}_{\nabla_{((\nabla_{\vec{e}_i}\vec{e}_j)^{\top}\vec{w})^{\top}}}\vec{w} \\
&= \Big(R(\vec{w},(\nabla_{\vec{e}_i}\vec{e}_j)^{\top})\vec{w}\Big)^{\perp} \\
&\quad - \sum_{k=1}^{2}\Big\langle \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}}\vec{w}, \vec{e}_k\Big\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w}.
\end{aligned}
\tag{2.16}
$$

Finally, we easily have

$$
\begin{aligned}
\nabla^{\perp}_{\frac{d}{dt}}\left(R(\vec{w},\vec{e}_i)\vec{e}_j\right)^{\perp} &= \Big(\nabla_{\frac{d}{dt}}R(\vec{w},\vec{e}_i)\vec{e}_j\Big)^{\perp} - \sum_{k=1}^{2}\left\langle R(\vec{w},\vec{e}_i)\vec{e}_j, \vec{e}_k\right\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w} \\
&= \left(\nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_j + R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_j + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w}\right)^{\perp} \\
&\quad - \sum_{k=1}^{2}\left\langle R(\vec{w},\vec{e}_i)\vec{e}_j, \vec{e}_k\right\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w}
\end{aligned}
\tag{2.17}
$$

Gathering (2.15), (2.16) and (2.17) we obtain

$$
\begin{aligned}
\nabla^{\perp}_{\frac{d}{dt}}\nabla^{\perp}_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j} = \Big(&R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_j)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_j)\vec{w} \\
&+ R(\vec{w},\nabla_{\vec{e}_i}\vec{e}_j)\vec{w} + R(\vec{w},\vec{e}_j)\nabla_{\vec{e}_i}\vec{w} - R(\vec{w},(\nabla_{\vec{e}_i}\vec{e}_j)^{\top})\vec{w} \\
&+ \nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_j + R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_j + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w}\Big)^{\perp} \\
&- \sum_{k=1}^{2}\Big\langle\Big(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^{\top}}\Big)\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j, \vec{e}_k\Big\rangle \nabla^{\perp}_{\vec{e}_k}\vec{w}
\end{aligned}
$$

$$= \Bigg(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_j)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_j)\vec{w} + R(\vec{w},\vec{\mathbb{I}}_{i,j})\vec{w} + R(\vec{w},\vec{e}_j)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_j + R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_j + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_j}\vec{w} - \sum_{k=1}^{2}\left\langle \left(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top}\right)\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j, \vec{e}_k\right\rangle \nabla_{\vec{e}_k}\vec{w}\Bigg)^\perp.$$

This concludes the proof of the lemma. □

2.3.3. Second variation of the mean curvature. Now, making as earlier the shift of notation $\vec{e}_i = e^{-\lambda}\partial_{x_i}\vec{\Phi}$, we obtain

$$\begin{aligned}
2\nabla^\perp_{\frac{d}{dt}}\nabla^\perp_{\frac{d}{dt}}\vec{H}_{g_t} &= \sum_{i,j=1}^{2}\left(\frac{d^2}{dt^2}g^{i,j}\right)\vec{\mathbb{I}}_{i,j} + 2\left(\frac{d}{dt}g^{i,j}\right)\left(\nabla^\perp_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j}\right) + \delta_{i,j}e^{-2\lambda}\nabla^\perp_{\frac{d}{dt}}\nabla^\perp_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j}\\
&= \sum_{i,j=1}^{2}\Bigg\{ \left(2\left(-\langle\nabla_{\vec{e}_i}\vec{w},\nabla_{\vec{e}_j}\vec{w}\rangle + \langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_j\rangle\right)\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&\quad + 4\left(\left(\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle\nabla_{\vec{e}_j}\vec{w},\vec{e}_i\rangle\right)\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle + \langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\right)\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&\quad - 2\left(\langle\nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle\nabla_{\vec{e}_j}\vec{w},\vec{e}_i\rangle\right)\left((\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j\right)^\perp\Bigg\}\\
&\quad - \sum_{i=1}^{2}2\left(4\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle - \left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle + \langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle\right)^2\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&\quad + \sum_{i=1}^{2}\Bigg(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_i)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_i)\vec{w} + R(\vec{w},\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_i))\vec{w}\\
&\qquad + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_i + R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_i + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w}\\
&\qquad - \sum_{k=1}^{2}\left\langle(\nabla_{\vec{e}_i}\nabla_{\vec{e}_i} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_i)^\top})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_i,\vec{e}_k\right\rangle\nabla_{\vec{e}_k}\vec{w}\Bigg)^\perp
\end{aligned}$$

Then

$$\frac{d^2}{dt^2}|\vec{H}_g|^2 = 2\left\langle\nabla^\perp_{\frac{d}{dt}}\nabla^\perp_{\frac{d}{dt}}\vec{H}_g, H_g\right\rangle + 2\left|\nabla^\perp_{\frac{d}{dt}}\vec{H}_g\right|^2$$

and

$$\begin{aligned}
\frac{d^2}{dt^2}\int_\Sigma |H_{g_t}|^2 d\mathrm{vol}_{g_t} &= \int_\Sigma 2\left\langle\nabla^\perp_{\frac{d}{dt}}\nabla^\perp_{\frac{d}{dt}}\vec{H}_g, H_g\right\rangle + 2\left|\nabla^\perp_{\frac{d}{dt}}\vec{H}_g\right|^2 d\mathrm{vol}_g\\
&\quad + 4\int_\Sigma\left\langle\nabla^\perp_{\frac{d}{dt}}H_g,\vec{H}_g\right\rangle\left(\frac{d}{dt}d\mathrm{vol}_g\right) + \int_\Sigma|\vec{H}_g|^2\left(\frac{d^2}{dt^2}d\mathrm{vol}_g\right).
\end{aligned}$$

As

$$\nabla^{\perp}_{\frac{d}{dt}} H_{g_t} = \frac{1}{2}\left(\Delta_g^{\vec{n}}\vec{w} + \mathscr{R}_1^{\vec{n}}(\vec{w}) - \sum_{i,j=1}^{2}\langle \nabla_{\vec{e}_i}\vec{w}, \vec{e}_j\rangle \vec{\mathbb{I}}_{i,j}\right)$$

we get

(2.18)

$$\begin{aligned}
D^2W(\vec{\Phi})[\vec{w},\vec{w}] = \int_\Sigma \Bigg\langle \sum_{i,j=1}^{2}\Bigg\{ &\left(2\left(-\langle \nabla_{\vec{e}_i}\vec{w}, \nabla_{\vec{e}_j}\vec{w}\rangle + \langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_j\rangle\right)\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&+ 4\left(\langle \nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle \nabla_{\vec{e}_j}\vec{w},\vec{e}_i\rangle\right)\left(\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle + \langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)\\
&- 2\left(\langle \nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle + \langle \nabla_{\vec{e}_j}\vec{w},\vec{e}_i\rangle\right)\Big((\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_j\Big)^{\perp}\Bigg\}\\
&- \sum_{i=1}^{2} 2\left(4\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle - \left(\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle + \langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle\right)^2\right)\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_i)\\
&+ \sum_{i=1}^{2}\Big(R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{e}_i}R(\vec{w},\vec{e}_i)\vec{w} + R(\nabla_{\vec{e}_i}\vec{w},\vec{e}_i)\vec{w} + R(\vec{w},\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_i))\vec{w}\\
&\quad + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w} + \nabla_{\vec{w}}R(\vec{w},\vec{e}_i)\vec{e}_i + R(\vec{w},\nabla_{\vec{e}_i}\vec{w})\vec{e}_i + R(\vec{w},\vec{e}_i)\nabla_{\vec{e}_i}\vec{w}\\
&\quad - \sum_{k=1}^{2}\Big\langle (\nabla_{\vec{e}_i}\nabla_{\vec{e}_i} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_i)^\top})\vec{w} + R(\vec{w},\vec{e}_i)\vec{e}_i, \vec{e}_k\Big\rangle \nabla_{\vec{e}_k}\vec{w}\Big)^{\perp}, \vec{H}_g\Bigg\rangle d\mathrm{vol}_g\\
&+ \frac{1}{2}\int_\Sigma \left|\Delta_g^{\vec{n}}\vec{w} + \mathscr{R}_1^{\vec{n}}(\vec{w}) - \sum_{i,j=1}^{2}\langle \nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle \vec{\mathbb{I}}_{i,j}\right|^2 d\mathrm{vol}_g\\
&+ 2\int_\Sigma \left\langle \Delta_g^{\vec{n}}\vec{w} + \mathscr{R}_1^{\vec{n}}(\vec{w}) - \sum_{i,j=1}^{2}\langle \nabla_{\vec{e}_i}\vec{w},\vec{e}_j\rangle \vec{\mathbb{I}}_{i,j}, \vec{H}_g\right\rangle \left\langle \nabla\vec{w}, d\vec{\Phi}\right\rangle d\mathrm{vol}_g\\
&+ \int_\Sigma |\vec{H}_g|^2\Big(|\nabla\vec{w}|^2 - \mathscr{R}_2(\vec{w},\vec{w}) - 2\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle^2\\
&\quad - 2\langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle^2 - \left(\langle \nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle + \langle \nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right)^2\Big)d\mathrm{vol}_g.
\end{aligned}$$

where

$$\mathscr{R}_1^{\vec{n}}(\vec{w}) = \left(\sum_{i=1}^{2} R(\vec{w},\vec{e}_i)\vec{e}_i\right)^{\perp}, \quad \text{and} \quad \mathscr{R}_2(\vec{w},\vec{w}) = \sum_{i=1}^{2}\langle R(\vec{e}_i,\vec{w})\vec{w},\vec{e}_i\rangle .$$

Analogous computations for the second derivative of the Willmore energy in different special cases were already present in the literature (see [11], [15], [21]).

We remark that this formula makes sense for $\vec{\Phi} \in W^{2,2}(\Sigma, N^3) \cap W^{1,\infty}(\Sigma, N^3)$, and $\vec{w} \in W^{2,2}(\Sigma, TN^3) \cap W^{1,\infty}(\Sigma, TN^3)$. This formula does not use the fact the N is 3-dimensional, and as mentioned above, it remains valid in every C^3 Riemannian manifold (N^n, h) (this regularity is necessary, as ∇R is only continuous in a C^3 manifold).

In particular, if Σ is a closed Riemann surface, $p_1, \dots, p_m \in \Sigma$ are fixed distinct points, and $\vec{\Phi} : \Sigma \setminus \{p_1, \dots, p_m\} \to \mathbb{R}^n$ is a complete minimal surface with finite total curvature, and $\vec{w}$ is a variation compactly supported in $\Sigma \setminus \{p_1, \dots, p_m\}$, then

$$D^2 W(\vec{\Phi})[\vec{w}, \vec{w}] = \frac{1}{2} \int_\Sigma \left| \Delta_g^{\vec{n}} \vec{w} + \mathscr{R}_1^{\vec{n}}(\vec{w}) - \sum_{i,j=1}^2 \langle \nabla_{\vec{e}_i} \vec{w}, \vec{e}_j \rangle \, \vec{\mathbb{I}}_{i,j} \right|^2 d\mathrm{vol}_g \geq 0. \tag{2.19}$$

This inequality (2.19) shows the obvious fact that a minimal surface, which is an absolute minimiser of the Willmore functional, is stable. For a normal variation, *i.e.* such that $\vec{w} = \pi_{\vec{n}}(\vec{w})$, we have

$$\langle \nabla_{\vec{e}_i} \vec{w}, \vec{e}_j \rangle = - \langle \vec{w}, \nabla_{\vec{e}_i} \vec{e}_j \rangle = - \left\langle \vec{w}, \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j) \right\rangle$$

so

$$D^2 W(\vec{\Phi})[\vec{w}, \vec{w}] = \frac{1}{2} \int_\Sigma \left| \Delta_g^{\vec{n}} \vec{w} + \mathscr{R}_1^{\perp}(\vec{w}) + \mathscr{A}(\vec{w}) \right|^2 d\mathrm{vol}_g$$

where $\mathscr{A}$ is Simon's operator, defined by

$$\mathscr{A}(\vec{w}) = \sum_{i,j=1}^2 \left\langle \vec{w}, \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j) \right\rangle \vec{\mathbb{I}}(\vec{e}_i, \vec{e}_j).$$

In the case of a surface with trivial normal bundle, this equation gets even simpler, as there exists $w \in W^{2,2}(\Sigma, \mathbb{R})$, such that $\vec{w} = w \, \vec{n}$, and

$$D^2 W(\vec{\Phi})[\vec{w}, \vec{w}] = \frac{1}{2} \int_\Sigma \left(\Delta_g w + (|\vec{\mathbb{I}}_g|^2 + \mathrm{Ric}_{N^3}(\vec{n}, \vec{n})) w \right)^2 d\mathrm{vol}_g. \tag{2.20}$$

Indeed, if $\nabla = \vec{\Phi}^* \nabla$ is the tangent connection defined on Σ, then we define a second order operator $\overline{\nabla}^2_{i,j}$ (see [9], 5.4.12 for example), acting on smooth function on Σ, such that

$$\overline{\nabla}^2_{i,j} = \overline{\nabla}_{\vec{e}_i}\overline{\nabla}_{\vec{e}_j} - \overline{\nabla}_{\overline{\nabla}_{\vec{e}_i}\vec{e}_j}$$

If $f \in C^2(\Sigma)$, then we get the following expression for the Laplacian Δ_g on Σ

$$\Delta_g f = \mathrm{Tr}\overline{\nabla}^2 f = \sum_{i=1}^{2}\overline{\nabla}^2_{i,i} f.$$

We deduce that for a normal variation $\vec{w} = w\vec{n}$, as $(\nabla_{\vec{e}_i}\vec{e}_j)^\top = \overline{\nabla}_{\vec{e}_i}\vec{e}_j$, we have as $\vec{n}$ is a unit vector-field,

$$\begin{aligned}\left(\left(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top}\right)\vec{w}\right)^\perp &= \left(\overline{\nabla}^2_{i,j} w\right)\vec{n} + w\left\langle \nabla_{\vec{e}_i}\nabla_{\vec{e}_j}\vec{n}, \vec{n}\right\rangle \vec{n} \\ &= \left(\overline{\nabla}^2_{i,j} w - w\left\langle \nabla_{\vec{e}_i}\vec{n}, \nabla_{\vec{e}_j}\vec{n}\right\rangle\right)\vec{n} \\ &= \left(\overline{\nabla}^2_{i,j} w - we^{-4\lambda}\left(\mathbb{I}_{i,1}\mathbb{I}_{j,1} + \mathbb{I}_{i,2}\mathbb{I}_{j,2}\right)\right)\vec{n}\end{aligned}$$

so

$$\Delta_g^{\vec{n}}\vec{w} = (\Delta_g w + w^2|\mathbb{I}_g|^2)\vec{n}$$

while

$$\mathscr{R}_2(\vec{w},\vec{w}) = w^2\sum_{i=1}^{2}\langle R(\vec{e}_i,\vec{n})\vec{n},\vec{e}_i\rangle = w^2\,\mathrm{Ric}_{N^3}(\vec{n},\vec{n})$$

So we also have

$$\begin{aligned}\frac{d^2}{dt^2}d\mathrm{vol}_g &= \Big(|dw|_g^2 + w^2|\vec{\mathbb{I}}_g|^2 - w^2\,\mathrm{Ric}_{N^3}(\vec{n},\vec{n}) \\ &\quad - 2w^2(\mathbb{I}(\vec{e}_1,\vec{e}_1)^2 + \mathbb{I}(\vec{e}_2,\vec{e}_2)^2 + 2\mathbb{I}(\vec{e}_1,\vec{e}_2)^2)\Big)d\mathrm{vol}_g \\ &= \left((|dw|_g^2 - \left(|\vec{\mathbb{I}}_g|^2 + \mathrm{Ric}_{N^3}(\vec{n},\vec{n})\right)w^2\right)d\mathrm{vol}_g.\end{aligned} \tag{2.21}$$

Now let A be the area functional. By (2.21), if $\vec{\Phi}$ is a minimal surface, we have

$$\begin{aligned}D^2A(\vec{\Phi})[\vec{w},\vec{w}] &= \int_\Sigma \left(|dw|_g^2 - \left(|\vec{\mathbb{I}}_g|^2 + \mathrm{Ric}_{N^3}(\vec{n},\vec{n})\right)w^2\right)d\mathrm{vol}_g \\ &= -\int_\Sigma w\left(\Delta_g w + \left(|\vec{\mathbb{I}}_g|^2 + \mathrm{Ric}_{N^3}(\vec{n},\vec{n})\right)w\right)d\mathrm{vol}_g \\ &= -\int_\Sigma w\,L_g w\,d\mathrm{vol}_g\end{aligned}$$

so for a minimal surface $\vec{\Phi}$, if L_g is the Jacobi operator of $\vec{\Phi}$, we obtain

$$D^2W(\vec{\Phi})[\vec{w},\vec{w}]=\frac{1}{2}\int_\Sigma (L_g w)^2 d\mathrm{vol}_g.$$

An other interesting case is the second variation of the conformal Willmore $\mathscr{W}=\mathscr{W}_{S^3}$ for a minimal surface in S^3, already present in the paper of Joel Weiner ([31]) presenting first the Euler-Lagrange equation of Willmore functional in arbitrary Riemannian manifolds. If $\vec{\Phi}:\Sigma\to S^3$ is a minimal surface, then

$$\begin{aligned}
D^2\mathscr{W}(\vec{\Phi})[\vec{w},\vec{w}] &= D^2A(\vec{\Phi})[\vec{w},\vec{w}]+D^2W(\vec{\Phi})[\vec{w},\vec{w}]\\
&=\int_\Sigma |dw|_g^2-\left(|\vec{\mathbb{I}}_g|^2+\mathrm{Ric}_{S^3}(\vec{n},\vec{n})\right)w^2 d\mathrm{vol}_g\\
&\quad+\frac{1}{2}\left(\Delta_g w+\left(|\vec{\mathbb{I}}_g|^2+\mathrm{Ric}_{S^3}(\vec{n},\vec{n})\right)w\right)^2 d\mathrm{vol}_g\\
&=-\int_\Sigma w\left(\Delta_g w+\left(|\vec{\mathbb{I}}_g|^2+2\right)w\right)d\mathrm{vol}_g\\
&\quad+\frac{1}{2}\left(\Delta_g w+\left(|\vec{\mathbb{I}}_g|^2+2\right)w\right)^2 d\mathrm{vol}_g\\
&=\frac{1}{2}\int_\Sigma \left(\Delta_g w+\left(|\vec{\mathbb{I}}_g|^2+2\right)w\right)\left(\Delta_g w+|\vec{\mathbb{I}}_g|^2 w\right)d\mathrm{vol}_g\\
&=\frac{1}{2}\int_\Sigma w\left(L_g\circ(L_g-2)w\right)d\mathrm{vol}_g
\end{aligned}\tag{2.22}$$

so the index (for the Willmore energy) of a minimal surface in S^3 is equal to the (finite) number of negative eigenvalues of the strongly elliptic operator $L_g\circ(L_g-2)$. Therefore, if λ is a positive eigenvalue of L_g, E_λ the eigenspace associated to λ we define by

$$\dim E_\lambda$$

the dimension of the eigenspace. Therefore, we deduce that

$$\mathrm{Ind}_{\mathscr{W}}(\vec{\Phi})=\sum_{0<\lambda<2}\dim E_\lambda.\tag{2.23}$$

which was already contained in the paper of Joel Weiner [31] (notice the different sign convention which we used here).

3. First and second variation of Gauss curvature

In this Section, we compute the first and the second variation of the Gauss curvature. This may seem at first useless according to the Gauss-Bonnet theorem, as for every closed surface Σ, for every smooth metric g on Σ we have

$$\int_\Sigma K_g d\mathrm{vol}_g = 2\pi\,\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. However, this formula is not valid for non-closed surface. And when we perform variations, the total curvature does not stay constant in general. The necessity to consider non-closed surfaces will be clarified in the next Section, and the reader may first skip this technical part to get first some motivation for performing these computations. We fix an arbitrary connected Riemann surface Σ, which is *not* supposed to be closed.

3.1. First variation of K

Lemma 3.1. *For all smooth immersions $\vec{\Phi}:\Sigma\to\mathbb{R}^n$, for any admissible variation $\{\vec{\Phi}_t\}_{t\in I}$ (in the sense of Definition 2.1) of $\vec{\Phi}$, we have for all $t\in I$,*

$$\begin{aligned}\frac{d}{dt}\left(K_{g_t}d\mathrm{vol}_{g_t}\right) = d\bigg(\frac{1}{\sqrt{\det g_t}}\bigg(&\Big(-\left\langle \vec{\mathbb{I}}_{1,1}(t),\nabla_{\vec{e}_2(t)}\vec{w}_t\right\rangle + \left\langle \vec{\mathbb{I}}_{1,2}(t),\nabla_{\vec{e}_1(t)}\vec{w}_t\right\rangle\Big)dx_1\\ &+\Big(\left\langle \vec{\mathbb{I}}_{2,2}(t),\nabla_{\vec{e}_1(t)}\vec{w}_t\right\rangle - \left\langle \vec{\mathbb{I}}_{2,1}(t),\nabla_{\vec{e}_2(t)}\vec{w}_t\right\rangle\Big)dx_2\bigg)\bigg),\end{aligned}$$

where for all $t\in I$,

$$\vec{w}_t = \frac{d}{dt}\vec{\Phi}_t.$$

Proof. We have in conformal coordinates

$$K_{g_t}d\mathrm{vol}_{g_t} = e^{-4\lambda(t)}\left(\left\langle \vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle - |\vec{\mathbb{I}}_{1,2}|^2\right)e^{2\lambda(t)}dx_1\wedge dx_2,$$

where we recall the notation $e^{4\lambda(t)} = \det g_t = |\partial_{x_1}\vec{\Phi}_t\wedge\partial_{x_2}\vec{\Phi}_t|^2$. Therefore, we need to compute

$$\begin{aligned}(3.1)\qquad &\frac{d}{dt}\left((\det g_t)^{-\frac{1}{2}}\left(\left\langle \vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle-|\vec{\mathbb{I}}_{1,2}|^2\right)\right)\\ &=\left(\frac{d}{dt}(\det g_t)^{-\frac{1}{2}}\right)\left(\left\langle \vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle-|\vec{\mathbb{I}}_{1,2}|^2\right)\\ &\quad+(\det g_t)^{-\frac{1}{2}}\frac{d}{dt}\left(\left\langle \vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle-|\vec{\mathbb{I}}_{1,2}|^2\right)\\ &=(\mathrm{I})+(\mathrm{II}).\end{aligned}$$

We have by Lemma 2.2,

$$\begin{aligned}\frac{d}{dt}(\det g_t)^{-\frac{1}{2}}&=-\frac{1}{2}e^{-6\lambda}\Big(2e^{2\lambda_2}\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle+2e^{2\lambda_1}\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\\ &\quad-2\mu\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle+\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right)\Big)\\ &=-e^{-6\lambda}\Big(e^{2\lambda_2}\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle+e^{2\lambda_1}\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\\ &\quad-\mu\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle+\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right)\Big)\end{aligned}$$

and

$$\begin{aligned}(3.2)\qquad (\mathrm{I})&=\left(\frac{d}{dt}(\det g_t)^{-\frac{1}{2}}\right)K_{g_t}\\ &=-e^{-2\lambda}\Big(e^{2\lambda_2}\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle+e^{2\lambda_1}\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle\\ &\quad-\mu\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle+\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right)\Big)K_{g_t}.\end{aligned}$$

Recall that

$$\nabla^{\perp}_{\frac{d}{dt}}\vec{\mathbb{I}}_{i,j}=\left(\left(\nabla_{\vec{e}_i}\nabla_{\vec{e}_j}-\nabla_{(\nabla_{\vec{e}_i}\vec{e}_j)^\top}\right)\vec{w}\right)^{\perp}.$$

If $e^{2\lambda_i(t)}=|\partial_{x_i}\vec{\Phi}_t|^2$, we introduce the notation

$$\alpha_{i,j}^k(t)=\left\langle\nabla_{\vec{e}_i(t)}\vec{e}_j(t),\vec{e}_k(t)\right\rangle=\left\langle\partial^2_{x_i,x_j}\vec{\Phi}_t,\partial_{x_k}\vec{\Phi}_t\right\rangle.$$

To simplify notations, we drop the t index in the following. We recall that

$$(\nabla_{\vec{e}_i}\vec{e}_j)^\top = e^{-4\lambda}\left(e^{2\lambda_2}\langle\nabla_{\vec{e}_i}\vec{e}_j,\vec{e}_1\rangle - \mu\,\langle\nabla_{\vec{e}_i}\vec{e}_j,\vec{e}_2\rangle\right)\vec{e}_1 + e^{-4\lambda}\left(e^{2\lambda_1}\langle\nabla_{\vec{e}_i}\vec{e}_j,\vec{e}_2\rangle - \mu\,\langle\nabla_{\vec{e}_i}\vec{e}_j,\vec{e}_1\rangle\right)\vec{e}_2$$
$$= e^{-4\lambda}\left(e^{2\lambda_2}\alpha^1_{i,j} - \mu\,\alpha^2_{i,j}\right)\vec{e}_1 + e^{-4\lambda}\left(e^{2\lambda_1}\alpha^2_{i,j} - \mu\,\alpha^1_{i,j}\right)\vec{e}_2$$
$$= e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{i,j} - \mu\,\alpha^{l+1}_{i,j}\right)\vec{e}_l \tag{3.3}$$

while

$$\frac{d}{dt}\left\langle\vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle = \left\langle\nabla^2_{\vec{e}_1}\vec{w} - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{1,1} - \mu\,\alpha^{l+1}_{1,1}\right)\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle + \left\langle\nabla^2_{\vec{e}_2}\vec{w} - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{2,2} - \mu\,\alpha^{l+1}_{2,2}\right)\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle$$

and

$$\frac{d}{dt}|\vec{\mathbb{I}}_{1,2}|^2 = 2\left\langle\nabla_{\vec{e}_1}\nabla_{\vec{e}_2}\vec{w} - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{1,2} - \mu\,\alpha^{l+1}_{1,2}\right)\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle.$$

Therefore, we obtain

$$(\mathrm{II}) = e^{-2\lambda}\left(\left\langle\nabla^2_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle + \left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle - 2\left\langle\nabla_{\vec{e}_1}\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle\right) - e^{-6\lambda}\sum_{l=1}^{2}\left(\left(e^{2\lambda_{l+1}}\alpha^l_{1,1} - \mu\alpha^{l+1}_{1,1}\right)\left\langle\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle + \left(e^{2\lambda_{l+1}}\alpha^l_{2,2} - \mu\alpha^{l+1}_{2,2}\right)\left\langle\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle\right) \tag{3.4}$$
$$+ 2e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{1,2} - \mu\alpha^{l+1}_{1,2}\right)\left\langle\nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle. \tag{3.5}$$

We first compute (3.4), and we first make some remarks about covariant derivatives . We recall that the covariant derivative $\overline{\nabla}$ is the orthogonal projection $T\mathbb{R}^n \to \vec{\Phi}_{t*}(T\Sigma)$ of the flat connection on $\mathbb{R}^n$. By the definition of the covariant derivative ∇, if X, Y, Z are tangent vectors, we have

$$\nabla^\perp_X\vec{\mathbb{I}}(Y,Z) = \nabla^\perp_X(\vec{\mathbb{I}}(Y,Z)) - \vec{\mathbb{I}}(\overline{\nabla}_X Y, Z) - \vec{\mathbb{I}}(Y,\overline{\nabla}_X Z)$$

while Codazzi-Mainardi identity reads, as $R=0$

$$\nabla^{\perp}_X \vec{\mathbb{I}}(Y,Z)=\nabla^{\perp}_Y\vec{\mathbb{I}}(X,Z).$$

By compatibility of the connection ∇ with the metric, we get for $i=1,2$

$$\begin{aligned}e^{-2\lambda}\left\langle \nabla^2_{\vec{e}_k}\vec{w},\vec{\mathbb{I}}_{i,j}\right\rangle &= \nabla_{\vec{e}_k}\left(\left\langle \nabla_{\vec{e}_k}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{i,j}\right\rangle\right)\\ &-e^{-2\lambda}\left\langle \nabla_{\vec{e}_k}\vec{w},\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right)\right\rangle\\ &-\partial_{x_k}e^{-2\lambda}\left\langle\nabla_{\vec{e}_k}\vec{w},\vec{\mathbb{I}}_{i,j}\right\rangle.\end{aligned}\tag{3.6}$$

Now, thanks to the orthogonality of $\vec{\mathbb{I}}_{i,j}$, we have

$$\begin{aligned}\nabla^{\top}_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right)&=e^{-4\lambda}\left(e^{2\lambda_2}\left\langle\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right),\vec{e}_1\right\rangle-\mu\left\langle\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right),\vec{e}_2\right\rangle\right)\vec{e}_1\\ &+e^{-4\lambda}\left(e^{2\lambda_1}\left\langle\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right),\vec{e}_2\right\rangle-\mu\left\langle\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right),\vec{e}_1\right\rangle\right)\vec{e}_2\\ &=-e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l}\right\rangle-\mu\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l+1}\right\rangle\right)\vec{e}_l.\end{aligned}$$

Therefore, by definition of ∇, and Codazzi-Mainardi identity, we get

$$\begin{aligned}\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right)&=\nabla^{\perp}_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right)+\nabla^{\top}_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right)\\ &=\nabla^{\perp}_{\vec{e}_k}\vec{\mathbb{I}}(\vec{e}_i,\vec{e}_j)+\vec{\mathbb{I}}(\overline{\nabla}_{\vec{e}_k}\vec{e}_i,\vec{e}_j)+\vec{\mathbb{I}}(\vec{e}_i,\overline{\nabla}_{\vec{e}_k}\vec{e}_j)\\ &\quad-e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l}\right\rangle-\mu\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l+1}\right\rangle\right)\vec{e}_l\\ &=\nabla^{\perp}_{\vec{e}_i}\vec{\mathbb{I}}(\vec{e}_k,\vec{e}_j)+e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{i,k}-\mu\alpha^{l+1}_{i,k}\right)\vec{\mathbb{I}}_{j,l}\\ &\quad+e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{j,k}-\mu\alpha^{l+1}_{j,k}\right)\vec{\mathbb{I}}_{i,l}\\ &\quad-e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l}\right\rangle-\mu\left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l+1}\right\rangle\right)\vec{e}_l\end{aligned}$$

and now

$$\begin{aligned}\nabla^{\perp}_{\vec{e}_i}\vec{\mathbb{I}}(\vec{e}_j,\vec{e}_k) &= \nabla_{\vec{e}_i}\left(\vec{\mathbb{I}}_{j,k}\right) - \nabla^{\top}_{\vec{e}_i}\left(\vec{\mathbb{I}}_{j,k}\right) - \vec{\mathbb{I}}(\overline{\nabla}_{\vec{e}_i}\vec{e}_j,\vec{e}_k) - \vec{\mathbb{I}}(\vec{e}_j,\overline{\nabla}_{\vec{e}_i}\vec{e}_k)\\ &= \nabla_{\vec{e}_i}\left(\vec{\mathbb{I}}_{j,k}\right) - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{i,j} - \mu\alpha^{l+1}_{i,j}\right)\vec{\mathbb{I}}_{k,l}\\ &\quad - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{i,k} - \mu\alpha^{l+1}_{i,k}\right)\vec{\mathbb{I}}_{j,l}\\ &\quad + e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\left\langle\vec{\mathbb{I}}_{j,k},\vec{\mathbb{I}}_{i,l}\right\rangle - \mu\left\langle\vec{\mathbb{I}}_{j,k},\vec{\mathbb{I}}_{i,l+1}\right\rangle\right)\vec{e}_l\end{aligned}$$

so bringing together both expression, the two sums cancel, and we get

$$\begin{aligned}\nabla_{\vec{e}_k}\left(\vec{\mathbb{I}}_{i,j}\right) &= \nabla_{\vec{e}_i}\left(\vec{\mathbb{I}}_{i,j}\right) + e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{j,k} - \mu\alpha^{l+1}_{j,k}\right)\vec{\mathbb{I}}_{i,l}\\ &\quad - e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha^l_{i,j} - \mu\alpha^{l+1}_{i,j}\right)\vec{\mathbb{I}}_{k,l}\\ &\quad + e^{-4\lambda}\sum_{l=1}^{2}\Big(e^{2\lambda_{l+1}}\left(\left\langle\vec{\mathbb{I}}_{j,k},\vec{\mathbb{I}}_{i,l}\right\rangle - \left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l}\right\rangle\right)\\ &\qquad - \mu\left(\left\langle\vec{\mathbb{I}}_{j,k},\vec{\mathbb{I}}_{i,l+1}\right\rangle - \left\langle\vec{\mathbb{I}}_{i,j},\vec{\mathbb{I}}_{k,l+1}\right\rangle\right)\Big)\vec{e}_l.\end{aligned}\tag{3.7}$$

We deduce that

$$\begin{aligned}&e^{-2\lambda}\left\langle\nabla^2_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle + e^{-2\lambda}\left\langle\nabla^2_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle - 2e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle\\ &= \nabla_{\vec{e}_1}\left(\left\langle\nabla_{\vec{e}_1}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{2,2}\right\rangle\right) + \nabla_{\vec{e}_2}\left(\left\langle\nabla_{\vec{e}_2}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,1}\right\rangle\right)\\ &\quad - \nabla_{\vec{e}_1}\left(\left\langle\nabla_{\vec{e}_2}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,2}\right\rangle\right) - \nabla_{\vec{e}_2}\left(\left\langle\nabla_{\vec{e}_1}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,2}\right\rangle\right)\\ &\quad - e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w},\nabla_{\vec{e}_1}\left(\vec{\mathbb{I}}_{2,2}\right)\right\rangle - e^{-2\lambda}\left\langle\nabla_{\vec{e}_2}\vec{w},\nabla_{\vec{e}_2}\left(\vec{\mathbb{I}}_{1,1}\right)\right\rangle\\ &\quad + e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w},\nabla_{\vec{e}_2}\left(\vec{\mathbb{I}}_{1,2}\right)\right\rangle + e^{-2\lambda}\left\langle\nabla_{\vec{e}_2}\vec{w},\nabla_{\vec{e}_1}\left(\vec{\mathbb{I}}_{1,2}\right)\right\rangle\\ &\quad - \partial_{x_1}e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle - \partial_{x_2}e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle\\ &\quad + \partial_{x_1}e^{-2\lambda}\left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle + \partial_{x_2}e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle\\ &= \text{(i)} + \text{(ii)} + \text{(iii)}\end{aligned}$$

where (i), (ii) and (iii) correspond to the first, second and third line respectively. Now, we see that

$$\begin{aligned}\text{(i)}dx_1\wedge dx_2=\bigg(\frac{1}{\sqrt{\det g_t}}\Big(\Big(-\left\langle\vec{\mathbb{I}}_{1,1}(t),\nabla_{\vec{e}_2(t)}\vec{w}_t\right\rangle+\left\langle\vec{\mathbb{I}}_{1,2}(t),\nabla_{\vec{e}_1(t)}\vec{w}_t\right\rangle\Big)\,dx_1\\+\Big(\left\langle\vec{\mathbb{I}}_{2,2}(t),\nabla_{\vec{e}_1(t)}\vec{w}_t\right\rangle-\left\langle\vec{\mathbb{I}}_{2,1}(t),\nabla_{\vec{e}_2(t)}\vec{w}_t\right\rangle\Big)\,dx_2\Big)\bigg).\end{aligned}$$

Indeed, we have by (3.6)

$$\begin{aligned}&\Big\{\nabla_{\vec{e}_1}\left(\left\langle\nabla_{\vec{e}_1}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{2,2}\right\rangle\right)+\nabla_{\vec{e}_2}\left(\left\langle\nabla_{\vec{e}_2}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,1}\right\rangle\right)\\&\quad-\nabla_{\vec{e}_1}\left(\left\langle\nabla_{\vec{e}_2}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,2}\right\rangle\right)-\nabla_{\vec{e}_2}\left(\left\langle\nabla_{\vec{e}_1}\vec{w},e^{-2\lambda}\vec{\mathbb{I}}_{1,2}\right\rangle\right)\Big\}dx_1\wedge dx_2\\&=d\Big(e^{-2\lambda}\left(-\left\langle\vec{\mathbb{I}}_{1,1},\nabla_{\vec{e}_2}\vec{w}\right\rangle+\left\langle\vec{\mathbb{I}}_{1,2},\nabla_{\vec{e}_1}\vec{w}\right\rangle\right)dx_1\\&\qquad+e^{-2\lambda}\left(\left\langle\vec{\mathbb{I}}_{2,2},\nabla_{\vec{e}_1}\vec{w}\right\rangle-\left\langle\vec{\mathbb{I}}_{2,1},\nabla_{\vec{e}_2}\vec{w}\right\rangle\right)dx_2\Big).\end{aligned}$$

Therefore, we simply need to verify that all remaining terms cancel. Using the Codazzi-Mainardi identity (3.7) for the first two terms of (ii) cancels the last two terms and we get

$$\begin{aligned}\text{(ii)}=-e^{-2\lambda}\Big\langle\nabla_{\vec{e}_1}\vec{w},\,&e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{12}^{l}-\mu\alpha_{12}^{l+1}\right)\vec{\mathbb{I}}_{2,l}\\&-e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{2,2}^{l}-\mu\alpha_{2,2}^{l+1}\right)\vec{\mathbb{I}}_{1,l}\Big\rangle\\-e^{-2\lambda}\Big\langle\nabla_{\vec{e}_1}\vec{w},\,&e^{-4\lambda}\sum_{l=1}^{2}\Big(e^{2\lambda_{l+1}}\left(\left\langle\vec{\mathbb{I}}_{1,2},\vec{\mathbb{I}}_{2,l}\right\rangle-\left\langle\vec{\mathbb{I}}_{2,2},\vec{\mathbb{I}}_{1,l}\right\rangle\right)\\&-\mu\left(\left\langle\vec{\mathbb{I}}_{1,2},\vec{\mathbb{I}}_{2,l+1}\right\rangle-\left\langle\vec{\mathbb{I}}_{2,2},\vec{\mathbb{I}}_{1,l+1}\right\rangle\right)\Big)\vec{e}_l\Big\rangle\\-e^{-2\lambda}\Big\langle\nabla_{\vec{e}_2}\vec{w},\,&e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,2}^{l}-\mu\alpha_{1,2}^{l+1}\right)\vec{\mathbb{I}}_{1,l}\\&-e^{-4\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,1}^{l}-\mu\alpha_{1,1}^{l+1}\right)\vec{\mathbb{I}}_{2,l}\Big\rangle\\-e^{-2\lambda}\Big\langle\nabla_{\vec{e}_2}\vec{w},\,&e^{-4\lambda}\sum_{l=1}^{2}\Big(e^{2\lambda_{l+1}}\left(\left\langle\vec{\mathbb{I}}_{1,2},\vec{\mathbb{I}}_{1,l}\right\rangle-\left\langle\vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,l}\right\rangle\right)\\&-\mu\left(\left\langle\vec{\mathbb{I}}_{1,2},\vec{\mathbb{I}}_{1,l+1}\right\rangle-\left\langle\vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,l+1}\right\rangle\right)\Big)\vec{e}_l\Big\rangle\end{aligned}$$

The sum of the even lines of this expression gives

$$\begin{aligned}
&- e^{-2\lambda}\left\langle\nabla_{\vec{e}_1}\vec{w}, e^{-4\lambda}\Big(e^{2\lambda_2}(|\vec{\mathbb{I}}_{1,2}|^2 - \left\langle\vec{\mathbb{I}}_{2,2},\vec{\mathbb{I}}_{1,1}\right\rangle)\vec{e}_1\right.\\
&\qquad\qquad\left. - \mu((|\vec{\mathbb{I}}_{1,2}|^2 - \left\langle\vec{\mathbb{I}}_{2,2},\vec{\mathbb{I}}_{1,1}\right\rangle))\vec{e}_2\Big)\right\rangle\\
&- e^{-2\lambda}\left\langle\nabla_{\vec{e}_2}\vec{w}, e^{-4\lambda}\Big(e^{2\lambda_1}(|\vec{\mathbb{I}}_{1,2}|^2 - \left\langle\vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle)\vec{e}_2\right.\\
&\qquad\qquad\left. - \mu((|\vec{\mathbb{I}}_{1,2}|^2 - \left\langle\vec{\mathbb{I}}_{1,1},\vec{\mathbb{I}}_{2,2}\right\rangle))\vec{e}_1\Big)\right\rangle\\
&= e^{-2\lambda}\left(e^{2\lambda_2}\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_1\rangle + e^{2\lambda_1}\langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_2\rangle - \mu\left(\langle\nabla_{\vec{e}_1}\vec{w},\vec{e}_2\rangle + \langle\nabla_{\vec{e}_2}\vec{w},\vec{e}_1\rangle\right)\right)K_{g_t}\\
&= -\left(\frac{d}{dt}(\det g_t)^{-\frac{1}{2}}\right)K_{g_t}
\end{aligned}$$

which cancels with (3.2). Now, we have

$$\partial_{x_i}e^{4\lambda} = 2\left(\alpha_{i,1}^1 e^{2\lambda_2} + \alpha_{i,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{i,1}^2 + \alpha_{i,2}^1\right)\right)$$

so

$$\partial_{x_i}e^{-2\lambda} = -e^{-6\lambda}\left(\alpha_{i,1}^1 e^{2\lambda_2} + \alpha_{i,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{i,1}^2 + \alpha_{i,2}^1\right)\right).$$

The remaining terms is the sum of the odd lines of the last expression of (ii), (iii) and (3.5):

$$\begin{aligned}
&- e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,2}^l - \mu\alpha_{1,2}^{l+1}\right)\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{2,l}\right\rangle\\
&+ e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{2,2}^l - \mu\alpha_{2,2}^{l+1}\right)\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{1,l}\right\rangle\\
&- e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,2}^l - \mu\alpha_{1,2}^{l+1}\right)\left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,l}\right\rangle\\
&+ e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,1}^l - \mu\alpha_{1,1}^{l+1}\right)\left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{2,l}\right\rangle\\
&+ e^{-6\lambda}\left(\alpha_{1,1}^1 e^{2\lambda_2} + \alpha_{1,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{1,1}^2 + \alpha_{1,2}^1\right)\right)\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle\\
&+ e^{-6\lambda}\left(\alpha_{1,2}^1 e^{2\lambda_2} + \alpha_{2,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{2,1}^2 + \alpha_{2,2}^1\right)\right)\left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle\\
&- e^{-6\lambda}\left(\alpha_{1,1}^1 e^{2\lambda_2} + \alpha_{1,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{1,1}^2 + \alpha_{1,2}^1\right)\right)\left\langle\nabla_{\vec{e}_2}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle\\
&- e^{-6\lambda}\left(\alpha_{1,2}^1 e^{2\lambda_2} + \alpha_{2,2}^2 e^{2\lambda_1} - \mu\left(\alpha_{2,1}^2 + \alpha_{2,2}^1\right)\right)\left\langle\nabla_{\vec{e}_1}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle +
\end{aligned}$$

$$
\begin{aligned}
&- e^{-6\lambda}\sum_{l=1}^{2}\Bigg(\left(e^{2\lambda_{l+1}}\alpha_{1,1}^{l}-\mu\alpha_{1,1}^{l+1}\right)\left\langle \nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{2,2}\right\rangle\\
&+\left(e^{2\lambda_{l+1}}\alpha_{2,2}^{l}-\mu\alpha_{2,2}^{l+1}\right)\left\langle \nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,1}\right\rangle\Bigg)\\
&+2e^{-6\lambda}\sum_{l=1}^{2}\left(e^{2\lambda_{l+1}}\alpha_{1,2}^{l}-\mu\alpha_{1,2}^{l+1}\right)\left\langle \nabla_{\vec{e}_l}\vec{w},\vec{\mathbb{I}}_{1,2}\right\rangle=0
\end{aligned}
$$

which concludes the proof of the lemma. □

3.2. Second variation of K

Lemma 3.2. *For every minimal immersion $\vec{\Phi}:\Sigma\to\mathbb{R}^3$, and for every normal admissible variation $\{\vec{\Phi}_t\}_{t\in I}$ of $\vec{\Phi}$ with variation vector $\vec{w}=w\vec{n}$, we have*

$$
\frac{d^2}{dt^2}\left(K_{g_t}d\mathrm{vol}_{g_t}\right)_{|t=0}=d\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right), \tag{3.8}
$$

where $\star$ is the Hodge star operator.

Proof. Thanks to Lemma 3.1, we have

$$
\begin{aligned}
\frac{d}{dt}(K\,d\mathrm{vol}_g)=d\Big(&e^{-2\lambda}\left(-\left\langle\vec{\mathbb{I}}_{1,1},\partial_{x_2}\vec{w}\right\rangle+\left\langle\vec{\mathbb{I}}_{1,2},\partial_{x_1}\vec{w}\right\rangle\right)dx_1\\
&+e^{-2\lambda}\left(\left\langle\vec{\mathbb{I}}_{2,2},\partial_{x_1}\vec{w}\right\rangle-\left\langle\vec{\mathbb{I}}_{1,2},\partial_{x_2}\vec{w}\right\rangle\right)dx_2\Big),
\end{aligned}
$$

where $e^{4\lambda}=e^{4\lambda(t)}=\det g_t=|\partial_{x_1}\vec{\Phi}_t\wedge\partial_{x_2}\vec{\Phi}_t|^2$. We have as $\vec{w}=w\vec{n}$ the identity

$$
e^{-2\lambda}\left\langle\vec{\mathbb{I}}_{i,j},\partial_{x_k}\vec{w}\right\rangle=e^{-2\lambda}\mathbb{I}_{i,j}\left\langle\vec{n},\partial_{x_k}\vec{w}\right\rangle=e^{-2\lambda}\left\langle\partial^2_{x_i,x_j}\vec{\Phi},\vec{n}\right\rangle\partial_{x_k}w.
$$

Now, as $\vec{\Phi}$ is minimal, the mean curvature $\vec{H}_g$ vanished identically, so we obtain

$$
\begin{aligned}
\frac{d}{dt}e^{2\lambda}_{\ |t=0}&=\left\langle\partial_{x_1}\vec{\Phi},\partial_{x_1}\vec{w}\right\rangle+\left\langle\partial_{x_2}\vec{\Phi},\partial_{x_2}\vec{w}\right\rangle\\
&=\left\langle\partial_{x_1}\vec{\Phi},\partial_{x_1}\vec{n}\right\rangle w+\left\langle\partial_{x_2}\vec{\Phi},\partial_{x_2}\vec{n}\right\rangle w=-2e^{2\lambda}wH_g=0.
\end{aligned} \tag{3.9}
$$

Therefore, (3.9) implies that

$$
\frac{d}{dt}\left(e^{-2\lambda}\left\langle\vec{\mathbb{I}}_{i,j},\partial_{x_k}\vec{w}\right\rangle\right)_{|t=0}=e^{-2\lambda}\frac{d}{dt}\left(\left\langle\partial^2_{x_i,x_j}\vec{\Phi},\vec{n}\right\rangle\left\langle\partial_{x_k}\vec{w},\vec{n}\right\rangle\right)_{|t=0}.
$$

Choosing conformal coordinates, we compute

$$\begin{aligned}
\frac{d}{dt}\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\vec{n}\right\rangle_{|t=0} &= \left\langle \partial^2_{x_i,x_j}\vec{w},\vec{n}\right\rangle + \left\langle \partial^2_{x_i,x_j}\vec{\Phi}, -\sum_{l=1}^{2} e^{-2\lambda}\left\langle \partial_{x_l}\vec{w},\vec{n}\right\rangle \partial_{x_l}\vec{\Phi}\right\rangle \\
&= \left\langle (\partial^2_{x_i,x_j}w)\vec{n} + \partial_{x_i}w\partial_{x_j}\vec{n} + \partial_{x_j}w\partial_{x_i}\vec{n} + w\partial^2_{x_i,x_j}\vec{n},\vec{n}\right\rangle \\
&\quad - \sum_{l=1}^{2} e^{-2\lambda}\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\partial_{x_l}\vec{\Phi}\right\rangle \partial_{x_l} \\
&= \partial^2_{x_i,x_j}w - w\left\langle \partial_{x_i}\vec{n},\partial_{x_j}\vec{n}\right\rangle - e^{-2\lambda}\sum_{l=1}^{2}\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\partial_{x_l}\vec{\Phi}\right\rangle \partial_{x_l}w
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\frac{d}{dt}\left\langle \partial_{x_k}\vec{w},\vec{n}\right\rangle_{|t=0} &= \left\langle \partial_{x_k}\vec{w}, \frac{d}{dt}\vec{n}_{|t=0}\right\rangle = \sum_{l=1}^{2}\left\langle \partial_{x_k}\vec{w}, -e^{-2\lambda}\left\langle \vec{n},\partial_{x_l}\vec{w}\right\rangle \partial_{x_l}\vec{\Phi}\right\rangle \\
&= -e^{-2\lambda}\sum_{l=1}^{2}\left\langle \partial_{x_k}\vec{n},\partial_{x_l}\vec{\Phi}\right\rangle w\,\partial_{x_l}w = e^{-2\lambda}\sum_{l=1}^{2}\mathbb{I}_{k,l}\; w\,\partial_{x_l}w.
\end{aligned} \tag{3.11}$$

Therefore, , we have by (3.10) and (3.11)

$$\begin{aligned}
&\frac{d}{dt}\left(\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\vec{n}\right\rangle \left\langle \partial_{x_k}\vec{w},\vec{n}\right\rangle\right)_{|t=0} \\
&= \left(\partial^2_{x_i,x_j}w - w\left\langle \partial_{x_i}\vec{n},\partial_{x_j}\vec{n}\right\rangle - e^{-2\lambda}\sum_{l=1}^{2}\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\partial_{x_l}\vec{\Phi}\right\rangle \partial_{x_l}w\right)\partial_{x_k}w \\
&\quad + e^{-2\lambda}\mathbb{I}_{i,j}\sum_{l=1}^{2}\mathbb{I}_{k,l}\; w\,\partial_{x_l}w
\end{aligned}$$

and finally,

$$\begin{aligned}
e^{-2\lambda}\frac{d}{dt}\left\langle \vec{\mathbb{I}}_{i,j},\partial_{x_k}\vec{w}\right\rangle_{|t=0} &= e^{-2\lambda}(\partial^2_{x_i,x_j}w)(\partial_{x_k}w) - e^{-2\lambda}w\left\langle \partial_{x_i}\vec{n},\partial_{x_j}\vec{n}\right\rangle \partial_{x_k}w \\
&\quad - e^{-4\lambda}\sum_{l=1}^{2}\left\langle \partial^2_{x_i,x_j}\vec{\Phi},\partial_{x_l}\vec{\Phi}\right\rangle (\partial_{x_k}w)(\partial_{x_l}w) \\
&\quad + w\,e^{-4\lambda}\sum_{l=1}^{2}\mathbb{I}_{i,j}\mathbb{I}_{k,l}\partial_{x_l}w.
\end{aligned}$$

By minimality, we have

$$|\nabla_{e_1}\vec{n}|_g^2=|\nabla_{\vec{e}_2}\vec{n}|_g^2=\frac{|\vec{\mathbb{I}}_g|^2}{2}=-K_g,\quad \langle\nabla_{\vec{e}_1}\vec{n},\nabla_{\vec{e}_2}\vec{n}\rangle=0.$$

As a consequence, we get

$$\begin{aligned}
&e^{-2\lambda}\frac{d}{dt}\left(-\left\langle\vec{\mathbb{I}}_{1,1},\partial_{x_2}\vec{w}\right\rangle+\left\langle\vec{\mathbb{I}}_{1,2},\partial_{x_1}\vec{w}\right\rangle\right)\\
&=-\left(e^{-2\lambda}\partial_{x_1}^2 w-we^{-2\lambda}|\partial_{x_1}\vec{n}|^2\right)\partial_{x_2}w\\
&\quad+e^{-4\lambda}\left(\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&\quad-we^{-4\lambda}(\mathbb{I}_{1,1}\mathbb{I}_{1,2}\partial_{x_1}w+\mathbb{I}_{1,1}\mathbb{I}_{2,2}\partial_{x_2}w)\\
&\quad+\left(e^{-2\lambda}\partial_{x_1,x_2}^2 w-we^{-2\lambda}\left\langle\partial_{x_1}\vec{n},\partial_{x_2}\vec{n}\right\rangle\right)\partial_{x_1}w\\
&\quad-e^{-4\lambda}\left(\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\\
&\quad+we^{-4\lambda}(\mathbb{I}_{1,2}\mathbb{I}_{1,1}\partial_{x_1}w+\mathbb{I}_{1,2}\mathbb{I}_{1,2}\partial_{x_2}w)\\
&=-e^{-2\lambda}\partial_{x_1}^2 w\,\partial_{x_2}w+e^{-2\lambda}\partial_{x_1,x_2}^2 w\,\partial_{x_1}w\\
&\quad-e^{-4\lambda}w\partial_{x_2}w\left(\mathbb{I}_{1,1}\mathbb{I}_{2,2}-\mathbb{I}_{1,2}^2\right)+w\partial_{x_2}w|\nabla_{\vec{e}_1}\vec{n}|^2\\
&\quad+e^{-4\lambda}\Big\{\left(\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&\qquad-\left(\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\Big\}\\
&=-e^{-2\lambda}\partial_{x_1}^2 w\,\partial_{x_2}w+e^{-2\lambda}\partial_{x_1,x_2}^2 w\,\partial_{x_1}w-2K\,w\partial_{x_2}w\\
&\quad+e^{-4\lambda}\Big\{\left(\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&\qquad-\left(\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\Big\}.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
&e^{-2\lambda}\frac{d}{dt}\left(\left\langle\vec{\mathbb{I}}_{2,2},\partial_{x_1}\vec{w}\right\rangle-\left\langle\vec{\mathbb{I}}_{1,2},\partial_{x_2}\vec{w}\right\rangle\right)\\
&=e^{-2\lambda}\partial_{x_2}^2 w\partial_{x_1}w-w\partial_{x_1}w|\nabla_{\vec{e}_2}\vec{n}|^2+e^{-4\lambda}w(\mathbb{I}_{2,2}\mathbb{I}_{1,1}\partial_{x_1}w+\mathbb{I}_{2,2}\mathbb{I}_{1,2}\partial_{x_2}w)\\
&\quad-e^{-4\lambda}\left(\left\langle\partial_{x_2}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_2}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}\vec{w}\right)\partial_{x_1}w-e^{-2\lambda}\partial_{x_1,x_2}^2 w\\
&\quad-e^{-4\lambda}w\left(\mathbb{I}_{1,2}\mathbb{I}_{1,2}\partial_{x_1}w+\mathbb{I}_{1,2}\mathbb{I}_{2,2}\partial_{x_2}w\right)\\
&\quad+e^{-4\lambda}\left(\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial_{x_1,x_2}^2\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w
\end{aligned}$$

$$
\begin{aligned}
&= e^{-2\lambda}\left(\partial^2_{x_2}w\,\partial_{x_1}w-\partial^2_{x_1,x_2}w\,\partial_{x_2}w\right)+2K_g\,w\partial_{x_1}w\\
&\quad+e^{-4\lambda}\bigg\{-\left(\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}\vec{w}\right)\partial_{x_1}w\\
&\quad+\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\bigg\}.
\end{aligned}
$$

Now let $\{\alpha_t\}_{t\in I}$ be the family of 1-form such that for all $t\in I$, one has

$$
\frac{d}{dt}\left(K_{g(t)}d\mathrm{vol}_{g(t)}\right)=d\,\alpha_t.
$$

We have

$$
\begin{aligned}
\frac{d}{dt}(\alpha_t)_{t=0}&=\bigg\{-e^{-2\lambda}\partial^2_{x_1}w\,\partial_{x_2}w+e^{-2\lambda}\partial^2_{x_1,x_2}w\,\partial_{x_1}w-2K_g\,w\partial_{x_2}w\\
&\quad+e^{-4\lambda}\bigg\{\left(\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&\quad-\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\bigg\}\bigg\}dx_1\\
&\quad+\bigg\{e^{-2\lambda}\left(\partial^2_{x_2}w\,\partial_{x_1}w-\partial^2_{x_1,x_2}w\,\partial_{x_2}w\right)+2K_g\,w\partial_{x_1}w\\
&\quad+e^{-4\lambda}\bigg\{-\left(\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}\vec{w}\right)\partial_{x_1}w\\
&\quad+\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\bigg\}\bigg\}dx_2.
\end{aligned}
$$

Now, as $M=\vec{\Phi}(\Sigma)$ is minimal, $\vec{\Phi}:\Sigma\to\mathbb{R}^3$ is harmonic for the metric g, so in our conformal chart, we deduce that

$$
\Delta_g\vec{\Phi}=e^{-2\lambda}(\partial^2_{x_1}\vec{\Phi}+\partial^2_{x_2}\vec{\Phi})=0
$$

thus $\partial^2_{x_2}\vec{\Phi}=-\partial^2_{x_1}\vec{\Phi}$. And as $\vec{\Phi}$ is conformal, we have

$$
\left\langle\partial_{x_i}\vec{\Phi},\partial_{x_j}\vec{\Phi}\right\rangle=e^{2\lambda}\delta_{i,j},\quad\left\langle\partial^2_{x_i,x_j}\vec{\Phi},\partial_{x_j}\vec{\Phi}\right\rangle=\frac{1}{2}\partial_{x_i}e^{2\lambda},\quad 1\leq i,j\leq 2.
$$

We deduce that

$$\begin{aligned}
&\left(\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&-\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\\
&=\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle(\partial_{x_1}w)(\partial_{x_2}w)-\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle(\partial_{x_2}w)^2\\
&-\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle(\partial_{x_1}w)^2-\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle(\partial_{x_1}w)(\partial_{x_2}w)\\
&=\frac{1}{2}\bigg(\partial_{x_1}e^{2\lambda}(\partial_{x_1}w)(\partial_{x_2}w)-\partial_{x_2}e^{2\lambda}(\partial_{x_2}w)^2\\
&\quad-\partial_{x_2}e^{2\lambda}(\partial_{x_1}w)^2-\partial_{x_1}e^{2\lambda}(\partial_{x_1}w)(\partial_{x_2}w)\bigg)\\
&=-\frac{1}{2}\partial_{x_2}e^{2\lambda}\left((\partial_{x_1}w)^2+(\partial_{x_2}w)^2\right).
\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
&\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&+\left(\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w-\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}\vec{w}\right)\partial_{x_1}w\\
&=\frac{1}{2}\partial_{x_1}e^{2\lambda}\left((\partial_{x_1}w)^2+(\partial_{x_2}w)^2\right),
\end{aligned}$$

so

$$\begin{aligned}
&e^{-4\lambda}\bigg\{\left(\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\\
&-\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_1}w\bigg\}dx_1\\
&+e^{-4\lambda}\bigg\{-\left(\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}\vec{w}\right)\partial_{x_1}w\\
&+\left(\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_1}\vec{\Phi}\right\rangle\partial_{x_1}w+\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\partial_{x_2}\vec{\Phi}\right\rangle\partial_{x_2}w\right)\partial_{x_2}w\bigg\}dx_2\\
&=\frac{1}{2}e^{-4\lambda}|dw|^2\left(\partial_{x_1}e^{2\lambda}dx_2-\partial_{x_2}e^{2\lambda}dx_1\right)\\
&=\frac{1}{2}e^{-4\lambda}|dw|^2\star de^{2\lambda}=\frac{1}{2}(e^{-2\lambda}\star de^{2\lambda})|dw|^2_g.
\end{aligned}$$

The remaining terms are

$$
\begin{aligned}
&e^{-2\lambda}\Big\{ \Big(-\partial^2_{x_1} w\,\partial_{x_2} w + \partial^2_{x_1,x_2} w\,\partial_{x_1} w\Big)\,dx_1 \\
&\quad + \Big(\partial^2_{x_2} w\,\partial_{x_1} w - \partial^2_{x_1,x_2} w\,\partial_{x_2} w\Big)\,dx_2\Big\} \\
&= e^{-2\lambda}\left(\partial^2_{x_1} w(-\partial_{x_2} w dx_1) + \partial^2_{x_2} w(\partial_{x_1} w dx_2)\right) \\
&\quad + \frac{1}{2} e^{-2\lambda}\left(\partial_{x_2}|\partial_{x_1} w|^2 dx_1 - \partial_{x_1}|\partial_{x_2} w|^2 dx_2\right) \\
&= e^{-2\lambda}\left(\partial^2_{x_1} w(\star dw - \partial_{x_1} w dx_2) + \partial^2_{x_2} w(\star dw + \partial_{x_2} w dx_1)\right) \\
&\quad + \frac{1}{2} e^{-2\lambda}\left(\partial_{x_2}|\partial_{x_1} w|^2 dx_1 - \partial_{x_1}|\partial_{x_2} w|^2 dx_2\right) \\
&= \Delta_g\, w(\star dw) \\
&\quad + \frac{1}{2} e^{-2\lambda}\left(\partial_{x_2}(|\partial_{x_1} w|^2 + |\partial_{x_2} w|^2) dx_1 - \partial_{x_1}(|\partial_{x_1} w|^2 + |\partial_{x_2} w|^2) dx_2\right) \\
&= \Delta_g w \,\star dw - \frac{1}{2} e^{-2\lambda} \star d|dw|^2
\end{aligned}
$$

and

$$
2K_g w(-\partial_{x_2} w dx_1 + \partial_{x_1} w dx_2) = 2K_g w \star dw.
$$

As we have

$$
e^{-2\lambda} \star d|dw|^2 = e^{-2\lambda} \star d(e^{2\lambda}|dw|^2_g) = (e^{-2\lambda} \star de^{2\lambda})|dw|^2_g + \star d|dw|^2_g,
$$

we finally deduce that

$$
\begin{aligned}
\frac{d}{dt}\,(\alpha_t)_{|t=0} &= \frac{1}{2}(e^{-2\lambda} \star de^{2\lambda})|dw|^2_g + \Delta_g w \star dw \\
&\quad - \frac{1}{2}\left(\left(e^{-2\lambda} \star de^{2\lambda}\right)|dw|^2_g + \star d|dw|^2_g\right) + 2K_g w \star dw \\
&= (\Delta_g w + 2K_g w) \star dw - \frac{1}{2} \star d|dw|^2_g.
\end{aligned}
$$

This concludes the proof of the lemma. □

4. Index estimate

4.1. Introduction and definitions

According to the classification established by Robert Bryant in [3], Willmore spheres in S^3 are either minimal equatorial spheres or inverse stereographic projections of a certain class of complete minimal surfaces in $\mathbb{R}^3$. Let $\vec{\Phi}:\overline{\Sigma}\to\mathbb{R}^3$ be a complete minimal isometric immersion from an connected Riemann surface $\overline{\Sigma}$. By minimal we mean that the mean curvature tensor $\vec{H}_g$ (where g is the pull-back by $\vec{\Phi}$ of the euclidean metric on $\mathbb{R}^3$) of $\vec{\Phi}$ is identically zero. We say that $M=\vec{\Phi}(\overline{\Sigma})$ has finite total curvature if

$$C(M)=\int_{\overline{\Sigma}}|K_g|d\mathrm{vol}_g=\int_{\overline{\Sigma}}-K_g d\mathrm{vol}_g<\infty.$$

We first recall the following theorem.

Theorem 4.1. *Let Σ be an orientable surface and $\vec{\Phi}:\overline{\Sigma}\to\mathbb{R}^3$ be a complete minimal immersion with finite total curvature.*

1) ([12]) *$\overline{\Sigma}$ is conformally diffeomorphic to a closed Riemann surface Σ with a finite number a points removed, called the ends of $\overline{\Sigma}$. We say that an end $p\in\Sigma$ is embedded, if there exists a radius $r>0$, such that the restriction $\vec{\Phi}|D^2(p,r)\setminus\{p\}$ is an embedding.*
2) ([28]) *If $p\in\Sigma$ is an embedded end, and $r>0$ is such that $\vec{\Phi}|D^2(p,r)\setminus\{p\}$ is an embedding, then there exists $a,b\in\mathbb{R}$ and $c\in\mathbb{R}^2$ such that, up to rotation and translation*

$$\vec{\Phi}(D^2(p,r)\setminus\{p\})=\left\{(x,y):y=a\log|x|+b+\frac{c\cdot x}{|x|^2}+O\left(\frac{1}{|x|^2}\right),\right.$$
$$\left. x\in\mathbb{R}^2\setminus D^2(0,1)\right\}.$$

We say that the end is planar (or has zero logarithmic growth) if $a=0$.

3) ([22]) *$C(M)$ is an integer multiple of 4π, and furthermore, if all ends of $\overline{\Sigma}$ are embedded, and $\overline{\Sigma}$ has m ends, then*

$$C(M)=\int_{\Sigma}-K_g d\mathrm{vol}_g=4\pi\left(m+\gamma-1\right). \tag{4.1}$$

if the genus of Σ is γ.

We recall the fundamental theorem of Robert Bryant.

Theorem 4.2. *([3]) Let $\vec{\Psi}:S^2\to\mathbb{R}^3$ be a Willmore immersion, then either $\vec{\Psi}$ is totally umbilic, either $\vec{\Psi}$ is the inversion of a complete minimal surface with finite total curvature and planar ends.*

4.2. Second variation of conformal Willmore functional

We first define the index for the Willmore functional.

Definition 4.3. Let $\vec{\Psi} \in W_\iota^{2,2}(\Sigma, N^n) \cap C^\infty(\Sigma, N^n)$ be a Willmore surface. Then the second variation D^2W of $W = W_{N^n}$ is well defined by

$$D^2W(\vec{\Psi})[\vec{w}, \vec{w}] = \frac{d^2}{dt^2} W(\vec{\Psi}_t)_{|t=0}.$$

Here, $\{\vec{\Psi}_t\}_{t\in I} \in C^2(I, W_\iota^{2,2}(\Sigma, N^n) \cap C^\infty(\Sigma, N^n))$ is a C^2 family of immersions such that $\vec{\Psi}_0 = \vec{\Psi}$, and $\vec{w} = \frac{d}{dt}(\vec{\Psi}_t)_{|t=0}$. The index of a critical point $\vec{\Psi} \in W_\iota^{2,2}(\Sigma, N^n) \cap W^{1,\infty}(\Sigma, N^n)$ of the Willmore functional W, denoted by $\mathrm{Ind}_W(\vec{\Psi})$, is defined as the dimension of the maximal subspace of $W_{\vec{\Psi}}^{2,2}(\Sigma, TN^n) \cap W^{1,\infty}(\Sigma, TN^n)$ where the second derivative $D^2W(\vec{\Psi})$ is negative definite. We define in an analogous way the $\mathscr{W}$ index, denoted by $\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi})$.

Let us recall (2.20) and Lemma 3.2, we have the following formula.

Lemma 4.4. *Let Σ be a connected Riemann surface, an $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ be a complete minimal immersion. For every C^2 family of immersions $\{\vec{\Phi}_t\}_{t\in I} \in C^2(I, W_\iota^{2,2}(\Sigma, \mathbb{R}^n) \cap C^\infty(\Sigma, \mathbb{R}^n))$, we have*

$$\begin{aligned}(4.2)\qquad &\frac{d^2}{dt^2}\left(\left(H_{g_t}^2 - K_{g_t}\right) d\mathrm{vol}_{g_t}\right)_{|t=0}\\ &= \left\{\frac{1}{2}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - d\left((\Delta_g w + 2K_g w) \star dw - \frac{1}{2} \star d|dw|_g^2\right)\right\}\end{aligned}$$

if for $t \in I$, the pull-back metric g_t is defined by $g_t = \vec{\Phi}_t^* g_{\mathbb{R}^n}$.

Let Σ be a closed Riemann surface and $\vec{\Psi} : \Sigma \to \mathbb{R}^3$ be a (smooth) non-branched Willmore immersion which is the inversion of a complete minimal surface with planar ends (see the remark page 47 in [3]). Then up to translation there exists an integer m, a finite set of m distinct points $\{p_1, \ldots, p_m\} \subset \Sigma$, and a complete minimal isometric immersion $\vec{\Phi} : \Sigma \to \mathbb{R}^3$, such that $M = \vec{\Phi}(\overline{\Sigma})$ ($\overline{\Sigma} = \Sigma \setminus \{p_1, \ldots, p_m\}$) is a complete minimal surface with finite total curvature, and planar ends, and if $i : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ is the inversion at 0, given by

$$i(x) = \frac{x}{|x|^2}, \quad \forall x \in \mathbb{R}^3 \setminus \{0\},$$

then we have $\vec{\Psi} = i \circ \vec{\Phi}$. By conformal invariance, noting $\mathscr{W} = \mathscr{W}_{\mathbb{R}^3}$, we have

$$\mathscr{W}(\vec{\Phi}) = \mathscr{W}(\vec{\Psi}).$$

So by Theorem 4.1 and Gauss-Bonnet formula, we have

$$W(\vec{\Psi}) = \mathscr{W}(\vec{\Psi}) + 2\pi\chi(\Sigma) = C(M) + 2\pi\chi(\Sigma) = 4\pi m,$$

which shows that the Willmore energy is quantized by 4π. Another interpretation of this integer m is given by a theorem of Peter Li and Shing-Tung Yau [16]. They proved that if $x \in \mathbb{R}^3$, and $\vec{\Psi}^{-1}(\{x\}) = k$ for some $k \in \mathbb{N}$, then

$$W(\vec{\Psi}) \geq 4\pi k$$

so m is the maximum number if pre-images under $\vec{\Psi}$ of points in $\mathbb{R}^3$. Furthermore, thanks to our normalisation, we see that 0 has m pre-images under $\vec{\Psi}$. The next proposition shows that the conformal invariance leads to a much simpler formula for the second derivative of inversions of minimal surfaces.

Proposition 4.5. *Let Σ be a closed Riemann surface and $\vec{\Psi} : \Sigma \to \mathbb{R}^3$ be a smooth Willmore immersion which is the inversion of a complete minimal surface $\vec{\Phi} : \Sigma \setminus \{p_1, \dots, p_m\} \to \mathbb{R}^3$. Then for every normal variation $\vec{v} = v\, n_{\vec{\Psi}}$ of $\vec{\Psi}$, we have*

$$D^2 W(\vec{\Psi})[\vec{v}, \vec{v}] = \int_\Sigma \Big\{ \frac{1}{2}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - d\left((\Delta_g w + 2K_g w) \star dw - \frac{1}{2} \star d|dw|_g^2 \right) \Big\} \tag{4.3}$$

if $w = |\vec{\Phi}|^2 v$ and $g = \vec{\Phi}^* g_{\mathbb{R}^3}$.

Proof. Let $\{\vec{\Psi}_t\}_{t\in I}$ (where I is an open interval of $\mathbb{R}$ containing 0) an admissible variation of $\vec{\Psi}$ such that

$$\frac{d}{dt}\vec{\Psi}_t|_{t=0} = \vec{v} = v\vec{n}_{\vec{\Psi}}$$

for some $v \in C^\infty(\Sigma)$. We have

$$\mathscr{W}(\vec{\Psi}_t) = \int_\Sigma (H_{g_t}^2 - K_{g_t}) d\mathrm{vol}_{g_t} = \int_\Sigma H_{g_t}^2 d\mathrm{vol}_{g_t} - 2\pi\chi(\Sigma)$$

so, by the conformal invariance of $\mathscr{W}$,we have

$$D^2 W(\vec{\Psi})[\vec{v},\vec{v}] = D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}] = \frac{d^2}{dt^2}\mathscr{W}(\vec{\Psi}_t)|_{t=0} = \frac{d^2}{dt^2}\mathscr{W}(i\circ\vec{\Psi}_t)|_{t=0}. \tag{4.4}$$

The remainder of the proof is dedicated to the verification of the fact that

$$\frac{d^2}{dt^2}\mathscr{W}(i\circ\vec{\Psi}_t)|_{t=0} = D^2\mathscr{W}(\vec{\Phi})[\vec{w},\vec{w}]$$

if $\vec{w} = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}$. First, we have

$$\frac{d}{dt}\left(\vec{\Psi}_t\right)_{|t=0} = \vec{v} = v\,\vec{n}_{\vec{\Psi}},$$

so as

$$\vec{n}_{\vec{\Phi}} = \vec{n}_{\vec{\Psi}} - 2\left(\vec{\Psi}\cdot\vec{n}_{\vec{\Psi}}\right)\frac{\vec{\Psi}}{|\vec{\Psi}|^2} \tag{4.5}$$

we obtain

$$\begin{aligned}\vec{w} &= \frac{d}{dt}\left(\frac{\vec{\Psi}_t}{|\vec{\Psi}_t|^2}\right)_{|t=0} = \frac{\vec{v}}{|\vec{\Psi}|^2} - 2\frac{\vec{\Psi}\cdot\vec{v}}{|\vec{\Psi}|^4}\vec{\Psi} \\ &= \frac{v}{|\vec{\Psi}|^2}\left(\vec{n}_{\vec{\Psi}} - 2\left(\vec{\Psi}\cdot\vec{n}_{\vec{\Psi}}\right)\frac{\vec{\Psi}}{|\vec{\Psi}|^2}\right) = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}.\end{aligned} \tag{4.6}$$

Let $\vec{\Phi}_t$ denote the function $i\circ\vec{\Psi}_t$. We deduce from (4.6) that

$$\frac{d}{dt}\left(\vec{\Phi}_t\right)_{|t=0} = w\,\vec{n}_{\vec{\Phi}} \tag{4.7}$$

where $w = |\vec{\Phi}|^2 v$.

We now use the more precise result on the conformal invariance of the Willmore energy which implies that the 2-form

$$\alpha(\vec{\Psi})=(H_h^2-K_h)d\mathrm{vol}_h$$

if $h=\vec{\Psi}^*g_{\mathbb{R}^3}$ is the pull-back metric of the Willmore immersion $\vec{\Psi}:\Sigma\to\mathbb{R}^3$, is conformally invariant. As $\{\vec{\Psi}_t\}_{t\in I}$ is an admissible variation of $\vec{\Psi}$, the map

$$t\mapsto\alpha(\vec{\Psi}_t)$$

is a C^2 map with values into the 2-forms on Σ, we deduce by compactness of Σ that for all $\varepsilon>0$ such that $[-\varepsilon,\varepsilon]\subset I$, there exists a constant $0<C(\varepsilon)<\infty$ such that

$$\max\left\{\sup_{-\varepsilon\leq t\leq\varepsilon}\left|\alpha(\vec{\Psi}_t)\right|,\ \sup_{-\varepsilon\leq t\leq\varepsilon}\left|\frac{d}{dt}\alpha(\vec{\Psi}_t)\right|,\ \sup_{-\varepsilon\leq t\leq\varepsilon}\left|\frac{d^2}{dt^2}\alpha(\vec{\Psi}_t)\right|\right\}\leq C(\varepsilon)<\infty \tag{4.8}$$

where the norm of a 2-form β given locally in a local complex coordinate $z=x_1+ix_2$ by $\beta=f\,dx_1\wedge dx_2$ is defined as

$$|\beta|^2=|f|^2.$$

In particular, by the point-wise conformal invariance of $\alpha(\vec{\Psi})$, if $\vec{\Phi}=i\circ\vec{\Psi}$ is the inversion of $\vec{\Psi}$, recalling that $\vec{\Phi}_t=i\circ\vec{\Psi}_t$ for all $t\in I$, we have

$$\alpha(\vec{\Phi}_t)=\alpha(\vec{\Psi}_t). \tag{4.9}$$

In particular, we deduce from (4.8) and (4.9) that

$$\max\left\{\sup_{-\varepsilon\leq t\leq\varepsilon}\left|\alpha(\vec{\Phi}_t)\right|,\ \sup_{-\varepsilon\leq t\leq\varepsilon}\left|\frac{d}{dt}\alpha(\vec{\Phi}_t)\right|,\ \sup_{-\varepsilon\leq t\leq\varepsilon}\left|\frac{d^2}{dt^2}\alpha(\vec{\Phi}_t)\right|\right\}\leq C(\varepsilon)<\infty \tag{4.10}$$

We now have thanks to Lemma 4.4 the pointwise equality everywhere on $\Sigma\setminus\{p_1,\dots,p_n\}$ (and therefore Lebesgue almost everywhere on Σ)

$$\begin{aligned}\frac{d^2}{dt^2}\alpha(\vec{\Phi}_t)|_{t=0}&=\frac{1}{2}\left(\Delta_g w-2K_g\right)d\mathrm{vol}_g\\&\quad-d\left((\Delta_g w+K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)\end{aligned} \tag{4.11}$$

if $g = \vec{\Phi}^* g_{\mathbb{R}^3}$, and $w = |\vec{\Phi}|^2 v$, if the variation $\vec{\Psi}$ is normal and given by

$$\frac{d}{dt}\left(\vec{\Psi}_t\right)_{t=0} = v\,\vec{n}_{\vec{\Psi}},$$

where $\vec{n}_{\vec{\Psi}}$ is the unit normal of $\vec{\Psi}$. In particular, thanks to (4.48) and (4.49), we can apply Lebesgue's bounded convergence theorem to obtain that

$$\begin{aligned}(4.12)\qquad \frac{d^2}{dt^2}\mathscr{W}(\vec{\Psi}_t)_{t=0} &= \frac{d^2}{dt^2}\mathscr{W}(\vec{\Phi}_t)_{t=0} = \int_\Sigma \frac{d^2}{dt^2}\left(\alpha(\vec{\Phi}_t)\right)_{t=0}\\ &= \int_\Sigma \bigg\{\frac{1}{2}\left(\Delta_g w - 2K_g\right)^2 d\mathrm{vol}_g\\ &\quad - d\left((\Delta_g w + K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right)\bigg\}.\end{aligned}$$

This concludes the proof of the proposition. ☐

This proposition shows that the index $\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi})$ of $\vec{\Psi}$ is equal to the index of its inversion $\vec{\Phi} = i \circ \vec{\Psi}$ for normal variations of the form $\vec{w} = |\vec{\Phi}|^2 v\,\vec{n}_{\vec{\Phi}}$. A remarkable fact is that one can estimate the index by computing explicitly an integral involving a residue term.

4.3. Explicit formula for the second derivative of inversions of minimal surfaces

We begin with a definition of residues of complete minimal surfaces with embedded planar ends.

Definition-Proposition 4.6. *Let Σ be a closed Riemann surface, and let $\vec{\Phi} : \Sigma \setminus \{p_1, \dots, p_m\} \to \mathbb{R}^3$ be a complete minimal immersion with m planar embedded end. For all $1 \leq j \leq m$, for all (mutually disjoint) small enough open subsets $U_1, \dots, U_m \subset \Sigma$, for any complex coordinate $u_j : U_j \to D^2$ such that $u_j(p_j) = 0$, the limit*

$$(4.13)\qquad \lim_{r\to 0}\left(-\frac{r^2}{4\pi}\int_{\partial D^2_\Sigma(p_j,r)} \star\, d\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right),$$

where $D^2_\Sigma(p_j, r) = u_j^{-1}(D^2_{\mathbb{C}}(0,r))$ (and $0 < r < 1$), exists and is a finite positive number, independent of the chart $u_j : U_j \to D^2$ such that $u_j(p_j) = 0$. We call it the residue of $\vec{\Phi}$ at (p_j, U_j) and denote it by $\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)$.

Proof. We fix some $1 \leq j \leq m$. Thanks to a theorem of Richard Schoen ([28], see also the paper of Robert Osserman [22] about the Weierstrass parametrisation), as the end is embedded and planar, provided U_j is small enough, there exists a complex chart $u_j : U_j \to D^2_{\mathbb{C}}(0,1)$, such that $u_j(p_j) = 0$, and which sends p_j to 0 such that, for $\vec{\Phi}_{u_j} = \vec{\Phi} \circ u_j^{-1}$, one has

$$\vec{\Phi}_{u_j}(z) = \mathrm{Re} \int^z (\varphi_1, \varphi_2, \varphi_3) d\zeta, \tag{4.14}$$

where $z = x_1 + i\, x_2 \in D^2_{\mathbb{C}}(0,1) \setminus \{0\}$, and for $1 \leq j \leq 3$, φ_j is a holomorphic function with a pole of order at most 2 at 0, and finally

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0 \tag{4.15}$$

Assuming up to a rotation that the asymptotic normal is $(0,0,1)$, this translates to

$$\varphi_1(z) = \frac{a_1}{z^2} + \beta_1 + O(|z|), \quad \varphi_2(z) = \frac{a_2}{z^2} + \beta_2 + O(|z|), \quad \varphi_3(z) = \gamma + O(|z|)$$

where $a_1, a_2 \in \mathbb{C} \setminus \{0\}, \beta_1, \beta_2, \gamma \in \mathbb{C}$. The conformality condition (4.15) shows that

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \frac{a_1^2 + a_2^2}{z^4} + \frac{2(a_1\beta_1 + a_2\beta_2)}{z^2} + O\left(\frac{1}{z}\right). \tag{4.16}$$

In particular, we deduce that up to a rotation, we have for some $\alpha_j \in \mathbb{R} \setminus \{0\}$ with $\alpha_j > 0$ the identities

$$a_1 = -\alpha_j, \quad a_2 = i a_2 = -i\alpha_j, \quad \beta_2 = i\beta_1.$$

Therefore, we have by (4.14)

$$\vec{\Phi}_{u_j}(z) = \mathrm{Re}\left(\frac{\vec{A}_0}{z} + \vec{B}_0 z\right) + O(|z|^2) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3 \tag{4.17}$$

where $\vec{A}_0, \vec{B}_0 \in \mathbb{C}^3$ are defined by

$$\vec{A}_0 = \alpha_j\,(1, i, 0)\,, \quad \vec{B}_0 = \beta_j\,(1, i, 0) + (0, 0, b_3)$$

Notice that we have

$$\left\langle \vec{A}_0, \vec{A}_0 \right\rangle = \left\langle \vec{A}_0, \vec{B}_0 \right\rangle = 0, \quad |\vec{A}_0|^2 = 2\alpha_j^2, \quad \left\langle \vec{A}_0, \overline{\vec{B}_0} \right\rangle = 2\alpha_j \overline{\beta_j}. \tag{4.18}$$

(4.19) **From now on, we drop the u_j index for $\vec{\Phi}_{u_j}$ for the simplicity of notations**.

By (4.17) and (4.18) we have

$$|\vec{\Phi}(z)|^2 = \frac{\alpha_j^2}{|z|^2} + \alpha_j \operatorname{Re}\left(\overline{\beta}_j \frac{\overline{z}}{z}\right) + O\left(|z|\right). \tag{4.20}$$

Furthermore, an immediate computation shows that for every smooth function $\alpha : D^2 \to \mathbb{R}$, and for every smooth contour $\gamma \subset D^2$, we have

$$\frac{1}{4\pi}\int_{\gamma} \star\, d\alpha = \operatorname{Im}\left(\frac{1}{2\pi}\int_{\gamma} \partial\alpha\right) \tag{4.21}$$

Then, by (4.20),

$$\begin{aligned}\partial|\vec{\Phi}(z)|^2 &= -\frac{\alpha_j^2}{|z|^2}\frac{dz}{z} - \frac{1}{2}\alpha_j\overline{\beta_j}\frac{\overline{z}}{z^2}dz + \frac{1}{2}\alpha_j\beta_j\frac{dz}{\overline{z}} + O(1)\\ &= -\frac{\alpha_j^2}{|z|^2}\left(\frac{1}{z} + \frac{1}{2}\frac{\overline{\beta_j}}{\alpha_j}\frac{\overline{z}^2}{z} - \frac{1}{2}\frac{\beta_j}{\alpha_j}z + O(|z|^2)\right).\end{aligned} \tag{4.22}$$

Therefore, thanks to (4.22), we have

$$\begin{aligned}|\partial_z|\vec{\Phi}|^2|^2 &= \frac{\alpha_j^4}{|z|^4}\left(\frac{1}{|z|^2} + \operatorname{Re}\left(\frac{1}{\overline{z}}\cdot\frac{\overline{\beta_j}}{\alpha_j}\frac{\overline{z}^2}{z}\right) - \operatorname{Re}\left(\frac{1}{\overline{z}}\cdot\frac{\beta_j}{\alpha_j}\frac{z}{\overline{z}}\right) + O\left(|z|\right)\right)\\ &= \frac{\alpha_j^2}{|z|^4}\left(\frac{1}{|z|^2} + \operatorname{Re}\left(\frac{\overline{\beta_j}}{\alpha_j}\frac{\overline{z}}{z}\right) - \operatorname{Re}\left(\frac{\beta}{\alpha_j}\frac{z}{\overline{z}}\right) + O(|z|)\right)\\ &= \frac{\alpha_j^2}{|z|^6}\left(1 + O\left(|z|^3\right)\right).\end{aligned} \tag{4.23}$$

Turning now to the conformal parameter, we have

$$\partial_z\vec{\Phi} = -\frac{1}{2}\frac{\vec{A}_0}{z^2} + \frac{1}{2}\vec{B}_0 + O(|z|)$$

so by (4.18)

$$e^{2\lambda}=2|\partial_z\vec{\Phi}|^2=\frac{1}{2}\left|-\frac{\vec{A}_0}{z^2}+\vec{B}_0+O(|z|)\right|^2 \tag{4.24}$$
$$=\frac{1}{2}\left(\frac{|\vec{A}_0|^2}{|z|^4}-2\,\mathrm{Re}\left(\frac{\langle\vec{A}_0,\overline{\vec{B}_0}\rangle}{z^2}\right)+O\left(\frac{1}{|z|}\right)\right)$$
$$=\frac{\alpha_j^2}{|z|^4}-2\alpha_j\,\mathrm{Re}\left(\frac{\overline{\beta}_j}{z^2}\right)+O\left(\frac{1}{|z|}\right)$$
$$=\frac{\alpha_j^2}{|z|^4}\left(1-2\alpha_j^{-1}\,\mathrm{Re}\left(\beta_j z^2\right)+O(|z|^3)\right).$$

Thanks to (4.23) and (4.24), we obtain

$$|d|\vec{\Phi}|^2|_g^2=e^{-2\lambda}|2\,\partial_z|\vec{\Phi}|^2|^2 \tag{4.25}$$
$$=\frac{|z|^4}{\alpha_j^2}\left(1+2\alpha_j^{-1}\,\mathrm{Re}\left(\beta_j z^2\right)+O(|z|^3)\right)$$
$$\times\left(\frac{4\alpha_j^4}{|z|^6}\left(1+O(|z|^3)\right)\right)$$
$$=\frac{4\alpha_j^2}{|z|^2}+8\alpha_j\mathrm{Re}\left(\overline{\beta_j}\frac{\overline{z}}{z}\right)+O\left(|z|\right)$$

so that by (4.20) and (4.25)

$$4|\vec{\Phi}|^2-\frac{1}{2}|d|\vec{\Phi}|^2|_g^2=4\left(\frac{\alpha_j^2}{|z|^2}+\alpha_j\,\mathrm{Re}\left(\overline{\beta}_j\frac{\overline{z}}{z}\right)\right) \tag{4.26}$$
$$-\frac{1}{2}\left(\frac{4\alpha_j^2}{|z|^2}+8\alpha_j\mathrm{Re}\left(\overline{\beta_j}\frac{\overline{z}}{z}\right)\right)+O(|z|)$$
$$=\frac{2\alpha_j^2}{|z|^2}+O(|z|).$$

Therefore, we get the following expression

$$\partial\left(4|\vec{\Phi}|^2-\frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)=-\frac{2\alpha_j^2}{|z|^2}\frac{dz}{z}+O\left(1\right) \tag{4.27}$$

Finally, thanks to (4.21) with $\alpha = 4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2$, we have

$$
\begin{aligned}
(4.28)\qquad & \frac{1}{4\pi}\int_{S^1(0,r)} \star d\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right) \\
&= \mathrm{Im}\left(\frac{1}{2\pi}\int_{S^1(0,r)} \partial\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right) \\
&= \mathrm{Im}\left(\frac{1}{2\pi}\int_{S^1(0,r)} \frac{-2\alpha_j^2}{|z|^2}\frac{dz}{z}\right) + O(1) = -\frac{2\alpha_j^2}{r^2} + O(r).
\end{aligned}
$$

And we obtain thanks to (4.19)

$$
\lim_{r\to 0}\left(-\frac{r^2}{4\pi}\int_{\partial D^2_\Sigma(p_j,r)} \star d\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right) = 2\alpha_j^2 > 0,
$$

which concludes the proof that the limit in (4.13) is well-defined.

Now, to show the independence of the complex chart $u_j : U_j \to D^2$ such that $u_j(p_j) = 0$, let $\widetilde{u}_j : U_j \to D^2$ be another complex chart such that $\widetilde{u}_j(p_j) = 0$. The map $f = u_j \circ \widetilde{u}_j^{-1} : D^2 \to D^2$ is a holomorphic diffeomorphism and $f(0) = 0$, so $f(z) = e^{i\theta}z$ for some $\theta \in S^1$. Therefore, as

$$
\begin{aligned}
(4.29)\qquad \partial\left(4|\vec{\Phi}_{u_j}|^2 - \frac{1}{2}|d|\vec{\Phi}_{u_j}|^2|_g^2\right) &= (u_j^{-1})^*\left(\partial\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right) \\
&= -2\frac{\alpha_j^2}{|z|^2}\frac{dz}{z} + O(|z|),
\end{aligned}
$$

and f is a rotation, we have

$$
(4.30)\qquad f^*\left(\frac{1}{|z|^2}\frac{dz}{z}\right) = \frac{1}{|z|^2}\frac{dz}{z},
$$

Finally, we deduce from the two identities (4.29) and (4.30) that

$$
\begin{aligned}
f^*(u_j^{-1})^*\left(\partial\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right) &= (\widetilde{u_j}^{-1})^*\left(\partial\left(4|\vec{\Phi}|^2 - \frac{1}{2}|d|\vec{\Phi}|^2|_g^2\right)\right) \\
&= -\frac{2\alpha_j^2}{|z|^2}\frac{dz}{z} + O(|z|),
\end{aligned}
$$

and by (4.28), the residue is independent of the complex chart $u_j : U_j \to D^2$ such that $u_j(p_j) = 0$. □

We now state the main result of this Section, from which the main Theorem 1.1 will be an easy consequence.

Theorem 4.7. *Let Σ be a closed Riemann surface and $\vec{\Phi}:\Sigma\setminus\{p_1,\dots,p_m\}\to\mathbb{R}^3$ be a complete minimal immersion with m embedded planar ends, such that $\vec{\Psi}=i\circ\vec{\Phi}:\Sigma\to\mathbb{R}^3$ is a non-branched Willmore immersion. For all $1\leq j\leq m$, we fix some small enough disjoint charts (U_j,u_j), where u_j is a complex coordinate $u_j:U_j\to D^2\subset\mathbb{C}$ such that $u_j(p_j)=0$. For all normal variations $\vec{v}=v\,\vec{n}_{\vec{\Psi}}$ of $\vec{\Psi}$, we have*

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}]=\lim_{R\to 0}\left\{\frac{1}{2}\int_{\Sigma_R}(\Delta_g w-2K_g w)^2d\mathrm{vol}_g\right.\\ \left.-4\pi\sum_{j=1}^{m}\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}\,v^2(p_j)\right\}. \tag{4.31}$$

Here, $w=|\vec{\Phi}|^2v$, $g=\vec{\Phi}^*g_{\mathbb{R}^3}$, the set Σ_R is defined by

$$\Sigma_R=\Sigma\setminus\bigcup_{j=1}^{m}D^2_{\Sigma}(p_j,R),$$

where for all $1\leq j\leq n$, we have $D^2_\Sigma(p_j,R)=u_j^{-1}(D^2_{\mathbb{C}}(0,R))$ (for $0<R<1$), and finally, $\mathrm{Res}_{p_j}(\vec{\Phi},U_j)$ is given by the Definition 4.6 (it is independent of u_j, but not of *U_j). In particular, the limit on the right-hand side of* (4.31) *exists and is a well-defined real number for every normal variation*

$$\vec{v}\in W^{2,2}_{\vec{\Psi}}(\Sigma,\mathbb{R}^3)\cap W^{1,\infty}_{\vec{\Psi}}(\Sigma,\mathbb{R}^3).$$

Remark 4.8. *One of the main points of the theorem is to show that the limit of the right-hand side exists, is finite, and coincides with the second derivative of the inverted immersion $\vec{\Psi}$ of the minimal surface $\vec{\Phi}$ at some normal variation $\vec{v}=v\,\vec{n}_{\vec{\Psi}}$. Indeed, if for some $1\leq j\leq m$, $v(p_j)\neq 0$, as $\vec{\Phi}$ is complete, we have*

$$|\vec{\Phi}(z)|^2\underset{z\to p_j}{\longrightarrow}+\infty.$$

In particular, this is easy to see that

$$\frac{1}{2}\int_{\Sigma_R}(\Delta_g w-2K_g w)^2d\mathrm{vol}_g\\ =\frac{1}{2}\int_{\Sigma_R}\left(\Delta_g\left(|\vec{\Phi}|^2v\right)-2K_g\left(|\vec{\Phi}|^2v\right)\right)d\mathrm{vol}_g\underset{R\to 0}{\longrightarrow}+\infty$$

while we have trivially

$$4\pi\sum_{j=1}^{m}\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}v^2(p_j)\underset{R\to 0}{\longrightarrow}+\infty.$$

However, we will show that the limit of the difference

$$\lim_{R\to 0}\left\{\frac{1}{2}\int_{\Sigma_R}(\Delta_g w-2K_g w)^2 d\mathrm{vol}_g-4\pi\sum_{j=1}^{m}\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}\,v^2(p_j)\right\}$$

exists, is finite, and coincides with $D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}]$*, where* $\vec{\Psi}=\dfrac{\vec{\Phi}}{|\vec{\Phi}|^2}$ *is a compact smooth Willmore surface. We will also check explicitly after the proof of the theorem that we can show a priori that the quantity*

$$\frac{1}{2}\int_{\Sigma_R}(\Delta_g w-2K_g w)^2\,d\mathrm{vol}_g-4\pi\sum_{j=1}^{m}\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}v^2(p_j)$$

is bounded independently of R *as* $R\to 0$*.*

Proof. (of the theorem) By Proposition 4.5 if $\vec{v}=v\,\vec{n}_{\vec{\Psi}}$ and $\vec{w}=w\,\vec{n}_{\vec{\Phi}}$, where $w=|\vec{\Phi}|^2v$, we have

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}]=\int_{\Sigma}\left\{\frac{1}{2}(\Delta_g w-2K_g w)^2 d\mathrm{vol}_g\right.$$
$$\left.-\,d\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)\right\}.$$

As we saw in the remark, the two quantities under the integral are infinite when taken separately. In particular, we deduce from the proof of Proposition 4.5 that

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}]=\lim_{R\to 0}\int_{\Sigma_R}\left\{\frac{1}{2}(\Delta_g w-2K_g w)^2 d\mathrm{vol}_g\right.$$
$$\left.-\,d\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)\right\}. \tag{4.32}$$

Now, for all $R>0$, each component of the integral

$$\int_{\Sigma_R}\left\{\frac{1}{2}(\Delta_g w-2K_g w)^2 d\mathrm{vol}_g-d\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)\right\}$$

is finite, and by Stokes theorem

$$\begin{aligned}&\int_{\Sigma_R} d\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right)\\&=\int_{\partial\Sigma_R}\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right),\end{aligned}$$

so the point of the theorem is to express this last boundary integral as a function of R and the residues of $\vec{\Phi}$ as defined in the Definition 4.6. In particular, the equations (4.3) and (4.4) will then show that the limit exists and coincides with $D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}]$.

Now, as $u_j(\partial D^2_\Sigma(p_j,R)) = \partial D^2_{\mathbb{R}^2}(0,R)$, we have

$$\begin{aligned}&\int_{\partial D^2_\Sigma(p_j,R)}\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right)\\&=\int_{\partial D^2_{\mathbb{R}^2}(0,R)}(u_j^{-1})^*\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right),\end{aligned}$$

and thanks to the proof of Definition-Proposition 4.6, there exists (up to a rotation) $\alpha_j\in\mathbb{R}\setminus\{0\}$, such that for $\vec{\Phi}_{u_j}=\vec{\Phi}\circ u_j^{-1}$, we have

$$\vec{\Phi}_{u_j}(z)=\left(\alpha_j\frac{z}{|z|^2},0\right)+O(|z|),\quad z\in D^2.$$

For the sake of simplicity of notations, we will write $\vec{\Phi}$ instead of $\vec{\Phi}_{u_j}$ in the rest of the proof. Making the change of variable

$$(x_1,x_2,x_3)=\vec{\Phi}(z)$$

we get

$$z=\alpha_j\frac{x}{|x|^2}+O\left(\frac{1}{|x|^3}\right), \tag{4.33}$$

so in (x_1,x_2) coordinates, we obtain for some $a\in\mathbb{R}$ and $b\in\mathbb{R}^2$ the parametrisation

$$\vec{\Phi}(x)=\left(x,a+\frac{b\cdot x}{|x|^2}+O\left(\frac{1}{|x|^2}\right)\right), \tag{4.34}$$

where $x\in\mathbb{R}^2\setminus K$, and K is some compact set containing 0. The components of the induced metric g in these coordinates are

$$g_{i,j}(x)=\left\langle\partial_{x_i}\vec{\Phi}(x),\partial_{x_j}\vec{\Phi}(x)\right\rangle,\quad i,j=1,2.$$

and

$$\partial_{x_1}\vec{\Phi}(x) = \left(1, 0, \frac{b_1}{|x|^2} - 2\frac{(b\cdot x)x_1}{|x|^4} + O\left(\frac{1}{|x|^3}\right)\right)$$
$$\partial_{x_2}\vec{\Phi}(x) = \left(0, 1, \frac{b_2}{|x|^2} - 2\frac{(b\cdot x)x_2}{|x|^4} + O\left(\frac{1}{|x|^3}\right)\right).$$

We can differentiate under the O sign, as the Weierstrass parametrization shows that $\vec{\Phi}$ is locally the real part of a meromorphic function, therefore is it analytic outside of branched points, and the rest can be differentiated as the rest of a convergent series of power of $|x|^{-1}$. These expressions show that we have a conformally flat metric at infinity, as

$$g_{1,2}(x) = \frac{b_1 b_2}{|x|^4} - 2(b_1 x_2 + b_2 x_1)\frac{b\cdot x}{|x|^6} + 4x_1 x_2 \frac{(b\cdot x)^2}{|x|^8} + O\left(\frac{1}{|x|^5}\right) = O\left(\frac{1}{|x|^4}\right)$$
$$g_{i,i}(x) = 1 + \frac{\left(b_i|x|^2 - 2x_i(b\cdot x)\right)^2}{|x|^8} + O\left(\frac{1}{|x|^5}\right) = 1 + O\left(\frac{1}{|x|^4}\right), \quad i = 1,2$$

and

$$\det g(x) = 1 + O\left(\frac{1}{|x|^4}\right).$$

So we have

$$\begin{cases} g_{i,j} = \delta_{i,j} + O\left(\dfrac{1}{|x|^4}\right) \\ d\mathrm{vol}_g(x) = \sqrt{\det(g(x))}dx_1\wedge dx_2 = \left(1 + O\left(\dfrac{1}{|x|^4}\right)\right)dx_1\wedge dx_2 \end{cases} \tag{4.35}$$

which proves the asymptotic flatness.

Now, we have by Stokes theorem

$$\begin{aligned} &\int_{\Sigma_R} d\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right) \\ &= -\sum_{j=1}^m \int_{S^1_\Sigma(p_j,R)}\left((\Delta_g w + 2K_g w)\star dw - \frac{1}{2}\star d|dw|_g^2\right) \end{aligned} \tag{4.36}$$

where the circles $S^1_\Sigma(p_j,R) = \partial D^2_\Sigma(p_j,R)$ are *positively* oriented (which explains the negative sign in front of the sum). Setting $r = \alpha_j(R)^{-1}$, the change of variable (4.33) shows that

$$\int_{S^1_{\Sigma}(p_j,R)}\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)$$
$$=\int_{S^1_r}\left((\Delta_g w+2K_g w)\star dw-\frac{1}{2}\star d|dw|_g^2\right)+o(1)$$

where S^1_r is the circle in $\mathbb{R}^2$ of radius $r>0$. Indeed, if we make the change of variable

$$z=\alpha_j\frac{x}{|x|^2}, \tag{4.37}$$

we obtain

$$\vec{\Phi}(x)=\left(x+O\left(\frac{1}{|x|}\right),\tilde{a}+\frac{\tilde{b}\cdot x}{|x|^2}+O\left(\frac{1}{|x|^2}\right)\right)$$

for some $\tilde{a}\in\mathbb{R}$, $\tilde{b}\in\mathbb{R}^2$. As we will see in the following, the error term in these coordinates where $\vec{\Phi}$ is almost a graph is irrelevant, so we discard it and use (4.33) instead. Now thanks to (4.33), we have

$$w(x)=\frac{v\left(\frac{x}{|x|^2}\right)}{|\vec{\Psi}(x)|^2}=v\left(\frac{x}{|x|^2}\right)|\vec{\Phi}(x)|^2=v\left(\frac{x}{|x|^2}\right)\left(|x|^2+a^2+O\left(\frac{1}{|x|^2}\right)\right)$$

We now have

$$\begin{aligned}\star dw(x)&=\partial_{x_1}\left(\left(|x|^2+a^2+O(|x|^{-2})\right)v\left(\frac{x}{|x|^2}\right)\right)dx_2\\&\quad-\partial_{x_2}\left(\left(|x|^2+a^2+O(|x|^{-2})\right)v\left(\frac{x}{|x|^2}\right)\right)dx_1\\&=2v(i(x))\left(x_1dx_2-x_2dx_1\right)+O(|x|^{-3})v(i(x))(dx_2-dx_1)\\&\quad+(|x|^2+a^2+O(|x|^{-2}))\\&\quad\times\left\{\left(\left(\frac{1}{|x|^2}-\frac{2x_1^2}{|x|^4}\right)\partial_1 v(i(x))-\frac{2x_1x_2}{|x|^4}\partial_2 v(i(x))\right)dx_2\right.\\&\quad\left.-\left(\left(\frac{1}{|x|^2}-\frac{2x_2^2}{|x|^4}\right)\partial_2 v(i(x))-\frac{2x_1x_2}{|x|^4}\partial_1 v(i(x))\right)dx_1\right\}\\&=2v(i(x))(x_1dx_2-x_2dx_1)+\left(\frac{x_2^2-x_1^2}{|x|^2}\partial_1 v-\frac{2x_1x_2}{|x|^2}\partial_2 v\right)dx_2\\&\quad-\left(\frac{x_1^2-x_2^2}{|x|^2}\partial_2 v-\frac{2x_1x_2}{|x|^4}\partial_1 v\right)dx_1+O\left(\frac{1}{|x|^2}\right)\end{aligned}$$

Now, by (4.35), we have

$$\begin{aligned}\Delta_g w(x) &= \frac{1}{\sqrt{g(x)}}\sum_{i,j=1}^{2}\partial_{x_i}\left(\sqrt{g(x)}g^{i,j}(x)\partial_{x_j}\left(|\vec{\Phi}(x)|^2 v(i(x))\right)\right)\\ &= \frac{1}{\sqrt{g(x)}}\Big(\partial_{x_1}\left((1+O(|x|^{-4})\partial_{x_1}((|x|^2+a^2+O(|x|^{-2}))v(i(x))))\right)\\ &\quad+\partial_{x_2}\left((1+O(|x|^{-4})\partial_{x_2}((|x|^2+a^2+O(|x|^{-2}))v(i(x))))\right)\\ &\quad+\partial_{x_1}\left(O(|x|^{-4})\partial_{x_2}((|x|^2+a^2+O(|x|^{-2}))v(i(x)))\right)\\ &\quad+\partial_{x_2}\left(O(|x|^{-4})\partial_{x_1}((|x|^2+a^2+O(|x|^{-2}))v(i(x)))\right)\Big)\end{aligned}$$

Now, we notice that there holds

$$v(i(x)) = O(1), \quad dv(i(x)) = O(|x|^{-2})$$

so that

$$\begin{aligned}&\partial_{x_i}\left(O(|x|^{-4})\partial_{x_j}\left((|x|^2+a^2+O(|x|^{-2})v(i(x))\right)\right)\\ &= \partial_{x_i}\left(O(|x|^{-4})\left(2x_j v(i(x)) + (|x|^2+a^2+|x|^{-2})\partial_{x_j}v(i(x))\right)\right)\\ &= \partial_{x_i}(O(|x|^{-4})(O(|x|+O(1)))) = O(|x|^{-4}),\end{aligned}$$

and only the flat Laplacian will remain in the end. We have

$$\begin{aligned}&\Delta\left(|x|^2+a^2+O(|x|^{-2})v(i(x))\right)\\ &= (4+O(|x|^{-4}))v(i(x)) + 4\left\langle x+O(|x|^{-3}), \nabla v(i(x))\right\rangle\\ &\quad+(|x|^2+a^2+O(|x|^{-2}))\Delta v(i(x)).\end{aligned}$$

A simple computation will show that $\Delta v(i(x)) = O(|x|^{-3})$, so we can discard all error terms, and the constant term a^2, as we integrate on a circle of radius r. So it suffices to compute the flat Laplacian of w. We have

$$\Delta w(i(x)) = 4v(i(x)) + 2\left(2x_1\partial_{x_1}v(i(x)) + 2x_2\partial_{x_2}v(i(x))\right) + |x|^2\Delta v(i(x))$$

and

$$\begin{aligned}
&x_1\partial_{x_1}v(i(x))+x_2\partial_{x_2}v(i(x))\\
&=x_1\left(\frac{x_2^2-x_1^2}{|x|^4}\partial_1 v-\frac{2x_1x_2}{|x|^4}\partial_2 v\right)+x_2\left(\frac{x_1^2-x_2^2}{|x|^4}\partial_2 v-\frac{2x_1x_2}{|x|^4}\partial_1 v\right)\\
&=\frac{1}{|x|^4}\left(x_1(x_2^2-x_1^2)-2x_1x_2^2\right)\partial_1+\frac{1}{|x|^4}\left(-2x_1^2x_2+x_2(x_1^2-x_2^2)\right)\\
&=-\frac{1}{|x|^2}\left(x_1\partial_1 v+x_2\partial_2 v\right)\\
&=-\left\langle dv(i(x)),i(x))\right\rangle.
\end{aligned}$$

By the conformal invariance of the Laplacian in dimension 2, we have

$$\Delta\left(v(i(x))\right)=\frac{1}{|x|^4}\Delta v(i(x)),$$

so we get

$$\begin{aligned}
\Delta_g w(i(x))&=4v(i(x))-4\left\langle dv(i(x)),i(x)\right\rangle\\
&\quad+\frac{1}{|x|^2}\Delta v(i(x))+O\left(\frac{1}{|x|^3}\right).
\end{aligned}\tag{4.38}$$

Furthermore, we have

$$\partial_{x_1}\vec{\Phi}(x)\wedge\partial_{x_2}\vec{\Phi}(x)=\left(O\left(\frac{1}{|x|^2}\right),O\left(\frac{1}{|x|^2}\right),1\right),$$

so

$$\vec{n}(x)=\left(O\left(\frac{1}{|x|^2}\right),O\left(\frac{1}{|x|^2}\right),1\right),$$

and for $1\leq i,j\leq 2$, by (4.34)

$$\partial_{x_i,x_j}\vec{\Phi}(x)=\left(0,0,O\left(\frac{1}{|x|^3}\right)\right).$$

In particular, the components of the second fundamental form $\vec{\mathbb{I}}$ have the following decay

$$\mathbb{I}_{i,j}=\left\langle\partial^2_{x_1,x_2}\vec{\Phi},\vec{n}\right\rangle=O\left(\frac{1}{|x|^3}\right)\quad 1\leq i,j\leq 2$$

and we deduce the following asymptotic estimate for the Gauss curvature

$$K(x) = O\left(\frac{1}{|x|^6}\right)$$

and as $\star dw(i(x)) = O(|x|^2)$, we have

$$\begin{aligned}(4.39)\quad 2K(x)w(i(x))\star dw(x) &= 2K(x)|x|^2 v(i(x))O(|x|^2)\\ &= O\left(\frac{1}{|x|^6}\right)|x|^2 v(i(x))O(|x|^2) = O\left(\frac{1}{|x|^2}\right).\end{aligned}$$

Therefore, we obtain

$$\int_{S^1_r} 2K(x)w(i(x))\star dw(i(x)) = O\left(\frac{1}{r}\right).$$

Now, the error terms of order less than $O(|x|^{-2})$ in $|dw|_g^2$ will vanish when $r\to\infty$. By (4.35), as $Dw(x) = O(|x|)$, for $i\neq j$,

$$g^{i,j}(x)\partial_{x_i}w(x)\partial_{x_j}w(x) = O\left(\frac{1}{|x|^4}\right)O(|x|^2) = O\left(\frac{1}{|x|^2}\right).$$

Therefore, we will omit all these error terms, and we do not write the $O(|x|^{-2})$ in equalities

(4.40)

$$\begin{aligned}
&|dw(x)|_g^2 = |\partial_{x_1}w(x)|^2 + |\partial_{x_2}w(x)|^2\\
&= \Big\{\left(2x_1 + O(|x|^{-3})\right)v(i(x))\\
&\quad + (|x|^{-2} + a^2|x|^{-4} + O(|x|^{-6}))\left((x_2^2 - x_1^2)\partial_1 v(i(x)) - 2x_1x_2\partial_2 v(i(x))\right)\Big\}^2\\
&\quad + \Big\{\left(2x_2 + O(|x|^{-3})\right)v(i(x))\\
&\quad + (|x|^{-2} + a^2|x|^{-4} + O(|x|^{-6}))\left((x_2^2 - x_1^2)\partial_2 v(i(x)) - 2x_1x_2\partial_1 v(i(x))\right)\Big\}^2\\
&= \left(2x_1 v(i(x)) + |x|^{-2}((x_2^2 - x_1^2)\partial_1 v(i(x)) - 2x_1x_2\partial_2 v(i(x))\right)^2\\
&\quad + \left(2x_2 v(i(x)) + |x|^{-2}((x_1^2 - x_2^2)\partial_2 v(i(x)) - 2x_1x_2\partial_2 v_1(i(x))\right)^2\\
&= 4|x|^2 v(i(x))^2 + (\partial_1 v(i(x)))^2 + (\partial_2 v(i(x)))^2 - 4v(i(x))\left\langle x, Dv(i(x))\right\rangle
\end{aligned}$$

Now, $D^k v(i(x)) = O(r^{-2k})$ for all $k \geq 0$, so

$$\star d\left((\partial_1 v(i(x)))^2 + (\partial_2 v(i(x)))^2\right) = O\left(\frac{1}{|x|^2}\right),$$

and we can drop these terms in the $O(|x|^{-2})$ error. We have by (4.38), (4.39), (4.40)

$$\begin{aligned}
&\Delta_g w(i(x)) - \frac{1}{2}\star d|dw|_g^2\\
&= \Bigg(4v(i(x)) - 4\langle dv(i(x)), i(x)\rangle\\
&\quad + \frac{1}{|x|^2}\Delta v(i(x)) + O\left(\frac{1}{|x|^3}\right)\Bigg)\star d\left(|x|^2 v(i(x))\right)\\
&\quad - \frac{1}{2}\star d\left(4|x|^2 v(i(x))^2 - 4v(i(x))\langle dv(i(x)), x\rangle\right) + O\left(\frac{1}{|x|^2}\right)
\end{aligned}$$

Now we remark that

$$\star d\left(|x|^2 v(i(x))^2\right) = 2|x|^2 v(i(x))\star d\left(v(i(x))\right) + v(i(x))^2\star d\left(|x|^2\right),$$

so that

$$4v(i(x))\star d\left(|x|^2 v(i(x))\right) - \frac{1}{2}\star d\left(4|x|^2 v(i(x))\right) = 2v(i(x))^2\star d(|x|^2).$$

Then, we have

$$\begin{aligned}
&\star d\left(v(i(x))\langle dv(i(x)), x\rangle\right)\\
&= \star d\left(|x|^2 v(i(x))\langle dv(i(x)), i(x)\rangle\right)\\
&= |x|^2 v(i(x))\star d\left(\langle dv(i(x)), i(x)\rangle\right) + \langle dv(i(x)), i(x)\rangle\star d\left(|x|^2 v(i(x))\right),
\end{aligned}$$

so

$$\begin{aligned}
&-4\langle dv(i(x)), i(x)\rangle\star d\left(|x|^2 v(i(x))\right) + 2\star d\left(v(i(x))\langle dv(i(x)), x\rangle\right)\\
&= 2|x|^2 v(i(x))\star d\left(\langle v(i(x)), i(x)\rangle\right) - 2\langle dv(i(x)), i(x)\rangle\star d\left(|x|^2 v(i(x))\right)
\end{aligned}$$

Finally, we get

$$
\begin{aligned}
(4.41) \qquad & (\Delta_g w + 2Kw) \star dw - \frac{1}{2} \star d|dw|_g^2 \\
&= 2v^2(i(x)) \star d(|x|^2) + \frac{1}{|x|^2}\Delta v(i(x)) \star d(|x|^2 v(i(x))) \\
&\quad + 2|x|^2 v(i(x)) \star d\left(\langle dv(i(x)), i(x)\rangle\right) \\
&\quad - 2\langle dv(i(x)), i(x)\rangle \star d\left(|x|^2 v(i(x))\right) + O\left(\frac{1}{|x|^2}\right) \\
&= (1) + (2) + (3) + (4).
\end{aligned}
$$

We are now able to compute the boundary integral up to a vanishing error term as $r \to \infty$. We can develop v at 0 up to order two to get precise estimates of the boundary integral. We write for some real coefficients $\{a_{i,j}\}_{i,j\geq 0} \subset \mathbb{R}$

$$
(4.42) \qquad v(x) = \sum_{i,j=0}^{k} a_{i,j} x_1^i x_2^j + O(|x|^{k+1}).
$$

where

$$
(4.43) \qquad a_{i,j} = \frac{\partial_1^i \partial_2^j v(0)}{i!j!}.
$$

We recall the three following formulae, valid for $x, y \in \mathbb{R}$

$$
\begin{aligned}
2\cos x \cos x &= \cos(x+y) + \cos(x-y), \\
2\sin x \sin y &= \cos(x-y) - \cos(x+y), \\
2\cos x \sin y &= \sin(x+y) - \sin(x-y).
\end{aligned}
$$

In particular, if $a, b \in \mathbb{N}$, with $a \neq b$, we have

$$
\begin{aligned}
(4.44) \qquad \int_0^{2\pi} \cos(ax)\sin(bx)dx &= \int_0^{2\pi} \cos(ax)\cos(bx)dx \\
&= \int_0^{2\pi} \sin(ax)\sin(bx)dx = 0.
\end{aligned}
$$

So by (4.43),

$$\begin{aligned}
&\int_{S^1_r} 2v^2(i(x))\star(d|x|^2) = 4\int_{S^1_r} v^2(i(x))(x_1dx_2 - x_2dx_1)\\
&= 4\int_0^{2\pi} r^2\Bigg(a_{0,0} + a_{1,0}\frac{\cos(\theta)}{r} + a_{0,1}\frac{\sin(\theta)}{r} + a_{2,0}\left(\frac{\cos(\theta)}{r}\right)^2\\
&\qquad + a_{0,2}\left(\frac{\sin(\theta)}{r}\right)^2 + a_{1,1}\frac{\cos(\theta)\sin(\theta)}{r^2} + O(r^{-3})\Bigg)^2 d\theta\\
&= 4\int_0^{2\pi}\left(a_{0,0}^2r^2 + 2a_{0,0}\left(a_{2,0}\cos^2(\theta) + a_{0,2}\sin^2(\theta)\right)\right.\\
&\qquad \left. + a_{1,0}^2\cos^2(\theta) + a_{0,1}^2\sin^2(\theta)\right)d\theta + O(r^{-1})\\
&= 8\pi a_{0,0}^2 + 8\pi a_{0,0}\left(a_{2,0} + a_{0,2}\right) + 4\pi(a_{1,0}^2 + a_{0,1}^2) + O(r^{-1})\\
&= 8\pi r^2 v(p)^2 + 4\pi\left(v(p)\Delta v(p) + |dv(p)|^2\right) + O(r^{-1})
\end{aligned}$$

therefore

$$\int_{S^1_r}(1) = 8\pi r^2 v(p)^2 + 4\pi\left(v(p)\Delta v(p) + |dv(p)|^2\right) + O(r^{-1}). \tag{4.45}$$

Then, there holds

$$\begin{aligned}
&\int_{S^1_r}|x|^2\Delta(v(i(x)))\star d\left(|x|^2v(i(x))\right)\\
&= \int_0^{2\pi} 2v(i(x))\Delta v(i(x))\frac{1}{|x|^2}\left(x_1dx_2 - x_2dx_1\right)\\
&+ \int_0^{2\pi} 2v(i(x))\Delta v(i(x))\frac{1}{|x|^2}\Bigg\{|x|^2\left(\frac{x_2^2 - x_1^2}{|x|^4}\partial_1 v(i(x)) - \frac{2x_1x_2}{|x|^4}\partial_2 v(i(x))\right)dx_2\\
&- \left(\frac{x_1^2 - x_2^2}{|x|^4}\partial_2 v(i(x)) - \frac{2x_1x_2}{|x|^4}\partial_1 v(i(x))\right)dx_1\Bigg\}\\
&= 4\pi v(p)\Delta v(p) + O(r^{-1}).
\end{aligned}$$

and

$$\int_{S^1_r}(2) = 4\pi v(p)\Delta v(p) + O(r^{-1}). \tag{4.46}$$

Now, we compute

$$\begin{aligned}
&|x|^2v(i(x))\star d\left(\langle dv(i(x)),i(x)\rangle\right)\\
&=|x|^2v(i(x))\bigg\{\partial_{x_1}\left(\frac{x_1}{|x|^2}\partial_1 v+\frac{x_2}{|x|^2}\partial_2 v\right)dx_2-\partial_{x_2}\left(\frac{x_1}{|x|^2}\partial_1 v+\frac{x_2}{|x|^2}\partial_2 v\right)dx_1\bigg\}\\
&=|x|^2v(i(x))\Bigg\{\left(\frac{x_2^2-x_1^2}{|x|^4}\partial_1 v-\frac{2x_1x_2}{|x|^4}\partial_2 v\right)dx_2-\left(\frac{x_1^2-x_2^2}{|x|^4}\partial_2 v-\frac{2x_1x_2}{|x|^4}\partial_1 v\right)\\
&\qquad+\frac{x_1}{|x|^2}\left(\frac{x_2^2-x_1^2}{|x|^4}\partial_1^2 v-\frac{2x_1x_2}{|x|^4}\partial_{12}^2 v\right)dx_2\\
&\qquad+\frac{x_2}{|x|^2}\left(\frac{x_2^2-x_1^2}{|x|^4}\partial_{12}^2 v-\frac{2x_1x_2}{|x|^4}\partial_2^2 v\right)dx_2\\
&\qquad-\frac{x_1}{|x|^2}\left(\frac{x_1^2-x_2^2}{|x|^4}\partial_{12}^2 v-\frac{2x_1x_2}{|x|^4}\partial_1^2 v\right)dx_1\\
&\qquad-\frac{x_2}{|x|^2}\left(\frac{x_1^2-x_2^2}{|x|^4}\partial_2^2 v-\frac{2x_1x_2}{|x|^4}\partial_{12}^2 v\right)dx_1\Bigg\}\\
&=\text{(i)}+\text{(ii)}.
\end{aligned}$$

We first see that

$$\begin{aligned}
\text{(i)}&=|x|^2v(i(x))\Bigg\{\left(\frac{x_2^2-x_1^2}{|x|^4}\partial_1 v-\frac{2x_1x_2}{|x|^4}\partial_2 v\right)dx_2\\
&\qquad-\left(\frac{x_1^2-x_2^2}{|x|^4}\partial_2 v-\frac{2x_1x_2}{|x|^4}\partial_1 v\right)\Bigg\}\\
&=|x|^2v(i(x))\star dv(i(x))
\end{aligned}$$

and we have already computed this integral, so we find

$$\int_{S_r^1}\text{(i)}=-\pi\left(|dv(p)|^2+v(p)\Delta v(p)\right),\tag{4.47}$$

and

$$\begin{aligned}
\int_{S_r^1}\text{(ii)}=\int_{S_r^1}v(i(x))\bigg\{&\frac{1}{|x|^4}\left(x_1^2(x_2^2-x_1^2)-2x_1^2x_2^2\right)\partial_1^2 v\\
&+\left(-2x_1^2x_2^2+x_2^2(x_1^2-x_2^2)\right)\partial_2^2 v\bigg\}+O(r^{-1})
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{2\pi} v\left(\frac{\cos(\theta)}{r},\frac{\sin(\theta)}{r}\right)\left(\cos^2(\theta)\partial_1^2 v\left(\frac{\cos(\theta)}{r},\frac{\sin(\theta)}{r}\right)\right.\\
&\qquad\left.+\sin^2(\theta)\partial_2^2 v\left(\frac{\cos(\theta)}{r},\frac{\sin(\theta)}{r}\right)\right)d\theta + O(r^{-1})\\
&= -\pi v(p)\Delta v(p).
\end{aligned}
$$

Finally, we have

$$
\int_{S_r^1}(3) = 2\int_{S_r^1}(\mathrm{i})+(\mathrm{ii}) = -2\pi\left(|dv(p)|^2 + 2v(p)\Delta v(p)\right). \tag{4.48}
$$

We have only left (4):

$$
\begin{aligned}
\int_{S_r^1}(4) &= -2\int_{S_r^1}\langle dv(i(x)), i(x)\rangle \star d\left(|x|^2 v(i(x))\right)\\
&= -2\int_{S_r^1}\langle dv(i(x)), i(x)\rangle\, v(i(x))2(x_1dx_2 - x_2dx_1)\\
&\quad -2\int_{S_r^1}\langle dv(i(x)), x\rangle \star dv(i(x))\\
&= -2\left\{(\mathrm{iii})+(\mathrm{iv})\right\}.
\end{aligned}
$$

Now, we easily compute

$$
\begin{aligned}
(\mathrm{iii}) &= \int_0^{2\pi}\left(\partial_1 v\frac{\cos(\theta)}{r} + \partial_2 v\frac{\sin(\theta)}{r}\right)2vr^2 d\theta\\
&= 2\int_0^{2\pi}\left(ra_{0,0} + a_{1,0}\cos(\theta) + a_{0,1}\sin(\theta)\right)\\
&\quad\times\left\{\cos(\theta)\left(a_{1,0} + 2a_{2,0}\frac{\cos(\theta)}{r} + 2a_{1,1}\frac{\sin(\theta)}{r}\right)\right.\\
&\quad\left.+\sin(\theta)\left(a_{0,1} + 2a_{0,2}\frac{\sin(\theta)}{r} + 2a_{1,1}\frac{\cos(\theta)}{r}\right)\right\}d\theta + O(r^{-1})\\
&= 2\int_0^{2\pi}\left(a_{1,0}^2\cos^2(\theta) + a_{0,1}^2\sin^2(\theta)\right.\\
&\quad\left.+2a_{0,0}\left(a_{2,0}\cos^2(\theta) + a_{0,2}\sin^2(\theta)\right)\right)d\theta + O(r^{-1})\\
&= 2\pi(|dv(p)|^2 + v(p)\Delta v(p)) + O(r^{-1}),
\end{aligned}
$$

while

$$\begin{aligned}(\mathrm{iv}) &= -\int_0^{2\pi}(r\cos(\theta)\partial_1 v + r\sin(\theta)\partial_2 v)\\ &\quad\times\left(\frac{-1}{r}\left(\partial_{x_1}v\cos(\theta)+\partial_{x_2}v\sin(\theta)\right)\right)+O(r^{-1})\\ &= -\int_0^{2\pi}(\partial_1 v\ \cos(\theta)+\partial_2\sin(\theta))^2 d\theta + O(r^{-1})\\ &= -\pi|dv(p)|^2+O(r^{-1}).\end{aligned}$$

Gathering estimates, we get

$$\int_{S^1_r}(4) = -2\pi\left(|dv(p)|^2+2v(p)\Delta v(p)\right)+O(r^{-1}) \tag{4.49}$$

and we finally obtain by (4.41), (4.45), (4.46), (4.48), (4.48), (4.49)

$$\int_{S^1_r}\left((\Delta_g w+2Kw)\star dw-\frac{1}{2}\star d|dw|_g^2\right) = -8\pi r^2 v^2(p)+O\left(\frac{1}{r}\right). \tag{4.50}$$

By (4.6), we get the correct multiplicative factor of α^2 in front of this expression, which gives the correct expression as thanks to Definition 4.6, as $r^2 = 2\alpha_j^2 R^{-2}$. This concludes the proof of the theorem thanks to (4.32). ☐

Remark 4.9. *As $|\vec{\Phi}|^2$ is unbounded at the points p_j for $1\leq j\leq m$, the first term under the limit of the right-hand side of* (4.31) *goes to $+\infty$ as $R\to 0$, while the second term goes to $-\infty$, as long as v is* not *compactly supported in $\Sigma\setminus\{p_1,\dots,p_m\}$. We can also check directly that this quantity is bounded, while the existence of the limit is a consequence of the preceding remark 4.8. To do so, we fix some $R_0>0$ such that the disks $D^2_\Sigma(p_j,R_0)$ (for $1\leq j\leq m$) are disjoint and we need to estimate the integral*

$$\frac{1}{2}\int_{D^2(p_j,R_0)\setminus D^2(p_j,R)}\left(\Delta_g w-2K_g w\right)^2 d\mathrm{vol}_g,\quad 1\leq j\leq m$$

as $R\to 0$, and $w=|\vec{\Phi}|^2 v$. The asymptotic flatness (4.35) *shows that*

$$\begin{aligned}&\frac{1}{2}\int_{D^2(0,r)\setminus D^2(0,1)}(\Delta_g w-2K_g w)^2 d\mathrm{vol}_g\\ &=\frac{1}{2}\int_{D^2(0,r)\setminus D^2(0,1)}\left(\Delta\left(|x|^2 v\left(\frac{x}{|x|^2}\right)\right)-2K_g(x)|x|^2 v\left(\frac{x}{|x|^2}\right)\right)^2 d\mathscr{L}^2(x)+O(1).\end{aligned}$$

As

$$\Delta\left(|x|^2v\left(\frac{x}{|x|^2}\right)\right)=4v\left(\frac{x}{|x|^2}\right)+\frac{4}{|x|^2}\left(x_1\partial_1v\left(\frac{x}{|x|^2}\right)+x_2\partial_2v\left(\frac{x}{|x|^2}\right)\right)$$
$$+\frac{1}{|x|^2}\Delta v\left(\frac{x}{|x|^2}\right)=O(1),$$

and $K_g(x)=O(|x|^{-6})$ *by* (4.35), *and by the previous computation*

$$w(x)=O(|x|^2),\quad \Delta w(x)=O(1)$$

we have for some $C>0$

$$\int_{D^2(0,r)\setminus D^2(0,1)}|\Delta_g w\cdot 2K_g w|\,dx \leq C\int_{D^2(0,r)\setminus D^2(0,1)}\frac{dx}{|x|^4}=C\pi\left(1-\frac{1}{r^2}\right)\leq \pi C \tag{4.51}$$

which is bounded as $r\to\infty$. *Likewise,*

$$\int_{D^2(0,r)\setminus D^2(0,1)}|K_g(x)w(x)|^2dx \leq C\int_{D^2(0,r)\setminus D^2(0,1)}\frac{dx}{|x|^8}=C\frac{\pi}{3}\left(1-\frac{1}{r^6}\right)\leq C\frac{\pi}{3} \tag{4.52}$$

which is also bounded as $r\to\infty$. *Therefore, by* (4.51) *and* (4.52), *we have*

$$\frac{1}{2}\int_{D^2(0,r)\setminus D^2(0,1)}(\Delta_g w-2K_g)^2d\mathrm{vol}_g =\frac{1}{2}\int_{D^2(0,r)\setminus D^2(0,1)}(\Delta_g w)^2d\mathrm{vol}_g+O(1). \tag{4.53}$$

so the only singular terms will come from the integration of $(\Delta_g w)^2$.

As we can suppose that the variation v *is smooth, we have for some* $\{a_{i,j}\}_{i,j\geq 0}\subset\mathbb{R}$

$$v(x)=\sum_{i,j=0}^k a_{i,j}x_1^ix_2^j+O(|x|^{k+1})$$

for all $k\in\mathbb{N}$.

Using polar coordinates $(x_1, x_2) = (\rho\cos(\theta), \rho\sin(\theta))$, $(\rho, \theta) \in [1, r] \times S^1$,

$$\begin{aligned}\Delta\left(|x|^2 v\left(\frac{x}{|x|^2}\right)\right) &= 4\Bigg(a_{0,0} + a_{1,0}\frac{\cos(\theta)}{\rho} + a_{0,1}\frac{\sin(\theta)}{\rho} \\ &\qquad + a_{2,0}\frac{\cos^2(\theta)}{\rho^2} + \frac{\sin^2(\theta)}{\rho^2} + a_{1,1}\frac{\sin(2\theta)}{2\rho^2}\Bigg) \\ &\quad - 4\frac{\cos(\theta)}{\rho}\left(a_{1,0} + 2a_{2,0}\frac{\cos(\theta)}{\rho} + a_{1,1}\frac{\sin(\theta)}{\rho}\right) \\ &\quad - 4\frac{\sin(\theta)}{\rho}\left(a_{0,1} + 2a_{0,2}\frac{\sin(\theta)}{\rho} + a_{1,1}\frac{\cos(\theta)}{\rho}\right) \\ &\quad + \frac{2(a_{2,0} + a_{0,2})}{\rho^2} + O\left(\frac{1}{\rho^3}\right) \\ &= 4\left(a_{0,0} - a_{2,0}\frac{\cos^2(\theta)}{\rho^2} - a_{0,2}\frac{\sin^2(\theta)}{\rho^2} - a_{1,1}\frac{\sin(2\theta)}{2\rho^2}\right) \\ &\quad + \frac{2(a_{2,0} + a_{0,2})}{\rho^2} + O\left(\frac{1}{\rho^3}\right)\end{aligned}$$

so by (4.53)

$$\begin{aligned}&\frac{1}{2}\int_{D^2(0,r)\setminus D^2(0,1)} (\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g \\ &= \frac{1}{2}\int_1^r \int_{S^1}\Bigg(16a_{0,0}^2 - 32a_{0,0}\left(a_{2,0}\frac{\cos^2(\theta)}{\rho^2} + a_{0,2}\frac{\sin^2(\theta)}{\rho^2}\right) \\ &\qquad + 16\frac{a_{0,0}(a_{2,0} + a_{0,2})}{\rho^2}\Bigg)\rho\, d\rho d\theta + O(1) \\ &= 16\pi a_{0,0}^2 \int_1^r \rho^2 d\rho - 16a_{0,0}\Bigg(a_{2,0}\left(\int_1^r \frac{d\rho}{\rho}\right)\left(\int_{S^1}\cos^2(\theta)d\theta\right) \\ &\qquad + a_{0,2}\left(\int_1^r \frac{d\rho}{\rho}\right)\left(\int_{S^1}\sin^2(\theta)d\theta\right)\Bigg) \\ &\quad + 16\pi a_{0,0}(a_{2,0} + a_{0,2})\int_1^r \frac{d\rho}{\rho} \\ &= 16\pi a_{0,0}^2\frac{(r^2 - 1)}{2} - 16\pi a_{0,0}(a_{2,0} + a_{0,2})\log r \\ &\quad + 16\pi a_{0,0}(a_{2,0} + a_{0,2})\log r + O(1) \\ &= 8\pi a_{0,0}^2 r^2 + O(1) = 8\pi r^2 v^2(p_j) + O(1).\end{aligned} \tag{4.54}$$

As $\mathrm{Res}_{p_j}(\vec{\Phi}, U_j) = 2\alpha_j^2$*, and* $r^2 = \dfrac{\alpha_j^2}{R^2}$*, we obtain*

$$4\pi\frac{\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)}{R^2}v^2(p_j) = 8\pi r^2 v^2(p_j), \tag{4.55}$$

and by (4.54) *and* (4.53)

$$\frac{1}{2}\int_{D^2_\Sigma(p_j,R_0)\setminus D^2_\Sigma(p_j,R)}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g = 4\pi\frac{\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)}{R^2}v^2(p_j) + O(1).$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\int_{\Sigma_R}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - 4\pi\sum_{j=1}^m \frac{\mathrm{Res}_{p_j}(\vec{\Phi}, U_j)}{R^2}v^2(p_j)\\ &= \frac{1}{2}\int_{\Sigma_{R_0}}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g + O(1)\end{aligned}$$

which is bounded independently of $R \to 0$ *as* $R_0 > 0$ *is fixed and the metric and the variation* $w = |\vec{\Phi}|^2 v$ *are smooth and bounded on*

$$\Sigma_{R_0} = \Sigma \setminus \bigcup_{j=1}^m D^2_\Sigma(p_j, R_0).$$

This concludes the remarks on the well-definitiveness of (4.31).

4.4. Proof of the main Theorem : Theorem 1.1

Proof. Suppose first that $\Psi : S^2 \to S^3$ is a non-umbilic Willmore sphere such that

$$m = \frac{1}{4\pi}\mathscr{W}(\Psi)$$

then it is the inversion at 0 (after translation if necessary), after a stereographic projection in $\mathbb{R}^3$, of a complete minimal surface with m embedded ends with zero logarithmic growth and finite total curvature. Furthermore, for every normal variation $\vec{v} = v\vec{n}$ of Ψ such that $v \in W^{2,2}(S^2, \mathbb{R})$, if $\vec{\Phi} = i \circ \Psi$,

$w = |\vec{\Phi}|^2 v$, we have by (4.31) (for some $U_1, \dots, U_m$ as in Definition (4.6)),

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}] = \lim_{R\to 0}\frac{1}{2}\int_{S^2_R}(\Delta_g w - 2K_g w)^2 d\mathrm{vol}_g - 4\pi\sum_{j=1}^m \frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}\, v^2(p_j). \tag{4.56}$$

In particular, if $v(p_j) = 0$ for all $1 \leq j \leq m$, then

$$D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{v}] \geq 0.$$

Now, we introduce the continuous map (thanks of the Sobolev embedding $W^{2,2}(S^2,\mathbb{R}) \hookrightarrow C^0(S^2,\mathbb{R})$)

$$\begin{aligned} V : W^{2,2}(S^2,\mathbb{R}) &\to \mathbb{R}^m \\ v &\mapsto (v(p_1),\dots,v(p_m)) \end{aligned}$$

We see that $V^{-1}(0) \subset W^{2,2}(S^2,\mathbb{R})$ is a closed sub-space of codimension at most m. As the second variation of W can only be negative on the complementary of $V^{-1}(0) \subset W^{2,2}(S^2,\mathbb{R})$ by (4.56), the dimension of the subspace of $W^{2,2}(S^2,\mathbb{R})$ where $D^2\mathscr{W}(\vec{\Psi})$ is negative is bounded by m. Therefore, $\mathrm{Ind}_{\mathscr{W}}(\Psi) \leq m$.

Now, we treat the umbilical case (where $m = 1$). Thanks to the result of Bryant ([3]), if there exists an umbilical point, then $\vec{\Psi} : S^2 \to S^3$ is totally umbilical, and is a geodesic 2-sphere. In particular $\vec{\Psi} : S^2 \to S^3$ is a minimal immersion. Therefore, as $\vec{\Psi}$ is an absolute minimiser of $\mathscr{W}$, it should have index 0. As the Gauss map of a minimal surface is holomorphic, it is *a fortiori* harmonic, so

$$\Delta_g \vec{n} + |d\vec{n}|_g^2\, \vec{n} = 0$$

if $g = \vec{\Psi}^* g_{S^3}$, and $\vec{n} : S^2 \to S^2$ is the Gauss map of $\vec{\Psi}$. The Jacobi operator of the minimal surface $\vec{\Psi} : S^2 \to S^3$ is simply

$$L_g = \Delta_g + (|\vec{\mathbb{I}}_g|^2 + 2) = \Delta_g + (|d\vec{n}|_g^2 + 2)$$

so for all $a \in \mathbb{R}^4$, we have

$$L_g(a\cdot\vec{n}) = a\cdot\left(\Delta_g\vec{n} + |d\vec{n}|_g^2\,\vec{n}\right) + 2a\cdot\vec{n} = 2a\cdot\vec{n}$$

so 2 is a positive eigenvalue of the Jacobi operator, and one can show that it is the only one. Furthermore, the associated eigenspace is 1-dimensional (see the paper of Frederick Almgren [1]) so $\vec{\Psi}:S^2\to S^3$ is of index 1. Therefore, by (2.22) and (2.23) $\vec{\Psi}$ is stable *i.e.* $\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi})=0$, and the bound is trivially verified. ☐

Remark 4.10. *This proof shows in particular that whenever in the classification of Bryant, we have an Willmore immersion $\vec{\Psi}:\Sigma\to S^3$ from a closed Riemann surface Σ which is not totally umbilic, with Bryant's quartic form $\mathscr{Q}_{\vec{\Psi}}$ identically equal to 0 (theorem E in [3]), then for some stereographic projection $\pi:S^3\to\mathbb{R}^3$, $\pi\circ\vec{\Psi}:\Sigma\to\mathbb{R}^3$ is a branched complete minimal immersion with embedded planar ends. In particular $\mathscr{W}(\vec{\Psi})$ is quantized by 4π and*

$$\mathrm{Ind}_{\mathscr{W}}(\vec{\Psi})\leq\frac{1}{4\pi}\mathscr{W}_{S^3}(\vec{\Psi}).$$

However, when the genus of Σ is larger than 1, then there are Willmore immersions with non-zero quartic form $\mathscr{Q}_{\vec{\Psi}}$. One example is furnished by the Clifford torus in S^3 with energy $2\pi^2$, which cannot be the inverse stereographic projection of a minimal surface in $\mathbb{R}^3$, as its Willmore energy (or area) is not an integer multiple of 4π.

4.5. Index and Schrödinger operators

Let $\overline{\Sigma}$ be a closed Riemann surface and $\Sigma=\overline{\Sigma}\setminus\{p_1,\dots,p_m\}$ be a connected Riemann surface with m punctured points $\{p_1,\dots,p_m\}\subset\overline{\Sigma}$, and fix some $U_1,\dots,U_m$ as in Definition 4.6. If $\vec{\Phi}:\Sigma\to\mathbb{R}^3$ is a complete minimal immersion, recall that for a normal variation $\vec{w}=w\vec{n}$, we have

$$\begin{aligned}D^2A(\vec{\Phi})[\vec{w},\vec{w}]&=\int_\Sigma\left(|dw|_g^2+2K_gw^2\right)d\mathrm{vol}_g\\&=-\int_\Sigma w\,(L_gw)\,d\mathrm{vol}_g\end{aligned}$$

if $L_g=\Delta_g-2K_g=\Delta_g+|dN|_g^2$ if $N:\Sigma\to S^2$ is the Gauss map of $\vec{\Phi}$. If $\vec{\Phi}$ conformal, and if $d\mathrm{vol}_\Sigma$ is a canonical volume form on Σ, we have,

$$\begin{aligned}d\mathrm{vol}_g&=\frac{|d\vec{\Phi}|^2}{2}d\mathrm{vol}_\Sigma\\-2K_g&=|dN|_g^2=2|d\vec{\Phi}|^{-2}|dN|^2\end{aligned}$$

so if $M=\vec{\Phi}(\Sigma)$ is a finite curvature minimal surface with m embedded planar ends, and $\vec{\Psi}:\Sigma\to\mathbb{R}^3$ is the inversion at 0 of $\vec{\Phi}$, we have if $\vec{v}=v\vec{n}$ is a normal variation, by (4.31)

$$
\begin{aligned}
&D^2\mathscr{W}(\vec{\Psi})[\vec{v},\vec{w}]\\
&=\lim_{R\to 0}\frac{1}{2}\int_{\Sigma_R}\left(L_g(|\vec{\Phi}|^2 v)\right)^2 d\mathrm{vol}_g-4\pi\sum_{j=1}^m\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}v^2(p_j)\\
&=\lim_{R\to 0}\frac{1}{2}\int_{\Sigma_R}\left(2|d\vec{\Phi}|^{-2}\Delta_\Sigma(|\vec{\Phi}|^2 v)+2|d\vec{\Phi}|^{-2}|dN|^2|\vec{\Phi}|^2v^2\right)^2\frac{|d\vec{\Phi}|^2}{2}d\mathrm{vol}_\Sigma\\
&\quad-4\pi\sum_{j=1}^m\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}v^2(p_j)\\
&=\lim_{R\to 0}\int_{\Sigma_R}\left(L_{\vec{\Phi}}(|\vec{\Phi}|^2 v)\right)^2\frac{d\mathrm{vol}_\Sigma}{|d\vec{\Phi}|^2}-4\pi\sum_{j=1}^m\frac{\mathrm{Res}_{p_j}(\vec{\Phi},U_j)}{R^2}v^2(p_j)
\end{aligned}
$$

if $L_{\vec{\Phi}}=\Delta_\Sigma+|dN|^2$. Moreover, the Gauss map $N:\Sigma\to S^2$ is a holomorphic map, so it is in particular harmonic, *i.e.*

$$\Delta_\Sigma N+|dN|^2N=0.$$

Therefore, (recalling (4.6) for the definition of the residue) we can study in general the problem of finding the index of the following quadratic form

$$
\begin{aligned}
Q_f(v,v)=&\lim_{R\to 0}\int_{\Sigma_R}\left(L_f(|\mathrm{Re}\,f|^2v)\right)^2\frac{d\mathrm{vol}_\Sigma}{|d\mathrm{Re}\,f|^2}\\
&-4\pi\sum_{j=1}^m\frac{\mathrm{Res}_{p_j}(\mathrm{Re}\,(f),U_j)}{R^2}v^2(p_j)
\end{aligned}
$$

where $f:\Sigma\to\mathbb{C}^3$ is a meromorphic immersion with at most simple poles at each end $p_j\in\overline{\Sigma}$ $(1\leq j\leq m)$, and $N:\Sigma\to S^2$ is the holomorphic Gauss map of $\mathrm{Re}\,f:\Sigma\to\mathbb{R}^3$.

Remark 4.11. *We remark that for every conformal transformation of* $\varphi:\overline{\Sigma}\to\overline{\Sigma}$*, we have*

$$Q_{f\circ\varphi}(v\circ\varphi,v\circ\varphi)=Q_f(v,v)$$

This is easily seen by the conformal invariance of the Laplacian in dimension 2 (see the book of Frédéric Hélein [13]).

References

[1] F. Almgren Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Annals of Mathematics **84** (1966).

[2] W. J. E. Blaschke, Vorlesungen Über Differentialgeometrie III: Differentialgeometrie der Kreise und Kugeln, Springer-Verlag, coll. Grundlehren der mathematischen Wissenschaften (1929).

[3] R. L. Bryant, *A duality theorem for Willmore surfaces*, Journal of Differential Geometry **20** (1984), 23–53.

[4] B.-Y. Chen, *Some conformal invariants of submanifolds and their applications*, Boll. Un. Mat. Ital. **10** (1974), no. 4, 380–385.

[5] B.-Y. Chen, Total Mean Curvature and Submanifolds of Finite Type, second Edition, Series in Pure Mathematics (2015).

[6] S.-Y. Cheng and J. Tysk, *Schrödinger operators and index bounds for minimal submanifolds*, Rocky Mountain Journal of Mathematics, **24** (Summer 1994), no. 3.

[7] O. Chodosh and D. Maximo, *The topology and index of minimal surfaces*, Journal of Differential Geometry **104** (2016), no. 3, 399–418.

[8] N. Ejiri and M. Kotani, *Index and flats ends of minimal surfaces*, Tokyo J. Math. **16** (1993), no. 1.

[9] H. Federer, Geometric Measure Theory, Springer-Verlag (1969).

[10] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three manifolds*, Inventiones Mathematicae **82** (1985), 121–132.

[11] Z. Guo, H. Li, and C. Wang, *The second variational formula for Willmore submanifolds in* S^n, Results in Mathematics **40** (2001), no. 1-4, 205–225.

[12] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[13] F. Hélein, Applications Harmoniques, Lois de Conservation, et Repères Mobiles, Diderot éditeur, Sciences et Arts (1996).

[14] R. Kusner, *On the minimax sphere eversion*, unpublished note (1995).

[15] T. Lamm, J. Metzger, and F. Schulze, *Foliations of asymptotically flat manifolds by surfaces of Willmore type*, Mathematische Annalen **350** (2011), no. 1, 1–78.

[16] P. Li and S.-T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Inventiones Mathematicae **69** (1982), 269–291.

[17] F. J. López, *The classification of complete minimal surfaces with total curvature greater than* -12π, Transactions of American Mathematical Society **334** (1992), no. 1.

[18] F. J. López and F. Martín, *Complete minimal surfaces in* $\mathbb{R}^3$, Publicacions Matemátiques **43** (1999), no. 2, 341–449.

[19] F. C. Marques and A. Neves, *Morse index and multiplicity of min-max minimal hypersurfaces*, Cambridge Journal of Mathematics **4** (2016), no. 4, 463–511.

[20] S. Montiel and A. Ros, *Schrödinger operators associated to a holomorphic map*, in: Global Differential Geometry and Global Analysis, pp. 147–174, Volume 1481 of the series Lecture Notes in Mathematics, (2006).

[21] C. Ndiaye and R. Schätzle, *Explicit conformally constrained Willmore minimizers in arbitrary codimension*, Calc. Var. Partial Differential Equations **51** (2014), no. 1-2, 291–314.

[22] R. Osserman, *On complete minimal surfaces*, Archive for Rational Mechanics and Analysis **13** (December 1963), no. 1, 392–404.

[23] F. Paulin, Groupes et Géométries, Lecture Notes (2014).

[24] T. Rivière, *Analysis aspects of Willmore surfaces*, Inventiones Mathematicae **174** (2008), 1–45.

[25] T. Rivière, *Weak immersions of surfaces with* L^2*-bounded second fundamental form*, PCMI Graduate Summer School (2013).

[26] T. Rivière, *Willmore minimax surfaces and the cost of the sphere eversion*, `arXiv:1512.08918`, (2015).

[27] T. Rivière, *Variational principles for immersed surfaces with* L^2*-bounded second fundamental form*, Journal für die reine und angewandte Mathematik **695** (2014), 41–98.

[28] R. M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, Journal of Differential Geometry **18** (1983), 791–809.

[29] G. Thomsen, *Über konforme Geometrie, I: Grundlagen der konformen Flächentheorie*, Hamb. Math. Abh. **3** (1923).

[30] J. Tysk, *Eigenvalue estimates with applications to minimal surfaces*, Pacific Journal of Mathematics **128** (1987), no. 2.

[31] J. L. Weiner, *On a problem of Chen, Willmore, et al*, Indiana University Mathematics Journal **27** (1978), no. 1.

[32] T. J. Willmore, Total Curvature in Riemannian Geometry, Ellis Horwood Series Mathematics and its applications (1982).

Department of Mathematics, ETH Zentrum
CH-8093 Zürich, Switzerland
E-mail address: alexis.michelat@math.ethz.ch

Received May 14, 2016
Accepted February 27, 2018