Initial data in general relativity described by expansion, conformal deformation and drift

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The conformal method is a technique for finding Cauchy data in general relativity solving the Einstein constraint equations, and its parameters include a conformal class, a conformal momentum (as measured by a densitized lapse), and a mean curvature. Although the conformal method is successful in generating constant mean curvature (CMC) solutions of the constraint equations, it is unknown how well it applies in the non-CMC setting, and there have been indications that it encounters difficulties there. We are therefore motivated to investigate alternative generalizations of the CMC conformal method.

Introducing a densitized lapse into the ADM Lagrangian, we find that solutions of the momentum constraint can be described in terms of three parameters. The first is conformal momentum as it appears in the standard conformal method. The second is volumetric momentum, which appears as an explicit parameter in the CMC conformal method, but not in the non-CMC formulation. We have called the third parameter drift momentum, and it is the conjugate momentum to infinitesimal motions in superspace that preserve conformal class and volume form up to independent diffeomorphisms. This decomposition of solutions of the momentum constraint leads to extensions of the CMC conformal method where conformal and volumetric momenta both appear as parameters. There is more than one way to treat drift momentum, in part because of an interesting duality that emerges, and we identify three candidates for incorporating drift into a variation of the conformal method.

1. Introduction

An initial data set in general relativity consists of the geometry and matter distribution of the universe at an instant in time, along with the instantaneous rate of change of these quantities. The associated Cauchy problem is to determine an ambient spacetime for the initial data set that satisfies the Einstein equations as well as the applicable matter field equations. In contrast to Newtonian gravity, initial data cannot be freely specified, and must satisfy certain underdetermined compatibility conditions known as the Einstein constraint equations. These constraint PDEs admit a wide variety of solutions, and as a consequence we have enormous flexibility, but not complete freedom, in specifying initial conditions. One would therefore like to find intrinsic parameters describing the set of solutions of the constraint equations.

This problem is already difficult, and not yet understood, for vacuum spacetimes with a vanishing cosmological constant, in which case an initial data set consists of a Riemannian manifold (M^n, g_{ab}) and a symmetric tensor

K_{ab} representing the second fundamental form of the embedding of M^n into its ambient spacetime. Vacuum spacetimes are Ricci flat, and hence the Gauss and Codazzi equations imply the following relations between g_{ab} and K_{ab}:

(1.1a) $\qquad R_g - |K|_g^2 + (\mathrm{tr}_g K)^2 = 0 \qquad$ [Hamiltonian constraint]

(1.1b) $\qquad \mathrm{div}_g K - \mathrm{d}(\mathrm{tr}_g K) = 0 \qquad$ [momentum constraint]

where d is the exterior derivative, R_g is the scalar curvature, div_g is the divergence, and tr_g is the trace operator of g_{ab}. Equations (1.1) are the vacuum Einstein constraint equations, and the fact that they are underdetermined reflects the physical property that gravitational waves can propagate in vacuum, as well as the gauge property that we have freedom to choose coordinates in spacetime.

There are a number of approaches for finding solutions of the constraint equations in specific circumstances, and we note in particular the examples provided by gluing methods [23][24][13] [12][15][14][11] [7], as well as the density and perturbation techniques of [20][21]. These constructions provide deep insight into the diversity of solutions of the constraint equations and their properties, but they do not yield parameterizations. Indeed, as far as concrete parameterizations are concerned, there is presently only a single general purpose candidate, the conformal method, and it occurs in the literature in two principal variations. The original conformal method was initiated by Lichnerowicz [28] and later extended by York to construct constant-mean curvature (CMC) solutions [39] and, along with O'Murchadha, to construct non-CMC solutions of the constraint equations [35]. It provides an essentially ideal parameterization of CMC solutions of the constraint equations on compact manifolds [22]. Subsequently York developed the Lagrangian conformal thin-sandwich (CTS) method [40] and then with Pfeiffer presented the Hamiltonian form of the CTS method [36]. It turns out that the standard and CTS conformal methods are two different ways to write down the same parameterization of the constraint equations [33], and we will refer to all these techniques collectively as the conformal method. Using the language of [33] that emphasizes the role of conformal geometry, the Hamiltonian form of the conformal method has four parameters:

- A conformal class $\mathbf{g}$, represented by the choice of a metric $g_{ab} \in \mathbf{g}$.

- A conformal momentum $\boldsymbol{\sigma}$, represented by a pair $(g_{ab}; \sigma_{ab})$ where σ_{ab} is trace-free and divergence free. Writing $q = 2n/(n-2)$ for the

critical Sobolev exponent, if $\phi > 0$ is a conformal factor then the pair $(\phi^{q-2} g_{ab};\ \phi^{-2} \sigma_{ab})$ represents the same conformal momentum $\boldsymbol{\sigma}$.

- An arbitrary function τ dictating a mean curvature.

- A so-called densitized lapse represented by a pair $(g_{ab};\ N)$ where N is a positive function. If $\phi > 0$ is a conformal factor, $(\phi^{q-2} g_{ab};\ \phi^q N)$ represents the same densitized lapse.

The choice of a densitized lapse $\mathbf{N}$ allows for a notion of conformal momentum to be assigned to a solution of the constraint equations, and after fixing a densitized lapse every solution of the constraint equations uniquely determines conformal parameters $(\mathbf{g}, \boldsymbol{\sigma}, \tau, \mathbf{N})$. The central question for the conformal method is the extent to which this map is a bijection.

Suppose for concreteness that M is compact. If we restrict our attention to CMC solutions of the constraint equations (i.e. solutions with $\tau \equiv \tau_0$ for some constant τ_0) then the map from solutions of the constraint equations onto conformal parameters is indeed a bijection [22], with the following caveats based on the sign of the Yamabe invariant $Y_{\mathbf{g}}$ of the conformal class $\mathbf{g}$:

- If $Y_g > 0$, then $\boldsymbol{\sigma} = 0$ is impossible.

- If $Y_g < 0$, then $\tau_0 = 0$ is impossible.

- If $Y_g = 0$, then $\boldsymbol{\sigma} = 0$ is impossible and $\tau_0 = 0$ is impossible, except that there is a homothety family of solutions corresponding to the case where both $\boldsymbol{\sigma} = 0$ and $\tau_0 = 0$.

Moreover, these same results largely extend into the near-CMC regime: see, e.g., [25] and [1] as augmented by [33] for existence and uniqueness theorems, and see [26] for non-existence results when $Y_{\mathbf{g}} \geq 0$ and $\boldsymbol{\sigma} = 0$. Indeed, the theory for near-CMC solutions is satisfactory and complete, except that existence is not understood if $\mathbf{g}$ admits nontrivial conformal Killing fields.

On the other hand, the properties of the conformal method when τ is far-from-CMC are largely unknown. On compact manifolds we have a single far-from CMC existence theorem [19][30]: given a Yamabe positive conformal class $\mathbf{g}$ and an arbitrary mean curvature τ, if $\boldsymbol{\sigma} \neq 0$ is close to zero (with closeness depending on τ), there exists *at least* one associated solution of the constraint equations. This foray into far-from-CMC territory can, moreover, be thought of as a perturbation off of a CMC solution with $\tau_0 = 0$ [18]. And although the far-from-CMC existence result is consistent with the possibility

that the good properties of the CMC conformal method extend to far-from-CMC solutions, subsequent case studies in [31] and [34] show that at least sometimes they do not.

The work in [31] exhibits a family of symmetric conformal data on the torus such that in the far-from-CMC regime there are multiple solutions when $\boldsymbol{\sigma}$ is small, no solutions with the symmetry of the data when $\boldsymbol{\sigma}$ is large, and certain rare cases that lead to exceptional one-parameter families of non-CMC solutions. The mean curvatures studied in [31] are discontinuous but piecewise smooth. Although it not known if similar difficulties occur for smooth mean curvatures, the follow-up study in [34] shows that at least the exceptional one-parameter families of [31] persist for smooth mean curvatures.

The conformal parameters considered in [34] have the form $(\mathbf{g}, \mu\boldsymbol{\sigma}^{\flat}, \tau, \mathbf{N})$ where $\mathbf{g}$ is the conformal class of a flat product metric g_{ab} on the torus, $\boldsymbol{\sigma}^{\flat}$ is a particular conformal momentum, μ is a constant, and where τ and $\mathbf{N} = (g_{ab};\, N)$ are arbitrary, except that τ and N depend on only one factor of the torus. Writing

$$(1.2) \qquad \tau^* = \frac{\int_M N\tau\,\omega_g}{\int_M N\,\omega_g}$$

where ω_g is the volume form of g_{ab}, [34] shows that if μ and τ^* have the same sign, then the conformal parameters generate a slice of a flat spacetime (typically a Kasner solution, with certain other spacetimes occurring non-generically). The case where $\tau^* = 0$ is special, however: if μ and τ^* both vanish, then the conformal parameters construct a one parameter family of solutions of the constraint equations. Note that if $\tau = \tau_0$ for some constant τ_0, then $\tau^* = \tau_0$ and the CMC one-parameter families occur when $\tau_0 = 0$. But if τ is not constant then the computation of τ^* involves a particular choice of representative of $\mathbf{g}$, and the condition $\tau^* = 0$ is not readily computed in advance. Indeed, τ^* can be computed with respect to the physical metric that solves the constraint equations, but to compute τ^* when working with some other background metric, one must first conformally transform to a flat metric, at which point one has all but solved the constraint equations [34]. Hence we have an example of non-uniqueness for certain non-CMC conformal parameters where the non-uniqueness is difficult to detect *a priori*.

The success of the conformal method in the CMC setting has physical consequences including, for example, Fischer and Moncrief's program of Hamiltonian reduction [17]. In contrast, failures of the conformal method for non-CMC conformal parameters may not imply anything in particular

about general relativity. The set of solutions of the constraint equations has, when given a suitable topology, a manifold structure [12][5] at generic points, and there are many possible choices of charts for this manifold. Although the conformal method provides a useful and successful chart in a neighborhood of CMC solutions, we interpret the evidence to date as suggesting that this chart simply breaks down outside of this neighborhood. If this is indeed the case, the details of this breakdown may be meaningful facts about the conformal method, but perhaps not about the constraint equations.

In this article we examine the possibility that the CMC conformal method admits an extension, other than the standard conformal method, that potentially has better properties for non-CMC solutions of the constraint equations. In particular, we identify geometrically and physically motivated alternatives that replace the mean curvature parameter τ with two independent quantities: the constant τ^* from equation (1.2) along with a second parameter, described below, that we will call a drift. The guiding principle leading to these alternatives is to treat the densitized lapse as a fundamental object, and to apply it uniformly to both conformal and volumetric degrees of freedom.

Densitized lapses first appeared in the context of the constraint equations in York's development of the conformal thin sandwich method [40], where they occur as lapses that conformally transform according to $N \mapsto \phi^q N$ when we change $g_{ab} \mapsto \phi^{q-2} g_{ab}$. Although densitized lapses arrived somewhat late in the development of the conformal method, because the original conformal method and the CTS methods are equivalent, densitized lapses have been a part of the conformal method all along. In this work we represent a densitized lapse by a choice of volume form α on M. To every metric g_{ab} we then assign a lapse according to

$$(1.3) \qquad\qquad N_{g,\alpha} = \frac{\omega_g}{\alpha}$$

where ω_g is the volume form of g_{ab}. Since volume forms conformally transform according to $\omega_g \mapsto \phi^q \omega_g$ we recover York's transformation law, and in terms of our earlier notation the volume form α corresponds to the densitized lapse $\mathbf{N}$ represented by $(g_{ab}; \omega_g/\alpha)$. Note that if we interpret α as 'coordinate area', then equation (1.3) expresses the lapse as the ratio of physical to coordinate area in addition to its standard interpretation as the ratio of physical to coordinate time. Using equation (1.3) to rewrite the usual Arnowitt-Deser-Misner (ADM) Lagrangian[3] so that it depends on α instead of the standard lapse, we find that the following features emerge.

- The densitized lapse assigns each pair (g_{ab}, K_{ab}), regardless of whether it solves the constraint equations or not, a conformal velocity and a conformal momentum of motion in $\mathcal{C}/\mathcal{D}_0$, where $\mathcal{C}$ is the set of conformal classes on M and $\mathcal{D}_0$ is the connected component of the identity of the diffeomorphism group. These dynamical quantities are associated with their standard ADM counterparts as described in diagram (4.2), but doing so requires a densitized lapse rather than the standard ADM lapse. For CMC solutions of the constraints, the measurement of conformal momentum is independent of the choice of densitized lapse, but this is not true for non-CMC solutions. The conformal method uses conformal velocity or conformal momentum as one of its parameters depending on whether we use the Lagrangian or the Hamiltonian formulation, and these quantities are connected to each via a Legendre transformation associated with a Lagrangian (conformal kinetic energy) on the tangent bundle $T\,\mathcal{C}/\mathcal{D}_0$. Sections 3 and 4 describe these results.

- The densitized lapse assigns each pair (g_{ab}, K_{ab}) a volumetric velocity and momentum of motion in $\mathcal{V}/\mathcal{D}_0$, where $\mathcal{V}$ is the set of volume forms. Volumetric velocity and momentum are associated with ADM velocity and momentum as described in diagram 7.2, and again this relationship uses a densitized lapse. Volumetric momentum is a single number, and if $g^{ab}K_{ab} = \tau_0$ for some constant τ_0, the volumetric momentum is $-2\kappa\tau_0$ where $\kappa = (n-1)/n$. For non-constant mean curvature the measurement of conformal momentum depends on the choice of densitized lapse and equals $-2\kappa\tau^*$ where τ^* is the quantity (1.2) identified previously in [34]. In the CMC conformal method, the volumetric momentum is one of the explicit parameters, but this is not the case for the non-CMC conformal method. Volumetric velocity and momentum are connected to each via a Legendre transformation associated with a Lagrangian (volumetric kinetic energy) on the tangent bundle $T\,\mathcal{V}/\mathcal{D}_0$. Sections 6 and 7 describe these results, and we see in these sections that the volumetric parameters have a structure that completely parallels that of the conformal parameters, but that is ignored in the standard conformal method where the mean curvature is specified explicitly.

- Conformal momentum at a metric g_{ab} is related to the York decomposition of trace-free tensors A_{ab}

(1.4)
$$A_{ab} = \sigma_{ab} + \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g W)_{ab}$$

where σ_{ab} is transverse traceless, $\mathbf{L}_g$ is the conformal Killing operator of g_{ab}, and W^a is a vector field. Volumetric momentum is associated with a York-like splitting of mean curvature functions τ:

(1.5)
$$\tau = \tau^* + \frac{1}{N_{g,\alpha}}\operatorname{div} V$$

where τ^* is a constant and V^a is a vector field. In this way, τ^* plays the same role for volumetric degrees of freedom that σ_{ab} plays for conformal degrees of freedom.

- Let $\mathcal{M}$ be the space of metrics. Instantaneous motion in $\mathcal{M}/\mathcal{D}_0$ can be decomposed into three components: conformal, volumetric, and drift. The decomposition depends on the choice of a densitized lapse, and the conformal and volumetric components of this decomposition agree with the notions of conformal and volumetric velocity just discussed. A drift is an instantaneous motion in $\mathcal{M}/\mathcal{D}_0$ that preserves both conformal class (modulo diffeomorphisms) and volume form (modulo diffeomorphisms). Although a metric is uniquely determined by its conformal class and volume form, there are nontrivial drifts, and indeed the drifts at a metric g_{ab} can be identified with the space of vector fields on M, modulo the divergence-free vector fields and conformal Killing fields of g_{ab}. Section 9 contains basic results concerning drifts.

- It is well known that solutions of the momentum constraint correspond to the momenta of motion in $\mathcal{M}/\mathcal{D}_0$. In Section 10 we show that after selection of a densitized lapse, such momenta can be decomposed into three components: conformal, volumetric, and drift. The conformal and volumetric momenta are the quantities identified previously, and a drift momentum at g_{ab} can be described by a pair of linked drifts $(\mathbf{W}, \mathbf{V})$. The drifts $\mathbf{W}$ and $\mathbf{V}$ can be represented by vector fields W^a and V^a solving the drift equation

(1.6)
$$\operatorname{div}_g\left[\frac{1}{2N_{g,\alpha}}\mathbf{L}_g W\right] = \kappa\, \mathrm{d}\left[\frac{1}{N_{g,\alpha}}\operatorname{div}_g V\right]$$

where $\mathbf{L}_g$ is the conformal Killing operator of g_{ab}. Equation (1.6) has a remarkable symmetry between the conformal and volumetric parameters W^a and V^a. We can specify V^a and solve for W^a, in which case we can add an arbitrary divergence-free vector field to V^a, but equation (1.6) is only solvable after adding a specific choice of conformal Killing field to V^a. Conversely, we can specify W^a and solve for V^a, in which case we can add an arbitrary conformal Killing field to W^a and we must additionally add a particular divergence-free vector field to ensure that equation (1.6) is solvable. So a pair $(\mathbf{W}, \mathbf{V})$ representing a drift momentum is uniquely determined by either its conformal drift $\mathbf{W}$ or its volumetric drift $\mathbf{V}$. Section 10 describes these results in detail.

- The CMC solutions of the constraint equations are the solutions with zero drift momentum.

- Although solutions of the momentum constraint correspond to momenta in $\mathcal{M}/\mathcal{D}_0$, solutions of the constraint equations are not well described in terms of velocities in $\mathcal{M}/\mathcal{D}_0$. There exist distinct solutions of the vacuum constraint equations, generating distinct spacetimes, that nevertheless have identical geometries and velocities in $\mathcal{M}/\mathcal{D}_0$. This phenomenon occurs because the drift momentum of a pair $(\mathbf{W}, \mathbf{V})$ corresponds to a velocity $\mathbf{V} - \mathbf{W}$ in $\mathcal{M}/\mathcal{D}_0$, and this can vanish even if $\mathbf{W}$ and $\mathbf{V}$ do not. Either the conformal drift $\mathbf{W}$ or the volumetric drift $\mathbf{V}$ can be taken as a parameter of motion that determines the other, but using the difference $\mathbf{V} - \mathbf{W}$ leads to non-uniqueness. Section 11 describes how we can take either factor $\mathbf{W}$ or $\mathbf{V}$ to be the drift velocity corresponding to drift momentum, and that in either case we can construct a Lagrangian (conformal or volumetric drift kinetic energy) whose Legendre transformation connects drift velocity and momentum.

- The kinetic energy term of the ADM Lagrangian, when restricted to solutions of the momentum constraint, decomposes into three independent terms corresponding to conformal, volumetric, and drift kinetic energy.

These main results effectively comprise a study of the interaction of densitized lapses with the momentum constraint. In Section 12 we then propose variations of the conformal method where the parameters include a conformal class, a conformal momentum, a volumetric momentum, and a vector field determining a drift momentum. There is more than one way

to do this, however, and we present three candidates that each include the CMC conformal method as a special case. The resulting equations are technically more challenging than those of the standard conformal method, and we therefore postpone their analysis for future work. Although we hope that features of the momentum constraint documented here will assist those efforts, it remains to be seen the extent to which these drift parameterizations, or perhaps some variation, outperform the conformal method. Regardless, drifts have the potential to play a role in understanding any variation of the CMC conformal method. For example, the one-parameter families discovered for the standard conformal method in [34] all have the property that they have zero conformal momentum and zero volumetric momentum, but not-necessarily zero drift momentum. Moreover, drifts are related to past difficulties in applying the standard conformal method to construct non-CMC solutions of the constraints with metrics having nontrivial conformal Killing fields, and we discuss in Section 12 how the standard conformal method might be adjusted to account for conformal Killing fields.

Our main goal is to find well-motivated alternatives to the conformal method, and in order minimize distraction we work under hypotheses that reduce the number of technical details. In particular, we work only on oriented compact manifolds, and we work only in the smooth category. Smoothness comes with the attendant complexity of Fréchet manifolds, and we have emphasized linear algebra over topology when working with their tangent spaces. For example, direct sums and isomorphisms are always meant in the sense of linear algebra, although in many cases it is obvious that the subspaces involved are closed and the maps involved are at least continuous. We adopt an intuitive (but precise) approach to working with tangent and cotangent spaces to infinite dimensional spaces such as $\mathcal{C}$ and $\mathcal{C}/\mathcal{D}_0$. Sections 1.1, 3 and 6 contain the related definitions and details, and it is important to note that the simplicity of our approach comes with the penalty that objects such as $T^*\mathcal{C}/\mathcal{D}_0$ appearing in the theorems are to be understood rather formally. We also adopt some helpful but non-standard notations regarding the trace/trace-free decomposition of $T\mathcal{M}$ and its interaction with the numerous quotient spaces we work with. Again, Sections 1.1, 3 and 6 contain the details.

1.1. Notation and conventions

Throughout we assume that M is a smooth, compact, connected, oriented n-manifold with $n \geq 3$. The set of smooth functions on M is $C^\infty(M)$ and if E is a smooth bundle over M, then $C^\infty(M, E)$ is the set of smooth sections

of E. We write TM and T^*M for the tangent and cotangent bundles of M, S^2M and S_2M for the bundles of symmetric $(2,0)$ and $(0,2)$ tensors, and $\Lambda^n M$ for the bundle of n-forms. All tensors are assumed to be smooth unless otherwise noted; in Section 10 we work with L^2 Sobolev spaces $W^{k,2}$ where $k \in \mathbb{Z}$ denotes the order of differentiability.

We have the following sets of interest:

$\mathcal{M}$, the smooth metrics on M,

$\mathcal{C}$, the conformal classes of smooth metrics,

$\mathcal{V}$, the smooth volume forms (i.e., the positively oriented elements of $C^\infty(M, \Lambda^n)$),

$\mathcal{K}$, the space $C^\infty(M, S_2M)$ of second fundamental forms.

Three constants derived from the dimension n arise sufficiently frequently that we use the notation

$$(1.7) \qquad q = \frac{2n}{n-2} \qquad \kappa = \frac{n-1}{n} \qquad a = 2\kappa q = \frac{4(n-1)}{n-2}.$$

We also use the symbol a as an abstract index, but there should be no confusion since the constant a defined above is never used as an exponent.

1.1.1. The space $\mathcal{M}$ of metrics.   The set $\mathcal{M}$ of smooth metrics over M is the open subset of positive definite elements of the Fréchet vector space $C^\infty(M, S_2M)$. Hence $\mathcal{M}$ is a Fréchet manifold, and if $g_{ab} \in \mathcal{M}$, then $T_g\mathcal{M} = C^\infty(M, S_2M)$. Note that we use abstract index notation in this paper with the understanding that indices can be dropped freely if they clutter notation or are otherwise intrusive.

Let $g_{ab} \in \mathcal{M}$. The dual space $(T_g\mathcal{M})^*$ contains a wide variety of distributions, and it will be convenient to work with a smaller subspace. Recalling that M is oriented, we define

$$(1.8) \qquad\qquad T_g^*\mathcal{M} = C^\infty(M, S^2M \otimes \Lambda^n M).$$

If $h_{ab} \in T_g(M)$ and $F^{ab}\omega \in T_g^*\mathcal{M}$, then F^{ab} acts on h_{ab} via

$$(1.9) \qquad\qquad \left\langle F^{ab}\omega, h_{ab} \right\rangle = \int_M F^{ab} h_{ab}\, \omega.$$

One readily verifies that with this action, $T_g^*\mathcal{M} \subseteq (T_g\mathcal{M})^*$.

There is a natural L^2 metric $\mathcal{G}$ on $\mathcal{M}$ defined by

$$\mathcal{G}(h_{ab}, \widehat{h}_{ab}) = \int_M \left\langle h, \widehat{h} \right\rangle_g \omega_g$$

for all h_{ab} and $\widehat{h}_{ab} \in T_g\mathcal{M}$. Here and elsewhere ω_g is the oriented volume form of g_{ab}. The metric $\mathcal{G}$ determines a map from $T_g\mathcal{M}$ to $(T_g\mathcal{M})^*$ defined by

(1.10) $h_{ab} \mapsto \mathcal{G}(h_{ab}, \cdot)$

and it is easy to see that $T_g^*\mathcal{M}$ is the image of $T_g\mathcal{M}$ under this map. Thus we have a natural identification of $T_g\mathcal{M}$ with $T_g^*\mathcal{M}$.

The trace-free and pure-trace subspaces of $T_g\mathcal{M}$ play an important role in this paper and it will be helpful to have special notation to work with them. Suppose β is a function and u_{ab} is symmetric and trace-free with respect to g_{ab}, We define

(1.11) $(g_{ab}; u_{ab}, \beta) = u_{ab} + \frac{2}{n}\beta\, g_{ab} \in T_g\mathcal{M}.$

It is easy to see that any $h_{ab} \in T_g\mathcal{M}$ admits a unique decomposition of the form (1.11). Similarly, if f is a function and A^{ab} is symmetric and trace-free with respect to g_{ab} we define

(1.12) $(g_{ab}; A^{ab}, f)^* = (A^{ab} + \frac{1}{2}fg^{ab})\omega_g \in T_g^*\mathcal{M}.$

Note that

(1.13) $\left\langle (g_{ab}; A^{ab}, f)^*, (g_{ab}; u_{ab}, \beta) \right\rangle = \int \left[A^{ab}u_{ab} + f\beta \right] \omega_g.$

It is sometimes convenient to work with elements of $T_g^*\mathcal{M}$ represented by covariant tensors, so if B_{ab} is symmetric and trace-free with respect to g_{ab} we define

(1.14) $(g_{ab}; B_{ab}, f)^* = (g_{ab}; g^{ac}g^{bd}B_{bd}, f)^*.$

1.1.2. The space $\mathcal{M}/\mathcal{D}_0$ of geometries. Let $\mathcal{D}_0$ be connected component of the identity e in the group of smooth diffeomorphisms from M to M. Then $\mathcal{M}/\mathcal{D}_0$ is the set of equivalence classes of metrics where g_{ab} is related to $\widehat{g}_{ab}$ if there exists $\Phi \in \mathcal{D}_0$ with $\widehat{g}_{ab} = \Phi^* g_{ab}$. We write $\{g_{ab}\}$ for the equivalence class of g_{ab} in $\mathcal{M}/\mathcal{D}_0$.

Recall that $\mathcal{D}_0$ is a Fréchet manifold and $T_e\mathcal{D}_0 = C^\infty(M, TM)$ [27]. Suppose Φ_t is a path of diffeomorphisms with $\Phi_0 = e$, and let X^a be its infinitesimal generator. Given a metric g_{ab}, the path of metrics $\gamma_{ab}(t) = \Phi_t^* g_{ab}$ remains in $\{g_{ab}\}$ and satisfies

$$(1.15) \qquad \gamma'_{ab}(0) = (\text{Lie}_g X)_{ab} = \nabla_a X_b + \nabla_b X_a$$

where ∇ is the Levi-Civita connection of g_{ab}. Since γ_{ab} is stationary in $\mathcal{M}/\mathcal{D}_0$, the directions $\text{Im}\,\text{Lie}_g \subseteq T_g\mathcal{M}$ become null directions in $\mathcal{M}/\mathcal{D}_0$, which motivates the formal definition

$$(1.16) \qquad T_g\mathcal{M}/\mathcal{D}_0 = T_g\mathcal{M}/\text{Im}\,\text{Lie}_g .$$

By working formally and infinitesimally, we avoid details concerning the structure of $\mathcal{M}/\mathcal{D}_0$ as a stratified space. One can often think of $T_g\mathcal{M}/\mathcal{D}_0$ as a proxy for an actual tangent space $T_{\{g\}}\mathcal{M}/\mathcal{D}_0$ that we have not defined [16]; doing so requires demonstrating that $\text{Im}\,\text{Lie}_g$ is a closed subspace.

Let $(g_{ab};\, u_{ab}, \beta) \in T_g\mathcal{M}$. We continue the practice of denoting quotients by $\mathcal{D}_0$ using curly braces and define

$$(1.17) \qquad \{g_{ab};\, u_{ab}, \beta\} = (g_{ab};\, u_{ab}, \beta) + \text{Im}\,\text{Lie}_g \in T_g\mathcal{M}/\mathcal{D}_0.$$

It is helpful to think of the projection

$$(1.18) \qquad (g_{ab};\, u_{ab}, \beta) \mapsto \{g_{ab};\, u_{ab}, \beta\}$$

as the pushforward from $T_g\mathcal{M}$ to $T_g\mathcal{M}/\mathcal{D}_0$.

The **conformal Killing operator** of a metric g_{ab} acts on vector fields X^a by

$$(1.19) \qquad (\mathbf{L}_g X)_{ab} = (\text{Lie}_g X)_{ab} - \frac{2}{n}\,\text{div}_g X$$

where $\text{div}_g X = \nabla_a X^a$. An element of the kernel of $\mathbf{L}_g$ is a **conformal Killing field**. Note that in trace/trace-free notation

$$(1.20) \qquad (\text{Lie}_g X)_{ab} = (g_{ab};\, (\mathbf{L}_g X)_{ab}, \text{div}_g X)$$

and hence

$$(1.21) \qquad \{g_{ab};\, (\mathbf{L}_g X)_{ab}, \text{div}_g X\} = 0.$$

Since $T_g\mathcal{M}/\mathcal{D}_0$ is a quotient of $T_g\mathcal{M}$ by $\mathrm{Im}\,\mathrm{Lie}_g$ we formally define

(1.22) $$T_g^*\mathcal{M}/\mathcal{D}_0 = (\mathrm{Im}\,\mathrm{Lie}_g)^\perp = \left\{ A \in T_g^*\mathcal{M} : A|_{\mathrm{Im}\,\mathrm{Lie}_g} = 0 \right\}.$$

Consequently, $T_g^*\mathcal{M}/\mathcal{D}_0 \subseteq T_g^*\mathcal{M}$ and an integration by parts exercise shows that $F^{ab}\omega_g \in T_g^*\mathcal{M}/\mathcal{D}_0$ if and only if

$$(\mathrm{div}_g F)^b = \nabla_a F^{ab} = 0.$$

If $(g_{ab};\ A^{ab}, f)^* \in T_g\mathcal{M}$, then the divergence-free condition is

(1.23) $$\nabla_a A^{ab} + \frac{1}{2}\nabla^b f = 0$$

and we write

(1.24) $$\{g_{ab};\ A^{ab}, f\}^*$$

for any $(g_{ab};\ A^{ab}, f)^* \in T_g^*\mathcal{M}$ that satisfies equation (1.23). Elements $\mathbf{F} \in T_g^*\mathcal{M}/\mathcal{D}_0$ are functionals on $T_g\mathcal{M}/\mathcal{D}_0$ according to to the rule

(1.25) $$\langle \mathbf{F}, \{\mathbf{h}\} \rangle = \langle \mathbf{F}, \mathbf{h} \rangle,$$

and we see that the natural embedding $T_g^*\mathcal{M}/\mathcal{D}_0 \hookrightarrow T_g^*\mathcal{M}$ is the pullback associated with the pushforward (1.18).

2. The ADM Lagrangian with densitized lapse

In the traditional approach to the ADM $n + 1$ decomposition of general relativity, on each slice of constant coordinate time we select a positive function N (the lapse) and a vector field X^a (the shift) that describe the layout of a coordinate system in spacetime. A metric and second fundamental form $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$ determine the ADM momentum

(2.1) $$\Pi^{ab} = \left[K^{ab} - \mathrm{tr}_g K g^{ab} \right] \omega_g \in T_g^*\mathcal{M}$$

and also determine, in conjunction with the lapse and shift, the ADM velocity

(2.2) $$\dot{g}_{ab} = 2N K_{ab} + (\mathrm{Lie}_g X)_{ab} \in T_g\mathcal{M}.$$

From these maps we obtain an isomorphism $i_{N,X^a} : T\mathcal{M} \to T^*\mathcal{M}$

(2.3)

$$
\begin{array}{ccc}
 & \mathcal{M} \times \mathcal{K} & \\
 {\scriptstyle (N,X^a)}\nearrow & & \nwarrow \\
 T\,\mathcal{M} \xleftarrow{\hspace{3cm}} & & T^*\,\mathcal{M} \\
 & {\scriptstyle i_{N,X^a}} &
\end{array}
$$

where the notation (N, X^a) denotes a nameless function that depends on the lapse and shift. Recalling the definition of a Legendre transformation from, e.g., [29] that relates velocities and momenta, the map i_{N,X^a} is the Legendre transformation associated with the ADM Lagrangian

(2.4) $L_{\mathrm{ADM}}(g_{ab}, \dot{g}_{ab};\ N, X^a)$

$$
= \int_M N \left(R_g + \frac{1}{4N^2} |\dot{g} - \mathrm{Lie}_g X|_g^2 - \frac{1}{4N^2} (\mathrm{tr}_g \dot{g} - 2\,\mathrm{div}_g X)^2 \right) \omega_g.
$$

Writing $\dot{g}_{ab} = (g_{ab};\ u_{ab}, \beta)$ in trace/trace-free notation we have

(2.5) $L_{\mathrm{ADM}}(g_{ab}, u_{ab}, \beta;\ N, X^a)$

$$
= \int_M N \left(R_g + \frac{1}{4N^2} |u - \mathbf{L}_g X|_g^2 - \frac{\kappa}{N^2} (\beta - \mathrm{div}_g X)^2 \right) \omega_g
$$

and the Legendre transformation can be written
(2.6)
$$
i_{N,X^a}(g_{ab};\ u_{ab}, \beta) = \left(g_{ab};\ \frac{1}{2N} (u_{ab} - (\mathbf{L}_g X)_{ab}), -2\kappa\frac{1}{N} (\beta - \mathrm{div}_g X) \right)^*
$$

with inverse

(2.7) $i_{N,X^a}^{-1}((g_{ab};\ A_{ab}, f)^*) = \left(g_{ab};\ 2NA_{ab} + (\mathbf{L}_g X)_{ab}, -\frac{N}{2\kappa}f + \mathrm{div}_g X \right).$

It will also be helpful to have trace/trace-free expressions for the conversion from a second fundamental form to ADM velocity or momentum. If $K_{ab} = A_{ab} + (\tau/n)g_{ab}$ where A_{ab} is trace-free, then the ADM velocity is

(2.8) $(g_{ab};\ 2NA_{ab} + (\mathbf{L}_g X)_{ab}, N\tau + \mathrm{div}_g X)$

and the ADM momentum is

(2.9) $(g_{ab};\ A_{ab}, -2\kappa\tau)^*.$

We will work with a modified form of the ADM Lagrangian where the lapse is prescribed indirectly using a construct called a densitized lapse. Densitized lapses appear in various contexts in general relativity [10] [2] [38] and were introduced to the constraint equations in York's conformal thin sandwich method [40]. As mentioned in the introduction, we will represent a densitized lapse by a choice of volume form α, and given a metric g_{ab}, the lapse associated with g_{ab} and α is

$$(2.10) \qquad\qquad N_{g,\alpha} = \frac{\omega_g}{\alpha}.$$

Note that if $\widehat{g}_{ab} = \phi^{q-2} g_{ab}$ for some conformal factor ϕ then

$$(2.11) \qquad\qquad N_{\widehat{g},\alpha} = \phi^q N_{g,\alpha}.$$

We will call α a **lapse form**.

Replacing N with $N_{g,\alpha}$ and using equation (2.10) to rewrite ω_g in terms of α, the Lagrangian (2.5) becomes

$$
\begin{aligned}
(2.12) \qquad & L_{\mathrm{ADM}'}(g_{ab}, u_{ab}, \beta;\ \alpha, X^a) \\
& = \int_M \left(N_{g,\alpha}^2 R_g + \frac{1}{4}|u - \mathbf{L}_g X|_g^2 - \kappa(\beta - \mathrm{div}_g\, X)^2 \right) \alpha.
\end{aligned}
$$

For the remainder of this paper we work with the densitized-lapse ADM Lagrangian (2.12). The Legendre transformation associated with this Lagrangian is the standard transformation (2.6) with the substitution $N = N_{g,\alpha}$ and we have the commutative diagram

$$(2.13)$$

The distinction between the standard and densitized-lapse Legendre transformations is perhaps subtle. Given a metric g_{ab} and a lapse form α, there always exists a lapse N such that the maps i_{N,X^a} and i_{α,X^a} agree as maps from $T_g\mathcal{M}$ to $T_g^*\mathcal{M}$. But if we consider a second metric $\widehat{g}_{ab}$ with volume form $\omega_{\widehat{g}}$ different from ω_g, then the two Legendre transformations as maps from $T_{\widehat{g}}\mathcal{M}$ to $T_{\widehat{g}}^*\mathcal{M}$ are no longer the same. This difference is important when thinking of the Legendre transformation as a map between the total bundles $T\mathcal{M}$ and $T^*\mathcal{M}$, and we will find that the densitized lapse is particularly compatible with the product structure $\mathcal{M} = \mathcal{C} \times \mathcal{V}$. For example,

given a lapse form α, we will be able to assign a notion of conformal velocity, momentum, and kinetic energy measured by α to each curve in $\mathcal{M}$ in such a way that if two curves in $\mathcal{M}$ descend to the same curve in $\mathcal{C}$ or even $\mathcal{C}/\mathcal{D}_0$, then these conformal quantities are preserved. The next several sections make these ideas precise, and we start by recalling the definitions of tangent and cotangent spaces of $\mathcal{C}$ and $\mathcal{C}/\mathcal{D}_0$ from [33].

3. Conformal tangent and cotangent spaces

If $g_{ab} \in \mathcal{M}$, we write $[g_{ab}]$ for its conformal class, and we use bold type to denote a conformal class when we do not wish to emphasize any particular representative. So $[g_{ab}] = \mathbf{g} \in \mathcal{C}$ if and only if $g_{ab} \in \mathbf{g}$. By convention we use conformal transformations of the form $\widehat{g}_{ab} = \phi^{q-2} g_{ab}$ since the exponent $q - 2$ leads to a simple conformal transformation law for scalar curvature:

$$(3.1) \qquad R_{\widehat{g}} = \phi^{1-q}(-a\,\Delta_g\,\phi + R_g\phi).$$

Following [33], if $\mathbf{g} \in \mathcal{C}$ we define $T_{\mathbf{g}}\mathcal{C}$ to be the set of equivalence classes of pairs $(g_{ab};\, u_{ab})$ where $g_{ab} \in \mathbf{g}$, u_{ab} is trace-free with respect to g_{ab}, and where

$$(3.2) \qquad (g_{ab};\, u_{ab}) \sim (\phi^{q-2} g_{ab};\, \phi^{q-2} u_{ab}).$$

The trace-free condition arises because we identify $\mathcal{C}$ with the set of metrics with a fixed volume form, and the equivalence relation reflects the arbitrary choice of volume form. We will write

$$(3.3) \qquad [g_{ab};\, u_{ab}]$$

for the element of $T_{\mathbf{g}}\mathcal{C}$ determined by $(g_{ab};\, u_{ab})$, and we will write $\mathbf{u}$ for a conformal tangent vector when we do not wish to emphasize a particular representative. At $\mathbf{g} \in \mathcal{C}$, we define the conformal killing operator $\mathbf{L_g}$ acting on a vector field X^a

$$(3.4) \qquad \mathbf{L_g}\,X^a = [g_{ab};\, (\mathbf{L}_g\,X)_{ab}]$$

where g_{ab} is any representative of $\mathbf{g}$; the conformal transformation law $\mathbf{L}_{\widehat{g}} = \phi^{q-2}\,\mathbf{L}_g$ if $\widehat{g}_{ab} = \phi^{q-2} g_{ab}$ ensures that $\mathbf{L_g}$ is well-defined.

The cotangent space $T_{\mathbf{g}}^*\mathcal{C}$ is also defined as a set of equivalence classes of pairs $(g_{ab};\, A_{ab})$ where $g_{ab} \in \mathbf{g}$, $g^{ab}A_{ab} = 0$, but the equivalence relation

differs. Now

(3.5) $(g_{ab}; A_{ab}) \sim (\phi^{q-2} g_{ab}; \phi^{-2} A_{ab}).$

and we write

(3.6) $[g_{ab}; A_{ab}]^*$

for the element of $T^*_{[g]}\mathcal{C}$ determined by $(g_{ab}; A_{ab})$. As before, we use bold face when no representative is preferred. If $\mathbf{u} \in T_{\mathbf{g}}\mathcal{C}$ and $\mathbf{A} \in T^*_{\mathbf{g}}\mathcal{C}$ we define

(3.7) $\langle \mathbf{A}, \mathbf{u} \rangle = \int_M \langle A, u \rangle_g \, \omega_g$

where g_{ab} is any representative of $\mathbf{g}$ and where u_{ab} and A_{ab} are the representatives such that

(3.8) $\mathbf{A} = [g_{ab}; A_{ab}]^* \quad \text{and} \quad \mathbf{u} = [g_{ab}; u_{ab}].$

The equivalence relations for conformal tangent and cotangent vectors ensures that this action is well defined.

   We have a natural map from $T_g \mathcal{M}$ to $T_{[g]}\mathcal{C}$ given by

(3.9) $(g_{ab}; u_{ab}, \beta) \mapsto [g_{ab}; u_{ab}]$

that can be thought of as the pushforward. From equation (3.7) we have the corresponding pullback $T^*_{[g]}\mathcal{C} \to T^*_g \mathcal{M}$ which can be written in the notation of equation (1.14) as

(3.10) $\mathbf{A} \mapsto (g_{ab}; A_{ab}, 0)^*$

if $\mathbf{A} = [g_{ab}; A_{ab}]^*$.

   Sitting below the space of conformal classes is the space $\mathcal{C}/\mathcal{D}_0$ of conformal geometries. Two conformal classes $\mathbf{g}$ and $\widehat{\mathbf{g}}$ are equivalent at the level of conformal geometries if there is a diffeomorphism $\Phi \in \mathcal{D}_0$ such that $\Phi^* \mathbf{g} = \widehat{\mathbf{g}}$. Concretely, two metrics g_{ab} and $\widehat{g}_{ab}$ determine the same conformal geometry if there is a diffeomorphism $\Phi \in \mathcal{D}_0$ and a smooth positive function ϕ such that $\widehat{g}_{ab} = \phi^{q-2} \Phi^* g_{ab}$. We write $\{\mathbf{g}\}$ for the conformal geometry determined by the conformal class $\mathbf{g}$.

   In defining the tangent spaces to $\mathcal{M}/\mathcal{D}_0$ we quotiented by the directions $\operatorname{Im} \operatorname{Lie}_g$. The pushforward of $\operatorname{Im} \operatorname{Lie}_g$ into $T_{[g]}\mathcal{C}$ is $\operatorname{Im} \mathbf{L}_{[g]}$ and we therefore

formally define for any $\mathbf{g} \in \mathcal{C}$

$$(3.11) \qquad T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0 = (T_{\mathbf{g}}\mathcal{C})/\operatorname{Im}\mathbf{L_g}.$$

We write $\{\mathbf{u}\}$ for the equivalence class $\mathbf{u} + \operatorname{Im}\mathbf{L_g} \in T_{\mathbf{g}}\,\mathcal{C}/\mathcal{D}_0$. Correspondingly, we define

$$(3.12) \qquad T_{\mathbf{g}}^*\mathcal{C}/\mathcal{D}_0 = (\operatorname{Im}\mathbf{L_g})^{\perp} = \{\mathbf{A} \in T_{\mathbf{g}}^*\mathcal{C} : \mathbf{A}|_{\operatorname{Im}\mathbf{L_g}} = 0\}$$

and elements $\boldsymbol{\sigma} \in T^*\mathcal{C}/\mathcal{D}_0$ are then well-defined functionals on $T\mathcal{C}/\mathcal{D}_0$ according to

$$(3.13) \qquad \langle \boldsymbol{\sigma}, \{\mathbf{u}\} \rangle = \langle \boldsymbol{\sigma}, \mathbf{u} \rangle.$$

An integration by parts exercise shows that $[g_{ab};\, \sigma_{ab}]^* \in T_{\mathbf{g}}^*\mathcal{C}$ if and only if σ_{ab} is divergence-free with respect to g_{ab}. Since σ_{ab} is trace-free as well, it is a so-called transverse traceless (TT) tensor. We will write $\{g_{ab};\, \sigma_{ab}\}^*$ if we wish to emphasize that $[g_{ab};\, \sigma_{ab}]^*$ belongs to $T^*\mathcal{C}/\mathcal{D}_0$.
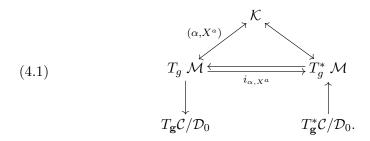
We define the pushforward $T_{\mathbf{g}}\mathcal{C} \to T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0$ by

$$(3.14) \qquad \mathbf{u} \mapsto \{\mathbf{u}\} = \mathbf{u} + \operatorname{Im}\mathbf{L_g}.$$

Its corresponding pullback is the natural embedding $T_{\mathbf{g}}^*\mathcal{C}/\mathcal{D}_0 \hookrightarrow T_{\mathbf{g}}^*\mathcal{C}$.

4. Conformal velocity, momentum and kinetic energy

Let $\mathbf{g}$ be a conformal class and let g_{ab} be any representative. Given a lapse form α and a shift X^a we can combine diagram (2.13) with the pushforward and pullback maps described in the previous section to obtain the following diagram at the fixed metric g_{ab}:

$$(4.1)$$

The principal goal of this section is to show that the Legendre transformation i_{α,X^a} descends to an isomorphism $j_\alpha^{\mathcal{C}} : T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0 \to T_{\mathbf{g}}^*\mathcal{C}/\mathcal{D}_0$ that such that for

every $g_{ab} \in \mathbf{g}$, the diagram

(4.2)

$$
\begin{array}{ccc}
 & \mathcal{K} & \\
\scriptstyle(\alpha, X^a) \nearrow & & \nwarrow \\
T_g\mathcal{M} & \xleftrightarrow{\;\;i_{\alpha,X^a}\;\;} & T_g^*\mathcal{M} \\
\downarrow & & \uparrow \\
T_\mathbf{g}\mathcal{C}/\mathcal{D}_0 & \xleftrightarrow{\;\;j_\alpha^\mathcal{C}\;\;} & T_\mathbf{g}^*\mathcal{C}/\mathcal{D}_0
\end{array}
$$

nearly commutes. The failure of commutativity comes from the fact that the projection $T_g\mathcal{M} \to T_\mathbf{g}\mathcal{C}/\mathcal{D}_0$ loses information that cannot be recovered. Instead, we will see that traversing the lower loop of diagram (4.2) when starting from the middle row is a projection.

With the isomorphism $j_\alpha^\mathcal{C}$ in hand, we assign to a pair $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$ a conformal velocity and momentum as follows. The conformal velocity is obtained by forming diagram (4.2) for g_{ab} and then mapping K_{ab} through the left-hand side of the diagram starting from $\mathcal{K}$ to obtain a velocity in $T_{[g]}\mathcal{C}/\mathcal{D}_0$. Note that this is a velocity modulo diffeomorphsims, and strictly speaking it is a 'conformal geometric velocity'. The conformal momentum is constructed from the conformal velocity by applying $j_\alpha^\mathcal{C}$. In this sense, $j_\alpha^\mathcal{C}$ behaves like a Legendre transformation, and we show in Proposition 4.9 that it arises from a Lagrangian on $T\,\mathcal{C}/\mathcal{D}_0$ that we will call conformal kinetic energy.

To construct $j_\alpha^\mathcal{C}$ it turns out that it is easiest to construct $(j_\alpha^\mathcal{C})^{-1}$ first. Diagram (4.1) defines a map from $T_\mathbf{g}^*\mathcal{C}/\mathcal{D}_0$ to $T_\mathbf{g}\mathcal{C}/\mathcal{D}_0$ given by traveling from the lower-right corner to the lower left corner. In principle this map depends on α, X^a, and the choice $g_{ab} \in \mathbf{g}$, and we provisionally call this map $j^{-1}_{\alpha, X^a, g}$. The first order of business is to show that this map is independent of X^a (because we are reducing to a quotient modulo $\mathcal{D}_0$) and g_{ab} (because we are using a densitized lapse) to obtain a map j_α^{-1}. We then show that j_α^{-1} is, as the notation suggests, the inverse of a map $j_\alpha^\mathcal{C}$.

Lemma 4.1. *Let $\mathbf{g} \in \mathcal{C}$ and let α be a fixed lapse form. For any pair of shifts X^a and $\widehat{X}^a$, and any pair of representatives g_{ab} and $\widehat{g}_{ab}$ of $\mathbf{g}$,*

(4.3)
$$
j^{-1}_{\alpha, X^a, g} = j^{-1}_{\alpha, \widehat{X}, \widehat{g}}
$$

and we call the common map j_α^{-1}. Moreover, for all $\boldsymbol{\sigma} \in T_{\mathbf{g}}^* \mathcal{C}/\mathcal{M}$,

$$(4.4) \qquad j_\alpha^{-1}(\boldsymbol{\sigma}) = \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab}\}$$

where σ_{ab} is the representative of $\boldsymbol{\sigma}$ with respect to g_{ab}.

Proof. Let $\boldsymbol{\sigma} \in T_{\mathbf{g}}^* \mathcal{C}/\mathcal{D}_0$. To compute $j_{\alpha,X^a,g}^{-1}(\boldsymbol{\sigma})$, let σ_{ab} be the representative of $\boldsymbol{\sigma}$ with respect to g_{ab}. From equation (3.10) the pullback of $\boldsymbol{\sigma}$ to $T_g\mathcal{M}$ is $(g_{ab};\ \sigma_{ab}, 0)^*$. Applying i_{α,X^a}^{-1} from equation (2.7) we arrive at $(g_{ab};\ 2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g X)_{ab}, 0)$. Finally, we apply the pushforward from equation (3.9) to conclude

$$(4.5) \qquad j_{\alpha,X^a,g}^{-1}(\boldsymbol{\sigma}) = \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g X)_{ab}\} = \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab}\}$$

since $\{g_{ab};\ (\mathbf{L}_g X)_{ab}\} = 0$.

At this stage, it is clear that j_{α,g,X^a}^{-1} is independent of the shift X^a. Now suppose $\widehat{g}_{ab}$ is another representative of $\mathbf{g}$ with $\widehat{g}_{ab} = \phi^{q-2}g_{ab}$ for some conformal factor ϕ. The representative of $\boldsymbol{\sigma}$ with respect to $\widehat{g}_{ab}$ is $\widehat{\sigma}_{ab} = \phi^{-2}\sigma_{ab}$ and we have $N_{\alpha,\widehat{g}} = \phi^q N_{\alpha,g}$. Recalling equation (3.2) we find

$$(4.6) \qquad \begin{aligned} j_{\alpha,\widehat{g},X^a}^{-1}(\boldsymbol{\sigma}) &= \{\widehat{g}_{ab};\ 2N_{\alpha,\widehat{g}}\widehat{\sigma}_{ab}\} \\ &= \{\phi^{q-2}g_{ab};\ \phi^{q-2}2N_{\alpha,g}\sigma_{ab}\} \\ &= \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab}\} \\ &= j_{\alpha,g,X^a}^{-1}(\boldsymbol{\sigma}). \end{aligned}$$

Hence $j_{\alpha,X^a,g}^{-1} = j_{\alpha,X^a,\widehat{g}}^{-1}$ as claimed, and equation (4.4) follows from equation (4.5). $\qquad\square$

To show j_α^{-1} is the inverse of a function $j_\alpha^{\mathcal{C}}$ we require York splitting [39], which we use in the following form.

Proposition 4.2 (York Splitting). *Let $g_{ab} \in \mathcal{M}$ and let $N > 0$ be a lapse. If A_{ab} is symmetric and trace-free, then there is a g_{ab}-TT tensor σ_{ab} and a vector field W^a such that*

$$(4.7) \qquad A_{ab} = \sigma_{ab} + \frac{1}{2N}(\mathbf{L}_g W)_{ab}.$$

This decomposition is unique up to addition of a conformal Killing field to W^a.

Equivalently, if u_{ab} is symmetric and trace-free, there is a unique g_{ab}-TT tensor σ_{ab} and a vector field W^a, unique up to addition of a conformal Killing field, such that

$$(4.8) \qquad u_{ab} = 2N\sigma_{ab} + (\mathbf{L}_g W)_{ab}.$$

When $N \equiv 1/2$, Proposition 4.2 is classical York splitting, and the result for arbitrary lapses is equivalent to classical York splitting [33]; see also [36]. The decomposition from Proposition 4.2 defines a projection from symmetric trace-free tensors to transverse-traceless tensors and we introduce the following notation for it.

Definition 4.3. Let g_{ab} be a metric and let α be a lapse form. Given a symmetric, trace-free tensor A_{ab}, its **York projection** is

$$(4.9) \qquad Y_{g,\alpha}(A_{ab}) = \sigma_{ab}$$

where σ_{ab} is the unique g_{ab}-TT tensor such that equation (4.7) holds with $N = N_{g,\alpha}$.

We now show that the formal notation j_α^{-1} is justified by constructing an inverse $j_\alpha^{\mathcal{C}}$.

Lemma 4.4. For all $\mathbf{g} \in \mathcal{C}$, $j_\alpha^{-1} : T_{\mathbf{g}}^*\mathcal{C}/\mathcal{D}_0 \to T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0$ is a linear isomorphism. If $\{\mathbf{u}\} \in T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0$ then $j_\alpha^{\mathcal{C}}(\{\mathbf{u}\})$ is computed as follows. Pick any $g_{ab} \in \mathbf{g}$ and pick any u_{ab} such that

$$(4.10) \qquad \{\mathbf{u}\} = \{g_{ab};\ u_{ab}\}.$$

Let $\sigma_{ab} = Y_{g,\alpha}(1/(2N_{g,\alpha})u_{ab})$, so σ_{ab} is the unique g_{ab}-TT tensor such that

$$(4.11) \qquad u_{ab} = 2N_{g,\alpha}\sigma_{ab} + (\mathbf{L}_g W)_{ab}$$

for some vector field W^a. Then

$$(4.12) \qquad j_\alpha^{\mathcal{C}}(\{\mathbf{u}\}) = \{g_{ab};\ \sigma_{ab}\}^*.$$

Proof. To see that j_α^{-1} is injective, suppose $j_\alpha^{-1}(\boldsymbol{\sigma}) = 0$ for some $\boldsymbol{\sigma} = \{g_{ab};\ \sigma_{ab}\}^*$. From Lemma 4.1 it follows that

$$(4.13) \qquad\qquad 0 = j_\alpha^{-1}(\boldsymbol{\sigma}) = \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab}\}$$

and consequently $2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g W)_{ab} = 0$ for some vector field W^a. But 0 also admits the decomposition $0 = 2N_{\alpha,g}0 + \mathbf{L}_g 0$ and the uniqueness clause of Proposition 4.2 implies $\sigma_{ab} = 0$. Therefore $\boldsymbol{\sigma} = 0$.

To show j_α^{-1} is surjective, consider $\{\mathbf{u}\} \in T_{\mathbf{g}}\mathcal{C}/\mathcal{D}_0$. Let $g_{ab} \in \mathbf{g}$ and pick any u_{ab} such that

$$(4.14) \qquad\qquad \mathbf{u} = \{g_{ab};\ u_{ab}\}.$$

Let σ_{ab} be the unique g_{ab}-transverse traceless tensor given by Proposition 4.2 such that

$$(4.15) \qquad\qquad u_{ab} = 2N_{g,\alpha}\sigma_{ab} + (\mathbf{L}_g W)_{ab}.$$

for some vector field W^a. From Lemma 4.1 it follows that

$$(4.16)\quad j_\alpha^{-1}(\{g_{ab};\ \sigma_{ab}\}) = \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab}\}$$
$$= \{g_{ab};\ 2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g W)_{ab}\} = \{g_{ab};\ u_{ab}\} = \{\mathbf{u}\}.$$

This establishes the claimed surjectivity, so j_α^{-1} has an inverse $j_\alpha^{\mathcal{C}}$. Equation (4.12) follows from applying $j_\alpha^{\mathcal{C}}$ to both sides of equation (4.16). $\square$

Having constructed the isomorphism $j_\alpha^{\mathcal{C}}$, we obtain diagram (4.2), which commutes except perhaps when going around the lower loop. A straightforward exercise using Lemma 4.4 shows that traversing the loop starting at the level of $\mathcal{C}/\mathcal{D}_0$ is the identity, but traversing the loop starting at the level of $\mathcal{M}$ is a projection. In particular, if we start at $T_g^*\mathcal{M}$, then the projection is

$$(4.17) \qquad\qquad (g_{ab};\ A_{ab}, f)^* \mapsto (g_{ab};\ Y_{g,\alpha}(A_{ab}), 0)^*.$$

As mentioned previously, we assign a conformal velocity in $T_{[g]}\mathcal{C}/\mathcal{D}_0$ to $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$ by descending the left-hand side of diagram (4.2). In principle the velocity depends on both the lapse form α and the shift X^a, but in fact it is independent of the shift. To see this, let K_{ab} be a second-fundamental form which we write in trace/trace-free form as $K_{ab} = A_{ab} + (\tau/n)g_{ab}$. Proceeding down the left-hand side of diagram (4.2), we

first obtain $(g_{ab};\ 2N_{g,\alpha}A_{ab} + (\mathbf{L}_g X)_{ab}, N_{g,\alpha}\tau + \operatorname{div} X) \in T_g\mathcal{M}$ and then arrive at $\{g_{ab};\ 2N_{\alpha,g}A_{ab} + (\mathbf{L}_g X)_{ab}\} \in T_{[g]}\mathcal{C}/\mathcal{D}_0$. But $\{g_{ab};\ (\mathbf{L}_g X)_{ab}\} = 0$ and therefore the final result is $\{g_{ab};\ 2N_{\alpha,g}A_{ab}\}$, which is independent of X^a.

Definition 4.5. Let $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$, and let α be a lapse form. The **conformal velocity** of (g_{ab}, K_{ab}), as measured by α, is

$$(4.18) \qquad\qquad v_\alpha^{\mathcal{C}}(g_{ab}, K_{ab}) = \{g_{ab};\ 2N_{\alpha,g}A_{ab}\}$$

where A_{ab} is the g_{ab}-trace-free part of K_{ab}.

To obtain the corresponding conformal momentum, we convert the velocity to a momentum via $j_\alpha^{\mathcal{C}}$. Starting with $\{g_{ab};\ 2N_{\alpha,g}A_{ab}\} \in T_{[g]}\mathcal{C}/\mathcal{D}_0$, let $\sigma_{ab} = Y_{g,\alpha}(A_{ab})$ be the York projection, so

$$(4.19) \qquad\qquad 2N_{\alpha,g}A_{ab} = 2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g W)_{ab}$$

for some vector field W^a. Lemma 4.4 then implies

$$(4.20) \qquad\qquad j_\alpha^{\mathcal{C}}(\{g_{ab};\ 2N_{\alpha,g}A_{ab}\}) = \{g_{ab};\ \sigma_{ab}\}^*.$$

Definition 4.6. Let $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$, and let α be a lapse form. Let A_{ab} be the g_{ab}-trace-free part of K_{ab} and let $\sigma_{ab} = Y_{g,\alpha}(A_{ab})$ be its York projection. The **conformal momentum** of (g_{ab}, K_{ab}), as measured by α, is

$$(4.21) \qquad\qquad m_\alpha^{\mathcal{C}}(g_{ab}, K_{ab}) = \{g_{ab};\ \sigma_{ab}\}^*.$$

From the maps $v_\alpha^{\mathcal{C}}$ and $m_\alpha^{\mathcal{C}}$ we obtain the diagram

$$(4.22)$$

$$
\begin{array}{ccc}
 & \mathcal{M} \times \mathcal{K} & \\
{\scriptstyle v_\alpha^{\mathcal{C}}}\swarrow & & \searrow{\scriptstyle m_\alpha^{\mathcal{C}}} \\
T\,\mathcal{C}/\mathcal{D}_0 & \underset{j_\alpha^{\mathcal{C}}}{\longleftarrow\!\!\!\longrightarrow} & T^*\mathcal{C}/\mathcal{D}_0
\end{array}
$$

which should be compared with diagram (2.13). Note in particular that although the ADM momentum is computed without reference to the lapse or shift, both the conformal velocity and conformal momentum depend in general the choice of a lapse form. The CMC solutions of the constraint equations are an exception to this observation, however. If (g_{ab}, K_{ab}) is a CMC solution of the constraint equations, then $K_{ab} = \sigma_{ab} + (\tau_0/n)g_{ab}$ for

some transverse traceless tensor σ_{ab} and some constant τ_0. Hence the York projection of K_{ab} is σ_{ab} regardless of the choice of lapse form.

The map $j_\alpha^\mathcal{C}$ appeared previously in [33], where it was denoted j_α and where it was derived from purely geometric considerations. The approach taken here suggests that $j_\alpha^\mathcal{C}$ is a Legendre transformation, and our next task is to identify a Lagrangian on $T\,\mathcal{C}/\mathcal{D}_0$ for which $j_\alpha^\mathcal{C}$ is the associated Legendre transformation.

Consider the kinetic energy term of the densitized-lapse ADM Lagrangian:

$$(4.23) \qquad K(g_{ab}, u_{ab}, \beta; X^a, \alpha) = \int \left[\frac{1}{4} |u - \mathbf{L}_g X|_g^2 - \kappa(\beta - \mathrm{div}_g X)^2 \right] \alpha.$$

The first term on the right-hand side involves the kinetic energy due to conformal deformation. Let σ_{ab} be the g_{ab}-TT tensor such that

$$(4.24) \qquad\qquad j_\alpha^\mathcal{C}(\{g_{ab};\ u_{ab}\}) = \{g_{ab};\ \sigma_{ab}\}^*.$$

So there is a vector field W^a such that

$$(4.25) \qquad\qquad u_{ab} = 2N_{\alpha,g}\sigma_{ab} + (\mathbf{L}_g(W + X))_{ab}.$$

Then

$$\int \frac{1}{4} |u - \mathbf{L}_g X|_g^2 \alpha = \int \left[N_{\alpha,g}^2 |\sigma|_g^2 + \frac{N}{2} \langle \sigma, \mathbf{L}_g(W) \rangle_g + \frac{1}{4} |\mathbf{L}_g(W)|_g^2 \right] \alpha$$

$$= \int \left[N_{\alpha,g}^2 |\sigma|_g^2 + \frac{1}{4} |\mathbf{L}_g(W)|_g^2 \alpha + \frac{1}{2} \int \langle \sigma, \mathbf{L}_g(W) \rangle_g \right] \omega_g$$

$$(4.26) \qquad\qquad = \int \left[N_{\alpha,g}^2 |\sigma|_g^2 + \frac{1}{4} |\mathbf{L}_g(W)|_g^2 \right] \alpha$$

where we have used the fact that $N_{\alpha,g}\alpha = \omega_g$ as well as the L^2-orthogonality of σ_{ab} and $\mathbf{L}_g(W)$ with respect to g_{ab}. The conformal kinetic energy is the first term on the right-hand side of the final expression of equation (4.26).

Definition 4.7. Let α be a lapse form. The **conformal kinetic energy** of $(g_{ab};\ u_{ab}, \beta) \in T_g\mathcal{M}$, as measured by α, is

$$(4.27) \qquad\qquad K_\alpha^\mathcal{C}(g_{ab}, u_{ab}) = \int N_{\alpha,g}^2 |\sigma|_g^2 \alpha$$

where σ_{ab} is the g_{ab}-TT tensor such that $j_\alpha^\mathcal{C}(\{g_{ab};\ u_{ab}\}) = \{g_{ab};\ \sigma_{ab}\}^*$.

The following lemma shows conformal kinetic energy descends to a well defined function on $T\,\mathcal{C}/\mathcal{D}_0$.

Lemma 4.8. *Suppose* $\{g_{ab};\ u_{ab}\} = \{\widehat{g}_{ab};\ \widehat{u}_{ab}\}$. *Then*

$$K_\alpha^{\mathcal{C}}(g_{ab}, u_{ab}) = K_\alpha^{\mathcal{C}}(\widehat{g}_{ab}, \widehat{u}_{ab}).$$

Proof. Suppose $\{g_{ab}, u_{ab}\} = \{\widehat{g}_{ab}, \widehat{u}_{ab}\}$, and let ϕ be the conformal factor such that $\widehat{g}_{ab} = \phi^{q-2} g_{ab}$. Since $\{g_{ab}, u_{ab}\} = \{\widehat{g}_{ab}, \widehat{u}_{ab}\}$, the corresponding conformal momenta $\{g_{ab};\ \sigma_{ab}\}^*$ and $\{\widehat{g}_{ab};\ \widehat{\sigma}_{ab}\}^*$ are the same. So $\widehat{\sigma}_{ab} = \phi^{-2}\sigma_{ab}$. Since $N_{\widehat{g},\alpha} = \omega_{\widehat{g}}/\alpha = \phi^q \omega_g/\alpha = \phi^q N_{g,\alpha}$ we conclude

$$(4.28)\qquad K_\alpha^{\mathcal{C}}(\widehat{g}_{ab}, \widehat{u}_{ab}) = \int_M N_{\widehat{g},\alpha}^2 |\widehat{\sigma}|_{\widehat{g}}^2\ \alpha = \int_M \phi^{2q} N_{g,\alpha}^2 \phi^{4-2q} |\phi^{-2}\sigma|_g^2\ \alpha$$

$$= \int_M N_{g,\alpha}^2 |\sigma|_g^2\ \alpha = K_\alpha^{\mathcal{C}}(g_{ab}, u_{ab}).$$

$\square$

We will use the same notation $K_\alpha^{\mathcal{C}}$ to denote a function on $T\,\mathcal{C}/\mathcal{D}_0$ rather than $T\mathcal{M}$. Thinking of it as a Lagrangian, the following proposition shows that $j_\alpha^{\mathcal{C}}$ is its Legendre transformation.

Proposition 4.9. *Suppose* $g_{ab} \in \mathcal{M}$ *is a metric and* $u_{ab}(t)$ *is a smooth path of* g_{ab}*-trace-free tensors. Then*

$$(4.29)\qquad \left.\frac{d}{dt}\right|_{t=0} K_\alpha^{\mathcal{C}}(\{g_{ab}, u_{ab}(t)\}) = \left\langle j_\alpha^{\mathcal{C}}(\{g_{ab};\ u_{ab}(0)\}), \{g_{ab}, u_{ab}'(0)\} \right\rangle.$$

Proof. For each t, let $\sigma_{ab}(t)$ be the transverse-traceless tensor with

$$(4.30)\qquad j_\alpha^{\mathcal{C}}(\{g_{ab};\ u_{ab}(t)\}) = \{g_{ab};\ \sigma_{ab}(t)\}^*;$$

since $\sigma_{ab}(t) = Y_{g,\alpha}(u_{ab}(t)/(2N_{g,\alpha}))$, the curve $\sigma_{ab}(t)$ is smooth. For each t let $W^a(t)$ be a vector field such that

$$(4.31)\qquad u_{ab}(t) = 2N_{g,\alpha}\sigma_{ab}(t) + (\mathbf{L}_g W)_{ab}(t).$$

Then

$$(4.32)\qquad K_\alpha^{\mathcal{C}}(\{g_{ab}, u_{ab}(t)\}) = \int_M N_{g,\alpha}^2 |\sigma(t)|_g^2\ \alpha = \int_M N_{g,\alpha} |\sigma(t)|_g^2\ \omega_g$$

and

$$(4.33) \qquad \frac{d}{dt}\bigg|_{t=0} K_\alpha^\mathcal{C}(\{g_{ab}, u_{ab}(t)\}) = \int_M 2N_{g,\alpha} \left\langle \sigma(0), \sigma'(0) \right\rangle_g \, \omega_g$$

$$= \int_M \left\langle \sigma(0), u'(0) - (\mathbf{L}_g W)'(0) \right\rangle_g \, \omega_g$$

$$= \int_M \left\langle \sigma(0), u'(0) \right\rangle_g \, \omega_g.$$

since σ_{ab} is transverse traceless. But from equations (3.7) and (3.13) this last expression is precisely

$$(4.34) \quad \left\langle [g_{ab};\, \sigma_{ab}(0)], \{g_{ab};\, u'_{ab}(0)\} \right\rangle = \left\langle j_\alpha^\mathcal{C}(\{g_{ab}, u_{ab}(0)\}), \{g_{ab};\, u'_{ab}(0)\} \right\rangle.$$

$\square$

5. The conformal method

As presented in [33], the conformal method can be understood in terms of the conformal parameters discussed in the previous section. We have the following two formulations.

Problem 5.1 (Lagrangian conformal method). *Let* $\mathbf{g}$ *be a conformal class, let* α *be a lapse form, let* $\{\mathbf{u}\} \in T_\mathbf{g}\mathcal{C}/\mathcal{M}$ *be a conformal velocity, and let* τ *be a mean curvature. Find all solutions* $(\overline{g}_{ab}, \overline{K}_{ab})$ *of the vacuum constraint equations* (1.1) *such that*

$$(5.1) \qquad \begin{aligned} [\overline{g}_{ab}] &= \mathbf{g} \\ v_\alpha^\mathcal{C}(\overline{g}_{ab}, \overline{K}_{ab}) &= \{\mathbf{u}\} \\ \overline{g}^{ab}\overline{K}_{ab} &= \tau. \end{aligned}$$

Problem 5.2 (Hamiltonian conformal method). *Let* $\mathbf{g}$ *be a conformal class, let* α *be a lapse form, let* $\boldsymbol{\sigma} \in T_\mathbf{g}^*\mathcal{C}/\mathcal{M}$ *be a conformal momentum, and let* τ *be a mean curvature. Find all solutions* $(\overline{g}_{ab}, \overline{K}_{ab})$ *of the vacuum constraint equations* (1.1) *such that*

$$(5.2) \qquad \begin{aligned} [\overline{g}_{ab}] &= \mathbf{g} \\ m_\alpha^\mathcal{C}(\overline{g}_{ab}, \overline{K}_{ab}) &= \boldsymbol{\sigma} \\ \overline{g}^{ab}\overline{K}_{ab} &= \tau. \end{aligned}$$

The two problems differ only in whether the conformal velocity or momentum is prescribed, and they are equivalent: $(\overline{g}_{ab}, \overline{K}_{ab})$ is a solution of

Problem 5.1 for parameters $(\mathbf{g}, \{\mathbf{u}\}, \tau, \alpha)$ if and only if it is a solution of Problem 5.2 for parameters $(\mathbf{g}, \boldsymbol{\sigma}, \tau, \alpha)$ with $\boldsymbol{\sigma} = j_\alpha^{\mathcal{C}}(\{\mathbf{u}\})$.

In order to write down PDEs corresponding to these problems we choose representative tensors of the conformal parameters. In the Hamiltonian case, we can take conformal parameters to be a metric g_{ab}, a transverse traceless tensor σ_{ab}, a mean curvature τ, and a lapse N. These prescribe Hamiltonian conformal parameters

$$
\begin{aligned}
\mathbf{g} &= [g_{ab}] \\
\boldsymbol{\sigma} &= \{g_{ab};\ \sigma_{ab}\}^* \\
\tau &= \tau \\
\alpha &= \omega_g/N
\end{aligned}
$$

(5.3)

and the constraint equations become the CTS-H equations

$$
- a\,\Delta_g\,\phi + R_g\phi - \left|\sigma + \frac{1}{2N}\,\mathbf{L}_g\,W\right|_g^2 \phi^{-q-1} + \kappa\tau^2\phi^{q-1} = 0
$$

(5.4)
$$\quad \text{[CTS-H Hamiltonian constraint]}$$

$$
\operatorname{div}_g\left(\frac{1}{2N}\,\mathbf{L}_g\,W\right) - \kappa\phi^q\,\mathrm{d}\,\tau = 0
$$

$$\quad \text{[CTS-H momentum constraint]}$$

which first appeared, in a slightly different form, in [37]. These equations are to be solved for a conformal factor ϕ and a vector field W^a, and if a solution exists then

(5.5)
$$
\begin{aligned}
\overline{g}_{ab} &= \phi^{q-2}g_{ab} \\
\overline{K}_{ab} &= \phi^{-2}\left(\sigma_{ab} + \frac{1}{2N}(\mathbf{L}_g\,W)_{ab}\right) + \frac{\tau}{n}\overline{g}_{ab}
\end{aligned}
$$

solve the vacuum constraint equations. Note that in York's original formulation of the conformal method, there are three parameters $(g_{ab}, \sigma_{ab}, \tau)$ and N is implicitly $1/2$. This is not an essential restriction since one can control the lapse form α by moving g_{ab} within its conformal class while suitably adjusting σ_{ab}, but the requirement of tying the conformal class representative to α leads to some inflexibility. Hence we prefer the CTS-H equations to those of York's original conformal method. In the Lagrangian case, the parameter σ_{ab} is replaced with an arbitrary symmetric, trace-free tensor u_{ab}

which determines a conformal velocity

$$
(5.6) \qquad \{\mathbf{u}\} = \{g_{ab};\ u_{ab}\} = [g_{ab};\ u_{ab}] + \operatorname{Im} \mathbf{L_g} .
$$

and the CTS-H equations become the CTS-L equations for ϕ and W^a

$$
- a\, \Delta_g\, \phi + R_g \phi - \left| \frac{1}{2N}(u + \mathbf{L}_g\, W) \right|_g^2 \phi^{-q-1} + \kappa \tau^2 \phi^{q-1} = 0
$$

(5.7) [CTS-L Hamiltonian constraint]

$$
\operatorname{div}_g \left(\frac{1}{2N}\, \mathbf{L}_g\, W \right) - \kappa \phi^q\, \mathrm{d}\,\tau = - \operatorname{div}_g \left[\frac{1}{2N} u \right] .
$$

[CTS-L momentum constraint]

Since the conformal method specifies a conformal velocity or momentum (modulo diffeomorphisms), we would like to understand how the mean curvature is related to volumetric velocity or momentum (modulo diffeomorphisms). We have seen that if K_{ab} has trace/trace-free decomposition $K_{ab} = A_{ab} + (\tau/n)g_{ab}$, then the conformal momentum is obtained from a lapse-dependent York projection of A_{ab}. It turns out that volumetric momentum is a single number, and is obtained from an analogous lapse-dependent York-like projection of τ. Indeed, there is a way to treat the volumetric degrees of freedom in a fashion completely in parallel to the manner in which the conformal method treats the conformal degrees of freedom, and we described this in the next two sections.

6. Volumetric tangent spaces

The space $\mathcal{V}$ of volume forms is an open subset of $C^\infty(M, \Lambda^n M)$, so the tangent space at $\omega \in \mathcal{V}$ is

$$
(6.1) \qquad T_\omega \mathcal{V} = C^\infty(M, \Lambda^n M).
$$

We define

$$
(6.2) \qquad T_\omega^* \mathcal{V} = C^\infty(M)
$$

and identify $T_\omega^* \mathcal{V}$ as a subset of $(T_\omega \mathcal{V})^*$ by defining the action of $f \in T_\omega^* \mathcal{V}$ on $\eta \in T_\omega \mathcal{V}$ by

$$
(6.3) \qquad \langle f, \eta \rangle = \int_M f\, \eta.
$$

Suppose $\gamma(t)$ is a path of metrics with $\gamma(0) = g_{ab}$ and $\gamma'(0) = (g_{ab}; u_{ab}, \beta)$. A standard computation shows that the associated path of volume forms $\omega(t)$ satisfies

$$(6.4) \qquad\qquad\qquad\qquad \omega'(0) = \beta\omega_g.$$

Hence the pushforward $T_g\mathcal{M} \to T_{\omega_g}\mathcal{V}$ is

$$(6.5) \qquad\qquad\qquad\qquad (g_{ab}; u_{ab}, \beta) \mapsto \beta\omega_g.$$

To compute the pullback we note that if $f \in T^*_{\omega_g}\mathcal{V}$, equation (1.13) implies

$$(6.6) \qquad \langle f, \beta\omega_g \rangle = \int_M f\beta\,\omega_g = \langle (g_{ab};\, 0, f)^*, (g_{ab};\, u_{ab}, \beta) \rangle$$

and hence the pullback $T^*_{\omega_g}\mathcal{V} \to T^*_g\mathcal{M}$ is

$$(6.7) \qquad\qquad\qquad f \mapsto (g_{ab};\, 0, f)^* = \frac{f}{2}g^{ab}\omega_g.$$

We now consider volume forms modulo diffeomorphisms, $\mathcal{V}/\mathcal{D}_0$. Suppose Φ_t is a path of diffeomorphisms starting at the identity with infinitesimal generator X^a. If ω is a volume form and $\gamma(t) = \Phi_t^*\omega$, then

$$(6.8) \qquad\qquad\qquad\qquad \gamma'(0) = \mathrm{Div}_\omega(X)$$

where the divergence operator Div_ω applied to X^a is the Lie derivative $\mathcal{L}_X\omega$. Note that if g_{ab} is a metric then

$$(6.9) \qquad\qquad\qquad\qquad \mathrm{Div}_{\omega_g}(X) = \mathrm{div}_g(X)\omega_g.$$

Since γ is stationary in $\mathcal{V}/\mathcal{D}_0$, the directions $\mathrm{Div}_{\omega_g} X$ are null directions in $\mathcal{V}/\mathcal{D}_0$ and we make the formal definition

$$(6.10) \qquad\qquad\qquad\qquad T_\omega\mathcal{V}/\mathcal{D}_0 = T_\omega\mathcal{V}/\mathrm{Im}\,\mathrm{Div}_\omega.$$

The space $T_\omega\mathcal{V}/\mathrm{Im}\,\mathrm{Div}_\omega$ is much simpler than its conformal counterpart, and indeed is one dimensional.

Lemma 6.1. *The map* $\dot{\mathrm{Vol}} : T_\omega\mathcal{V}/\mathcal{D}_0 \to \mathbb{R}$ *given by*

$$(6.11) \qquad\qquad\qquad \dot{\mathrm{Vol}}(\eta + \mathrm{Im}\,\mathrm{Div}_\omega) = \int_M \eta$$

is well defined and is an isomorphism.

Proof. We claim that if η is an n-form, then $\int_M \eta = 0$ if and only if $\eta \in \operatorname{Im} \operatorname{Div}_\omega$.

To see this, let g_{ab} be any metric such that $\omega_g = \omega$. Now if $\eta \in \operatorname{Im} \operatorname{Div}_\omega$, then there is a vector field X^a such that $\eta = \operatorname{div}_g X \omega_g$ and hence $\int \eta = 0$.

Conversely, suppose $\int_M \eta = 0$. Then $\eta = f \omega_g$ for some zero-mean function. Since M is connected, there exists a unique zero-mean solution u of $\Delta_g u = f$. Setting $X^a = \nabla^a u$ we find that $\eta = \operatorname{div}_g(X) \omega_g = \operatorname{Div}_\omega(X)$.

Since the kernel of $\eta \mapsto \int_M \eta$ is $\operatorname{Im} \operatorname{Div}_\omega$, we conclude that integration descends to a map $\dot{\operatorname{Vol}}$ on the quotient space $T_\omega \mathcal{V} / \operatorname{Im} \operatorname{Div}_\omega = T_\omega \mathcal{V} / \mathcal{D}_0$. And since $\dot{\operatorname{Vol}}$ is surjective, the claimed isomorphism holds. $\qquad\square$

We will henceforth identify $T_\omega \mathcal{V} / \mathcal{D}_0$ with $\mathbb{R}$ using $\dot{\operatorname{Vol}}$. Note that with this identification, the pushforward $T_\omega \mathcal{V} \to T_\omega \mathcal{V} / \mathcal{D}_0$ is

$$(6.12) \qquad\qquad \eta \mapsto \int_M \eta.$$

Since $\mathbb{R}$ is naturally identified with its dual vector space we define $T_\omega^* \mathcal{V} / \mathcal{D}_0 = \mathbb{R}$. The pullback $T_\omega^* \mathcal{V} / \mathcal{D}_0 \to T_\omega^* \mathcal{V}$ takes the constant $c \in \mathbb{R}$ to the constant function $c \in C^\infty(M)$ since

$$(6.13) \qquad\qquad c \int_M \eta = \int_M c\eta = \langle c, \eta \rangle.$$

Note that the constant functions in $T_\omega^* \mathcal{V}$ are the annihilator of $\operatorname{Im} \operatorname{Div}_\omega$, and hence we could have equivalently defined $T_\omega^* \mathcal{V} / \mathcal{D}_0 = (\operatorname{Im} \operatorname{Div}_\omega)^\perp$ in an approach analogous to that of Section 3.

It will be helpful to have notation for the composite pushforward $T_g \mathcal{M} \to T_{\omega_g} \mathcal{V} / \mathcal{D}_0$. If $g_{ab} \in \mathcal{M}$ and $\beta \in C^\infty(M)$ we define

$$(6.14) \qquad\qquad \{g_{ab};\ \beta\} = \int_M \beta \omega_g.$$

From composition we obtain the following pushforwards and pullbacks associated with the projection $\mathcal{M} \to \mathcal{V} / \mathcal{D}_0$.

Lemma 6.2. *The pushforward $T_g \mathcal{M} \to T_{\omega_g} \mathcal{V} / \mathcal{D}_0$ is the map*

$$(6.15) \qquad\qquad (g_{ab};\ u_{ab}, \beta) \mapsto \{g_{ab};\ \beta\}.$$

The pullback $T_{\omega_g}^* \mathcal{V} / \mathcal{D}_0 \to T_g^* \mathcal{M}$ is

$$(6.16) \qquad\qquad c \mapsto (g_{ab};\ 0, c)^*.$$

Proof. Equation (6.15) is a consequence of equations (6.7), (6.5) and (6.14). Equation (6.16) follows from the formula $c \mapsto c$ for the pullback $T_\omega^* \mathcal{V}/\mathcal{D}_0 \to T_\omega^* \mathcal{V}$ and equation (6.7). $\square$

7. Volumetric velocity, momentum, and kinetic energy

Let ω be a volume form, and let g_{ab} be any metric with $\omega_g = \omega$. Starting from the diagram (2.13) and the pushforward/pullback maps from Lemma 6.2 we have the following diagram:

$$(7.1)$$

We wish to construct an isomorphism $j_\alpha^{\mathcal{V}} : T_\omega \mathcal{V}/\mathcal{D}_0 \to T_\omega^* \mathcal{V}/\mathcal{D}_0$, analogous to $j_\alpha^{\mathcal{C}}$, such that for every metric g_{ab} with $\omega_g = \omega$, the diagram

$$(7.2)$$

commutes (with the exception that traversal of the bottom loop starting at the middle row is a projection).

Recalling Lemma 6.1 and our identification of $T_\omega \mathcal{V}/\mathcal{D}_0$ and $T_\omega^* \mathcal{V}/\mathcal{D}_0$ with $\mathbb{R}$, we claim that

$$(7.3) \qquad j_\alpha^{\mathcal{V}}(v) = -\left(\frac{2\kappa}{\int_M N_{g,\alpha} \omega_g} \right) v$$

is the desired isomorphism. Evidently, $j_\alpha^{\mathcal{V}}$ is invertible, and

$$(7.4) \qquad (j_\alpha^{\mathcal{V}})^{-1}(p) = -\left(\frac{1}{2\kappa} \int_M N_{g,\alpha} \omega_g \right) p.$$

So to establish diagram (7.2) we need only show that traveling from the lower-right corner to the lower-left corner of diagram (7.1) is the same map as $(j_\alpha^\mathcal{V})^{-1}$, regardless of the choice of g_{ab} with $\omega_g = \omega$. To this end, let $p \in T_\omega^* \mathcal{V}/\mathcal{D}_0$. From equation (6.7) its pullback is $(g_{ab};\ 0, p)^* \in T_g^* \mathcal{M}$, and we apply i_{α, X^a}^{-1} from equation (2.7) to obtain $(g_{ab};\ L_g X_{ab}, -p N_{\alpha, g}/(2\kappa) + \mathrm{div}_g X)$. Finally, applying the pushforward from equations (6.15) and (6.14) we arrive at

$$(7.5) \quad \int_M \left[-(N_{\alpha, g}/(2\kappa))p + \mathrm{div}_g X \right] \omega_g = - \left(\frac{1}{2\kappa} \int_M N_{g,\alpha} \omega_g \right) p = (j_\alpha^\mathcal{V})^{-1}(p)$$

as desired. This establishes diagram (7.2), which evidently commutes except possibly when traversing the lower loop starting at the middle row. As in the conformal case, such a traversal is a projection, and to describe concisely it we introduce the volumetric equivalent of York splitting.

Lemma 7.1 (Volumetric York splitting). *Let $g_{ab} \in \mathcal{M}$ and let N be a positive function.*

If $\tau \in C^\infty(M)$, there is constant τ^* and a smooth vector field V^a such that

$$(7.6) \qquad\qquad \tau = \tau^* + \frac{1}{N} \mathrm{div}_g V.$$

The constant τ^* is uniquely given by

$$(7.7) \qquad\qquad \tau^* = \frac{\int_M N\tau\omega_g}{\int_M N\omega_g}$$

and V^a is unique up to addition of a (smooth) divergence-free vector field.
Equivalently, if $\beta \in C^\infty(M)$, there is a unique constant

$$(7.8) \qquad\qquad \tau^* = \frac{\int_M \beta\omega_g}{\int_M N\omega_g}$$

and a smooth vector field V^a, unique up to addition of a (smooth) divergence-free vector field, such that

$$(7.9) \qquad\qquad \beta = N\tau^* + \mathrm{div}_g V$$

Proof. Let $\tau \in C^\infty$ and let τ^* be given by equation (7.7). So

$$(7.10) \qquad \int_M [N\tau - N\tau^*]\,\omega_g = 0$$

and Lemma 6.1 implies there is a smooth vector field V^a such that

$$(7.11) \qquad N\tau - N\tau^* = \operatorname{div}_g V.$$

This establishes equation (7.6).

The uniqueness of τ^* follows from multiplying equation (7.6) by $N\omega_g$ and integrating. Moreover, we see that we can write $\tau = \tau^* + (1/N)\operatorname{div}_g \widehat{V}$ for some other smooth vector field $\widehat{V}^a$ if and only if the difference $V^a - \widehat{V}^a$ is smooth and divergence free. Finally, we note that the decomposition (7.9) is a trivial (but useful) rephrasing of equation (7.6). $\qquad\square$

Definition 7.2. Let g_{ab} be a metric and let $\tau \in C^\infty(M)$. The **volumetric York projection** of τ is

$$(7.12) \qquad Y_{g,\alpha}(\tau) = \frac{\int_M N_{g,\alpha}\tau\omega_g}{\int_M N_{g,\alpha}\omega_g}.$$

Equivalently, $Y_{g,\alpha}(\tau)$ is the unique constant τ^* given by Lemma 7.1 such that

$$(7.13) \qquad \tau = \tau^* + \frac{1}{N_{g,\alpha}}\operatorname{div}_g V$$

for some vector field V^a. Note that we use the same notation $Y_{g,\alpha}$ as conformal York projection, with the difference being that the argument is a function rather than a symmetric $(0,2)$-tensor.

Using the notation of Definition 7.2, a short computation shows that the projection obtained by traversing the lower loop of diagram (7.2) starting from $T_g^*\mathcal{M}$ is the map

$$(7.14) \qquad (g_{ab};\ A_{ab}, f)^* \mapsto (g_{ab};\ 0, Y_{g,\alpha}(f))^*.$$

We can also express $j_\alpha^\mathcal{V}$ in terms of volumetric York projection.

Lemma 7.3. *Suppose $\{g_{ab};\, \beta\} \in T_{\omega_g}\mathcal{V}/\mathcal{D}_0$. Then*

(7.15) $$j_\alpha^\mathcal{V}(\{g_{ab};\, \beta\}) = -2\kappa\tau^*$$

where $\tau^* = Y_{g,\alpha}(\beta/N_{g,\alpha})$, or equivalently where τ^* is the unique constant such that

(7.16) $$\beta = N_{g,\alpha}\tau^* + \mathrm{div}_g\, V$$

for some vector field V^a.

Proof. Let $\{g_{ab};\, \beta\} \in T_{\omega_g}\mathcal{V}/\mathcal{D}_0$. From equations (7.3) and (6.14) we find

(7.17) $$j_\alpha^\mathcal{V}(\{g_{ab};\, \beta\}) = -\frac{2\kappa}{\int_M N_{g,\alpha}\,\omega_g}\{g_{ab};\, \beta\} = -2\kappa\frac{\int_M \beta\omega_g}{\int_M N_{g,\alpha}\omega_g}.$$

Now let $\tau^* = Y_{g,\alpha}(\beta/N)$. Equation (7.13) implies equation (7.16) and integrating with respect to ω_g we find

(7.18) $$\tau^* = \frac{\int_M \beta\omega_g}{\int_M N_{g,\alpha}\omega_g}.$$

Equation (7.15) now follows from equations (7.17) and (7.18).   □

Given $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$, the volumetric velocity and momentum measured with respect to a lapse form α are defined analogously to their conformal counterparts. For the velocity we send K_{ab} down the left-hand side of diagram (7.2) starting at $\mathcal{K}$, and we convert the velocity into a momentum by applying $j_\alpha^\mathcal{V}$ in the form of Lemma 7.3. This leads to the following definitions.

Definition 7.4. Let $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$, and let α be a lapse form. Writing $\tau = g^{ab}K_{ab}$, the **volumetric velocity** of (g_{ab}, K_{ab}), as measured by α, is

(7.19) $$v_\alpha^\mathcal{V}(g_{ab}, K_{ab}) = \{g_{ab};\, N_{g,\alpha}\tau\} = \int_M N_{g,\alpha}\tau\omega_g.$$

The **volumetric momentum** of (g_{ab}, K_{ab}), as measured by α, is

(7.20) $$m_\alpha^\mathcal{V}(g_{ab}, K_{ab}) = -2\kappa\tau^*$$

where $\tau^* = Y_{g,\alpha}(\tau)$.

Note that the volumetric velocity is the rate of change of slice volume, as measured with respect to coordinate time. We also note that if $\tau \equiv \tau_0$ for some constant τ_0, then equation (7.12) shows that the volumetric momentum is simply $-2\kappa\tau_0$.

The volumetric kinetic energy is derived in a parallel fashion to conformal kinetic energy. Consider the kinetic energy terms of the densitized-lapse ADM Lagrangian:

$$(7.21) \qquad K(g_{ab}, u_{ab}, \beta; X^a, \alpha) = \int \left[\frac{1}{4} |u - \mathbf{L}_g X|_g^2 - \kappa(\beta - \mathrm{div}_g X)^2 \right] \alpha.$$

The second term on the right-hand side involves the kinetic energy due to expansion. Define τ^* by

$$(7.22) \qquad\qquad -2\kappa\tau^* = j_\alpha^{\mathcal{V}}(\{g_{ab};\ \beta\}).$$

From Lemma 7.3 we see that we can write

$$(7.23) \qquad\qquad \beta = N_{g,\alpha}\tau^* + \mathrm{div}_g(V + X)$$

for some vector field V^a. Then, since $N_{g,\alpha}\alpha = \omega_g$, we find

$$(7.24) \qquad -\kappa \int_M (\beta - \mathrm{div}_g X)^2 \alpha = -\kappa \int_M (N\tau^* + \mathrm{div}_g V)^2 \alpha$$

$$= -\kappa \int_M \left[N_{g,\alpha}^2 (\tau^*)^2 + (\mathrm{div}_g V)^2 \right] \alpha$$

$$- 2\kappa \int_M \tau^* \, \mathrm{div}_g V \omega_g$$

$$= -\kappa \int_M \left[N_{g,\alpha}^2 (\tau^*)^2 + (\mathrm{div}_g V)^2 \right] \alpha.$$

The volumetric kinetic energy, as measured by α, is the first term on the final right-hand side of equation (7.24).

Definition 7.5. Let α be a lapse form. The **volumetric kinetic energy** of $(g_{ab};\ u_{ab}, \beta) \in T_g\mathcal{M}$, as measured by α, is

$$(7.25) \qquad\qquad K_\alpha^{\mathcal{V}}(g_{ab}, \beta) = -\kappa \int_M N_{g,\alpha}^2 (\tau^*)^2 \alpha$$

where

$$(7.26) \qquad\qquad -2\kappa\tau^* = j_\alpha^{\mathcal{V}}(\{g_{ab};\ \beta\}).$$

From equation (7.26) and the definition of $j_\alpha^\mathcal{V}$ we see that

$$(7.27) \qquad \tau^* = \frac{1}{\int_M N_{g,\alpha}\omega_g}\{g_{ab};\ \beta\} = \frac{1}{\int_M N_{g,\alpha}^2\alpha}\{g_{ab};\ \beta\}$$

and hence we can also write

$$(7.28) \quad K_\alpha^\mathcal{V}(g_{ab},\beta) = -\frac{\kappa}{\int_M N_{g,\alpha}^2\alpha}\left(\{g_{ab};\ \beta\}\right)^2 = -\frac{\kappa}{\int_M N_{g,\alpha}\omega_g}\left(\{g_{ab};\ \beta\}\right)^2.$$

So $K_\alpha^\mathcal{V}$ descends to a Lagrangian on $T\,\mathcal{V}/\mathcal{D}_0$ (which we also call $K_\alpha^\mathcal{V}$), and the associated Legendre transformation of $\{g_{ab};\ \beta\}$ is the linearization

$$(7.29) \qquad (K_\alpha^\mathcal{V})'(\{g_{ab};\ \beta\}) = -\frac{2\kappa}{\int_M N_{g,\alpha}\ \omega_g}\{g_{ab};\ \beta\} = j_\alpha^\mathcal{V}(\{g_{ab};\ \beta\}).$$

8. Volumetric momentum and the standard conformal method

Consider Hamiltonian conformal method parameters $(\mathbf{g},\boldsymbol{\sigma},\tau,\alpha)$ and suppose $(\overline{g}_{ab},\overline{K}_{ab})$ is a solution of the vacuum Einstein constraint equations generated by it. So

$$(8.1) \qquad \begin{aligned} [\overline{g}_{ab}] &= \mathbf{g} \\ m_\alpha^\mathcal{C}(\overline{g}_{ab},\overline{K}_{ab}) &= \boldsymbol{\sigma} \\ \overline{g}^{ab}\overline{K}_{ab} &= \tau. \end{aligned}$$

The conformal momentum of the solution, as measured by α, is specified directly via $\boldsymbol{\sigma}$. But there is only an indirect connection between the conformal data and the volumetric momentum measured by α. Indeed, suppose g_{ab} is a representative of $\mathbf{g}$ and let σ_{ab} be the representative of $\boldsymbol{\sigma}$ with respect to g_{ab}. Equation (8.1) is equivalent to the existence of a conformal factor ϕ and a vector field W^a such that

$$(8.2) \qquad \begin{aligned} \overline{g}_{ab} &= \phi^{q-2}g_{ab} \\ \overline{K}_{ab} &= \phi^{-2}\left(\sigma_{ab} + \frac{1}{2N_{\alpha,g}}(\mathbf{L}_g W)_{ab}\right) + \frac{\tau}{n}\overline{g}_{ab}. \end{aligned}$$

and such that ϕ and W^a solve the CTS-H equations (5.4). The volumetric momentum of $(\overline{g}_{ab},\overline{K}_{ab})$ measured by α is $-2\kappa\tau^*$ where

$$(8.3) \qquad \tau^* = \frac{\int_M N_{\overline{g},\alpha}\tau\omega_{\overline{g}}}{\int_M N_{\overline{g},\alpha}\omega_{\overline{g}}} = \frac{\int_M \phi^{2q}N_{g,\alpha}\tau\omega_g}{\int_M \phi^{2q}N_{g,\alpha}\omega_g}.$$

Notice from the right-hand side of equation (8.3) that the computation of τ^* from $(g_{ab}, \sigma_{ab}, \tau, \alpha)$ appears to involve the unknown conformal factor ϕ in an essential way. Although we need not know ϕ exactly (one can compute τ^* from $c\phi$ for any positive constant c), it seems unlikely that one can compute τ^* without at least determining at least $c\phi$ and thereby effectively solving the CTS-H equations. Moreover, if the conformal data generates more than one solution of the constraints, as happens at least in some cases involving an L^∞ mean curvature that changes sign [31], there is no reason to believe that the volumetric momenta of the two solutions will agree.

Hence the conformal method treats the conformal and volumetric degrees of freedom differently, with the conformal degrees respecting a kind of diffeomorphism invariance, but not the volumetric degrees. On its face, this need not be problematic. But the discrepancy seems to negatively impact the effectiveness of the conformal method as a parameterization in the far-from-CMC setting. As mentioned in the introduction, the recent study in [34] presented a family $\mathcal{F}$ of smooth, non-CMC conformal data sets that generate certain U^{n-1}-symmetric slices of flat Kasner spacetimes. Given $(\mathbf{g}, \boldsymbol{\sigma}, \tau, \alpha) \in \mathcal{F}$, it either generates a single U^{n-1}-symmetric slice of a flat Kasner spacetime, or it generates a homothety family of U^{n-1}-symmetric slices. The homothety families appear precisely when $\tau^* = 0$, as computed with respect to one (and consequently any) of the generated U^{n-1}-symmetric solutions of the Einstein constraint equations. So the quantity τ^* that we seem to be unable to control directly from the conformal parameters determines, in the setting of [34], the multiplicity of solutions generated by the conformal parameters.

From the evidence of the role of τ^* from [34], along with the naturality of treating the conformal and volumetric degrees of freedom in the parallel ways discussed in Sections 4 and 7, we are therefore lead to consider conformal-like methods where the parameters include

1) a conformal class $\mathbf{g}$,

2) a lapse form α,

3) either a conformal velocity $\{\mathbf{u}\}$ or a conformal momentum $\boldsymbol{\sigma}$, with $\boldsymbol{\sigma} = j_\alpha^{\mathcal{C}}(\{\mathbf{u}\})$, and

4) either a volumetric velocity $v \in \mathbb{R}$ or a volumetric momentum $-2\kappa\tau^* \in \mathbb{R}$ with $-2\kappa\tau^* = j_\alpha^{\mathcal{V}}(v)$.
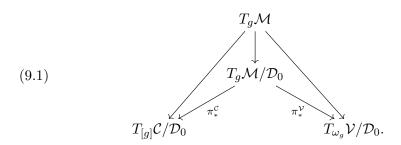
This list is evidently not comprehensive; the standard conformal method is successful in the near-CMC case but we have now replaced a function τ

with a scalar τ^*. In the remainder of the paper we examine alternatives for augmenting this list with geometrically natural degrees of freedom.

9. Drifts

Consider a path of metrics $g_{ab}(t)$ such that the conformal class of $g_{ab}(t)$ is constant along the curve *up to diffeomorphism*, and such that the volume form of $g_{ab}(t)$ is also constant up to potentially unrelated diffeomorphisms along the curve. By applying an appropriate path of diffeomorphisms, we could ensure that either the conformal class or the volume form is constant along the curve, but in general we cannot ensure both are constant. For example, suppose we apply diffeomorphisms to fix the conformal class. Since the diffeomorphism class of the volume form is constant, the volume will also remain constant along the curve, but we are free to smoothly reallocate the fixed volume by some flow. So although the conformal geometry and volume are constant, the conformal class and volume form can move relative to one another. Since the conformal class is a more rigid object than the volume form (e.g., the space of conformal Killing fields is finite dimensional, but the space of divergence-free vector fields is not), we visualize the volume form as drifting relative to the landmarks provided by the fixed conformal geometry. With this intuition in mind, we call an infinitesimal motion in $\mathcal{M}/\mathcal{D}_0$ that preserves conformal geometry and volume a drift.

To formalize these ideas, we first observe that the pushforwards from $T_g\mathcal{M}$ to $T_{[g]}\mathcal{C}/\mathcal{D}_0$ and $T_{\omega_g}\mathcal{V}/\mathcal{D}_0$ can be factored through $T_g\mathcal{M}/\mathcal{D}_0$ to obtain the maps $\pi_*^{\mathcal{C}}$ and $\pi_*^{\mathcal{V}}$ in the diagram

(9.1)

$$
\begin{array}{ccc}
 & T_g\mathcal{M} & \\
 & \downarrow & \\
 & T_g\mathcal{M}/\mathcal{D}_0 & \\
\swarrow \quad {}_{\pi_*^{\mathcal{C}}} & & {}_{\pi_*^{\mathcal{V}}} \quad \searrow \\
T_{[g]}\mathcal{C}/\mathcal{D}_0 & & T_{\omega_g}\mathcal{V}/\mathcal{D}_0.
\end{array}
$$

Indeed, we claim that

(9.2)
$$
\pi_*^{\mathcal{C}}(\{g_{ab};\ u_{ab}, \beta\}) = \{g_{ab};\ u_{ab}\}.
$$

First, note that the map $\pi^{\mathcal{C}}$ is well defined, for if X^a is a vector field, $\mathrm{Lie}_g\, X = (g_{ab};\ (\mathbf{L}_g\, X)_{ab}, \mathrm{div}_g\, X)$ and

(9.3) $$\pi_*^{\mathcal{C}}(\{g_{ab};\ \mathbf{L}_g\, X, \mathrm{div}_g\, X\}) = \{g_{ab};\ (\mathbf{L}_g\, X)_{ab}\} = 0.$$

Moreover, from equations (1.18) and (3.14) we see that equation (9.2) is exactly the statement that the left-hand triangle of diagram 9.1 commutes. Similar considerations show that

(9.4) $$\pi_*^{\mathcal{V}}(\{g_{ab};\ u_{ab}, \beta\}) = \{g_{ab};\ \beta\} = \int_M \beta\omega_g.$$

Definition 9.1. Let $g_{ab} \in \mathcal{M}$. A **drift** at g_{ab} is an element $\mathbf{U} \in T_g\mathcal{M}/\mathcal{D}_0$ such that $\pi_*^{\mathcal{C}}(\mathbf{U}) = 0$ and $\pi_*^{\mathcal{V}}(\mathbf{U}) = 0$. We denote the collection of drifts at g_{ab} by Drift_g.

Lemma 9.2. *Suppose $\mathbf{U} \in T_g\mathcal{M}/\mathcal{D}_0$. Then $\mathbf{U} \in \mathrm{Drift}_g$ if and only if there is a vector field R^a such that*

(9.5) $$\mathbf{U} = \{g_{ab}; 0, \mathrm{div}_g\, R\}.$$

Moreover, if $\widehat{R}^a$ is another vector field, then

(9.6) $$\{g_{ab};\ 0, \mathrm{div}_g\, R\} = \{g_{ab};\ 0, \mathrm{div}_g\, \widehat{R}\}$$

if and only if there is a divergence-free vector field E^a and a conformal Killing field Q^a such that

(9.7) $$\widehat{R}^a = R^a + E^a + Q^a.$$

Proof. Suppose $\mathbf{U} = \{g_{ab};\ u_{ab}, \beta\} \in T_g\mathcal{M}/\mathcal{D}_0$. From equation (9.2) we see that $\pi_*^{\mathcal{C}}(\mathbf{U}) = 0$ if and only if $u_{ab} \in \mathrm{Im}\,\mathbf{L}_g$. Hence there is a vector field W^a such that $u_{ab} = (\mathbf{L}_g\, W)_{ab}$. Similarly, from equation (9.4) and Lemma 6.1 we see that $\pi_*^{\mathcal{V}}(\mathbf{U}) = 0$ if and only if there is a vector field V^a such that $\beta = \mathrm{div}_g\, V$. Thus $\mathbf{U}$ is a drift if and only if there are vector fields W^a and V^a such that

(9.8) $$\mathbf{U} = \{g_{ab};\ (\mathbf{L}_g\, W)_{ab}, \mathrm{div}_g\, V\}.$$

Moreover,

$$(9.9) \qquad \{g_{ab};\ (\mathbf{L}_g\, W)_{ab}, \mathrm{div}_g\, V\} = \{g_{ab};\ (\mathbf{L}_g\, W)_{ab}, \mathrm{div}_g\, V\} - \mathrm{Lie}_g\, W$$
$$= \{g_{ab};\ (\mathbf{L}_g\, W)_{ab}, \mathrm{div}_g\, V\}$$
$$- \{g_{ab};\ (\mathbf{L}_g\, W)_{ab}, \mathrm{div}_g\, W\}$$
$$= \{g_{ab};\ 0, \mathrm{div}_g(V - W)\}.$$

Setting $R^a = V^a - W^a$ we see that $\mathbf{U}$ is a drift if and only if there is a vector field R^a such that equation (9.5) holds.

Now suppose R^a and $\widehat{R}^a$ are vector fields such that

$$(9.10) \qquad \{g_{ab};\ 0, \mathrm{div}_g\, R\} = \{g_{ab};\ 0, \mathrm{div}_g\, \widehat{R}\}.$$

Hence

$$(9.11) \qquad (g_{ab};\ 0, \mathrm{div}_g(\widehat{R} - R)) \in \mathrm{Im}\, \mathrm{Lie}_g\, .$$

and there is a vector field Q^a such that

$$(9.12) \qquad (g_{ab};\ 0, \mathrm{div}_g(\widehat{R} - R)) = \mathrm{Lie}_g\, Q = (g_{ab};\ (\mathbf{L}_g\, Q)_{ab}, \mathrm{div}_g\, Q)$$

Equation (9.12) implies $(\mathbf{L}_g\, Q)_{ab} = 0$ and hence Q^a is a conformal Killing field. Defining $E^a = \widehat{R}^a - R^a - Q^q$, equation (9.12) also implies that E^a is divergence free. Since

$$(9.13) \qquad \widehat{R}^a = R^a + Q^a + E^a$$

we see that if equation (9.6) holds then so does equation (9.7). Conversely, if R^a and $\widehat{R}^a$ are related via (9.7) we can reverse the previous argument to conclude (9.10). $\qquad\qquad\square$

Given a metric g_{ab}, let $\mathcal{Q}_g$ be the subgroup of $\mathcal{D}_0$ that preserves the conformal class of g_{ab} and let $\mathcal{E}_g$ be the subgroup that preserves the volume form of g_{ab}. We define $T_e\mathcal{Q}_g$ to be the set of conformal Killing fields of g_{ab} and $T_e\mathcal{E}_g$ to be the set of ω_g-divergence free vector fields. Lemma 9.2 provides an isomorphism

$$(9.14) \qquad \mathrm{Drift}_g \approx T_e\mathcal{D}_0/(T_e\mathcal{Q}_g \oplus T_e\mathcal{E}_g).$$

We wish to show that motion in $\mathcal{M}/\mathcal{D}_0$ can be completely described in terms of volume expansion, conformal deformation, and drift. If $\mathbf{U} \in$

$T_g\mathcal{M}/\mathcal{D}_0$, assigning a conformal velocity and volumetric velocity is straight-forward: simply apply $\pi_*^{\mathcal{C}}$ and $\pi_*^{\mathcal{V}}$. Assigning a drift to **U** requires, however, a choice of projection

$$(9.15) \qquad\qquad T_g\mathcal{M}/\mathcal{D}_0 \to \mathrm{Drift}_g$$

and we now construct a family of such projections that depend on the choice of a lapse form α.

Consider the lower loop of diagram (4.2) where we additionally factor the pushforward $T_g\mathcal{M} \to T_{[g]}\mathcal{C}/\mathcal{D}_0$ through $T_g\mathcal{M}/\mathcal{D}_0$:

(9.16)

$$
\begin{array}{ccc}
T_g\mathcal{M} & \xleftarrow{\quad i_{\alpha,X^a}\quad} & T_g^*\mathcal{M} \\
\downarrow & & \uparrow \\
T_g\mathcal{M}/\mathcal{D}_0 & & \\
{\scriptstyle \pi_*^c}\downarrow & & \uparrow \\
T_{[g]}\mathcal{C}/\mathcal{D}_0 & \xleftarrow{\quad j_\alpha^{\mathcal{C}}\quad} & T_{[g]}^*\mathcal{C}/\mathcal{D}_0
\end{array}
$$

Let $\iota^{\mathcal{C}} : T_{[g]}\mathcal{C}/\mathcal{D}_0 \to T_g\mathcal{M}/\mathcal{D}_0$ be the map obtained by nearly completing the loop in diagram (9.16). From Lemma 4.4 and equations (2.7) and (1.18) we find

$$(9.17) \qquad\qquad \iota_\alpha^{\mathcal{C}}(\{g_{ab};\ u_{ab}\}) = \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab}, 0\}$$

where σ_{ab} is the unique g_{ab}-TT tensor such that

$$(9.18) \qquad\qquad u_{ab} = 2N_{g,\alpha}\sigma_{ab} + (\mathbf{L}_g W)_{ab}$$

for some vector field W^a. The following lemma shows that $\iota_\alpha^{\mathcal{C}}$ selects an α-dependent representative in $T_g\mathcal{M}/\mathcal{D}_0$ of each conformal motion in $T_{[g]}\mathcal{C}/\mathcal{D}_0$.

Lemma 9.3. *The map* $\iota_\alpha^{\mathcal{C}}$ *satisfies*

$$(9.19) \qquad\qquad \begin{aligned} \pi_*^{\mathcal{C}} \circ \iota_\alpha^{\mathcal{C}} &= \mathrm{id} \\ \pi_*^{\mathcal{V}} \circ \iota_\alpha^{\mathcal{C}} &= 0. \end{aligned}$$

Proof. Note that $\pi_*^{\mathcal{C}} \circ \iota_\alpha^{\mathcal{C}}$ is the map obtained by traversing the bottom loop of diagram 4.2 starting at $T_{[g]}\mathcal{C}/\mathcal{D}_0$. In Section 4 we showed that this map is the identity. On the other hand, from equations (9.17) and (9.4) we see that $\pi_*^{\mathcal{V}} \circ \iota_\alpha^{\mathcal{C}} = 0$. □

Similarly, from the diagram

(9.20)

$$
\begin{array}{ccc}
T_g\mathcal{M} & \xleftarrow{\qquad i_{\alpha,X^a} \qquad} & T_g^*\mathcal{M} \\
\downarrow & & \uparrow \\
T_g\mathcal{M}/\mathcal{D}_0 & & \\
\pi_*^{\mathcal{V}} \downarrow & & \\
T_{\omega_g}\mathcal{V}/\mathcal{D}_0 & \xleftarrow{\qquad j_\alpha^{\mathcal{V}} \qquad} & T_{\omega_g}^*\mathcal{V}/\mathcal{D}_0
\end{array}
$$

we obtain a map $\iota_\alpha^{\mathcal{V}} : T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \to T_g\mathcal{M}/\mathcal{D}_0$ given by

(9.21)
$$
\iota_\alpha^{\mathcal{V}}(\{g_{ab};\ \beta\}) = \{g_{ab};\ 0, N_{g,\alpha}\tau^*\}
$$

where τ^* is the unique constant given by volumetric York splitting (Lemma 7.1) such that

(9.22)
$$
\beta = N_{g,\alpha}\tau^* + \mathrm{div}_g V
$$

for some vector field V^a. We have an analogue of Lemma 9.3 that shows that $\iota_\alpha^{\mathcal{V}}$ selects an α-dependent representative in $T_g\mathcal{M}/\mathcal{D}_0$ of each volumetric motion in $T_{\omega_g}\mathcal{V}/\mathcal{D}_0$; we omit the proof.

Lemma 9.4. *The map $\iota_\alpha^{\mathcal{V}}$ satisfies*

(9.23)
$$
\begin{aligned}
\pi_*^{\mathcal{C}} \circ \iota_\alpha^{\mathcal{V}} &= 0 \\
\pi_*^{\mathcal{V}} \circ \iota_\alpha^{\mathcal{V}} &= \mathrm{id}\,.
\end{aligned}
$$

Writing

(9.24)
$$
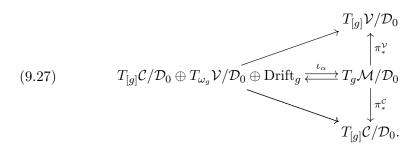\iota^{\mathrm{Drift}} : \mathrm{Drift}_g \to T_g\mathcal{M}/\mathcal{D}_0
$$

for the natural embedding we define

(9.25)
$$
\iota_\alpha : T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Drift}_g \to T_g\mathcal{M}/\mathcal{D}_0
$$

by

(9.26)
$$
\iota_\alpha = \iota_\alpha^{\mathcal{C}} \oplus \iota_\alpha^{\mathcal{V}} \oplus \iota^{\mathrm{Drift}}\,.
$$

Proposition 9.5. *Let g_{ab} be a metric and let α be a lapse form. Then ι_α is an isomorphism and the following diagram commutes:*

$$
\begin{array}{ccc}
 & & T_{[g]}\mathcal{V}/\mathcal{D}_0 \\
 & \nearrow & \big\uparrow{\scriptstyle \pi_*^{\mathcal{V}}} \\
(9.27) \qquad T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Drift}_g \xrightleftharpoons{\;\iota_\alpha\;} T_g\mathcal{M}/\mathcal{D}_0 \\
 & \searrow & \big\downarrow{\scriptstyle \pi_*^{\mathcal{C}}} \\
 & & T_{[g]}\mathcal{C}/\mathcal{D}_0.
\end{array}
$$

Moreover, if $\mathbf{R}$ is a drift,

$$(9.28) \qquad\qquad \iota_\alpha^{-1}(\mathbf{R}) = (0, 0, \mathbf{R}).$$

Proof. Note that $\mathrm{Drift}_g = \mathrm{Ker}\,\pi_*^{\mathcal{C}} \cap \mathrm{Ker}\,\pi_*^{\mathcal{V}}$, so $\pi_*^{\mathcal{C}} \circ \iota^{\mathrm{Drift}} = 0$ and $\pi_*^{\mathcal{V}} \circ \iota^{\mathrm{Drift}} = 0$. Using this fact and Lemmas 9.3 and 9.4 we conclude

$$(9.29) \qquad \pi_*^{\mathcal{C}} \circ \iota_\alpha = \pi_*^{\mathcal{C}} \circ \iota_\alpha^{\mathcal{C}} + \pi_*^{\mathcal{C}} \circ \iota_\alpha^{\mathcal{V}} + \pi_*^{\mathcal{C}} \circ \iota^{\mathrm{Drift}} = \mathrm{id} + 0 + 0.$$

This establishes the lower triangle of diagram (9.27) up to showing ι_α has an inverse. The upper triangle is established similarly, and we turn our attention to the invertibility of ι_α.

To see that ι_α is injective, notice that from the facts established thus far for diagram (9.27) that anything in the kernel of ι_α must be of the form $(0, 0, \mathbf{R})$ for some drift $\mathbf{R}$. But $\iota_\alpha(0, 0, \mathbf{R}) = \mathbf{R}$, so ι_α has trivial kernel.

To show that ι_α is surjective, let $\{g_{ab};\, u_{ab}, \beta\} \in T_g\mathcal{M}/\mathcal{D}_0$. Let σ_{ab} be the g_{ab}-TT tensor such that

$$(9.30) \qquad\qquad u_{ab} = 2N_{g,\alpha}\sigma_{ab} + (\mathbf{L}_g W)_{ab}$$

for some vector field W^a, and let τ^* be the constant such that

$$(9.31) \qquad\qquad \beta = N_{g,\alpha}\tau^* + \mathrm{div}_g V$$

for some vector field V^a. Let

$$(9.32) \qquad\qquad \mathbf{R} = \{g_{ab};\, (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\}$$

and observe that $\mathbf{R}$ is a drift. Then

(9.33)
$$\iota_\alpha^{\mathcal{C}}(\{g_{ab};\ u_{ab}\}) = \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab}, 0\}$$
$$\iota_\alpha^{\mathcal{V}}(\{g_{ab};\ \beta\}) = \{g_{ab};\ 0, N_{g,\alpha}\tau^*\}$$
$$\iota^{\mathrm{Drift}}(\mathbf{R}) = \{g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\}$$

so

(9.34)
$$\iota_\alpha(\{g_{ab};\ u_{ab}\}, \{g_{ab};\ \beta\}, \mathbf{R})$$
$$= \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab}, 0\} + \{g_{ab};\ 0, N_{g,\alpha}\tau^*\} + \{g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\}$$
$$= \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab} + (\mathbf{L}_g W)_{ab}, N_{g,\alpha}\tau^* + \mathrm{div}_g V\}$$
$$= \{g_{ab};\ u_{ab}, \beta\}$$

as desired.

Finally, we note that equation (9.28) follows from the invertibility of ι_α and the fact that $\iota_\alpha(0, 0, \mathbf{R}) = \mathbf{R}$ for any drift $\mathbf{R}$. $\qquad\square$

Proposition 9.5 is the formal assertion that motion in $\mathcal{M}/\mathcal{D}_0$ is characterized by conformal deformation, volume expansion, and drift. The conformal and volumetric velocities are unambiguously associated with $\mathbf{U} \in T_g\mathcal{M}/\mathcal{D}_0$ via $\pi_*^{\mathcal{C}}$ and $\pi_*^{\mathcal{V}}$, and Proposition 9.5 provides a lapse-form-dependent map from $T_g\mathcal{M}/\mathcal{D}_0$ to Drift_g: compute ι_α^{-1}, and extract the drift component. Let us call this map $\pi_\alpha^{\mathrm{Drift}}$.

Proposition 9.6. *The map $\pi_\alpha^{\mathrm{Drift}} : T_g\mathcal{M}/\mathcal{D}_0 \to \mathrm{Drift}_g$ is a projection and*

(9.35)
$$\pi_\alpha^{\mathrm{Drift}}(\{g_{ab};\ u_{ab}, \beta\}) = \{g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\}$$

where W^a and V^a are any vector fields obtained from York splitting

(9.36)
$$u_{ab} = 2N_{g,\alpha}\sigma_{ab} + \mathbf{L}_g W$$
$$\beta = 2N_{g,\alpha}\tau^* + \mathrm{div}_g V$$

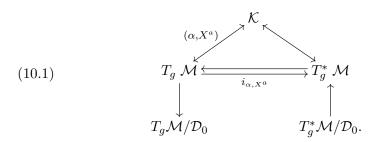*for some g_{ab}-TT tensor σ_{ab} and some constant τ^*.*

Moreover,

(9.37)
$$\iota_\alpha^{-1} = \pi_\alpha^{\mathcal{C}} \oplus \pi_\alpha^{\mathcal{V}} \oplus \pi_\alpha^{\mathrm{Drift}}.$$

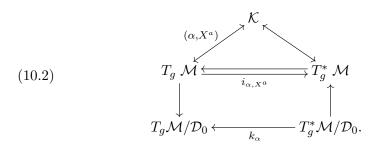*Proof.* That ι_α is a projection follows from equation (9.28), and formula (9.35) was established in the body of the proof of Proposition 9.5. Equation (9.37) follows from Proposition 9.5 and the definition of $\pi_\alpha^{\mathrm{Drift}}$. $\qquad\square$

10. Drifts and the momentum constraint

Consider a metric g_{ab} and a lapse form α. From diagram (2.13) and the pushforwards and pullbacks described in Section 1 we obtain the diagram

$$
(10.1)
$$

which is the analog of the conformal and volumetric equivalents (4.1) and (7.1). In the conformal and volumetric cases, the Legendre transformation i_{α,X^a} descended to a Legendre transformation after quotienting by diffeomorphisms. This is not the case for diagram (10.1), however. There is certainly a map $k_\alpha : T_g^*\mathcal{M}/\mathcal{D}_0 \to T_g\mathcal{M}/\mathcal{D}_0$ obtained by traveling from the lower-right to the lower-left of diagram (10.1):

$$
(10.2)
$$

But it turns out that k_α can fail to be an isomorphism, and this gives some insight about the configuration space for the Einstein equations. The elements of $T_g^*\mathcal{M}/\mathcal{D}_0$ are precisely the solutions of the momentum constraint, but is is not quite correct to think of these as momenta corresponding to the velocity of the system in $T_g\mathcal{M}/\mathcal{D}_0$, and the notion of drifts seems to play a key role here.

An arbitrary element of $T_g^*\mathcal{M}/\mathcal{D}_0$ can be written as

$$
(10.3) \qquad\qquad (g_{ab};\ A_{ab}, -2\kappa\tau)^*
$$

for some trace-free A_{ab} and some function τ that satisfy the momentum constraint

(10.4) $$\nabla^a A_{ab} = \kappa \nabla_b \tau.$$

From equations (2.7) and (1.18) we find

(10.5) $$k_\alpha((g_{ab};\ A_{ab}, -2\kappa\tau)^*)$$
$$= \{g_{ab}, 2N_{g,\alpha}A_{ab} + (\mathbf{L}_g X)_{ab}, N_{g,\alpha}\tau + \mathrm{div}_g X\}$$
$$= \{g_{ab}, 2N_{g,\alpha}A_{ab}, N_{g,\alpha}\tau\}.$$

Applying York splitting we can write

(10.6)
$$A_{ab} = \sigma_{ab} + \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g W)_{ab}$$
$$\tau = \tau^* + \frac{1}{N_{g,\alpha}}\,\mathrm{div}_g V$$

for a g_{ab}-TT tensor σ_{ab}, a constant τ^*, and vector fields W^a and V^a. Writing $\mathbf{U}$ for $k_\alpha((g_{ab};\ A_{ab}, -2\kappa\tau)^*)$ equation (10.5) becomes

(10.7) $\mathbf{U} = \{g_{ab}, 2N_{g,\alpha}\sigma_{ab}, 0\} + \{g_{ab}, 0, N_{g,\alpha}\tau^*\} + \{g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\}$,

so in the language of Proposition 9.6

(10.8)
$$\pi_\alpha^{\mathcal{C}}(\mathbf{U}) = \{g_{ab};\ 2N_{g,\alpha}\sigma_{ab}\}$$
$$\pi_\alpha^{\mathcal{V}}(\mathbf{U}) = \{g_{ab};\ N_{g,\alpha}\tau^*\}$$
$$\pi_\alpha^{\mathrm{Drift}}(\mathbf{U}) = \{g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V\} = \{g_{ab};\ 0, \mathrm{div}_g(V - W)\}.$$

The potential for difficulty lies in the cancellation in $\pi_\alpha^{\mathrm{Drift}}(\mathbf{U})$: although W^a and V^a might not be zero, it might be that $\pi_\alpha^{\mathrm{Drift}}(\mathbf{U})$ is zero (and hence k_α may have nontrivial kernel).

The momentum constraint (10.4) can be written in terms of the York-projected variables as

(10.9) $$\mathrm{div}_g\left(\sigma + \frac{1}{2N_{g,\alpha}}\mathbf{L}_g W\right) = \kappa\,\mathrm{d}\left(\tau^* + \frac{1}{N_{g,\alpha}}\mathrm{div}_g V\right)$$

or more simply

(10.10) $$\mathrm{div}_g\left(\frac{1}{2N}\mathbf{L}_g W\right) = \kappa\,\mathrm{d}\left(\frac{1}{N}\mathrm{div}_g V\right)$$

where $N = N_{g,\alpha}$. So σ_{ab} and τ^* are not involved in the momentum constraint, and we have only the relationship (10.10) between W^a and V^a that, for reasons explained below, we call the **drift equation**. One might hope that equation (10.10) prevents cancellation in $\pi_\alpha^{\text{Drift}}(\mathbf{U})$, but this is not always the case.

Suppose M^n is the torus $(S^1)^n$ equipped with the flat product metric $\bar{g}_{ab}$, and let s be the coordinate of the first factor of the torus. Consider vector fields $W^a = (w(s), 0, \ldots, 0)$ and $V^a = (v(s), 0, \ldots, 0)$, and suppose N is a lapse that depends only on s. A brief computation shows that equation (10.10) can be written

$$(10.11) \qquad\qquad \kappa \left(\frac{1}{2N} 2w' \right)' = \kappa \left(\frac{1}{N} v' \right)'$$

where primes denote differentiation with respect to s. Since (10.11) is an equation on the circle, $w(s)$ and $v(s)$ solve equation (10.11) if and only if $w = v + c$ for some constant c. Hence $W^a = V^a + K^a$ for some Killing field K^a and the associated drift from equation (10.8) vanishes identically. Thus, in this case, k_α has nontrivial kernel and is not an isomorphism.

The thin-sandwich conjecture [4] states that initial data is characterized by a metric g_{ab} and the projection of its ADM velocity into $T_g\mathcal{M}/\mathcal{D}_0$. It is not expected to hold in general [6], and the observation from the preceding paragraph appears to be another facet of its failure. Indeed, from [34] Proposition 6.2 and the discussion above it follows that there exist distinct solutions of the constraint equations, that generate distinct spacetimes, that nevertheless have the same metric and such that for some lapse form α

- the conformal velocities measured by α for both solutions are the same,

- the volumetric velocities measured by α for both solutions are the same, and

- the complementary drifts for both solutions are zero.

Hence the projections of the ADM velocities in to $T\mathcal{M}/\mathcal{D}_0$ for these distinct solutions of the constraint equations are identical.

Although k_α is not an isomorphism, it turns out that solutions of the momentum constraint can nevertheless be parameterized in terms of a conformal momentum, a volumetric momentum, and a drift. The key idea is to identify equation (10.10) as representing a relationship between two drifts, and we start by looking at the role of V^a .

Theorem 10.1. *Suppose g_{ab} is a metric, α is a lapse form, and $\mathbf{V} \in \mathrm{Drift}_g$. Let V^a be any vector field such that*

$$(10.12) \qquad\qquad \mathbf{V} = \{g_{ab};\ 0, \mathrm{div}_g V\}.$$

Then there is a conformal Killing field Q^a, unique up to addition of a proper Killing field, such that

$$(10.13) \qquad \mathrm{div}_g\left(\frac{1}{2N_{g,\alpha}} \mathbf{L}_g W \right) = \kappa\, \mathrm{d}\left(\frac{1}{N_{g,\alpha}} \mathrm{div}_g(V + Q) \right)$$

admits a solution W^a. The solution W^a is unique up to addition of a conformal Killing field, and the set of solutions does not depend on the choice of V^a representing $\mathbf{V}$ or on the subsequent choice of conformal Killing field Q^a such that equation (10.13) is solvable.

Proof. Let V^a be any representative of $\mathbf{V}$ and for brevity let $N = N_{g,\alpha}$. From elliptic theory the equation

$$(10.14) \qquad \mathrm{div}_g\left(\frac{1}{2N} \mathbf{L}_g W \right) = \kappa\, \mathrm{d}\left(\frac{1}{N} \mathrm{div}_g(V) \right)$$

admits a solution W^a if and only if

$$(10.15) \qquad \int_M \frac{1}{N} \mathrm{div}_g(V)\, \mathrm{div}_g(Q) \omega_g = 0$$

for all conformal Killing fields Q^a, in which case the solution W^a is unique up to addition of a conformal Killing field. Although V^a need not satisfy condition (10.15), we claim that there is a conformal Killing field $\widehat{Q}^a$ such that

$$(10.16) \qquad \int_M \frac{1}{N} \mathrm{div}_g(V + \widehat{Q})\, \mathrm{div}_g(Q) \omega_g = 0$$

for all conformal Killing fields Q^a, and that $\widehat{Q}^a$ is unique up to addition of a proper Killing field. Since proper Killing fields are divergence-free, the right-hand side of (10.10) is independent of the choice of admissible conformal Killing fields, as is the set of solutions of (10.10).

Consider the functional

(10.17) $$F(Q^a) = \int_M \frac{1}{N} \operatorname{div}_g(V+Q)^2 \omega_g$$

on the finite-dimensional space $T_e \mathcal{Q}_g$ of conformal Killing fields, and observe that $\widehat{Q}^a$ is stationary for F if and only if equation (10.16) holds. Moreover, since the highest order term of F is quadratic and non-negative definite, the stationary points of F correspond with its minimizers.

First suppose that g_{ab} does not admit any (nontrivial) proper Killing fields. Then every nontrivial conformal Killing field satisfies $\operatorname{div} Q \not\equiv 0$ and the quadratic term of F is positive definite. Hence F admits a unique minimizer. On the other hand, if g_{ab} admits a nontrivial space $\mathcal{X}$ of proper Killing fields, then F descends to a functional on the quotient $T_e \mathcal{Q}/\mathcal{X}$ and its quadratic order term is again positive definite. Hence we pick up a minimizer of F over the conformal Killing fields, and it is unique up to addition of a proper Killing field.

This establishes the main result up to independence of the solution set with respect to the choice of representative of $\mathbf{V}$. Let V^a and $\widetilde{V}^a$ be two representatives, so Lemma 9.2 implies that

(10.18) $$V^a = \widetilde{V}^a + \widetilde{Q}^a + \widetilde{E}^a$$

for some conformal Killing field $\widetilde{Q}^a$ and some divergence-free vector field $\widetilde{E}^a$. Let Q^a be a conformal Killing field and let W^a be a vector field such that

(10.19) $$\operatorname{div}_g \left(\frac{1}{2N} \mathbf{L}_g W \right) = \kappa \, \mathrm{d} \left(\frac{1}{N} \operatorname{div}_g(V+Q) \right).$$

We wish to show that there is a conformal Killing field $\overline{Q}^a$ such that

(10.20) $$\operatorname{div}_g \left(\frac{1}{2N} \mathbf{L}_g W \right) = \kappa \, \mathrm{d} \left(\frac{1}{N} \operatorname{div}_g(\widetilde{V}+\overline{Q}) \right)$$

as well. Since

(10.21) $$\operatorname{div}_g(V+Q) = \operatorname{div}_g(\widetilde{V}+\widetilde{E}+\widetilde{Q}+Q) = \operatorname{div}_g(\widetilde{V}+\widetilde{Q}+\widehat{Q})$$

we conclude that equation (10.20) holds with $\overline{Q}^a = \widetilde{Q}^a + Q^a$.   □

Theorem 10.1 provides a map $j_\alpha^{\mathrm{Drift}}$ from Drift_g to $T_g^* \mathcal{M}/\mathcal{D}_0$ as follows. Given a drift $\mathbf{V}$, let V^a be a representative and let Q^a and W^a be a conformal

Killing field and vector field respectively solving (10.10). We define

$$(10.22) \qquad j_\alpha^{\text{Drift}}(\mathbf{V}) = \left(g_{ab}; \; \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g\,W)_{ab}, -\frac{2\kappa}{N_{g,\alpha}}\,\text{div}_g(V+Q) \right)^*.$$

Note that j_α^{Drift} is well defined since $\text{div}_g(V+Q)$ and $(\mathbf{L}_g\,W)_{ab}$ are uniquely determined by $\{g;\, 0, \text{div}_g\,V\}$ even though V^a, Q^a and W^a need not be, and that equation (10.10) ensures that j_α^{Drift} maps into $T_g^*\mathcal{M}/\mathcal{D}_0$, not just $T_g^*\mathcal{M}$. The map j_α^{Drift} is injective for if

$$(10.23) \qquad \left(g_{ab}; \; \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g\,W)_{ab}, -\frac{2\kappa}{N_{g,\alpha}}\,\text{div}_g(V+Q) \right)^* = 0$$

then $\text{div}_g(V+Q) = 0$ and hence the source drift $\mathbf{V} = \{g_{ab};\, 0, \text{div}_g\,V\}$ satisfies

$$(10.24) \qquad \{g_{ab};\, 0, \text{div}_g\,V\} = \{g_{ab};\, \mathbf{L}_g\,Q, \text{div}_g(V+Q)\}$$
$$= \{g_{ab};\, 0, \text{div}_g(V+Q)\} = 0.$$

So $\text{Im}\,j_\alpha^{\text{Drift}}$ is isomorphic to Drift_g. The following result shows that $\text{Im}\,j_\alpha^{\text{Drift}}$ complements the conformal and volumetric momenta, which formalizes our earlier claim that solutions of the momentum constraint can be parameterized by the selection of a conformal momentum, a volumetric momentum, and a drift.

Proposition 10.2.

$$T_g^*\mathcal{M}/\mathcal{D}_0 = T_{[g]}^*\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}^*\mathcal{V}/\mathcal{D}_0 \oplus \text{Im}(j_\alpha^{\text{Drift}}).$$

Proof. Suppose $(g_{ab};\, A_{ab}, -2\kappa\tau)^* \in T_g^*\mathcal{M}/\mathcal{D}_0$. From York splitting there are vector fields W^a and V^a solving equation (10.10) as well as a TT-tensor σ_{ab} and a constant τ^* such that

$$(10.25)$$
$$A_{ab} = \sigma_{ab} + \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g\,W)_{ab}$$
$$\tau = \tau^* + \frac{1}{N_{g,\alpha}}\,\text{div}_g\,V.$$

So

$$(10.26) \quad (g_{ab};\, A_{ab}, -2\kappa\tau)^* = (g_{ab};\, \sigma_{ab}, 0)^* + (g_{ab};\, 0, \tau^*)^*$$
$$+ (g_{ab};\, (1/2N_{g,\alpha})(\mathbf{L}_g\,W)_{ab}, (1/N_{g,\alpha})\,\text{div}_g\,V)^*.$$

Since $(g_{ab}; A_{ab}, -2\kappa\tau)^*$ solves the momentum constraint (10.4), W^a and V^a solve equation (10.10) and hence

(10.27) $(g_{ab}; (1/2N_{g,\alpha})(\mathbf{L}_g W)_{ab}, (1/N_{g,\alpha})\operatorname{div}_g V)^* \in \operatorname{Im} j_\alpha^{\mathrm{Drift}}.$

Equation (10.26) therefore exhibits $(g_{ab}; A_{ab}, -2\kappa\tau)^*$ as the sum of a conformal momentum, a volumetric momentum, and term in the image of $j_\alpha^{\mathrm{Drift}}$.

To establish the direct sum decomposition (10.2) we need only show that the summands are mutually transverse. Now if W^a and V^a solve equation (10.10) and either of $(\mathbf{L}_g W)_{ab}$ or $\operatorname{div}_g V$ vanishes, an integration by parts exercise shows the other must as well. Hence $\operatorname{Im} j_\alpha^{\mathrm{Drift}}$ is transverse to $T_{[g]}^* \mathcal{C}/\mathcal{D}_0$ and $T_{\omega_g}^* \mathcal{V}/\mathcal{D}_0$, which are obviously transverse to each other. $\square$

We now show that Theorem 10.1 can be understood as describing a map R_α from drifts to drifts. Given a drift $\mathbf{V}$, let V^a be any representative and let W^a be a solution of equation (10.13). We define

(10.28) $R_\alpha(\mathbf{V}) = \{g_{ab}; (\mathbf{L}_g W)_{ab}, 0\} = \{g_{ab}; 0, -\operatorname{div}_g W\} \in \operatorname{Drift}_g$

and note that R_α is well-defined since W^a is uniquely determined up to adding a conformal Killing field. Proposition 10.2 shows that solutions of the momentum constraint are parameterized by a conformal momentum, a volumetric momentum, and a pair $(\mathbf{W}, \mathbf{V})$ of drifts that are joined at the hip by $\mathbf{W} = R_\alpha(\mathbf{V})$.

It turns out that R_α is invertible, and $\mathbf{W}$ determines $\mathbf{V}$ as well. The reverse process proceeds as follows: let W^a be a vector field such that $\mathbf{W} = \{g_{ab}; (\mathbf{L}_g W)_{ab}, 0\}$ and attempt to solve

(10.29) $\kappa \, \mathrm{d}\left(\frac{1}{N_{g,\alpha}} \operatorname{div}_g V\right) = \operatorname{div}_g\left(\frac{1}{2N_{g,\alpha}} \mathbf{L}_g W\right)$

for V^a. Now if equation (10.29) admits a solution, we can multiply the equation by an arbitrary divergence free vector field E^a and integrate by parts to find

(10.30) $\displaystyle\int_M \frac{1}{2N_{g,\alpha}} \langle \mathbf{L}_g W, \mathbf{L}_g E \rangle \, \omega_g = 0,$

which poses a compatibility condition on W^a. We will show that equation (10.29) admits a solution V^a if and only if the compatibility equation is satisfied, and the solution is unique up to adding a divergence free vector

field. Hence $\operatorname{div}_g V$ and the drift

$$(10.31) \qquad\qquad \mathbf{V} = \{g_{ab};\ 0, \operatorname{div}_g V\}$$

are uniquely determined by $\mathbf{W}$ once the compatibility condition (10.30) is met. In general an arbitrary representative W^a of $\mathbf{W}$ will fail the compatibility condition, but we will show that we can adjust W^a by adding a divergence-free vector field to remedy this deficiency. Note that adding a divergence-free vector field does not change the drift represented by W^a. The following three propositions contain the technical tools needed to carry out this procedure; we start by showing that equation (10.29) is solvable if the compatibility condition is met.

Proposition 10.3. *Suppose g_{ab} is a metric, N is a positive function and η_a is a 1-form. The equation*

$$(10.32) \qquad\qquad \kappa \, \mathrm{d}\left(\frac{1}{2N} \operatorname{div}_g V\right) = \eta$$

admits a smooth solution V^a if and only if

$$(10.33) \qquad\qquad \int_M \langle \eta, E \rangle \, \omega_g = 0$$

for all g-divergence-free vector fields E^a, in which case V^a is determined up to addition of a (smooth) divergence-free vector field.

Proof. Multiplying equation (10.33) by a divergence free vector field and integrating by parts shows that equation (10.33) is necessary for a solution to exist, and we henceforth assume η_a satisfies this condition. Applying the Helmholtz-Hodge decomposition we can write

$$(10.34) \qquad\qquad \eta_a = \nabla_a f + \mu_a$$

where f is a function, μ_a is divergence-free, and both are smooth. Multiplying equation (10.34) by μ^a, integrating, and using the compatibility condition we find that $\mu_a \equiv 0$ and hence

$$(10.35) \qquad\qquad \eta_a = \nabla_a f.$$

Since $\eta = \mathrm{d} f$, to solve equation (10.32) it suffices to find a smooth vector field V^a and a constant c such that

$$(10.36) \qquad\qquad \operatorname{div}_g V = \frac{2N}{\kappa}(f + c).$$

We pick c so that

$$(10.37) \qquad \int_M \frac{2N}{\kappa}(f+c)\omega_g = 0$$

and find a function u so that

$$(10.38) \qquad \Delta_g u = \frac{2N}{\kappa}(f+c).$$

Equation (10.36) is then solved taking $V^a = \nabla^a u$, and we see that V^a is smooth. If we add a smooth divergence-free vector field to V^a we obtain another solution, and we now show that all smooth solutions are obtained this way.

Suppose that V^a and $\widehat{V}^a$ are two solutions. It follows that

$$(10.39) \qquad \mathrm{d}\left(\frac{1}{N}\,\mathrm{div}_g(V - V^a)\right) = 0$$

and hence

$$(10.40) \qquad \mathrm{div}_g(V - \widehat{V}) = cN$$

for some constant c. Integrating over the manifold we conclude $c = 0$ and hence V^a and $\widehat{V}^a$ differ by a divergence-free vector field. And if V^a and $\widehat{V}^a$ are both smooth, so is the difference. $\qquad\square$

Adjusting the right-hand of equation (10.29) to meet the compatibility condition involves adding a suitable divergence-free vector field E^a, and we will see that E^a is the solution of a certain Stokes-like PDE. Let Lie_g be the Killing operator of g_{ab}, so $(\mathrm{Lie}_g X)_{ab} = \nabla_a X_b + \nabla_b X_a$, let $\mathrm{Lie}_g^* = -2\,\mathrm{div}_g$ be its adjoint, and let

$$(10.41) \qquad \mathcal{L}_{g,N} = \mathrm{Lie}_g^*\ 1/(2N)\ \mathrm{Lie}_g\,.$$

Given forcing terms η_a and h we form the Stokes equations

$$(10.42\mathrm{a}) \qquad \mathcal{L}_{g,N} E = \eta + \mathrm{d}\,p$$
$$(10.42\mathrm{b}) \qquad \mathrm{div}_g E = h$$

where the unknowns are E^a and the pressure p. In practice we will usually take $h = 0$ so that E^a is divergence-free, but it will aid a regularity bootstrap to consider a non-homogeneous term here.

Each of η_a and h must satisfy a compatibility condition in order for system (10.42) to be solvable. Multiplying the first equation of system (10.42) by a Killing field K^a we find

$$(10.43) \qquad\qquad \int_M \eta_a K^a \omega_g = 0,$$

and integrating the second equation of system (10.42) we have

$$(10.44) \qquad\qquad \int_M h \omega_g = 0.$$

These compatibility conditions are sufficient for there to exist a solution of the Stokes system.

Proposition 10.4. *Let g_{ab} be a smooth metric and let N be a smooth positive function. Let η_a be a 1-form in $W^{-1,2}$ that is L^2 orthogonal to the proper Killing fields and let h be a function in L^2 that is L^2 orthogonal to the constants. Then there exists a vector field $E^a \in W^{1,2}$ and a function $p \in L^2$ solving the Stokes system* (10.42) *in the sense of distributions, and the solution is unique up to adding a Killing field to E^a and a constant to p.*

Proof. We can reduce to the case where $h = 0$ by solving

$$(10.45) \qquad\qquad \Delta f = h$$

for a function $f \in W^{2,2}$, which is possible since h is orthogonal to the constants, and writing $E^a = \widehat{E}^a + \nabla^a f$ where $\widehat{E}^a$ is an unknown divergence-free function. Since $\nabla f \in W^{1,2}$ we have $\mathcal{L}_{g,N} \nabla f \in W^{-1,2}$ and we see that (E^a, p) solves the original system if and only if $(\widehat{E}^a, p)$ solves the system with η replaced with $\eta - \mathcal{L}_{g,N} \nabla f$ and h replaced with 0. Henceforth we assume $h = 0$, and we seek a divergence-free vector field E^a and a pressure p solving (10.42a).

First suppose g_{ab} has no nontrivial proper Killing fields, and let $J^{1,2}$ be subspace of divergence-free $W^{1,2}$ vector fields. For E^a and $F^a \in J^{1,2}$, define

$$(10.46) \qquad A(E^a, F^a) = \int_M \frac{1}{2N} \langle \mathrm{Lie}_g E, \mathrm{Lie}_g F \rangle_g \, \omega_g.$$

We claim that there is a constant c such that $A(E^a, E^a) \geq c \int_M |E|_g^2 \omega_g$ for all $E^a \in J^{1,2}$. Suppose not. Then we can find a sequence of divergence-free

vector fields $\{E^a_{(k)}\}_k$ such that $A(E^a_{(k)}, E^a_{(k)}) \to 0$ and such that each $E^a_{(k)}$ has norm 1 in in L^2. Recall Korn's inequality [8], which implies that there is a constant C such that

$$(10.47) \qquad ||E||^2_{W^{1,2}} \leq C \left[\int_M \langle \mathrm{Lie}_g E, \mathrm{Lie}_g E \rangle_g \, \omega_g + ||E||^2_{L^2} \right].$$

Since N is bounded above, a similar inequality holds replacing

$$\int_M \langle \mathrm{Lie}_g \cdot, \mathrm{Lie}_g \cdot \rangle_g \, \omega_g$$

with A and hence the sequence $\{E^a_{(k)}\}_k$ is bounded in $W^{1,2}$. So a subsequence converges weakly in $W^{1,2}$ and strongly in L^2 to a limit $\widetilde{E}^a$. The quadratic form $E \mapsto A(E, E)$ is non-negative definite, so it is weakly lower semicontinuous. Hence the weak limit satisfies $A(\widetilde{E}, \widetilde{E}) = 0$ and is a Killing field. Since $||\widetilde{E}||_{L^2} = 1$ as well, g_{ab} admits a nontrivial Killing field, which is a contradiction.

We have now established that $||E||^2_{L^2}$ is controlled by $A(E, E)$, and it then follows from inequality (10.47) that there is a constant c such that

$$(10.48) \qquad A(E^a, E^a) \geq c||E||^2_{W^{1,2}}$$

for all $E \in J^{1,2}$. So A is coercive over $J^{1,2}$ and the Lax-Milgram theorem implies there is a unique $E^a \in J^{1,2}$ such that

$$(10.49) \qquad \int_M \frac{1}{2N} \langle DE, DF \rangle_g \, \omega_g = \int_M \langle \eta, F \rangle \, \omega_g$$

for all $F^a \in J^{1,2}$. Now

$$(10.50) \qquad F^a \mapsto \int_M \frac{1}{2N} \langle DE, DF \rangle_g \, \omega_g - \int_M \langle \eta, F \rangle \, \omega_g$$

is a continuous functional on $W^{1,2}$ that vanishes on $J^{1,2}$. From the Helmholtz-Hodge decomposition of $W^{-1,2}$ there is a unique weakly divergence free G^a in $W^{-1,2}$ and function p in L^2, uniquely determined up to a constant, such that functional (10.50) is equal to

$$(10.51) \qquad G^a + \nabla^a p.$$

But since this functional vanishes on $J^{1,2}$ we conclude that $G^a = 0$ and hence

$$(10.52) \qquad \int_M \frac{1}{2N_{g,\alpha}} \langle DE, DF \rangle_g \, \omega_g - \int_M \langle \eta, F \rangle \, \omega_g = \int_M p \, \mathrm{div}_g F \omega_g$$

for all $F^a \in W^{1,2}$, which proves existence if g_{ab} admits no nontrivial Killing fields. Moreover, if $(\widehat{E}^a, \widehat{p})$ is any solution of the Stokes system, we see that $\widehat{E}^a$ also satisfies (10.49) and therefore equals E^a. But then $\widehat{p}$ satisfies the equation (10.52) for the pressure and is therefore equal to p plus a constant.

If g_{ab} admits nontrivial Killing fields, we replace the space $J^{1,2}$ in the proof above with the L^2 orthogonal complement in $J^{1,2}$ of the Killing fields; call this new space $\widehat{J}^{1,2}$. The proof above then finds $E^a \in \widehat{J}^{1,2}$ such that equation (10.49) holds for all F^a in $\widehat{J}^{1,2}$. Since η^a is L^2 orthogonal to the proper Killing fields, equation (10.49) holds for all $F^a \in J^{1,2}$ and the remainder of the proof continues without change. $\qquad\square$

Proposition 10.4 establishes existence of weak solutions of the Stokes system, and we now show that when the forcing terms are smooth, so are the solutions.

Proposition 10.5. *In Proposition 10.4, if $(\eta_a, h) \in W^{k-2,2} \times W^{k-1,2}$ for some integer $k \geq 2$, then $(E^a, p) \in W^{k,2} \times W^{k+1,2}$. In particular, if η_a and h are smooth, so are E^a and p.*

Proof. Suppose $(\eta_a, h) \in L^2 \times W^{1,2}$. Applying the divergence to equation (10.42a) we find that p is a weak solution of

$$
\begin{aligned}
\Delta\, p &= [\mathrm{div}_g, \mathcal{L}_{g,N}]E + \mathcal{L}_{g,N}\,\mathrm{div}_g E - \mathrm{div}_g \eta \\
&= [\mathrm{div}_g, \mathcal{L}_{g,N}]E + \mathcal{L}_{g,N}h - \mathrm{div}_g \eta.
\end{aligned}
$$
(10.53)

Since $[\mathrm{div}_g, \mathcal{L}_{g,N}]$ is a second-order operator and since $E^a \in W^{1,2}$, the first term on the right-hand side of equation (10.53) belongs to $W^{-1,2}$. It is easy to see that the remaining terms on the right-hand side of equation (10.53) also belong to $W^{-1,2}$ as well and hence $p \in W^{1,2}$. But then the right-hand side of equation (10.42a) is in L^2 and since $\mathcal{L}_{g,N}$ is elliptic, we conclude that $E^a \in W^{2,2}$.

To obtain higher regularity, we proceed by a bootstrap. For example, suppose $\eta_a \in W^{1,2}$ and $h \in W^{2,2}$. Let ∂ be a first order operator. Then $\widehat{E}^a = \partial E^a$ and $\widehat{p} = \partial p$ belong to $W^{1,2}$ and L^2 respectively and satisfy

$$
\begin{aligned}
\mathcal{L}_{g,N}\widehat{E} &= \partial\eta + [\mathcal{L}_{g,N}, \partial]\,E - [\mathrm{d}, \partial]\,p + \mathrm{d}\,\widehat{p} \\
\mathrm{div}_g\,\widehat{E} &= [\mathrm{div}_g, \partial]\,E + \partial h
\end{aligned}
$$
(10.54)

where $[\cdot, \cdot]$ is the commutator. Since $(E^a, p) \in W^{2,2} \times W^{1,2}$, and since $(\eta, h) \in W^{1,2} \times W^{2,2}$, we see that the right-hand sides of equations (10.54) belong to L^2 and $W^{1,2}$ respectively. Hence by our previous result, $(\partial E^a, \partial p) \in$

$W^{2,2} \times W^{1,2}$. So $(E^a, p) \in W^{3,2} \times W^{2,2}$, and the remainder of the bootstrap continues similarly. □

From Propositions 10.3 and 10.4 we obtain the following analogue of Theorem 10.1.

Theorem 10.6. *Suppose g_{ab} is a metric, α is a lapse form, and $\mathbf{W} \in$ Drift$_g$. Let W^a be any vector field such that*

$$(10.55) \qquad \mathbf{W} = \{g_{ab}; \ (\mathbf{L}_g W)_{ab}, 0\} = \{g_{ab}; \ 0, -\operatorname{div}_g W\}.$$

Then there is a divergence-free vector field E^a, unique up to addition of a proper Killing field, such that

$$(10.56) \qquad \kappa \, \mathrm{d}\left(\frac{1}{N_{g,\alpha}} \operatorname{div}_g(V)\right) = \operatorname{div}_g\left(\frac{1}{2N_{g,\alpha}}[\mathbf{L}_g(W + E)]\right).$$

admits a solution V^a. The solution V^a is unique up to addition of a divergence-free vector field, and this space of solutions does not depend on the choice of W^a or on the choice of divergence free vector field E^a such that equation (10.56) is solvable.

Proof. From Proposition 10.3 we know that equation (10.56) admits a solution so long as

$$(10.57) \qquad \int_M \left\langle \operatorname{div}_g\left(\frac{1}{2N} \mathbf{L}_g(W + E)\right), F \right\rangle_g \omega_g = 0$$

for all divergence-free vector fields F^a. Thinking of W^a as fixed and E^a as an unknown vector field we see that E^a satisfies

$$(10.58) \quad \int_M \left\langle \operatorname{div}_g\left(\frac{1}{2N} \mathbf{L}_g E\right), F \right\rangle_g \omega_g = -\int_M \left\langle \operatorname{div}_g\left(\frac{1}{2N} \mathbf{L}_g W\right), F \right\rangle_g \omega_g$$

for all divergence-free vector fields. Since E^a is divergence-free, $(\mathbf{L}_g E)_{ab} = \operatorname{Lie}_g E$ and E^a is a weak solution of the Stokes equation

$$(10.59) \qquad D_g^*\left(\frac{1}{2N} \operatorname{Lie}_g E\right) = -\operatorname{div}_g\left(\frac{1}{2N} \mathbf{L}_g W\right) + \nabla p$$

for some function p. Proposition 10.4 shows that there is a solution of (10.59), and that it is unique up to addition of a proper Killing field.

Thus we have shown there is a divergence-free vector field E^a, unique up to addition of a proper Killing field, such that equation (10.56) admits a solution V^a, and Proposition 10.3 shows that the solution is unique up to addition of a divergence-free vector field. The proof that this space of solutions is independent of the choice of W^a is analogous to the same step of Theorem 10.1. $\qquad\square$

11. Drift velocity, drift momentum, and drift kinetic energy

We saw in Section 10 that the map $k_\alpha : T_g^*\mathcal{M}/\mathcal{D}_0 \to T_g\mathcal{M}/\mathcal{D}_0$ described in diagram 10.1 can fail to be an isomorphism. In terms of the decompositions

$$(11.1) \qquad T_g^*\mathcal{M}/\mathcal{D}_0 = T_{[g]}^*\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}^*\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Im}(j_\alpha^{\mathrm{Drift}})$$

and

$$(11.2) \qquad T_g\mathcal{M}/\mathcal{D}_0 \approx T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Drift}_g .$$

given by Propositions 10.2 and 9.5 respectively, a computation shows

$$(11.3) \qquad k_\alpha(\boldsymbol{\sigma}, -2\kappa\tau^*, j_\alpha^{\mathrm{Drift}}(\mathbf{V})) = ((j_\alpha^{\mathcal{C}})^{-1}(\boldsymbol{\sigma}), (j_\alpha^{\mathcal{V}})^{-1}(-2\kappa\tau^*), \mathbf{V} - \mathbf{W})$$

where $\mathbf{W} = R_\alpha(\mathbf{V})$ and R_α is the map described in equation (10.28). Since the Legendre transformations $j_\alpha^{\mathcal{C}}$ and $j_\alpha^{\mathcal{V}}$ are isomorphisms, we see that k_α fails to be an isomorphism precisely when $\mathbf{V} \mapsto \mathbf{V} - R_\alpha(\mathbf{V})$ fails to be an isomorphism. We address this difficulty by treating the pair $(\mathbf{W}, \mathbf{V})$, linked by the equation $\mathbf{W} = R_\alpha(\mathbf{V})$, as the drift component of motion rather than the difference $\mathbf{V} - \mathbf{W}$. Since R_α is invertible, we can parameterize the pairs $(\mathbf{W}, \mathbf{V})$ in terms of either component, and we will refer to $\mathbf{W}$ as **conformal drift** and $\mathbf{V}$ as **volumetric drift**.

Suppose that we parameterize drift pairs in terms of their volumetric component. To this end, given a lapse form α and a shift X^a we define a projection

$$(11.4) \qquad \pi_{\alpha,X^a}^{\mathrm{Drift}} : T_g\mathcal{M} \to \mathrm{Drift}_g$$
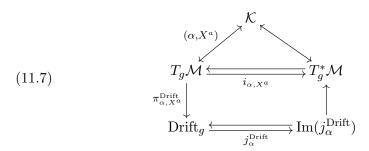
as follows. Given $(g_{ab}; u_{ab}, \beta) \in T_g\mathcal{M}$ we apply volumetric York decomposition to write

$$(11.5) \qquad \beta = N_{g,\alpha}\tau^* + \mathrm{div}_g(V + X)$$

for a unique constant τ^* and a vector field V^a that is unique up to adding a divergence-free vector field. Then

$$(11.6) \qquad \pi_{\alpha,X^a}^{\text{Drift}}((g_{ab};\ u_{ab},\beta)) = \{g_{ab};\ 0,\operatorname{div}_g V\}.$$

We claim that the diagram

$$(11.7)$$

commutes, except that traversing the bottom loop starting from the middle row is a projection. It is enough to establish the following.

Proposition 11.1. *The map obtained from diagram* (11.7) *starting at* Drift_g *and traversing the bottom loop back to* Drift_g *is the identity.*

Proof. Let $\mathbf{V} = \{g_{ab};\ 0,\operatorname{div}_g V\} \in \text{Drift}_g$. From the definition of j_α^{Drift} in equation (10.22) we find

$$(11.8) \qquad j_\alpha^{\text{Drift}}(\mathbf{V}) = \left(g_{ab};\ \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g W)_{ab}, -\frac{2\kappa}{N_{g,\alpha}} \operatorname{div}_g(V+Q) \right)^*$$

where the vector field W^a and the conformal Killing field Q^a are determined by Theorem 10.1. Applying i_{α,X^a}^{-1} from equation (2.7) we arrive at

$$(11.9) \qquad (g_{ab};(\mathbf{L}_g(W+X))_{ab},\operatorname{div}_g(V+Q+X))$$

and applying $\pi_{\alpha,X^a}^{\text{Drift}}$ we complete the loop at

$$(11.10) \qquad \{g_{ab};\ 0,\operatorname{div}_g(V+Q)\}.$$

Since Q^a is a conformal Killing field,

$$(11.11) \qquad \{g_{ab};\ 0,\operatorname{div}_g(Q)\} = \{g_{ab};\ \mathbf{L}_g Q,\operatorname{div}_g(Q)\} = 0$$

and hence

$$(11.12) \qquad \{g_{ab};\ 0,\operatorname{div}_g(V+Q)\} = \{g_{ab};\ 0,\operatorname{div}_g(V)\} = \mathbf{V}.$$

Thus traversing the loop is the identity. □

Drift velocity and momentum are defined following the pattern seen previously for conformal and volumetric quantities. Given $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$ we form diagram (11.7) and descend the left hand side from $\mathcal{K}$ to Drift_g. If K_{ab} has mean curvature τ, we apply volumetric York splitting to write

$$(11.13) \qquad \tau = \tau^* + \frac{1}{N_{g,\alpha}} \operatorname{div} V,$$

and equations (2.9), (11.5) and (11) show that the drift velocity is

$$(11.14) \qquad \{g_{ab};\ 0, \operatorname{div}_g V\}.$$

Note that although both maps on the left-hand side of diagram (11.7) depend on the shift X^a, their composition does not and only depends on the lapse form α. Drift momentum is obtained from drift velocity by applying $j_\alpha^{\mathrm{Drift}}$.

Definition 11.2. Let $(g_{ab}, K_{ab}) \in \mathcal{M} \times \mathcal{K}$ and let α be a lapse form. The **drift velocity** of (g_{ab}, K_{ab}), as measured by α, is

$$(11.15) \qquad v_\alpha^{\mathrm{Drift}}(g_{ab}, K_{ab}) = \{g_{ab};\ 0, \operatorname{div}_g V\}$$

where V^a is obtained by the splitting (11.13) of $\tau = g^{ab} K_{ab}$. The **drift momentum** of (g_{ab}, K_{ab}), as measured by α, is

$$(11.16) \quad m_\alpha^{\mathrm{Drift}}(g_{ab}, K_{ab}) = \left\{ g_{ab};\ \frac{1}{2N_{g,\alpha}} (\mathbf{L}_g W)_{ab}, -\frac{2\kappa}{N_{g,\alpha}} \operatorname{div}_g (V + Q) \right\}^*$$

where W^a and Q^a are the vector field and conformal Killing field obtained from Theorem 10.1.

Since $\mathrm{Drift}_g \subseteq T_g \mathcal{M}/\mathcal{D}_0$, every element of $T_g^* \mathcal{M}/\mathcal{D}_0$ defines a functional on Drift_g. Since $j_\alpha^{\mathrm{Drift}} : \mathrm{Drift}_g \to T_g^* \mathcal{M}/\mathcal{D}_0$, we can therefore consider $j_\alpha^{\mathrm{Drift}}$ as a map into $(\mathrm{Drift}_g)^*$ and it is then natural to identify a Lagrangian such that $j_\alpha^{\mathrm{Drift}}$ is its Legendre transformation.

Definition 11.3. Given $\mathbf{V} \in \mathrm{Drift}_g$ the **drift kinetic energy** of $\mathbf{V}$, as measured by α, is

$$(11.17) \qquad K_\alpha^{\mathrm{Drift}}(\mathbf{V}) = -\int \kappa (\operatorname{div}_g V)^2 \alpha$$

where V^a is any representative of $\mathbf{V}$ such that

$$(11.18) \qquad\qquad \int \operatorname{div}_g(V) \operatorname{div}_g Q\alpha = 0$$

for all conformal Killing fields Q^a; note that Theorem 10.1 ensures that $\operatorname{div}_g V$ (and hence drift kinetic energy) is uniquely determined by $\mathbf{V}$.

To show that $K_\alpha^{\mathrm{Drift}}$ is a Lagrangian, one ought to demonstrate a configuration space such that $K_\alpha^{\mathrm{Drift}}$ is a function on its tangent bundle. Clearly each tangent space must be isomorphic to Drift_g, but the right choice of base space is not clear. So we content ourselves by restricting our attention to each fibre Drift_g of the presumed total space and show that $j_\alpha^{\mathrm{Drift}}$ is the Legendre transformation of $K_\alpha^{\mathrm{Drift}}$ on that fibre. Consider a path $\mathbf{V}(t) = \{g_{ab};\ 0, \operatorname{div}_g V(t)\}$ of drifts where $V^a(t)$ is a path of vector fields satisfying condition (11.18). Since $V^a(t)$ satisfies the compatibility condition, there exists a path of vector fields $W^a(t)$ with

$$(11.19) \qquad \operatorname{div}_g\left(\frac{1}{2N_{g,\alpha}} \mathbf{L}_g W\right) = \kappa\,\mathrm{d}\left(\frac{1}{N_{g,\alpha}} \operatorname{div}_g V\right).$$

Then, recalling equations (1.25) and (1.13), we find

$$(11.20) \qquad \frac{d}{dt} K_\alpha^{\mathrm{Drift}}(\mathbf{V}) = -2\kappa \int_M \operatorname{div}_g V \operatorname{div}_g \dot{V}\alpha$$

$$= \int_M \left(\frac{-2\kappa}{N_{g,\alpha}} \operatorname{div}_g V\right) \operatorname{div}_g \dot{V} \omega_g$$

$$= \left\langle \left\{g_{ab};\ \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g W)_{ab}, -2\kappa\frac{1}{N_{g,\alpha}} \operatorname{div}_g V\right\}^*, \{g_{ab};\ 0, \operatorname{div}_g \dot{V}\}\right\rangle$$

$$= \left\langle j_\alpha^{\mathrm{Drift}}(\mathbf{V}), \dot{\mathbf{V}}\right\rangle$$

where the various dependencies on t in equation (11.20) have been suppressed. Thus $K_\alpha^{\mathrm{Drift}}$ is the desired Lagrangian.

The preceding discussion was based on parameterizing pairs $(\mathbf{W}, \mathbf{V})$ with $\mathbf{W} = R_\alpha(\mathbf{V})$ in terms of volumetric drift $\mathbf{V}$. If we use conformal drift $\mathbf{W}$ instead we obtain a dual notion of drift velocity and kinetic energy which we now summarize briefly. The drift velocity of (g_{ab}, K_{ab}) is

$$(11.21) \qquad \widehat{v}_\alpha^{\mathrm{Drift}}(g_{ab}, K_{ab}) = \{g_{ab};\ (\mathbf{L}_g W)_{ab}, 0\}$$

where W^a is any vector field arising from the conformal York decomposition of the trace free part A_{ab} of K_{ab}:

$$(11.22) \qquad A_{ab} = \sigma_{ab} + \frac{1}{2N_{g,\alpha}}(\mathbf{L}_g W)_{ab}.$$

The drift momentum is

$$(11.23) \quad \widehat{m}_\alpha^{\text{Drift}}(g_{ab}, K_{ab}) = \left\{ g_{ab}; \ \frac{1}{2N_{g,\alpha}}[\mathbf{L}_g(W+E)]_{ab}, -\frac{2\kappa}{N_{g,\alpha}}\operatorname{div}_g(V) \right\}^*$$

where the vector field V^a and the divergence-free vector field E^a are provided by Theorem 10.6. The drift kinetic energy of $\mathbf{W}$ is

$$(11.24) \qquad \widehat{K}_\alpha^{\text{Drift}}(\mathbf{W}) = \int_M \frac{1}{4}|\mathbf{L}_g W|^2 \alpha.$$

where W^a is any representative of $\mathbf{W}$ satisfying the compatibility condition (10.30) It is easy to see that

$$(11.25) \qquad m_\alpha^{\text{Drift}}(g_{ab}, K_{ab}) = \widehat{m}_\alpha^{\text{Drift}}(g_{ab}, K_{ab})$$

if and only if the pair (g_{ab}, K_{ab}) satisfies the momentum constraint, so the drift momentum of a solution of the constraint equations is well-defined regardless of which factor $\mathbf{W}$ or $\mathbf{V}$ we use to parameterize drift velocity. The choice of using $\mathbf{W}$ or $\mathbf{V}$ is one of emphasis between conformal and volumetric motion: we can apparently measure drift of the conformal class relative to the normal direction, or we can measure drift of the volume form relative to the normal direction, but these two motions are linked by the momentum constraint and are not independent. This class of linked motion can be parameterized in terms of Drift_g, but we have two distinct natural parameterizations.

Continuing with our choice to parameterize drift motion by its volumetric component we have the following.
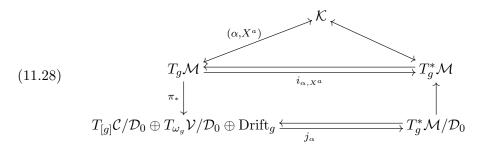
Theorem 11.4. *Let $g_{ab} \in \mathcal{M}$, let α be a lapse form, and let X^a be a shift. The map*

$$(11.26) \qquad j_\alpha : T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \text{Drift}_g \to T_g^*\mathcal{M}/\mathcal{D}_0$$

defined by

$$(11.27) \qquad j_\alpha = j_\alpha^\mathcal{C} \oplus j_\alpha^\mathcal{V} \oplus j_\alpha^{\text{Drift}}$$

is an isomorphism. Moreover, consider the diagram

(11.28)

$$
\begin{array}{ccc}
& \mathcal{K} & \\
{\scriptstyle(\alpha,X^a)}\nearrow & & \nwarrow \\
T_g\mathcal{M} & \underset{i_{\alpha,X^a}}{\overset{}{\rightleftarrows}} & T_g^*\mathcal{M} \\
\pi_* \downarrow & & \uparrow \\
T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Drift}_g & \underset{j_\alpha}{\rightleftarrows} & T_g^*\mathcal{M}/\mathcal{D}_0
\end{array}
$$

where the first two components of π_* are the natural pushforwards and the third component is $\pi_{\alpha,X^a}^{\mathrm{Drift}}$. Traversing the bottom loop of diagram (11.28) starting on the bottom row is the identity, and traversing the outermost loop starting at $\mathcal{K}$ is a projection onto second fundamental forms K_{ab} such that (g_{ab}, K_{ab}) solves the momentum constraint.

Proof. That j_α is an isomorphism follows from the fact that $j_\alpha^{\mathcal{C}}$, $j_\alpha^{\mathcal{V}}$, and $j_\alpha^{\mathrm{Drift}}$ are isomorphisms, along with Proposition 10.2. That traversing the bottom loop starting from the bottom row is the identity follows from the same fact for diagrams (4.2), (7.2) and (11.7). As a consequence, traversing the bottom loop starting from the middle row must be a projection. Since the maps i_{α,X^a}^{-1}, π_*, and j_α are surjective, the image of $T_g^*\mathcal{M}$ after traversing the bottom loop is the image of $T_g^*\mathcal{M}/\mathcal{D}_0$ under the natural pullback, i.e., the divergence-free elements. Hence traversing the outermost loop starting at $\mathcal{K}$ is a projection onto the second fundamental forms with divergence-free ADM momenta, i.e., the solutions of the momentum constraint. $\qquad\square$

Note that although Proposition 9.5 implies

(11.29) $$ T_g\mathcal{M}/\mathcal{D}_0 \approx T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \mathrm{Drift}_g, $$

the map π_* from Theorem 11.4 is *not* the pushforward from $T_g\mathcal{M}$ to $T_g\mathcal{M}/\mathcal{D}_0$. Indeed, if W^a and V^a are vector fields, the pushforward of

$$ (g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V) $$

is the drift $\{g_{ab};\ 0, \mathrm{div}_g(V - W)\}$, but the drift component of

$$ \pi_*((g_{ab};\ (\mathbf{L}_g W)_{ab}, \mathrm{div}_g V)) $$

is $\{g_{ab};\ 0, \mathrm{div}_g(V)\}$. This is the key distinction between diagrams (10.1) and (11.7) and is what ensures that j_α is always an isomorphism even though k_α

from diagram (10.1) can fail to be one. As always, the choice to make the drift component of $\pi_*((g_{ab}; (\mathbf{L}_g W)_{ab}, \operatorname{div}_g V))$ equal to $\{g_{ab}; 0, \operatorname{div}_g(V)\}$ rather than $\{g_{ab}; \mathbf{L}_g W, 0\} = \{g_{ab}; 0, -\operatorname{div}_g W\}$ is arbitrary, and a result analogous to Theorem 11.4 holds when using conformal drift.

On the other hand, if we do identify $T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \operatorname{Drift}_g$ with $T_g\mathcal{M}/\mathcal{D}_0$ (thinking of π_* as a projection into $T_g\mathcal{M}/\mathcal{D}_0$ that is not the push-forward), we can interpret j_α as being the Legendre transformation of the total kinetic energy

$$(11.30) \qquad \mathcal{K}_\alpha(\mathbf{u}, v, \mathbf{V}) = \mathcal{K}_\alpha^{\mathcal{C}}(\mathbf{u}) + \mathcal{K}_\alpha^{\mathcal{V}}(v) + \mathcal{K}_\alpha^{\operatorname{Drift}}(\mathbf{V})$$

where $(\mathbf{u}, v, \mathbf{V}) \in T_{[g]}\mathcal{C}/\mathcal{D}_0 \oplus T_{\omega_g}\mathcal{V}/\mathcal{D}_0 \oplus \operatorname{Drift}_g \approx T_g\mathcal{M}/\mathcal{D}_0$. Equation (11.30) can be obtained from the ADM kinetic energy, but we must account for the fact that we are representing drift velocity in terms of $\mathbf{V}$ not $\mathbf{W}$. Recall that if $(g_{ab}; u_{ab}, \beta) \in T_g\mathcal{M}$, the ADM kinetic energy is

$$(11.31) \qquad \int \frac{1}{4}|u - \mathbf{L}_g X|_g^2 - \kappa(\beta - \operatorname{div}_g X)^2 \alpha$$

Decomposing u_{ab} and β according to equations (4.25) and (7.23) we can rewrite the kinetic energy as

$$(11.32) \qquad \int N_{g,\alpha}^2 |\sigma|_g^2 - N_{g,\alpha}^2(\tau^*)^2 + \kappa\frac{1}{4}|\mathbf{L}_g W|_g^2 - \kappa(\operatorname{div}_g V)^2 \alpha$$

The first two terms are the conformal and volumetric kinetic energies. If the momentum constraint is satisfied, then V^a will satisfy the compatibility condition (11.18) and hence the term involving $\operatorname{div}_g V$ in expression (11.32) is the drift kinetic energy. So the total kinetic energy (11.30) is obtained from the ADM kinetic energy by dropping the $\mathbf{L}_g W$ term. In the dual treatment of drift velocity, the total kinetic energy would be obtained by keeping the $\mathbf{L}_g W$ term and dropping the $\operatorname{div}_g V$ term instead.

12. Drifts and the conformal method

Theorem 11.4 and Proposition 10.2 show that given a choice of lapse form α, solutions of the momentum constraint can be parameterized in terms of their conformal, volumetric, and drift momenta. Hence drift momentum naturally complements the candidate parameters for conformal-like methods discussed at the end of Section 8, and we consider in this section variations of the conformal method that incorporate drift as a parameter.

Suppose $(\overline{g}_{ab}, \overline{K}_{ab})$ is a solution of the momentum constraint with conformal momentum $\boldsymbol{\sigma} = \{\overline{g}_{ab};\ \overline{\sigma}_{ab}\}^*$, volumetric momentum $-2\kappa\tau^*$, and drift momentum

$$(12.1) \qquad \left\{ g_{ab};\ \frac{1}{2N_{\overline{g}}}\,\mathbf{L}_{\overline{g}}\,W_{ab}, -\frac{2\kappa}{N_{\overline{g},\alpha}}\,\operatorname{div}_{\overline{g}} V \right\}^*$$

where W^a and V^a are vector fields solving the drift equation

$$(12.2) \qquad \operatorname{div}_{\overline{g}}\left[\frac{1}{2N_{\overline{g},\alpha}}\,\mathbf{L}_{\overline{g}}\,W\right] = \kappa\ \mathrm{d}\left[\frac{1}{N_{\overline{g},\alpha}}\,\operatorname{div}_{\overline{g}} V\right].$$

Then

$$(12.3) \qquad \overline{K}_{ab} = \overline{\sigma}_{ab} + \frac{1}{2N_{\overline{g},\alpha}}\,\mathbf{L}_{\overline{g}}\,W + \frac{1}{n}\left(\tau^* + \frac{1}{N_{\overline{g},\alpha}}\,\operatorname{div}_{\overline{g}} V\right)\overline{g}_{ab}$$

and K_{ab} has mean curvature

$$(12.4) \qquad \tau = \overline{g}^{ab}\overline{K}_{ab} = \tau^* + \frac{1}{N_{\overline{g},\alpha}}\,\operatorname{div}_{\overline{g}} V.$$

Now suppose $\overline{g}_{ab} = \phi^{q-2}g_{ab}$ for some conformally related metric g_{ab}. The conformal momentum $\boldsymbol{\sigma}$ is represented at g_{ab} by $\sigma_{ab} = \phi^2\overline{\sigma}_{ab}$ and the volumetric momentum is still $-2\kappa\tau^*$. Using the transformation law $N_{\overline{g},\alpha} = \phi^q N_{g,\alpha}$ and the conformal transformation laws for divergences and for the conformal Killing operator, equation (12.2) can be written in terms of g_{ab} as

$$(12.5) \qquad \operatorname{div}_g\left[\frac{1}{2N_{g,\alpha}}\,\mathbf{L}_g\,W\right] = \kappa\phi^q\,\mathrm{d}\left[\frac{\phi^{-2q}}{N_{g,\alpha}}\,\operatorname{div}_g(\phi^q V)\right].$$

Note that since

$$(12.6) \qquad \tau = \tau^* + \frac{\phi^{-2q}}{N_{g,\alpha}}\,\operatorname{div}_g(\phi^q V),$$

equation (12.5) is simply the CTS-H momentum constraint after substituting equation (12.6). The Hamiltonian constraint for $(\overline{g}_{ab}, \overline{K}_{ab})$ can also be written in terms of g_{ab} making this same substitution and we arrive at two conformal methods depending on whether we specify V^a or W^a in equation (12.5). First, using Theorem 10.1 we can specify V^a up to a conformal Killing field Q^a and we obtain the following modification of the CTS-H method.

Problem 12.1 (CTS-H with Volumetric Drift). *Let g_{ab} be a metric, σ_{ab} a transverse traceless tensor with respect to g_{ab}, τ^* a constant, V^a a vector field, and α a lapse form. Setting $N = \omega_g/\alpha$, find a conformal factor ϕ, a vector field W^a and a conformal Killing field Q^a such that*

$$
- a\,\Delta_g\,\phi + R_g\phi - \left| \sigma + \frac{1}{2N}\,\mathbf{L}_g\,W \right|_g^2 \phi^{-q-1}
$$

(12.7)
$$
+ \kappa \left(\tau^* + \frac{\phi^{-2q}}{N}\,\mathrm{div}(\phi^q(V+Q)) \right)^2 \phi^{q-1} = 0,
$$

$$
\mathrm{div}_g\left(\frac{1}{2N}\,\mathbf{L}_g\,W \right) - \kappa\phi^q\,\mathrm{d}\left(\frac{\phi^{-2q}}{N}\,\mathrm{div}_g(\phi^q(V+Q)) \right) = 0.
$$

Alternatively, we can apply Theorem 10.6 and specify W^a up to a $\bar{g}_{ab}$ divergence-free vector field. Since $\phi^{-q}E^a$ is divergence-free with respect to $\bar{g}_{ab}$ if and only if E^a is divergence-free with respect to g_{ab} we obtain the following.

Problem 12.2 (CTS-H with Conformal Drift). *Let g_{ab} be a metric, σ_{ab} a transverse traceless tensor with respect to g_{ab}, τ^* a constant, W^a a vector field, and α a lapse form. Setting $N = \omega_g/\alpha$, find a conformal factor ϕ, a vector field V^a and a divergence-free vector field E^a such that*

$$
- a\,\Delta_g\,\phi + R_g\phi - \left| \sigma + \frac{1}{2N}\,\mathbf{L}_g(W + \phi^{-q}E) \right|_g^2 \phi^{-q-1}
$$

(12.8)
$$
+ \kappa \left(\tau^* + \frac{\phi^{-2q}}{N}\,\mathrm{div}_g(\phi^q V) \right)^2 \phi^{q-1} = 0,
$$

$$
\mathrm{div}_g\left(\frac{1}{2N}\,\mathbf{L}_g(W + \phi^{-q}E) \right) - \kappa\phi^q\,\mathrm{d}\left(\frac{\phi^{-2q}}{N}\,\mathrm{div}_g(\phi^q(V)) \right) = 0.
$$

The drift parameterizations in systems (12.7) and (12.8) pose significant analytical challenges beyond those of the standard conformal method. For example, both equations of both systems are second order in ϕ, and the Hamiltonian constraint is no longer semilinear in ϕ. Although the equations for the standard conformal method are technically simpler, and therefore more attractive at first glance, it may be that more sophisticated equations are required to effectively parameterize non-CMC solutions of the constraint equations. We will return to the analysis systems (12.7) and (12.8) in future work. For now, we make some observations to suggest that these systems are not intractable. First, for fixed ϕ, the problem for the momentum constraint

is equivalent to solving one of the variations (10.13) or (10.29) of the drift equation (10.10) with respect to the metric $\phi^{q-2} g_{ab}$. These are well-posed problems and hence it is natural to consider iteration schemes, not unlike those for the standard conformal method, that alternate between solving the Hamiltonian and momentum constraints. Semilinearity of the Hamiltonian constraint could be restored in such an iteration scheme by constructing a sequence of mean curvatures according to equation (12.4). Moreover, since the CMC conformal method arises as the special case $V^a = 0$ in system (12.7) or $W^a = 0$ in system (12.8), a natural first step is to consider the near-CMC case where V^a or W^a is small. It seems likely that near-CMC results similar to those of the standard conformal method are feasible, and the harder work will be determining the extent to which the geometric and physical structures that motivate the drift parameterizations are sufficient to address the shortcomings of the standard conformal method for non-constant mean curvature.

There is also the possibility that Problems 12.1 and 12.2 require further refinement. We are representing drifts by vector fields, and this introduces a degeneracy not present in the standard conformal method. A solution of the constraint equations uniquely determines a conformal class and, after selecting a lapse form, a conformal, volumetric and drift momentum. But the drift momentum determines a subspace of vector fields: if $(g_{ab}, \sigma_{ab}, \tau^*, V^a, \alpha)$ is a tuple of conformal parameters for system (12.7) generating a solution $(\bar{g}_{ab}, \bar{K}_{ab})$ of the constraints, this same solution will be generated by $(g_{ab}, \sigma_{ab}, \tau^*, V^a + \bar{E}^a + \bar{Q}^a, \alpha)$ whenever $\bar{Q}^a$ is a conformal Killing field for $\bar{g}_{ab}$ and $\bar{E}^a$ is divergence-free with respect to $\bar{g}_{ab}$. The conformal Killing field is not problematic since the set of conformal Killing fields is a conformal invariant, but the divergence-free vector fields for $\bar{g}_{ab}$ are not known *a priori*, and this poses a difficulty in determining if two tuples of conformal parameters determine the same solution of the constraints. A successful analysis system (12.7) should exhibit an identifiable subset of vector fields V^a such that solutions of the constraint equations determine only one vector field from the subset, with a similar requirement holding for system (12.8). The main difficulty is that of uniquely representing drifts at $\bar{g}_{ab}$ using the conformally related metric g_{ab}, but without knowing the connecting conformal factor.

If g_{ab} does not admit nontrivial conformal Killing fields, there is a way to uniquely identify the drifts at g_{ab} with the drifts the conformally related metric $\hat{g}_{ab} = \phi^{q-2} g_{ab}$, and this leads to third, alternative, parameterization.

Let $\mathbf{V} \in \mathrm{Drift}_g$ and let V^a be any representative. We then send $\mathbf{V}$ to

$$(12.9) \qquad \widehat{\mathbf{V}} = \{\widehat{g}_{ab};\ 0, \mathrm{div}_{\widehat{g}}(\phi^{-q}V)\} \in \mathrm{Drift}_{\widehat{g}}.$$

The map is well defined, for if U^a is another representative of $\mathbf{V}$, there is a divergence-free vector field E^a such that $U^a = V^a + E^a$; this uses the fact that there are no nonzero conformal Killing fields. But then $\phi^{-q}U^a = \phi^{-q}V^a + \phi^{-q}E^a$, and since $\phi^{-q}E^a$ is divergence-free with respect to $\widehat{g}_{ab}$,

$$(12.10) \qquad \{\widehat{g}_{ab};\ 0, \mathrm{div}_{\widehat{g}}\,\phi^{-q}U\} = \{\widehat{g}_{ab};\ 0, \mathrm{div}_{\widehat{g}}\,\phi^{-q}V\}.$$

Hence the map is well defined, and since it has an analogous inverse we have established an identification of Drift_g with $\mathrm{Drift}_{\widehat{g}}$. Using this identification we make the substitution $V^a \mapsto \phi^{-q}V^a$ into equation (12.4) to obtain

$$(12.11) \qquad \tau = \tau^* + \frac{\phi^{-2q}}{N_{g,\alpha}}\,\mathrm{div}_g V$$

and then substitute this mean curvature into the CTS-H equations. Note, however, that $\mathrm{div}_g V$ is a zero-mean function with respect to g_{ab} and one can dispense with the vector field entirely.

Problem 12.3 (CTS-H with lapse-scaled mean curvature). *Let g_{ab} be a metric with no nontrivial conformal Killing fields, σ_{ab} a transverse traceless tensor with respect to g_{ab}, τ^* a constant, ξ a zero-mean function, and α a lapse form. Setting $N = \omega_g/\alpha$, find a conformal factor ϕ and a vector field W^a such that*

$$- a\,\Delta_g\,\phi + R_g\phi - \left|\sigma + \frac{1}{2N}\,\mathbf{L}_g\,W\right|_g^2 \phi^{-q-1}$$

$$(12.12) \qquad + \kappa\left(\tau^* + \frac{\phi^{-2q}}{N}\xi\right)^2 \phi^{q-1} = 0,$$

$$\mathrm{div}_g\left(\frac{1}{2N}\,\mathbf{L}_g\,W\right) - \kappa\phi^q d\left(\frac{\phi^{-2q}}{N}\xi\right) = 0.$$

One could also work with the substitution $W^a \mapsto \phi^{-q}W^a$ in system (12.8) and obtain an analogous version of system (12.12), but this seems somewhat unnatural.

The drift parameterization also has the potential to inform the standard conformal method when the background metric has nontrivial conformal Killing fields. Very little is known in this case: we have near-CMC existence

under the very strong restriction that the mean curvature is constant along each flow line of every conformal Killing field [9], and we have near-CMC nonexistence on Yamabe-non-negative manifolds if the conformal momentum is zero[26]. Moreover, one can show that conformal Killing fields pose a genuine obstacle for some near-CMC seed data, but not others [32]. The difficulty with conformal Killing fields arises since the CTS-H momentum constraint is not always solvable when conformal Killing fields are present. Using the ideas that led to system (12.1) one can adjust the standard conformal method to include a correction term involving a conformal Killing field to restore solvability of the momentum constraint, and we will address this modification of the CTS-H equations in future work.

13. Conclusion

In hindsight, York's original CMC conformal method can be thought of as having three parameters:

- a conformal class $\mathbf{g}$ in $\mathcal{C}$,
- a conformal momentum $\boldsymbol{\sigma}$ in $T^*\mathcal{C}/\mathcal{D}_0$, and
- a volumetric momentum $-2\kappa\tau_0$ in $T^*\mathcal{V}/\mathcal{D}_0$.

CMC data sets are special, however: their conformal and volumetric momenta are unambiguously defined, intrinsic properties. The extension of the conformal method to non-CMC initial data sets employs a fourth parameter, a densitized lapse, which is used to measure conformal momentum in a way that only depends on conformal properties of the solution. The conformal momentum is compatible with the ADM Lagrangian, as seen in diagram (4.2), and the resulting non-CMC conformal method has four parameters:

- a conformal class $\mathbf{g}$ in $\mathcal{C}$,
- a densitized lapse, represented by a lapse form α,
- a conformal momentum $\boldsymbol{\sigma}$ in $T^*\mathcal{C}/\mathcal{D}_0$ as measured by α, and
- a mean curvature τ.

In this formulation the mean curvature no longer directly controls the volumetric momentum of the solution. We saw in Section 8, however, that the mechanism used by the standard conformal method to interpret conformal momentum can be applied to the volumetric degrees of freedom, and volumetric momentum, as measured by a densitized lapse, emerges as a property of a non-CMC initial data set. The parallels between conformal and volumetric momenta are striking, and indeed the volumetric theory described

in Section 7 is completely analogous to the conformal theory of Section 4. We have therefore considered alternatives to the conformal method where the mean curvature is determined indirectly by specification of a volumetric momentum and some other ingredient, and we have identified drifts as playing a role in understanding these alternatives.

Indeed, every solution of the momentum constraint is a sum of a conformal momentum, a volumetric momentum, and a drift momentum, which is represented by a pair of vector fields W^a and V^a solving the drift equation

$$(13.1) \qquad \operatorname{div}_g \left(\frac{1}{2N_{g,\alpha}} \mathbf{L}_g W \right) = \kappa \, \mathrm{d} \left(\frac{1}{N_{g,\alpha}} \operatorname{div}_g V \right).$$

Section 10 showed that equation (13.1) is not really a relationship between vector fields, but is a relationship between a pair of drifts $(\mathbf{W}, \mathbf{V})$. Moreover, the relationship is symmetric: either of $\mathbf{W}$ or $\mathbf{V}$ determines the other, and each of $\mathbf{W}$ or $\mathbf{V}$ can be taken as the velocity representing drift motion. Section 11 described dual theories, depending on the choice of using $\mathbf{W}$ or $\mathbf{V}$, in which the ADM kinetic energy descends to a kinetic energy Lagrangian without constraints on a tangent space decomposed into conformal, volumetric, and drift motion. We were obligated, however, to pick either conformal or volumetric drift as representing drift velocity because the difference $\mathbf{V} - \mathbf{W}$, which is the drift component of ADM velocity projected into $T\mathcal{M}/\mathcal{D}_0$, is not always sufficient to detect distinct solutions of the constraint equations.

These results show that the introduction of a densitized lapse into the ADM Lagrangian leads to a rich structure. Although some of this structure is employed by the standard conformal method, some of it is ignored, and in Section 12 we saw that there are alternative extensions of the CMC conformal method that incorporate volumetric momentum and drift as parameters instead of mean curvature. Indeed there are a number of ways to do this, and it is not yet clear how to best work with drift. Nevertheless, future progress in applying the conformal method, or some variation, in the non-CMC setting will require new ideas. An improved understanding of the geometry of the conformal method, of the type sought here, may well assist with these efforts.

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