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Isomonodromic deformations of irregular connections and stability of bundles

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Let G be a reductive affine algebraic group defined over $\mathbb{C}$, and let ∇_0 be a meromorphic G-connection on a holomorphic principal G-bundle E_0, over a smooth complex projective curve X_0, with polar locus $P_0 \subset X_0$. We assume that ∇_0 is irreducible in the sense that it does not factor through some proper parabolic subgroup of G. We consider the universal isomonodromic deformation $(E_t \to X_t, \nabla_t, P_t)_{t\in\mathcal{T}}$ of $(E_0 \to X_0, \nabla_0, P_0)$, where $\mathcal{T}$ is a certain quotient of a certain framed Teichmüller space we describe. We show that if the genus g of X_0 satisfies $g \geq 2$, then for a general parameter $t \in \mathcal{T}$, the principal G-bundle $E_t \to X_t$ is stable. For $g \geq 1$, we are able to show that for a general parameter $t \in \mathcal{T}$, the principal G-bundle $E_t \to X_t$ is semistable.

1. Introduction

The natural correspondence between a flat connection on a principal bundle defined over a variety and its monodromy representation is a recurrent theme in mathematics, with a long history, as evidenced by the name, Riemann–Hilbert problem, given to one of the core questions of the subject. This basic problem consists in asking when a representation of the fundamental group of a punctured Riemann sphere can be realized by a flat connection on a holomorphically trivial bundle, with simple poles at the punctures; the answer, which is affirmative most of the time, but not always ([Pl], [De], [AB], [Bol1], [Ko]), is in itself an interesting chapter of the history of mathematics.

If one relaxes the condition of triviality, and asks whether the representation can be realized on a principal bundle, then the answer is always yes, and indeed the correspondence is quite natural. The question then becomes that of whether the bundle can be made trivial, either by some twists at the punctures (Schlesinger transformations) or by deforming the location of the punctures (isomonodromic deformations). The deformation theoretic version of the Riemann–Hilbert problem becomes:

Given a logarithmic connection (E_0, ∇_0) *on* $\mathbb{P}^1_{\mathbb{C}}$ *with polar divisor* D_0 *of degree* n*, is there a point* t *of the Teichmüller space* $\mathrm{Teich}_{0,n}$ *such that the underlying holomorphic vector bundle* $E_t = \mathcal{E}|_{\mathbb{P}^1_{\mathbb{C}} \times \{t\}}$ *in the universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *for* (E_0, ∇_0) *is trivial?*

A partial answer to this question is given, in the case of vector bundles of rank two, by the following theorem of Bolibruch:

Theorem 1.1 ([Bol2]). *Let* (E_0, ∇_0) *be an irreducible trace–free logarithmic rank two connection with* $n \geq 4$ *poles on* $\mathbb{P}^1_{\mathbb{C}}$ *such that each singularity is resonant. Then there is a proper closed complex analytic subset* $\mathcal{Y} \subset \mathrm{Teich}_{0,n}$ *such that for all* $t \in \mathrm{Teich}_{0,n} \setminus \mathcal{Y}$*, the holomorphic vector bundle* $E_t = \mathcal{E}|_{\mathbb{P}^1_{\mathbb{C}} \times \{t\}}$ *underlying the universal isomonodromic deformation* $(\mathcal{E}, \nabla)$ *of* (E_0, ∇_0) *is trivial.*

In [He2], it is shown that the resonance condition in Theorem 1.1 is redundant.

This gives an indication for the Riemann sphere; one can actually consider a similar problem for an arbitrary Riemann surface. Indeed, triviality of a vector bundle over the Riemann sphere is equivalent to being semi-stable of degree zero. On a general Riemann surface, the question of whether one can realize a representation by a semi-stable vector bundle of degree zero was considered in [EH, EV]. The version in the set-up of deformations, whether a logarithmic connection on a principal bundle over an arbitrary Riemann surface admits an isomonodromic deformation to a logarithmic connection on a stable or semi-stable principal bundle, was treated in [BHH1]; see also [He2] for rank two. We recall from [BHH1], [He2]:

Theorem 1.2 ([BHH1], [He2]). *Let* X *be a compact connected Riemann surface of genus* g*, and let* $D_0 \subset X$ *be an ordered subset of it of cardinality* n*. Let* G *be a reductive affine algebraic group defined over* $\mathbb{C}$*. Let* E_G *be a holomorphic principal* G*–bundle on* X *and* ∇ *a logarithmic connection on* E_G *with polar divisor* D_0*. Let* $(\mathcal{E}_G, \nabla)$ *be the universal isomonodromic deformation of* (E_G, ∇_0) *over the universal Teichmüller curve* $\tau : (\mathcal{X}, \mathcal{D}) \longrightarrow \mathrm{Teich}_{g,n}$*. For any point* $t \in \mathrm{Teich}_{g,n}$*, the restriction* $\mathcal{E}_G|_{\tau^{-1}(t)} \longrightarrow \mathcal{X}_t := \tau^{-1}(t)$ *will be denoted by* $\mathcal{E}^t_G$*.*

1) *Assume that* $g \geq 2$ *and* $n = 0$*. Then there is a closed complex analytic subset* $\mathcal{Y} \subset \mathrm{Teich}_{g,n}$ *of codimension at least* g *such that for any* $t \in \mathrm{Teich}_{g,n} \setminus \mathcal{Y}$*, the holomorphic principal* G*–bundle* $\mathcal{E}^t_G \longrightarrow \mathcal{X}_t$ *is semi-stable.*

2) *Assume that* $g \geq 1$*, and if* $g = 1$*, then* $n > 0$*. Also, assume that the initial monodromy representation for* ∇ *at* $t = 0$ *does not factor through some proper parabolic subgroup of* G*. Then there is a closed complex analytic subset* $\mathcal{Y}' \subset \mathrm{Teich}_{g,n}$ *of codimension at least* g *such that for any* $t \in \mathrm{Teich}_{g,n} \setminus \mathcal{Y}'$*, the holomorphic principal* G*–bundle* $\mathcal{E}_G^t$ *is semistable.*

3) *Assume that* $g \geq 2$*. Assume that the monodromy representation for* ∇ *at* $t = 0$ *does not factor through some proper parabolic subgroup of* G*. Then there is a closed complex analytic subset* $\mathcal{Y}'' \subset \mathrm{Teich}_{g,n}$ *of codimension at least* $g - 1$ *such that for any* $t \in \mathrm{Teich}_{g,n} \setminus \mathcal{Y}''$*, the holomorphic principal* G*–bundle* $\mathcal{E}_G^t$ *is stable.*

Related results were also proved in [BHH2] and [BHH3].

Our aim here is to extend this result to connections with irregular singularities, that is connections with higher order poles. Let us consider a triple

$$(E_G \longrightarrow X, D, \nabla),$$

where E_G is a holomorphic principal G–bundle over a compact connected Riemann surface X, and ∇ is an integrable holomorphic connection on E_G, with possibly irregular singularities bounded by a divisor D on X; that is, if ∇ has poles of order n_i at points p_i of X, we set $D = \sum_{i=1}^m n_i p_i$, and let $D_0 = \sum_{i=1}^m p_i$ denote the reduced divisor. We will suppose that the leading order term (i.e., coefficient of z^{-n_i}) of the connection at p_i is conjugate to a regular semisimple element $h_{i,-n_i}$ of a fixed Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G. By a gauge transformation at the poles, the polar part of the connection can be conjugated to

$$h_i(z)dz = (h_{i,-n_i} z^{-n_i} + h_{i,-n_i+1} z^{-n_i+1} + \cdots + h_{i,-1} z^{-1})dz.$$

Now if we allow a formal gauge transformation, then the connection itself can be put in this form at the puncture; the power series that effects this transformation though does not typically converge. Instead, there is additional monodromy data, given by Stokes matrices [JMU]. A good introduction to the theory can be found in [Sa], and the more advanced results we need have been established in [Boa1, Boa2]. We now give a brief outline of the basic ideas.

For each irregular singular point, it is necessary to choose disks Δ_i centered at p_i, $1 \le i \le m$. On Δ_i, as noted, there is a formal solution

$$H_i(z) = \exp\left(\int h_i(z)dz\right)$$

with a monodromy $\mu_i = \exp(2\pi\sqrt{-1}h_{i,-1})$. The disk Δ_i is partitioned into $2n_i - 2$ angular sectors $\mathcal{S}_{i,j}$ determined by the $h_{i,-n_i}$. Associated to the intersections $\mathcal{S}_{i,2j} \cap \mathcal{S}_{i,2j+1}$, there is a fixed (independent of j) unipotent radical $U_{+,i}$ of a Borel subgroup associated to $\mathfrak{h}$; to the intersections $\mathcal{S}_{i,2j+1} \cap \mathcal{S}_{i,2j+2}$ we have associated the opposite unipotent radical $U_{-,i}$. We also choose a base point q_i in $\mathcal{S}_{i,1}$.

Now we can then consider on each sector actual integrals $g_{i,j}(z) \in G$ of the connection asymptotic to the H_i, and passing from the sector $\mathcal{S}_{i,2j}$ to $\mathcal{S}_{i,2j+1}$, the two solutions are related by Stokes factors $u_{+,i,j}$ lying in $U_{+,i}$. In passing from $\mathcal{S}_{i,2j+1}$ to $\mathcal{S}_{i,2j+2}$, the two solutions are related by Stokes factors $u_{-,i,j}$ lying in $U_{-,i}$. The monodromy of the connection around the singularity is the product

$$\rho_i = \mu_i u_{-,i,n-1} \cdots u_{+,i,2} u_{-,i,1} u_{+,i,1}.$$

This monodromy and its decomposition into Stokes factors is defined up to the action of a torus.

For the deformations, there is a generalized Teichmüller space $\mathrm{Teich}_{\mathfrak{h},g,m}$, which combines the standard $\mathrm{Teich}_{g,m}$ with the irregular polar parts. Note that this combines parameters on the curves with parameters associated to the group; in addition to the standard Teichmüller parameters of the punctured curve, one considers the extra parameters of the "irregular type": in coordinates centred at the puncture. This is a polar part

$$Q_i(z) = q_{i,-1}z^{-1} + \cdots + q_{i,n_i-1}z^{n_i-1}$$

with

$$h_i(z)dz = dQ_i(z) + \text{lower order terms}$$

that is terms involving a simple pole and holomorphic terms. Note that the formal solution H_i is determined by Q_i and the monodromy μ_i. The generalized Teichmüller space involves varying these Q_i as well as the punctured curve parameters.

Lying above these deformations on the base, there is a theory of isomonodromic deformations of such connections, generalizing the one we have for the logarithmic case. Over the base parameters, in particular $H_i(z) = \exp(\int h_i(z)dz)$, which becomes an Abelian transition function at the puncture, to get a connection we need to fix the Stokes factors $u_{\pm,i,j}$ at the irregular singularities, and the representation $\pi_1(X \setminus D_0) \longrightarrow \mathrm{GL}(n,\mathbb{C})$ of the fundamental group. Fixing such isomonodromy data gives a lift of the Teichmüller deformations to a deformation of singular connections. By Malgrange's theorem, such isomonodromic deformations exist, and determine the connection up to gauge transformations [Ma] [He1]. Our aim will be to show the following:

Theorem 1.3. *Assume that the monodromy representation for ∇_0 is irreducible in the sense that it does not factor through some proper parabolic subgroup of G.*

1) *If $g \geq 1$, then there is a closed complex analytic subset $\mathcal{Y} \subset \mathrm{Teich}_{\mathfrak{h},g,m}$ of codimension at least g such that for any $t \in \mathrm{Teich}_{\mathfrak{h},g,m} \setminus \mathcal{Y}$, the holomorphic principal G–bundle $\mathcal{E}^t_G$ is semistable.*
2) *If $g \geq 2$, then there is a closed complex analytic subset $\mathcal{Y}' \subset \mathrm{Teich}_{\mathfrak{h},g,m}$ of codimension at least $g-1$ such that for any $t \in \mathrm{Teich}_{\mathfrak{h},g,m} \setminus \mathcal{Y}'$, the holomorphic principal G–bundle $\mathcal{E}^t_G$ is stable.*

2. The base space

We will describe the space $\mathrm{Teich}_{\mathfrak{h},g,m}$.

The Teichmüller space $\mathrm{Teich}_{g,m}$ for genus g curves with m marked points is a contractible complex manifold of complex dimension $3g-3+m$, assuming that $3g-3+m>0$. We first build a framed Teichmüller space, containing the necessary parametrization on the curve necessary to build an irregular type. If the singularity divisor is $D = \sum_{i=1}^m n_i p_i$, we can enrich the Teichmüller space $\mathrm{Teich}_{g,m}$ by adding to each point $(\Sigma, \sum_{i=1}^m p_i)$ of $\mathrm{Teich}_{g,m}$, the additional data of a coordinate z_i centered at p_i, defined to order $n_i - 1$ inclusively, for all $n_i > 1$. We note that this additional data at p_i is the choice of an isomorphism of the algebra $m_{p_i}/m_{p_i}^{n_i}$ with $z\mathbb{C}[z]/z^{n_i}\mathbb{C}[z]$, where m_{p_i} is the ring of holomorphic functions defined around p_i that vanish at p_i. The Teichmüller space $\mathrm{Teich}_{g,m}$ together with the above data produce a framed Teichmüller space $\mathrm{FTeich}_{g,m,n_1,\dots,n_m}$; it is of dimension $3g-3+m+\sum_i(n_i-1)$.

Now consider the extra data of the irregular types Q_i of the polar parts of the connection. We set our framed Teichmüller space for deformations of the irregular part of the connections plus punctured curves to simply be a product:

$$\mathrm{FTeich}_{\mathfrak{h},g,m,n_1,\ldots,n_m} = \mathrm{FTeich}_{g,m,n_1,\ldots,n_m} \times \prod_{i=1}^{m} (\mathfrak{h}^{(n_i-2)} \times \mathfrak{h}_0).$$

In other words, we are allowing our irregular types to deform freely (preserving their order), just asking that the leading order term remain within the regular locus $\mathfrak{h}_0$ of the Lie algebra $\mathfrak{h}$.

Our desired space of deformations $\mathrm{Teich}_{\mathfrak{h},g,m}$ will be the quotient of this space by the groups of germs of diffeomorphisms of neighbourhoods of p_i which fix p_i, acting diagonally on the factors. Indeed, the action of these deformations gives essentially trivial isomonodromy deformations, which we do not want to consider. As the action at each puncture is on truncated power series, one need only act by groups of jets

$$J_{p_i,n_i} = \left\{ z \mapsto a_1 z + a_2 z^2 + \cdots + a_{n_i} z^{n_i-1} \;\middle|\; a_j \in \mathbb{C},\ a_1 \neq 0 \right\};$$

one has

$$\mathrm{Teich}_{\mathfrak{h},g,m} = \mathrm{FTeich}_{\mathfrak{h},g,m,n_1,\ldots,n_m} / {\textstyle\prod_i} J_{p_i,n_i}.$$

(In fact, one would want to go to a universal cover, but for our purposes, this is not necessary as we are just considering the local deformations.)

Let us now see what this gives us for infinitesimal deformations. The tangent space of $\mathrm{T}_X(-D_0)$ at any element $(X, D_0) \in \mathrm{T}_X(-D_0)$ is

$$\mathrm{H}^1(X, \mathrm{T}_X(-D_0)).$$

We note that a 1- cocycle v can be thought of as giving an infinitesimal deformation of the coordinate changes from one patch to another; the 1-coboundaries must be taken with values in the vector fields vanishing at D_0. Such a coboundary, however, affects the form of the irregular polar parts at p_i.

Indeed, these are not well defined in themselves, as they are acted on by diffeomorphisms of the curve fixing p_i. This must be taken into account in the deformation theory. Consider an infinitesimal local diffeomorphism of the curve given at the puncture $z = 0$ by a vector field $v(z)\partial/\partial z$. As we are considering punctured curves, we want $v(0) = 0$. The changes in the function $Q_i(z)$ caused by a change in parametrization, infinitesimally a vector field, should be considered as trivial: in other words,

$$Q_i(z+\epsilon v(z)) = Q_i(z) + \epsilon Q_i'(z)v(z)).$$

should be a trivial deformation. Thus, for our deformations, we will be interested in the complex

$$\mathcal{C} : \mathrm{T}_X(-D_0) \xrightarrow{F} \mathcal{PP} = \mathcal{O}_{D-D_0} \otimes \mathfrak{h} = \bigoplus_i \mathfrak{h}^{\oplus(n_i-1)};$$

the second sheaf is a sum of skyscraper sheaves supported at the points of D_0; the homomorphism F sends a vector field v around p_i and vanishing at p_i to the irregular polar part $(\mathcal{PP})$ of the contraction of v with the connection matrix h_i:

$$F : v(z) \longmapsto \mathcal{PP}((h_i(z)v(z))) = \mathcal{PP}(Q_i'(z)v(z)).$$

The first order deformations of the marked curve are given by the cohomology (group $\mathrm{H}^1(X, \mathrm{T}_X(-D_0))$; adding in the deformations of the irregular polar parts gives us the global hypercohomology group

$$\mathbb{H}^1(X, \mathcal{C}) = \mathrm{T}_{(X,D,H)}\mathrm{Teich}_{\mathfrak{h},g,m}.$$

We note that $\mathbb{H}^1(X, \mathcal{C})$ gives a local version of the space of admissible deformations in [Boa3] of the irregular curve defined by the triple $(X, D_0, \bigoplus_i \mathfrak{h}^{\oplus(n_i-1)})$. In [Boa3], the space of objects, consisting of a Riemann surface, some marked points on it and irregular types at the marked points, are defined more intrinsically.

We have an exact sequence

$$\bigoplus_i \mathfrak{h}^{\oplus(n_i-1)} \longrightarrow \mathbb{H}^1(X, \mathcal{C}) \longrightarrow \mathrm{H}^1(X, \mathrm{T}_X(-D_0)).$$

The elements β of $\mathrm{H}^1(X, \mathrm{T}_X(-D_0))$ encode extensions

$$0 \longrightarrow \mathrm{T}_X(-D_0) \longrightarrow \mathcal{T} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

This can be viewed as the tangent bundle to the infinitesimal one-parameter family of bundles represented by the element β, with the structure sheaf $\mathcal{O}_X$ representing a trivial normal bundle. An element $\widehat{\beta}$ of $\mathbb{H}^1(X, \mathcal{C})$ mapping to β encodes a bit more, namely a diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{T}_X(-D_0) & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_X & \longrightarrow 0 \\ & {\scriptstyle F}\downarrow & & \downarrow & & & \\ & \mathcal{PP} & = & \mathcal{PP}\,. & & & \end{array}$$

3. Deforming the bundle

The Lie algebra of G will be denoted by $\mathfrak{g}$. Let $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ be the adjoint bundle for E_G over X. Let $\mathrm{At}(E_G)$ denote the Atiyah bundle for E_G; it fits in the Atiyah exact sequence over X

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \longrightarrow \mathrm{T}_X \longrightarrow 0$$

[At]. The Atiyah bundle for E_G represents over the base the G–invariant vector fields on the principal G–bundle E_G; the subbundle of invariant vector fields tangent to the fibers is $\mathrm{ad}(E_G)$. The Atiyah exact sequence produces a short exact sequence

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}_{D_0} := \mathrm{At}_{D_0}(E_G) \longrightarrow \mathrm{T}_X(-D_0) \longrightarrow 0,$$

where D_0 is the reduced singular locus of the connection. In [BHH1] it is shown that the deformations of the logarithmic connection, over a curve X that is also being deformed, are parametrized by $\mathrm{H}^1(X, \mathrm{At}_{D_0})$.

To deal with the higher order poles, we need to consider the sheaf $\mathrm{At}_{D_0}(D - D_0)$ of meromorphic sections of At_{D_0} with poles living only in the $\mathrm{ad}(E_G)$ factor, bounded by $D - D_0$:

$$0 \longrightarrow \mathrm{ad}(E_G)(D - D_0) \longrightarrow \mathrm{At}_{D_0}(D - D_0) \longrightarrow \mathrm{T}_X(-D_0) \longrightarrow 0.$$

Now let us consider deformations of these. We cover our Riemann surface away from the punctures with Stokes sectors $S_{i,j}$, as well as other contractible open sets V_ν; choose flat trivializations on these sets, with the ones on Stokes sectors being compatible with the formal asymptotics. The transition functions on the bundle for these trivializations are then constants, with those between the Stokes sectors being the Stokes matrices. For the puncture, we have the transition functions $H_i(z)$. Re-label the Stokes sectors as being in the set of V_ν; we then have constant transition functions Θ_{ν_1,ν_2} away from the puncture, and $H_i(z)$ at the puncture. Now take a variation $H_i(z)(1 + \epsilon \int k_i(z))$ and a cocycle v_{ν_1,ν_2} for $\mathrm{T}_X(-D_0)$ which corresponds to infinitesimal displacements of the coordinate patches with respect to each other; these together arise from a class $\hat{\beta}$ in $\mathbb{H}^1(X, \mathcal{C})$.

We are, in our isomonodromic deformations, deforming the bundle above the curve by keeping the same Θ_{ν_1,ν_2}, and modifying the transition function at the puncture by

$$H_i(z)\left(1 + \epsilon \int k_i(z)\right).$$

As a deformation of the Atiyah bundle, the former consists of considering the mapping

$$\nabla : \mathrm{T}_X(-D_0) \longrightarrow \mathrm{At}_{D_0}(D - D_0),$$

and taking the induced map on the cocycles, i.e., taking $\nabla(v_{\nu_1,\nu_2})$ as a cocycle for

$$\mathrm{At}_{D_0}(D - D_0),$$

which, as we are away from the punctures, we can take to be a cocycle in for At_{D_0}. To this, we add the element $k_i(z)$ as a cocycle for the deformation of the transition function from the disk around the puncture to the Stokes sectors, for the subbundle $\mathrm{ad}(E_G)$ of At_{D_0}; the sum of the cocycles gives a class γ in $\mathrm{H}^1(X, \mathrm{At}_{D_0})$.

As for the deformation of the curves, an element γ of $\mathrm{H}^1(X, \mathrm{At}_{D_0})$ defines an extension

$$0 \longrightarrow \mathrm{At}_{D_0} \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

mapping to corresponding extensions of $\mathrm{T}_X(-D_0)$, and so gives a diagram

$$\begin{array}{ccccccccc}
 & & \mathrm{ad}(E_G) & = & \mathrm{ad}(E_G) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{At}_{D_0} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{T}_X(-D_0) & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0\,.
\end{array}$$

This represents over $\epsilon = 0$ the G–invariant vector fields on our first order extension, the $\mathcal{O}_X$–quotient being the normal bundle.

4. Deformations of reductions

4.1. Extending a reduction

The stability of G-bundles concerns reductions to a parabolic subgroup: the bundle E_G is stable (respectively, semistable) if for all its reductions E_P to a parabolic subgroup P, the associated bundle

$$\mathrm{ad}(E_G)/\mathrm{ad}(E_P) = E_P(\mathfrak{g}/\mathfrak{p})$$

has positive (respectively, non-negative) degree, where $\mathfrak{g}$ and $\mathfrak{p}$ are the Lie algebras of G and P respectively. If we want to ensure that the set of non stable bundles is somehow small along the isomonodromic deformation, we must see how reductions to a parabolic extend along a deformation, and in particular try to understand the space of first order obstructions to such an extension.

Given a reduction E_P, we now have two Atiyah bundles $\mathrm{At}^G_{D_0}$ and $\mathrm{At}^P_{D_0}$ over the surface associated to E_G and E_P respectively. These fit into a diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{ad}(E_P) & \longrightarrow & \mathrm{At}^P_{D_0} & \xrightarrow{\beta} & \mathrm{T}X(-D_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \xi & & \| & & \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}^G_{D_0} & \xrightarrow{\sigma} & \mathrm{T}X(-D_0) & \longrightarrow & 0\,. \\ & & \downarrow \omega_1 & & \downarrow \omega & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{4.1}$$

Now assume that the reduction to P extends to first order along a first order deformation of the G-bundle over the curve. One then has extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{At}^P_{D_0} & \longrightarrow & \mathcal{A}^P & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathrm{At}^G_{D_0} & \longrightarrow & \mathcal{A}^G & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0\,. \end{array} \tag{4.2}$$

given by extension classes $\gamma^P \in \mathrm{H}^1(X, \mathrm{At}^P_{D_0})$ and $\gamma^G \in \mathrm{H}^1(X, \mathrm{At}^G_{D_0})$. One has the lemma

Lemma 4.1 ([BHH1, Lemma 3.1]). *The above extension classes γ^P, γ^G are related by*

$$\gamma^G = \xi(\gamma^P).$$

Consequently, there is an obstruction to extending the reductions for deformations γ^G given by $\omega(\gamma^G) \in \mathrm{H}^1(X, E_P(\mathfrak{g}/\mathfrak{p}))$.

4.2. A second fundamental form

Assume now that there is a connection ∇ on the principal bundle E_G. It does not of course, necessarily preserve the reduction to E_P. The failure to preserve E_P is measured by a second fundamental form: one takes the composition

$$S(\nabla) = \omega \circ \nabla : \mathrm{T}_X(-D_0) \longrightarrow \mathrm{At}_{D_0}(D - D_0) \longrightarrow E_P(\mathfrak{g}/\mathfrak{p})(D - D_0). \tag{4.3}$$

The connection preserves the reduction to P if and only if $S(\nabla) = 0$.

Assume that E_P satisfies the condition that $S(\nabla) \neq 0$. We define some line bundles. Let

$$\mathcal{M}(D - D_0) \subset \mathrm{At}_{D_0}(D - D_0)$$

be the holomorphic line subbundle generated by the image $\nabla(\mathrm{T}_X(-D_0))$ in (4.3), and let

$$\mathcal{L}(D - D_0) \subset E_P(\mathfrak{g}/\mathfrak{p})(D - D_0)$$

be the holomorphic line subbundle generated by the image $\omega(\nabla(\mathrm{T}_X(-D_0)))$ in (4.3). More precisely, $\mathcal{M}_{D-D_0}$ (respectively, $\mathcal{L}_{D-D_0}$) is the inverse image in $\mathrm{At}_{D_0}(D - D_0)$ (respectively, $E_P(\mathfrak{g}/\mathfrak{p})(D - D_0)$) of the torsion part of the quotient

$$\mathrm{At}_{D_0}(D - D_0)/\nabla(\mathrm{T}_X(-D_0))$$

(respectively, $E_P(\mathfrak{g}/\mathfrak{p})/(\omega \circ \nabla)(\mathrm{T}X(-D_0))$). Set

$$\mathcal{M} = \mathcal{M}_{D-D_0} \cap \mathrm{At}_{D_0}, \qquad \mathcal{L} = \mathcal{L}_{D-D_0} \cap E_P(\mathfrak{g}/\mathfrak{p}). \tag{4.4}$$

We then have the diagram of homomorphisms of line bundles, with the columns being exact:

$$\begin{array}{ccccc}
\mathrm{T}_X(-D) & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{T}_X(-D_0) & \longrightarrow & \mathcal{M}(D-D_0) & \longrightarrow & \mathcal{L}(D-D_0) \\
\downarrow & & \downarrow & & \downarrow \\
Q_1 & & Q_2 & & Q_3 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0\,.
\end{array}$$

Note that Q_1, Q_2 and Q_3 are isomorphic torsion sheaves supported on $D - D_0$.

Lemma 4.2. *The horizontal homomorphisms in this diagram induce surjective maps on the level of first cohomology.*

Proof. The proof consists in noting that the cokernels of each of these homomorphisms are torsion sheaves. $\square$

If one considers the homomorphism $\mathrm{T}_X(-D) \longrightarrow \mathcal{M} \subset \mathrm{At}_{D_0}$ given by the connection, one has that $\mathcal{M}$ lies in the "Cartan component" of the bundle to order $n_i - 1$ at p_i, as it is a multiple of $h_i(z)$. For the sheaf $\mathcal{PP}$ of polar parts of the connection, let us consider the subsheaf $\mathcal{PP}_{||}$ whose sections are multiples of $h_i(z)$; likewise, in our deformation space $\mathbb{H}^1(X, \mathcal{C})$, let us consider the subspace $\mathbb{H}^1_{||}(X, \mathcal{C})$ of classes where the principal part is parallel to (i.e., a multiple of) $h_i(z)$.

Proposition 4.3. *We have a diagram*

$$\begin{array}{ccc}
\mathcal{PP}_{||} & \longrightarrow & \mathrm{H}^0(Q_2) \\
\downarrow & & \downarrow \\
\mathbb{H}^1_{||}(X, \mathcal{C}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(X, \mathrm{T}_X(-D_0)) & \longrightarrow & \mathrm{H}^1(X, \mathcal{M}(D-D_0))\,.
\end{array}$$

The top horizontal homomorphism is an isomorphism, and the other two horizontal homomorphisms are surjective.

Proof. On the top row, the components of $\mathcal{PP}_{\|}$ are exactly those of the torsion sheaf Q. On the bottom row, one has elements β of $\mathrm{H}^1(X, \mathrm{T}_X(-D_0))$ mapped by ∇ to $\mathrm{H}^1(X, \mathcal{M}(D - D_0))$. As argued in Lemma 4.2, this map is surjective.

One now wants to see that the top and bottom fit together correctly in the middle term. Let $\widehat{\beta} \in \mathbb{H}^1_{\|}(X, \mathcal{C})$ be represented by elements k_i of $\mathcal{PP}_{\|}$ at each puncture, and a representative cocycle β of $\mathrm{H}^1(X, \mathrm{T}_X(-D_0))$. If one turns k_i in the natural way into a cocycle supported on a punctured disk Δ_i at p_i, it gives precisely the element of $\mathrm{H}^1(X, \mathcal{M})$ which is the coboundary of k_i thought of as an element of Q_2. In turn, the cocycle β is simply mapped to $\mathrm{H}^1(X, \mathcal{M})$ by the sheaf map; the total map from $\mathbb{H}^1_{\|}(X, \mathcal{C})$ is given by the sum of these two contributions, as in the definition of $\mathcal{A}$ above. Since the top map is an isomorphism, and the bottom one is surjective, the middle map is also surjective. □

We now have a surjective map $\mathbb{H}^1_{\|}(X, \mathcal{C}) \longrightarrow \mathrm{H}^1(X, \mathcal{M})$, which, when mapped on to $\mathrm{H}^1(X, \mathrm{At}_{X_0})$, defines the extension $\mathcal{A}$. We saw in turn that the map

$$\omega : \mathrm{H}^1(X, \mathrm{At}_{X_0}) \longrightarrow \mathrm{H}^1(X, E_P(\mathfrak{g}/\mathfrak{p}))$$

gave an obstruction to extending to first order a reduction to P. We have a diagram:

$$\begin{array}{ccccc} \mathbb{H}^1_{\|}(X, \mathcal{C}) & \longrightarrow & \mathrm{H}^1(X, \mathcal{M}) & \xrightarrow{\omega_{\mathcal{L}}} & \mathrm{H}^1(X, \mathcal{L}) \\ & & \downarrow & & \downarrow \\ & & \mathrm{H}^1(X, \mathrm{At}_{X_0}) & \xrightarrow{\omega} & \mathrm{H}^1(X, E_P(\mathfrak{g}/\mathfrak{p}))\,. \end{array} \tag{4.5}$$

We have, as in Proposition 4.3 of [BHH1], the following:

Proposition 4.4. *Let $\widehat{\beta} \in \mathbb{H}^1_{\|}(X, \mathcal{C})$ represent an isomonodromy deformation class, of a connection with non-vanishing second fundamental form, yielding a class*

$$\gamma_{\mathcal{M}} \in \mathrm{H}^1(X, \mathcal{M}),$$

and a class

$$\gamma \in \mathrm{H}^1(X, \mathrm{At}_{X_0})$$

representing the extension $\mathcal{A}$. The obstruction $\omega(\gamma)$ to extending a reduction to P factors through $\mathcal{L}$, as $\omega_{\mathcal{L}}(\gamma_{\mathcal{M}}) \in \mathrm{H}^1(X, \mathcal{L})$, and if the bundle reduces to P then $\omega_{\mathcal{L}}(\gamma_{\mathcal{M}}) \in \mathrm{H}^1(X, \mathcal{L})$ is also zero.

The proof in essence works by taking the restriction of the Atiyah bundle and its extension which live above $\mathcal{L}$.

We will want to estimate the dimension of the space spanned by the obstructions $\omega_L(\gamma_{\mathcal{M}})$, as this will give a bound on the codimensions of the stable locus, as explained in the next section.

5. Harder-Narasimhan filtrations

Let as before $\mathrm{Teich}_{\mathfrak{h},g,m}$ be our Teichmüller space; over it we have, locally, after choosing a section of $\mathrm{FTeich}_{\mathfrak{h},g,m} \to \mathrm{Teich}_{\mathfrak{h},g,m}$, a universal family $(\mathcal{C}, \mathcal{D}, \mathcal{H})$ whose fiber at q is a curve $\mathcal{C}(q)$, a divisor

$$\mathcal{D}(q) = \sum_i n_i p_i(q)$$

and a collection of formal solutions $H_i(q)$. Over this in turn the isomonodromy process, described in section 3, gives a $G-$bundle $\mathcal{E}_G \longrightarrow \mathcal{C}$, equipped with a flat connection, with the appropriate polar behavior at $\mathcal{D}$. For $\mathcal{E}_G$, one has a Harder-Narasimhan filtration for families of G-bundles, as propounded in [GN] (see also [Sh]); the filtration is trivial if and only if the bundle is semi-stable.

Lemma 5.1 ([GN]). *Let $\mathcal{E}_G \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}_{\mathfrak{h},g,m}$ be as above. For each Harder–Narasimhan type κ, the set*

$$\mathcal{Y}_\kappa := \{t \in \mathcal{T}_{\mathfrak{h},g,m} \ \mid \ \mathcal{E}_G|_{\mathcal{C}_t} \ \ \text{is of type} \ \ \kappa\}$$

is a (possibly empty) locally closed complex analytic subspace of $\mathcal{T}_{\mathfrak{h},g,m}$. More precisely, for each Harder–Narasimhan type κ, the union $\mathcal{Y}_{\leq\kappa} := \bigcup_{\kappa' \leq \kappa} \mathcal{Y}_{\kappa'}$ is a closed complex analytic subset of $\mathcal{T}_{\mathfrak{h},g,m}$. Moreover, the principal G–bundle

$$\mathcal{E}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)} \longrightarrow \tau^{-1}(\mathcal{Y}_\kappa)$$

possesses a canonical holomorphic reduction of structure group inducing the Harder–Narasimhan reduction of $\mathcal{E}_G|_{\mathcal{C}_t}$ for every $t \in \mathcal{Y}_\kappa$.

It is our aim to show that all of these strata except that corresponding to the trivial filtration (and hence to semi-stable bundles) are of codimension

g, by showing that there is a g-dimensional family of directions for which the reduction does not extend.

Theorem 5.2. *Assume that the monodromy representation for ∇_0 is irreducible in the sense that it does not factor through some proper parabolic subgroup of G.*

1) *If $g \geq 1$, then there is a closed complex analytic subset $\mathcal{Y} \subset \mathrm{Teich}_{\mathfrak{h},g,m}$ of codimension at least g such that for any $t \in \mathrm{Teich}_{\mathfrak{h},g,m} \setminus \mathcal{Y}$, the holomorphic principal G–bundle $\mathcal{E}_G^t$ is semi-stable.*
2) *If $g \geq 2$, then there is a closed complex analytic subset $\mathcal{Y}' \subset \mathrm{Teich}_{\mathfrak{h},g,m}$ of codimension at least $g-1$ such that for any $t \in \mathrm{Teich}_{\mathfrak{h},g,m} \setminus \mathcal{Y}'$, the holomorphic principal G–bundle $\mathcal{E}_G^t$ is stable.*

Proof. Let $g > 1$. Let $\mathcal{Y} \subset \mathcal{T}_{\mathfrak{h},g,m}$ denote the (finite) union of all Harder-Narasimhan strata $\mathcal{Y}_\kappa$ as in Lemma 5.1 with non-trivial Harder-Narasimhan type κ. From Lemma 5.1 we know that $\mathcal{Y}$ is a closed complex analytic subset of $\mathcal{T}_{\mathfrak{h},g,m}$.

Take any $t \in \mathcal{Y}_\kappa \subset \mathcal{Y}$. Let $E_G = \mathcal{E}_G|_{\mathcal{C}_t}$ be the holomorphic principal G–bundle on

$$X := \mathcal{C}_t.$$

The holomorphic connection on E_G obtained by restricting the universal isomonodromy connection will be denoted by ∇. Since E_G is not semistable, there is a proper parabolic subgroup $P \subsetneq G$ and a holomorphic reduction of structure group $E_P \subset E_G$ to P, such that E_P is the Harder–Narasimhan reduction [Be], [AAB]; the type of this Harder–Narasimhan reduction is κ. From Lemma 5.1 we know that E_P extends along its stratum to a holomorphic reduction of structure group of the principal G–bundle $\mathcal{E}_G|_{\tau^{-1}(\mathcal{Y}_\kappa)}$ to the subgroup P.

Let $\mu_{\max}$ be the maximal slope (degree/rank) of the terms of the Harder Narasimhan-filtration.

We have

$$\mu_{\max}(E_P(\mathfrak{g}/\mathfrak{p})) < 0 \tag{5.1}$$

[AAB, p. 705]. In particular

$$\mathrm{degree}(E_P(\mathfrak{g}/\mathfrak{p})) < 0. \tag{5.2}$$

Form the irreducibility of the connection, we know that the second fundamental form $S(\nabla)$ does not vanish, and so we can build the line bundle

$\mathcal{L}$ as above in (4.4).

$$\mathcal{L} \subset E_P(\mathfrak{g}/\mathfrak{p}).$$

From (5.1) we have

$$\text{degree}(\mathcal{L}) < 0. \tag{5.3}$$

Therefore, $\mathrm{h}^0(X, \mathcal{L}) = 0$, and $\mathrm{h}^1(X, \mathcal{L}) \geq g$.

On the other hand, Lemma 4.2 and Lemma 4.3 gave a surjective map from our deformation space $\mathbb{H}^1_{\|}(X, \mathcal{C})$ to our obstruction space $\mathrm{H}^1(X, \mathcal{L})$. The space $\mathcal{Y}$ is thus of codimension at least g.

For the second case, when stability fails, the degree of $\mathcal{L}$ is only less than or equal to zero, and so $\mathrm{h}^1(X, \mathcal{L}) \geq g - 1$, giving the announced codimension. □

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