Communications in
Analysis and Geometry
Volume 29, Number 2, 293–327, 2021

An APS index theorem for even-dimensional manifolds with non-compact boundary

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We study the index of the APS boundary value problem for a strongly Callias-type operator $\mathcal{D}$ on a complete Riemannian manifold M. We use this index to define the relative η-invariant $\eta(\mathcal{A}_1, \mathcal{A}_0)$ of two strongly Callias-type operators, which are equal outside of a compact set. Even though in our situation the η-invariants of $\mathcal{A}_1$ and $\mathcal{A}_0$ are not defined, the relative η-invariant behaves as if it were the difference $\eta(\mathcal{A}_1) - \eta(\mathcal{A}_0)$. We also define the spectral flow of a family of such operators and use it to compute the variation of the relative η-invariant.

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†Partially supported by the Simons Foundation collaboration grant #G00005104.

1. Introduction

In [17] we studied the index of the Atiyah–Patodi–Singer (APS) boundary value problem for a strongly Callias-type operator on a complete *odd-dimensional* manifold with non-compact boundary. We used this index to define the *relative η-invariant* $\eta(\mathcal{A}_1, \mathcal{A}_0)$ of two strongly Callias-type operators on *even-dimensional* manifolds, assuming that $\mathcal{A}_0$ and $\mathcal{A}_1$ coincide outside of a compact set.

In this paper we discuss an even-dimensional analogue of [17]. Many parts of the paper are parallel to the discussion in [17]. However, there are two important differences. First, the Atiyah–Singer integrand was, of course, equal to 0 in [17], which simplified many formulas. In particular, the relative η-invariant was an integer. As opposed to it, in the current paper the Atiyah–Singer integrand plays an important role and the relative η-invariant is a real number. More significantly, the proof of the main result in [17] was based on the application of the Callias index theorem, [2, 21]. This theorem is not available in our current setting. Consequently, a completely different proof is proposed in Section 3.

Our results provide a new tool to study anomalies in quantum field theory. Mathematical description of many anomalies is given by index theorems for boundary value problems, cf. [3, 8, 28, 39], [10, Ch. 11]. However, most mathematically rigorous descriptions of anomalies in the literature only work on compact manifolds. The results of the current paper allow to extend many of these descriptions to non-compact setting, thus providing a mathematically rigorous description of anomalies in more realistic physical situations. In particular, Bär and Strohmaier, in [7, 8], gave a mathematically rigorous description of chiral anomaly by considering an APS boundary problem for Dirac operator on a Lorentzian spatially compact manifold. In a recent preprint [12] the first author extended the results of [7] to spatially non-compact case. The analysis in [12] depends heavily on the results of the current paper. Another applications to the anomaly considered in [29] will appear in [18].

We now briefly describe our main results.

1.1. Strongly Callias-type operators

A Callias-type operator on a complete Riemannian manifold M is an operator of the form $\mathcal{D} = D + \Psi$ where D is a Dirac operator and Ψ is a self-adjoint potential which anticommutes with the Clifford multiplication and satisfies certain growth conditions at infinity. In this paper we impose slightly stronger growth conditions on Ψ and refer to the obtained operator $\mathcal{D}$ as a *strongly Callias-type operator*. Our conditions on the growth of Ψ guarantee that the spectrum of $\mathcal{D}$ is discrete.

The Callias-type index theorem, proven in different forms in [2, 11, 19, 21, 22], computes the index of a Callias-type operator on a complete *odd-dimensional* manifold as the index of a certain operator induced by $\mathcal{D}$ on a compact hypersurface. Several generalizations and applications of the Callias-type index theorem were obtained recently in [13, 16, 23, 31, 32, 38].

P. Shi, [37], proved a version of the Callias-type index theorem for the APS boundary value problem for Callias-type operators on a complete odd-dimensional manifold with compact boundary.

Fox and Haskell [26, 27] studied Callias-type operators on manifolds with non-compact boundary. Under rather strong conditions on the geometry of the manifold and the operator $\mathcal{D}$ they proved a version of the Atiyah–Patodi–Singer index theorem.

In [17] we studied the index of the APS boundary value problem on an arbitrary complete odd-dimensional manifold with non-compact boundary. In the current paper, we obtain an even-dimensional analogue of [17].

1.2. An almost compact essential support

A manifold C, whose boundary is a disjoint union of two complete manifolds N_0 and N_1, is called *essentially cylindrical* if outside of a compact set it is isometric to a cylinder $[0,\varepsilon]\times N$, where N is a non-compact manifold. It follows that manifolds N_0 and N_1 are isometric outside of a compact set.

We say that an essentially cylindrical manifold M_1, which contains ∂M, is an *almost compact essential support of* $\mathcal{D}$ if the restriction of $\mathcal{D}^*\mathcal{D}$ to $M\setminus M_1$ is strictly positive and the restriction of $\mathcal{D}$ to the cylinder $[0,\varepsilon]\times N$ is a product, cf. Definition 2.27. Every strongly Callias-type operator on M which is a product near ∂M has an almost compact essential support.

Theorem 2.29 states that *the index of the APS boundary value problem for a strongly Callias-type operator* $\mathcal{D}$ *on a complete manifold* M *is equal to the index of the APS boundary value problem of the restriction of* $\mathcal{D}$ *to its almost compact essential support* M_1.

1.3. Index on an essentially cylindrical manifold

Let M be an essentially cylindrical manifold and let $\mathcal{D}$ be a strongly Callias-type operator on M, whose restriction to the cylinder $[0,\varepsilon]\times\partial M$ is a product. Suppose $\partial M=N_0\sqcup N_1$ and denote the restrictions of $\mathcal{D}$ to N_0 and N_1

by $\mathcal{A}_0$ and $-\mathcal{A}_1$ respectively (the sign convention means that we think of N_0 as the "left boundary" and of N_1 as the "right boundary" of M). Let $\mathcal{D}_B$ denote the operator $\mathcal{D}$ with APS boundary conditions.

Let $\alpha_{\mathrm{AS}}(\mathcal{D})$ denote the Atiyah–Singer integrand of $\mathcal{D}$. This is a differential form on M which depends on the geometry of the manifold and the bundle. Since all structures are product outside of the compact set K, this form vanishes outside of K. Hence, $\int_M \alpha_{\mathrm{AS}}(\mathcal{D})$ is well-defined and finite. Our main result here is that

$$\operatorname{ind} \mathcal{D}_B - \int_M \alpha_{\mathrm{AS}}(\mathcal{D}) \tag{1.1}$$

depends only on the operators $\mathcal{A}_0$ *and* $\mathcal{A}_1$ and does not depend on the interior of the manifold M and the restriction of $\mathcal{D}$ to the interior of M, cf. Theorem 3.4.

1.4. The relative η-invariant

Suppose now that $\mathcal{A}_0$ and $\mathcal{A}_1$ are self-adjoint strongly Callias-type operators on complete manifolds N_0 and N_1. An *almost compact cobordism* between $\mathcal{A}_0$ and $\mathcal{A}_1$ is a pair $(M, \mathcal{D})$ where M is an essentially cylindrical manifold with $\partial M = N_0 \sqcup N_1$ and $\mathcal{D}$ is a strongly Callias-type operator on M, whose restriction to the cylindrical part of M is a product and such that the restrictions of $\mathcal{D}$ to N_0 and N_1 are equal to $\mathcal{A}_0$ and $-\mathcal{A}_1$ respectively. We say that $\mathcal{A}_0$ and $\mathcal{A}_1$ are *cobordant* if there exists an almost compact cobordism between them. In particular, this implies that $\mathcal{A}_0$ and $\mathcal{A}_1$ are equal outside of a compact set.

Let $\mathcal{D}$ be an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Let B_0 and B_1 be the APS boundary conditions for $\mathcal{D}$ at N_0 and N_1 respectively. Let $\operatorname{ind} \mathcal{D}_{B_0 \oplus B_1}$ denote the index of the APS boundary value problem for $\mathcal{D}$. We define the *relative η-invariant* by the formula

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = 2\left(\operatorname{ind} \mathcal{D}_{B_0 \oplus B_1} - \int_M \alpha_{\mathrm{AS}}(\mathcal{D})\right) + \dim\ker \mathcal{A}_0 + \dim\ker \mathcal{A}_1.$$

It follows from the result of the previous subsection that $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is independent of the choice of an almost compact cobordism.

If M is a compact manifold, then the Atiayh-Patodi-Singer index theorem [4] implies that $\eta(\mathcal{A}_1, \mathcal{A}_0) = \eta(\mathcal{A}_1) - \eta(\mathcal{A}_0)$. In general, for non-compact manifolds, the individual η-invariants $\eta(\mathcal{A}_1)$ and $\eta(\mathcal{A}_0)$ might not be defined. However, $\eta(\mathcal{A}_1, \mathcal{A}_0)$ behaves like it were a difference of two individual

η-invariants. In particular, cf. Propositions 4.9–4.10,

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = -\eta(\mathcal{A}_0, \mathcal{A}_1), \qquad \eta(\mathcal{A}_2, \mathcal{A}_0) = \eta(\mathcal{A}_2, \mathcal{A}_1) + \eta(\mathcal{A}_1, \mathcal{A}_0).$$

Under rather strong conditions on the manifolds N_0 and N_1 and on the operators $\mathcal{A}_0$, $\mathcal{A}_1$, Fox and Haskell [26, 27] showed that the heat kernel of $\mathcal{A}_j$ $(j = 0, 1)$ has a nice asymptotic expansion similar to the one for operators on compact manifolds. Then they were able to define the individual η-invariants $\eta(\mathcal{A}_j)$ $(j = 0, 1)$. In this situation, as expected, our relative η-invariant is equal to the difference of the individual η-invariants: $\eta(\mathcal{A}_1, \mathcal{A}_0) = \eta(\mathcal{A}_1) - \eta(\mathcal{A}_0)$.

Under much weaker (but still quite strong) assumptions, Müller, [35], suggested a definition of a relative η-invariant based on analysis of the relative heat kernel [20, 21, 25]. This invariant behaves very similar to our $\eta(\mathcal{A}_1, \mathcal{A}_0)$. The precise conditions under which these two invariants are equal are not clear yet. We note that our invariant is defined on a much wider class of manifolds, where the relative heat kernel is not of trace class and can not be used to construct an η-invariant.

1.5. The spectral flow

Consider a family $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ of self-adjoint strongly Callias-type operators on a complete Riemannian manifold. We assume that there is a compact set $K \subset M$ such that the restriction of $\mathcal{A}^s$ to $M \setminus K$ is independent of s. Since the spectrum of $\mathcal{A}^s$ is discrete for all s, the spectral flow $\mathrm{sf}(\mathbb{A})$ can be defined in a more or less usual way. By Theorem 5.10, if $\mathcal{A}_0$ is a self-adjoint strongly Callias-type operator which is cobordant to $\mathcal{A}^0$ (and hence, to all $\mathcal{A}^s$), then

$$\eta(\mathcal{A}^1, \mathcal{A}_0) - \eta(\mathcal{A}^0, \mathcal{A}_0) - \int_0^1 \left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0)\right) ds = 2\,\mathrm{sf}(\mathbb{A}). \tag{1.2}$$

The derivative $\frac{d}{ds}\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0)$ can be computed as an integral of the transgression form — a differential form canonically constructed from the symbol of $\mathcal{A}^s$ and its derivative with respect to s. Thus (1.2) expresses the change of the relative η-invariant as a sum of $2\,\mathrm{sf}(\mathbb{A})$ and a local differential geometric expression.

2. Boundary value problems for Callias-type operators

In this section we recall some results about boundary value problems for Callias-type operators on manifolds with non-compact boundary, [17], keeping in mind applications to even-dimensional case. The operators considered here are slightly more general than those discussed in [17], but all the definitions and most of the properties of the boundary value problems remain the same.

2.1. Self-adjoint strongly Callias-type operators

Let M be a complete Riemannian manifold (possibly with boundary) and let $E \to M$ be a Dirac bundle over M, cf. [33, Definition II.5.2]. In particular, E is a Hermitian bundle endowed with a Clifford multiplication $c : T^*M \to \mathrm{End}(E)$ and a compatible Hermitian connection ∇^E. Let $D : C^\infty(M,E) \to C^\infty(M,E)$ be the Dirac operator defined by the connection ∇^E. Let $\Psi \in \mathrm{End}(E)$ be a self-adjoint bundle map (called a *Callias potential*). Then

$$\mathcal{D} := D + \Psi$$

is a formally self-adjoint Dirac-type operator on E and

$$\mathcal{D}^2 = D^2 + \Psi^2 + [D,\Psi]_+, \tag{2.1}$$

where $[D,\Psi]_+ := D\circ\Psi + \Psi\circ D$ is the anticommutator of the operators D and Ψ.

Definition 2.2. We call $\mathcal{D}$ a *self-adjoint strongly Callias-type operator* if

(i) $[D,\Psi]_+$ is a zeroth order differential operator, i.e. a bundle map;

(ii) for any $R > 0$, there exists a compact subset $K_R \subset M$ such that

$$\Psi^2(x) - \big|[D,\Psi]_+(x)\big| \geq R \tag{2.2}$$

for all $x \in M \setminus K_R$. In this case, the compact set K_R is called an R-*essential support* of $\mathcal{D}$, or an *essential support* when we do not need to stress the associated constant.

Remark 2.3. Condition (i) of Definition 2.2 is equivalent to the condition that Ψ anticommutes with the Clifford multiplication: $\big[c(\xi),\Psi\big]_+ = 0$, for all $\xi \in T^*M$.

2.4. Graded self-adjoint strongly Callias-type operators

Suppose now that $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded Dirac bundle such that the Clifford multiplication $c(\xi)$ is odd and the Clifford connection is even with respect to this grading. Then

$$D := \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

is the $\mathbb{Z}_2$-graded Dirac operator, where $D^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$ are formally adjoint to each other. Assume that the Callias potential Ψ has odd grading degree, i.e.,

$$\Psi = \begin{pmatrix} 0 & \Psi^- \\ \Psi^+ & 0 \end{pmatrix},$$

where $\Psi^\pm \in \mathrm{Hom}(E^\pm, E^\mp)$ are adjoint to each other. Then we have

$$\mathcal{D} = D + \Psi = \begin{pmatrix} 0 & D^- + \Psi^- \\ D^+ + \Psi^+ & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}. \tag{2.3}$$

Definition 2.5. Under the same condition as in Definition 2.2, $\mathcal{D}$ is called a *graded self-adjoint strongly Callias-type operator*. In this case, we also call $\mathcal{D}^+$ and $\mathcal{D}^-$ strongly Callias-type operators. They are formally adjoint to each other. By an *R-essential support* (or *essential support*) of $\mathcal{D}^\pm$ we understand as an R-essential support (or essential support) of $\mathcal{D}$.

Remark 2.6. When M is an oriented even-dimensional manifold there is a natural grading of E induced by the volume form. We will consider this situation in the next section.

Remark 2.7. Suppose there is a skew-adjoint isomorphism $\gamma : E^\pm \to E^\mp$, $\gamma^* = -\gamma$, which anticommutes with multiplication $c(\xi)$ for all $\xi \in T^*M$, satisfies $\gamma^2 = -1$, and is flat with respect to the connection ∇^E, i.e. $[\nabla^E, \gamma] = 0$. Then $\xi \mapsto \gamma \circ c(\xi)$ defines a Clifford multiplication of T^*M on E^+ and the corresponding Dirac operator is $\tilde{D}^+ = \gamma \circ D^+$. Suppose also that γ commutes with Ψ. Then $\Phi^+ = -i\gamma \circ \Psi^+$ is a self-adjoint endomorphism of E^+. In this situation,

$$\tilde{D}^+ + i\Phi^+ = \gamma \circ \mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^+)$$

is a strongly Callias-type operator in the sense of [17, Definition 3.4].

2.8. Restriction to the boundary

Assume that the Riemannian metric g^M is *product near the boundary*, that is, there exists a neighborhood $U \subset M$ of the boundary which is isometric to the cylinder

$$Z_r := [0, r) \times \partial M. \tag{2.4}$$

In the following we identify U with Z_r and denote by t the coordinate along the axis of Z_r. Then the inward unit normal one-form to the boundary is given by $\tau = dt$.

Furthermore, we assume that the Dirac bundle E is *product near the boundary*. In other words we assume that the Clifford multiplication $c : T^*M \to \mathrm{End}(E)$ and the connection ∇^E have product structure on Z_r, cf. [17, §3.7].

Let D be a $\mathbb{Z}_2$-graded Dirac operator. In this situation the restriction of D to Z_r takes the form

$$D = c(\tau)(\partial_t + \hat{A}) = \begin{pmatrix} 0 & c(\tau) \\ c(\tau) & 0 \end{pmatrix} \begin{pmatrix} \partial_t + A & 0 \\ 0 & \partial_t + A^\sharp \end{pmatrix}, \tag{2.5}$$

where

$$A : C^\infty(\partial M, E^+_{\partial M}) \to C^\infty(\partial M, E^+_{\partial M})$$

and

$$A^\sharp = c(\tau) \circ A \circ c(\tau) : C^\infty(\partial M, E^-_{\partial M}) \to C^\infty(\partial M, E^-_{\partial M}) \tag{2.6}$$

are formally self-adjoint operators acting on the restrictions of $E^\pm$ to the boundary.

Remark 2.9. It would be more natural to use the notation A^+ and A^- instead of A and $A^\sharp$. But since in the future we only deal with the operator $A : C^\infty(\partial M, E^+_{\partial M}) \to C^\infty(\partial M, E^+_{\partial M})$ we remove the superscript "+" to simplify the notation.

Let $\mathcal{D} = D + \Psi$ be a graded self-adjoint strongly Callias-type operator. Then the restriction of $\mathcal{D}$ to Z_r is given by

$$\mathcal{D} = c(\tau)(\partial_t + \hat{\mathcal{A}}) = \begin{pmatrix} 0 & c(\tau) \\ c(\tau) & 0 \end{pmatrix} \begin{pmatrix} \partial_t + \mathcal{A} & 0 \\ 0 & \partial_t + \mathcal{A}^\sharp \end{pmatrix}, \tag{2.7}$$

where

$$\mathcal{A} := A - c(\tau)\Psi^+ : C^\infty(\partial M, E^+_{\partial M}) \to C^\infty(\partial M, E^+_{\partial M}). \tag{2.8}$$

and $\mathcal{A}^\sharp = A^\sharp - c(\tau)\Psi^-$. By Remark 2.3, $c(\tau)\Psi^\pm \in \mathrm{End}(E^\pm_{\partial M})$ are self-adjoint bundle maps. Therefore $\mathcal{A}$ and $\mathcal{A}^\sharp$ are formally self-adjoint operators. In fact, they are strongly Callias-type operators, cf. Lemma 3.12 of [17]. In particular, they have discrete spectrum. Also,

$$\mathcal{A}^\sharp = c(\tau) \circ \mathcal{A} \circ c(\tau).$$

Definition 2.10. We say that a graded self-adjoint strongly Callias-type operator $\mathcal{D}$ is *product near the boundary* if the Dirac bundle E is product near the boundary and the restriction of the Callias potential Ψ to Z_r does not depend on t. The operator $\mathcal{A}$ (resp. $\mathcal{A}^\sharp$) of (2.7) is called the *restriction of* $\mathcal{D}^+$ *(resp.* $\mathcal{D}^-$*) to the boundary.*

2.11. Sobolev spaces on the boundary

Consider a graded self-adjoint strongly Callias-type operator

$$\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E),$$

cf. (2.3). The restriction of $\mathcal{D}^+$ to the boundary is a self-adjoint strongly Callias-type operator

$$\mathcal{A} : C^\infty(\partial M, E^+_{\partial M}) \to C^\infty(\partial M, E^+_{\partial M}).$$

We recall the definition of Sobolev spaces $H^s_{\mathcal{A}}(\partial M, E^+_{\partial M})$ of sections over ∂M which depend on the boundary operator $\mathcal{A}$, cf. [17, §3.13].

Definition 2.12. Set

$$C^\infty_{\mathcal{A}}(\partial M, E^+_{\partial M}) := \Big\{ \mathbf{u} \in C^\infty(\partial M, E^+_{\partial M}) : \\ \big\|(\mathrm{id} + \mathcal{A}^2)^{s/2}\mathbf{u}\big\|^2_{L^2(\partial M, E_{\partial M})} < +\infty \text{ for all } s \in \mathbb{R} \Big\}.$$

For all $s \in \mathbb{R}$ we define the *Sobolev* $H^s_{\mathcal{A}}$*-norm* on $C^\infty_{\mathcal{A}}(\partial M, E^+_{\partial M})$ by

$$\|\mathbf{u}\|^2_{H^s_{\mathcal{A}}(\partial M, E_{\partial M})} := \big\|(\mathrm{id} + \mathcal{A}^2)^{s/2}\mathbf{u}\big\|^2_{L^2(\partial M, E_{\partial M})}. \tag{2.9}$$

The Sobolev space $H^s_{\mathcal{A}}(\partial M, E^+_{\partial M})$ is defined to be the completion of $C^\infty_{\mathcal{A}}(\partial M, E^+_{\partial M})$ with respect to this norm.

2.13. Generalized APS boundary conditions

The eigensections of $\mathcal{A}$ belong to $H^s_{\mathcal{A}}(\partial M, E^+_{\partial M})$ for all $s \in \mathbb{R}$, cf. [17, §3.17]. For $I \subset \mathbb{R}$ we denote by

$$H^s_I(\mathcal{A}) \subset H^s_{\mathcal{A}}(\partial M, E^+_{\partial M})$$

the span of the eigensections of $\mathcal{A}$ whose eigenvalues belong to I.

Definition 2.14. For any $a \in \mathbb{R}$, the subspace

$$B = B(a) := H^{1/2}_{(-\infty,a)}(\mathcal{A}). \tag{2.10}$$

is called the *the generalized Atiyah–Patodi–Singer boundary conditions* for $\mathcal{D}^+$. If $a = 0$, then the space $B(0) = H^{1/2}_{(-\infty,0)}(\mathcal{A})$ is called the *Atiyah–Patodi–Singer (APS) boundary condition.*

The spaces $\bar{B}(a) := H^{1/2}_{(-\infty,a]}(\mathcal{A})$ and $\bar{B}(0) := H^{1/2}_{(-\infty,0]}(\mathcal{A})$ are called the *dual generalized APS boundary conditions* and the *dual APS boundary conditions* respectively.

The space $B^{\mathrm{ad}} = B^{\mathrm{ad}}(a) := H^{1/2}_{(-\infty,-a]}(\mathcal{A}^\sharp)$ is called the *adjoint of the generalized APS boundary condition for* $\mathcal{D}^+$. One can see that it is a dual generalized APS boundary condition for $\mathcal{D}^-$.

Definition 2.15. If B is a generalized APS boundary condition for $\mathcal{D}^+$, we denote by $\mathcal{D}^+_B$ the operator $\mathcal{D}^+$ with domain

$$\operatorname{dom} \mathcal{D}^+_B := \{u \in \operatorname{dom} \mathcal{D}^+_{\max} : u|_{\partial M} \in B\},$$

where $\operatorname{dom} \mathcal{D}^+_{\max}$ denotes the domain of the maximal extension of $\mathcal{D}^+$ (cf. [17]). We refer to $\mathcal{D}^+_B$ as the *generalized APS boundary value problem* for $\mathcal{D}^+$.

Recall that $\mathcal{D}^-$ is the formal adjoint of $\mathcal{D}^+$. It is shown in Example 4.9 of [17] that the L^2-adjoint of $\mathcal{D}^+_{B(a)}$ is given by $\mathcal{D}^-$ with the dual APS boundary condition $B^{\mathrm{ad}}(a)$:

$$\big(\mathcal{D}^+_{B(a)}\big)^{\mathrm{ad}} = \mathcal{D}^-_{B^{\mathrm{ad}}(a)}. \tag{2.11}$$

Theorem 2.16. *Suppose that a graded strongly Callias-type operator* (2.3) *is product near ∂M. Then the operator $\mathcal{D}^+_B : \operatorname{dom} \mathcal{D}^+_B \to L^2(M, E^-)$ is Fredholm. In particular, it has finite dimensional kernel and cokernel.*

Proof. For the case discussed in Remark 2.7 this is proven in Theorem 5.4 of [17]. Exactly the same proof works in the general case. □

2.17. The index of generalized APS boundary value problems

By (2.11) the cokernel of $\mathcal{D}^+_{B(a)}$ is isomorphic to the kernel of $\mathcal{D}^-_{B^{\mathrm{ad}}(a)}$.

Definition 2.18. Let $\mathcal{D}^+$ be a strongly Callias-type operator on a complete Riemannian manifold M which is product near the boundary. Let $B = H^{1/2}_{(-\infty,a)}(\mathcal{A})$ be a generalized APS boundary condition for $\mathcal{D}^+$ and let $B^{\mathrm{ad}} = H^{1/2}_{(-\infty,-a]}(\mathcal{A}^\sharp)$ be the adjoint of the generalized APS boundary condition. The integer

$$\operatorname{ind}\mathcal{D}^+_B := \dim\ker\mathcal{D}^+_B - \dim\ker(\mathcal{D}^-)_{B^{\mathrm{ad}}} \in \mathbb{Z} \tag{2.12}$$

is called the *index of the boundary value problem* $\mathcal{D}^+_B$.

It follows directly from (2.11) that

$$\operatorname{ind}(\mathcal{D}^-)_{B^{\mathrm{ad}}} = -\operatorname{ind}\mathcal{D}^+_B. \tag{2.13}$$

2.19. More general boundary value conditions

Generalized APS and dual generalized APS boundary conditions are examples of *elliptic boundary conditions*, [17, Definition 4.7]. In this paper we don't work with general elliptic boundary conditions. However, in Section 5 we need a slight modification of APS boundary conditions, which we define now.

Definition 2.20. We say that two closed subspaces X_1, X_2 of a Hilbert space H are *finite rank perturbations* of each other if there exists a finite dimensional subspace $Y \subset H$ such that $X_2 \subset X_1 \oplus Y$ and the quotient space $(X_1 \oplus Y)/X_2$ has finite dimension.

The *relative index* of X_1 and X_2 is defined by

$$[X_1, X_2] := \dim(X_1 \oplus Y)/X_2 - \dim Y. \tag{2.14}$$

One easily sees that the relative index is independent of the choice of Y. We shall need the following analogue of [17, Proposition 5.8]:

Proposition 2.21. *Let $\mathcal{D}$ be a graded self-adjoint strongly Callias-type operator on M and let B be a generalized APS or dual generalized APS boundary condition for $\mathcal{D}^+$. If $B_1 \subset H^{1/2}_{\mathcal{A}}(\partial M, E^+_{\partial M})$ is a finite rank perturbation of B, then the operator $\mathcal{D}^+_{B_1}$ is Fredholm and*

$$\operatorname{ind} \mathcal{D}^+_B - \operatorname{ind} \mathcal{D}^+_{B_1} = [B, B_1]. \tag{2.15}$$

The proof of the proposition is a verbatim repetition of the proof of [6, Theorem 8.14].

As an immediate consequence of Proposition 2.21 we obtain the following

Corollary 2.22. *Let $\mathcal{A}$ be the restriction of $\mathcal{D}^+$ to ∂M and let $B_0 = H^{1/2}_{(-\infty,0)}(\mathcal{A})$ and $\bar{B}_0 = H^{1/2}_{(-\infty,0]}(\mathcal{A})$ be the APS and the dual APS boundary conditions respectively. Then*

$$\operatorname{ind} \mathcal{D}^+_{\bar{B}_0} = \operatorname{ind} \mathcal{D}^+_{B_0} + \dim \ker \mathcal{A}. \tag{2.16}$$

2.23. The splitting theorem

Let M be a complete manifold. Let $N \subset M$ be a hypersurface disjoint from ∂M such that cutting M along N we obtain a manifold M' (connected or not) with ∂M and two copies of N as boundary. So we can write $M' = (M \setminus N) \sqcup N_1 \sqcup N_2$.

Let $E = E^+ \oplus E^- \to M$ be a $\mathbb{Z}_2$-graded Dirac bundle over M and $\mathcal{D}^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^\mp)$ be strongly Callias-type operators as in Subsection 2.4. They induce $\mathbb{Z}_2$-graded Dirac bundle $E' = (E')^+ \oplus (E')^- \to M'$ and strongly Callias-type operators

$$(\mathcal{D}')^\pm : C^\infty(M', (E')^\pm) \to C^\infty(M', (E')^\mp)$$

on M'. We assume that all structures are product near N_1 and N_2. Let $\mathcal{A}$ be the restriction of $(\mathcal{D}')^+$ to N_1. Then $-\mathcal{A}$ is the restriction of $(\mathcal{D}')^+$ to N_2 and, thus, the restriction of $(\mathcal{D}')^+$ to $N_1 \sqcup N_2$ is $\mathcal{A}' = \mathcal{A} \oplus (-\mathcal{A})$. The following *Splitting Theorem* is an analogue of Theorem 5.11 of [17] with the same proof.

Theorem 2.24. *Suppose $M, \mathcal{D}^+, M', (\mathcal{D}')^+$ are as above. Let B_0 be a generalized APS boundary condition on ∂M. Let $B_1 = H^{1/2}_{(-\infty,0)}(\mathcal{A})$ and $B_2 =$*

$H^{1/2}_{(-\infty,0]}(-\mathcal{A}) = H^{1/2}_{[0,\infty)}(\mathcal{A})$ *be the APS and the dual APS boundary conditions for* $(\mathcal{D}')^+$ *along* N_1 *and* N_2*, respectively. Then* $(\mathcal{D}')^+_{B_0\oplus B_1\oplus B_2}$ *is a Fredholm operator and*

$$\operatorname{ind}\mathcal{D}^+_{B_0} = \operatorname{ind}(\mathcal{D}')^+_{B_0\oplus B_1\oplus B_2}.$$

2.25. Reduction of the index to an essentially cylindrical manifold

The study of the index of Callias-type operators on manifolds without boundary can be reduced to a computation on the essential support. For manifolds with boundary we want an analogous subset, but the one which contains the boundary. Such a set is necessarily non-compact, but we want it to be "similar to a compact set". In [17, Section 6], we introduce the notion of essentially cylindrical manifolds, which replace the role of compact subsets in our study of boundary value problems.

Definition 2.26. An *essentially cylindrical* manifold C is a complete Riemannian manifold whose boundary is a union of two disjoint manifolds, $\partial C = N_0 \sqcup N_1$, such that

(i) there exist a compact set $K \subset C$, an open Riemannian manifold N, and an isometry $C \setminus K \simeq [0,\delta] \times N$;

(ii) under the above isometry $N_0 \setminus K = \{0\} \times N$ and $N_1 \setminus K = \{\delta\} \times N$.

See Figure 1 for an example of essentially cylindrical manifolds.

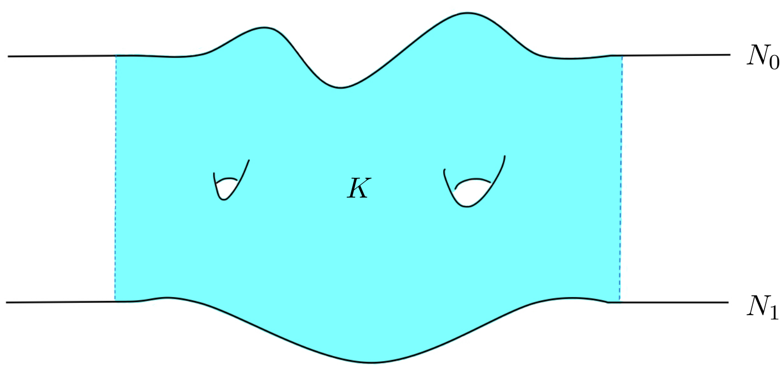

Figure 1. An essentially cylindrical manifold C.

Definition 2.27. Let $\mathcal{D}^+$ be a strongly Callias-type operator on M. An *almost compact essential support of* $\mathcal{D}^+$ is a smooth submanifold $M_1 \subset M$

with smooth boundary, which contains ∂M and such that there exist a (compact) essential support $K \subset M$ and $\varepsilon \in (0, r)$ such that

$$M_1 \setminus K = (\partial M \setminus K) \times [0, \varepsilon] \subset Z_r. \tag{2.17}$$

An almost compact essential support is a special type of essentially cylindrical manifolds. It is shown in [17, Lemma 6.5] that for any strongly Callias-type operator which is product near ∂M there exists an almost compact essential support. Figure 2 illustrates the idea of building an almost compact essential support.

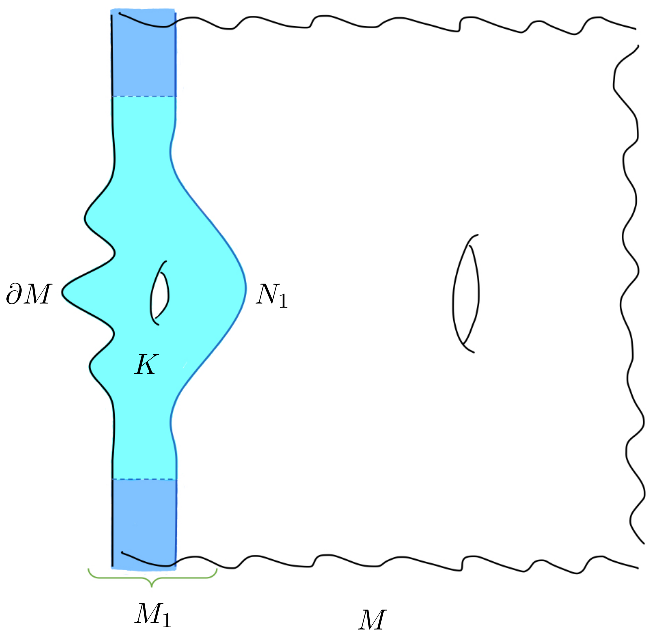

Figure 2. The idea to obtain an almost compact essential support M_1.

2.28. The index on an almost compact essential support

Suppose $M_1 \subset M$ is an almost compact essential support of $\mathcal{D}^+$. Set $N_0 := \partial M$ and let $N_1 \subset M$ be such that $\partial M_1 = N_0 \sqcup N_1$ as in Definition 2.26. The restriction of $\mathcal{D}$ to a neighborhood of N_1 need not be product. Since in this paper we only consider boundary value problems for operators which are product near the boundary, we first deform $\mathcal{D}$ to a product form. It is shown in [17, Lemma 6.8] that there exists a perturbation of all the structures such that

(i) the new structures are product near N_1. In particular, the corresponding Callias-type operator $\mathcal{D}'$ is product near N_1.

(ii) $\mathcal{D}'$ has a compact essential support, which is contained in M_1.

(iii) The difference $\mathcal{D} - \mathcal{D}'$ vanishes near ∂M and outside of a compact subset of M_1. In this situation we say that $\mathcal{D}'$ (or $(\mathcal{D}')^{\pm}$) is a *compact perturbation of* $\mathcal{D}$ *(or* $\mathcal{D}^{\pm}$*).*

Let $M_1 \subset M$ be an almost compact essential support of $\mathcal{D}^+$. Let $(\mathcal{D}')^+$ be a compact perturbation of $\mathcal{D}^+$ which is product near the boundary (cf. [17, §6.6]). Let $\mathcal{A}$ be the restriction of $\mathcal{D}^+$ to ∂M. It is also the restriction of $(\mathcal{D}')^+$. We denote by $-\mathcal{A}_1$ the restriction of $(\mathcal{D}')^+$ to N_1. Theorem 6.10 of [17] claims that the index of an elliptic boundary value problem on M can be reduced to an index on an almost compact essential support:

Theorem 2.29. *Suppose $M_1 \subset M$ is an almost compact essential support of $\mathcal{D}^+$ and let $\partial M_1 = \partial M \sqcup N_1$. Let $(\mathcal{D}')^+$ be a compact perturbation of $\mathcal{D}^+$ which is product near N_1 and such that there is a compact essential support for $(\mathcal{D}')^+$ which is contained in M_1. Let B_0 be a generalized APS boundary condition for $\mathcal{D}^+$. View $(\mathcal{D}')^+$ as an operator on M_1 and let*

$$B_1 = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_1) = H^{1/2}_{(0,\infty)}(\mathcal{A}_1)$$

be the APS boundary condition for $(\mathcal{D}')^+$ at N_1. Then

$$\operatorname{ind} \mathcal{D}^+_{B_0} = \operatorname{ind}(\mathcal{D}')^+_{B_0 \oplus B_1}. \tag{2.18}$$

3. The index of operators on essentially cylindrical manifolds

In this section we discuss the index of strongly Callias-type operators on even-dimensional essentially cylindrical manifolds. It is parallel to [17, Section 7], where the odd-dimensional case was considered.

From now on we assume that M is an *oriented even-dimensional* essentially cylindrical manifold whose boundary $\partial M = N_0 \sqcup N_1$ is a disjoint union of two non-compact manifolds N_0 and N_1. Let E be a Dirac bundle over M. As pointed out in Remark 2.6, there is a natural $\mathbb{Z}_2$-grading $E = E^+ \oplus E^-$ on E. Let $\mathcal{D}^+ : C^\infty(M, E^+) \to C^\infty(M, E^-)$ be a strongly Callias-type operator as in Definition 2.5 (these data might or might not come as a restriction of another operator to its almost compact essential support. In particular, we don't assume that the restriction of $\mathcal{D}^+$ to N_1 is invertible). Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to N_0 and N_1 respectively.

We first recall some definitions from [17, Section 7].

3.1. Compatible data

Let M be an essentially cylindrical manifold and let $\partial M = N_0 \sqcup N_1$. As usual, we identify a tubular neighborhood of ∂M with the product

$$Z_r := \big(N_0 \times [0,r)\big) \sqcup \big(N_1 \times [0,r)\big) \subset M.$$

Definition 3.2. We say that another essentially cylindrical manifold M' is *compatible* with M if there is a fixed isometry between Z_r and a neighborhood $Z_r' \subset M'$ of the boundary of M'.

Note that if M and M' are compatible then their boundaries are isometric.

Let M and M' be compatible essentially cylindrical manifolds and let Z_r and Z_r' be as above. Let $E \to M$ be a $\mathbb{Z}_2$-graded Dirac bundle over M and let $\mathcal{D}^+ : C^\infty(M,E^+) \to C^\infty(M,E^-)$ be a strongly Callias-type operator whose restriction to Z_r is product and such that M is an almost compact essential support of $\mathcal{D}^+$. This means that there is a compact set $K \subset M$ such that $M \setminus K = [0,\varepsilon] \times N$ and the restriction of $\mathcal{D}^+$ to $M \setminus K$ is product (i.e. is given by (2.7)). Let $E' \to M'$ be a $\mathbb{Z}_2$-graded Dirac bundle over M' and let $(\mathcal{D}')^+ : C^\infty(M',(E')^+) \to C^\infty(M',(E')^-)$ be a strongly Callias-type operator, whose restriction to Z_r' is product and such that M' is an almost compact essential support of $(\mathcal{D}')^+$.

Definition 3.3. In the situation discussed above we say that $\mathcal{D}^+$ and $(\mathcal{D}')^+$ are *compatible* if there is an isomorphism $E|_{Z_r} \simeq E'|_{Z_r'}$ of graded Dirac bundles which identifies the restriction of $\mathcal{D}^+$ to Z_r with the restriction of $(\mathcal{D}')^+$ to Z_r'.

Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to N_0 and N_1 respectively (the sign convention means that we think of N_0 as the "left boundary" and of N_1 as the "right boundary" of M). Let $B_0 = H^{1/2}_{(-\infty,0)}(\mathcal{A}_0)$ and $B_1 = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_1) = H^{1/2}_{(0,\infty)}(\mathcal{A}_1)$ be the APS boundary conditions for $\mathcal{D}^+$ at N_0 and N_1 respectively. Since $\mathcal{D}^+$ and $(\mathcal{D}')^+$ are equal near the boundary, B_0 and B_1 are also APS boundary conditions for $(\mathcal{D}')^+$.

We denote by $\alpha_{\rm AS}(\mathcal{D}^+)$ the Atiyah–Singer integrand of $\mathcal{D}^+$. It can be written as

$$\alpha_{\rm AS}(\mathcal{D}^+) := (2\pi i)^{-\dim M}\hat{A}(M)\mathrm{ch}(E/S)$$

where $\hat{A}(M)$ and $\mathrm{ch}(E/S)$ are the differential forms representing the $\hat{A}$-genus of M and the *relative Chern character* of E, cf. [9, §4.1]. Note that $\alpha_{\rm AS}(\mathcal{D}^+)$ depends only on the metric on M and the Clifford multiplication on E and thus is independent of the potential Ψ.

Since outside of a compact set K, M and E are product, the interior multiplication by $\partial/\partial t$ annihilates $\alpha_{\rm AS}$. Hence, the top degree component of $\alpha_{\rm AS}$ vanishes on $M \setminus K$. We conclude that the integral $\int_M \alpha_{\rm AS}(\mathcal{D}^+)$ is well-defined and finite. Similarly, $\int_{M'} \alpha_{\rm AS}((\mathcal{D}')^+)$ is well-defined and finite.

Theorem 3.4. *Suppose $\mathcal{D}^+$ is a strongly Callias-type operator on an oriented even-dimensional essentially cylindrical manifold M such that M is an almost compact essential support of $\mathcal{D}^+$. Suppose that the operator $(\mathcal{D}')^+$ is compatible with $\mathcal{D}^+$. Let $\partial M = N_0 \sqcup N_1$ and let $B_0 = H^{1/2}_{(-\infty,0)}(\mathcal{A}_0)$ and $B_1 = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_1) = H^{1/2}_{(0,\infty)}(\mathcal{A}_1)$ be the APS boundary conditions for $\mathcal{D}^+$ (and, hence, for $(\mathcal{D}')^+$) at N_0 and N_1 respectively. Then*

$$\mathrm{ind}\,\mathcal{D}^+_{B_0\oplus B_1} - \int_M \alpha_{\rm AS}(\mathcal{D}^+) = \mathrm{ind}(\mathcal{D}')^+_{B_0\oplus B_1} - \int_{M'} \alpha_{\rm AS}((\mathcal{D}')^+). \tag{3.1}$$

In particular, $\mathrm{ind}\,\mathcal{D}^+_{B_0\oplus B_1} - \int_M \alpha_{\rm AS}(\mathcal{D}^+)$ depends only on the restrictions $\mathcal{A}_0$ and $-\mathcal{A}_1$ of $\mathcal{D}^+$ to the boundary.

The rest of the section is devoted to the proof of this theorem.

Remark 3.5. In [17] the odd dimensional version of Theorem 3.4 was considered. Of course, in this case $\alpha_{\rm AS}$ vanishes identically and Theorem 7.5 of [17] states that the indexes of compatible operators are equal. The proof in [17] is based on an application of a Callias-type index theorem and can not be adjusted to our current situation. Consequently, a completely different proof is proposed below.

3.6. Gluing the data together

We follow [17, §7.6, §7.7] to glue M with M' and $\mathcal{D}^+$ with $(\mathcal{D}')^+$.

Let $-M'$ denote the manifold M' with the opposite orientation. We identify a neighborhood of the boundary of $-M'$ with the product

$$-Z_r' := \big(N_0 \times (-r,0]\big) \sqcup \big(N_1 \times (-r,0]\big)$$

and consider the union

$$\tilde{M} := M \cup_{N_0 \sqcup N_1} (-M').$$

Then $Z_{(-r,r)} := Z_r \cup (-Z_r')$ is a subset of $\tilde{M}$ identified with the product

$$\big(N_0 \times (-r,r)\big) \sqcup \big(N_1 \times (-r,r)\big).$$

We note that $\tilde{M}$ is a complete Riemannian manifold without boundary.

Let $E_{\partial M} = E^+_{\partial M} \oplus E^-_{\partial M}$ denote the restriction of $E = E^+ \oplus E^-$ to ∂M. The product structure on $E|_{Z_r}$ gives a grading-respecting isomorphism $\psi : E|_{Z_r} \to [0,r) \times E_{\partial M}$. Recall that we identified Z_r with Z_r' and fixed an isomorphism between the restrictions of E to Z_r and E' to Z_r'. By a slight abuse of notation we use this isomorphism to view ψ also as an isomorphism $E'|_{Z_r'} \to [0,r) \times E_{\partial M}$.

Let $\tilde{E} \to \tilde{M}$ be the vector bundle over $\tilde{M}$ obtained by gluing E and E' using the isomorphism $c(\tau) : E|_{\partial M} \to E'|_{\partial M'}$. This means that we fix isomorphisms

$$\phi : \tilde{E}|_M \to E, \quad \phi' : \tilde{E}|_{M'} \to E', \tag{3.2}$$

so that

$$\begin{aligned}
\psi \circ \phi \circ \psi^{-1} &= \mathrm{id} : [0,r) \times E_{\partial M} \to [0,r) \times E_{\partial M}, \\
\psi \circ \phi' \circ \psi^{-1} &= 1 \times c(\tau) : [0,r) \times E_{\partial M} \to [0,r) \times E_{\partial M}.
\end{aligned}$$

Note that the grading of E is preserved while the grading of E' is reversed in this gluing process. Therefore $\tilde{E} = \tilde{E}^+ \oplus \tilde{E}^-$ is a $\mathbb{Z}_2$-graded bundle.

We denote by $c' : T^*M' \to \mathrm{End}(E')$ the Clifford multiplication on E' and set $c''(\xi) := -c'(\xi)$. Then $\tilde{E}$ is a Dirac bundle over $\tilde{M}$ with the Clifford multiplication

$$\tilde{c}(\xi) := \begin{cases} c(\xi), & \xi \in T^*M; \\ c''(\xi) = -c'(\xi), & \xi \in T^*M'. \end{cases} \tag{3.3}$$

One readily checks that (3.3) defines a smooth odd-graded Clifford multiplication on $\tilde{E}$. Let $\tilde{D} : C^\infty(\tilde{M}, \tilde{E}) \to C^\infty(\tilde{M}, \tilde{E})$ be the $\mathbb{Z}_2$-graded Dirac operator. Then the isomorphism ϕ of (3.2) identifies the restriction of $\tilde{D}^{\pm}$ with $D^{\pm}$, the isomorphism ϕ' identifies the restriction of $\tilde{D}^{\pm}$ with $-(D')^{\mp}$, and isomorphism $\psi \circ \phi' \circ \psi^{-1}$ identifies the restriction of $\tilde{D}^{\pm}$ to $-Z'_r$ with

$$\tilde{D}^{\pm}|_{Z'_r} = -c'(\tau) \circ (D')^{\pm}_{Z'_r} \circ c'(\tau)^{-1}. \tag{3.4}$$

Let $(\Psi')^{\pm}$ denote the Callias potentials of $(\mathcal{D}')^{\pm}$, so that $(\mathcal{D}')^{\pm} = (D')^{\pm} + (\Psi')^{\pm}$. Consider the bundle maps $\tilde{\Psi}^{\pm} \in \mathrm{Hom}(\tilde{E}^{\pm}, \tilde{E}^{\mp})$ whose restrictions to M are equal to $\Psi^{\pm}$ and whose restrictions to M' are equal to $-(\Psi')^{\mp}$. The two pieces fit well on $Z_{(-r,r)}$ by Remark 2.3. To sum up the constructions presented in this subsection, we have

Lemma 3.7. *The operators $\tilde{\mathcal{D}}^{\pm} := \tilde{D}^{\pm} + \tilde{\Psi}^{\pm}$ are strongly Callias-type operators on $\tilde{M}$, formally adjoint to each other, whose restrictions to M are equal to $\mathcal{D}^{\pm}$ and whose restrictions to M' are equal to $-(D')^{\mp} - (\Psi')^{\mp} = -(\mathcal{D}')^{\mp}$.*

The operator $\tilde{\mathcal{D}}^{+}$ is a strongly Callias-type operator on a complete Riemannian manifold without boundary. Hence, [1], it is Fredholm. We again denote by $\alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^{+})$ the Atiyah–Singer integrand of $\tilde{\mathcal{D}}^{+}$. It is explained in the paragraph before Theorem 3.4 that the integral $\int_{\tilde{M}} \alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^{+})$ is well defined.

Lemma 3.8. $\mathrm{ind}\, \tilde{\mathcal{D}}^{+} = \int_{\tilde{M}} \alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^{+})$.

Proof. Since $\tilde{M}$ is a union of two essentially cylindrical manifolds, there exists a compact essential support $\tilde{K} \subset \tilde{M}$ of $\tilde{\mathcal{D}}$ such that $\tilde{M} \setminus \tilde{K}$ is of the form $S^1 \times N$. We can choose $\tilde{K}$ to be large enough so that the restriction of $\tilde{\mathcal{D}}$ to a neighborhood W of $\tilde{M} \setminus \tilde{K} \simeq S^1 \times N$ is a product of an operator on N and an operator on S^1. Then the restriction of α_{AS} to this neighborhood vanishes. We can also assume that $\tilde{K}$ has a smooth boundary $\Sigma = S^1 \times L$.

Let $\hat{\mathcal{D}}^{+}$ be a compact perturbation of $\tilde{\mathcal{D}}^{+}$ (cf. Subsection 2.28) in W which is product both near Σ and on W and whose essential support is contained in $\tilde{K}$. Then

$$\mathrm{ind}\, \tilde{\mathcal{D}}^{+} = \mathrm{ind}\, \hat{\mathcal{D}}^{+}.$$

We cut $\tilde{M}$ along Σ and apply the Splitting Theorem 2.24[1] to get

$$\mathrm{ind}\, \tilde{\mathcal{D}}^{+} = \mathrm{ind}\, \hat{\mathcal{D}}^{+}_{\tilde{K}} + \mathrm{ind}\, \hat{\mathcal{D}}^{+}_{\tilde{M}\setminus\tilde{K}}, \tag{3.5}$$

[1]Since Σ is compact we can also use the splitting theorem for compact hypersurfaces, [6, Theorem 8.17].

where $\operatorname{ind}\hat{\mathcal{D}}^+_{\tilde{K}}$ stands for the index of the restriction of $\hat{\mathcal{D}}^+$ to $\tilde{K}$ with APS boundary condition, and $\operatorname{ind}\hat{\mathcal{D}}^+_{\tilde{M}\setminus\tilde{K}}$ stands for the index of the restriction of $\hat{\mathcal{D}}^+$ to $\tilde{M}\setminus\tilde{K}$ with the dual APS boundary condition.

Since $\hat{\mathcal{D}}^+$ has an empty essential support in $\tilde{M}\setminus\tilde{K}$, by the vanishing theorem [17, Corollary 5.13], the second summand in the right hand side of (3.5) vanishes. The first summand in the right hand side of (3.5) is given by the Atiyah–Patodi–Singer index theorem [4, Theorem 3.10] (Note that Σ is outside of an essential support of $\hat{\mathcal{D}}^+$ and, hence, the restriction of $\hat{\mathcal{D}}^+$ to Σ is invertible. Hence, the kernel of the restriction of $\hat{\mathcal{D}}^+$ to Σ is trivial)

$$\operatorname{ind}\hat{\mathcal{D}}^+_{\tilde{K}} = \int_{\tilde{K}} \alpha_{\mathrm{AS}}(\hat{\mathcal{D}}^+) - \frac{1}{2}\eta(0),$$

where $\eta(0)$ is the η-invariant of the restriction of $\hat{\mathcal{D}}^+$ to Σ.

As $\alpha_{\mathrm{AS}}(\hat{\mathcal{D}}^+) = \alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^+) \equiv 0$ on W and $\hat{\mathcal{D}}^+ \equiv \tilde{\mathcal{D}}^+$ elsewhere, we have

$$\int_{\tilde{K}} \alpha_{\mathrm{AS}}(\hat{\mathcal{D}}^+) = \int_{\tilde{K}} \alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^+) = \int_{\tilde{M}} \alpha_{\mathrm{AS}}(\tilde{\mathcal{D}}^+).$$

To finish the proof of the lemma it suffices now to show $\eta(0) = 0$.

Let ω be the inward (with respect to $\tilde{K}$) unit normal one-form along Σ. Recall that $\Sigma = S^1 \times L$. We denote the coordinate along S^1 by θ. Suppose that $\{\omega, d\theta, e_1, \cdots, e_m\}$ forms a local orthonormal frame of $T^*\tilde{M}$ on Σ. Then the restriction of $\hat{\mathcal{D}}^+ = \hat{D}^+ + \hat{\Psi}^+$ to Σ can be written as

$$\hat{\mathcal{A}}^+_{\Sigma} = -\sum_{i=1}^{m} \tilde{c}(\omega)\tilde{c}(e_i)\nabla^{\tilde{E}}_{e_i} - \tilde{c}(\omega)\tilde{c}(d\theta)\partial_\theta - \tilde{c}(\omega)\hat{\Psi}^+$$

which maps $C^\infty(\Sigma, \tilde{E}^+|_\Sigma)$ to itself. We define a unitary isomorphism Θ on the space $C^\infty(\Sigma, \tilde{E}|_\Sigma)$ given by

$$\Theta u(\theta, y) := -\tilde{c}(\omega)\tilde{c}(d\theta)u(-\theta, y).$$

One can check that Θ anticommutes with $\hat{\mathcal{A}}^+_{\Sigma}$. As a result, the spectrum of $\hat{\mathcal{A}}^+_{\Sigma}$ is symmetric about 0. Therefore $\eta(0) = 0$ and lemma is proved. □

3.9. Proof of Theorem 3.4

Recall that we denote by B_0 and B_1 the APS boundary conditions for $\mathcal{D}^+ = \tilde{\mathcal{D}}^+|_M$ at N_0 and N_1 respectively. Let $(\mathcal{D}'')^+$ denote the restriction of $\tilde{\mathcal{D}}^+$

to $-M' = \tilde{M} \setminus M$. Let $\bar{B}_0$ and $\bar{B}_1$ be the dual APS boundary conditions for $(\mathcal{D}'')^+$ at N_0 and N_1 respectively. By the Splitting Theorem 2.24,

$$\operatorname{ind} \tilde{\mathcal{D}}^+ = \operatorname{ind} \mathcal{D}^+_{B_0 \oplus B_1} + \operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1}.$$

By Lemma 3.8, we obtain

$$\operatorname{ind} \mathcal{D}^+_{B_0 \oplus B_1} + \operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1} = \int_M \alpha_{\mathrm{AS}}(\mathcal{D}^+) + \int_{M'} \alpha_{\mathrm{AS}}((\mathcal{D}'')^+),$$

which means

$$\operatorname{ind} \mathcal{D}^+_{B_0 \oplus B_1} - \int_M \alpha_{\mathrm{AS}}(\mathcal{D}^+) = -\operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1} + \int_{M'} \alpha_{\mathrm{AS}}((\mathcal{D}'')^+). \tag{3.6}$$

By Lemma 3.7, $(\mathcal{D}'')^+ = -(\mathcal{D}')^-$. Thus $\bar{B}_0 \oplus \bar{B}_1$ is the adjoint of the APS boundary condition for $(-\mathcal{D}')^+$ (cf. Definition 2.14). Therefore,

$$\operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1} = \operatorname{ind}(-\mathcal{D}')^-_{\bar{B}_0 \oplus \bar{B}_1} = -\operatorname{ind}(-\mathcal{D}')^+_{B_0 \oplus B_1} = -\operatorname{ind}(\mathcal{D}')^+_{B_0 \oplus B_1},$$

where we used (2.13) in the middle equality. Also by the construction of local index density,

$$\alpha_{\mathrm{AS}}((\mathcal{D}'')^+) = \alpha_{\mathrm{AS}}((-\mathcal{D}')^-) = \alpha_{\mathrm{AS}}((\mathcal{D}')^-) = -\alpha_{\mathrm{AS}}((\mathcal{D}')^+).$$

Combining these equalities with (3.6) we obtain (3.1). □

4. The relative η-invariant

In the previous section we proved that on an essentially cylindrical manifold M the difference $\operatorname{ind} \mathcal{D}_{B_0 \oplus B_1} - \int_M \alpha_{\mathrm{AS}}(\mathcal{D})$ depends only on the restriction of $\mathcal{D}$ to the boundary, i.e., on the operators $\mathcal{A}_0$ and $-\mathcal{A}_1$. In this section we use this fact to define the *relative η-invariant* $\eta(\mathcal{A}_1, \mathcal{A}_0)$ and show that it has properties similar to the difference of η-invariants $\eta(\mathcal{A}_1) - \eta(\mathcal{A}_0)$ of operators on compact manifolds. For special cases, [27], when the index can be computed using heat kernel asymptotics, we show that $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is indeed equal to the difference of the η-invariants of $\mathcal{A}_1$ and $\mathcal{A}_0$. In the next section we discuss the connection between the relative η-invariant and the spectral flow.

In the case when $\mathcal{A}_0, \mathcal{A}_1$ are operators on even-dimensional manifolds, an analogous construction was proposed in [17, §8]. Even though the definition of the relative η-invariant for operators on odd-dimensional manifolds proposed in this section is slightly more involved than the definition in [17], we show that most of the properties of $\eta(\mathcal{A}_1, \mathcal{A}_0)$ remain the same.

4.1. Almost compact cobordisms

Let N_0 and N_1 be two complete *odd-dimensional* Riemannian manifolds and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be self-adjoint strongly Callias-type operators on N_0 and N_1 respectively, cf. Definition 2.2.

Definition 4.2. An *almost compact cobordism* between $\mathcal{A}_0$ and $\mathcal{A}_1$ is a pair $(M, \mathcal{D})$, where M is an essentially cylindrical manifold with $\partial M = N_0 \sqcup N_1$ and $\mathcal{D}$ is a graded self-adjoint strongly Callias-type operator on M such that

(i) M is an almost compact essential support of $\mathcal{D}$;

(ii) $\mathcal{D}$ is product near ∂M;

(iii) The restriction of $\mathcal{D}^+$ to N_0 is equal to $\mathcal{A}_0$ and the restriction of $\mathcal{D}^+$ to N_1 is equal to $-\mathcal{A}_1$.

If there exists an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$ we say that operator $\mathcal{A}_0$ is *cobordant* to operator $\mathcal{A}_1$.

Lemma 4.3. *An almost compact cobordism is an equivalence relation on the set of self-adjoint strongly Callias-type operators, i.e.,*

(i) *If $\mathcal{A}_0$ is cobordant to $\mathcal{A}_1$ then $\mathcal{A}_1$ is cobordant to $\mathcal{A}_0$.*

(ii) *Let $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ be self-adjoint strongly Callias-type operators on odd-dimensional complete Riemannian manifolds N_0, N_1 and N_2 respectively. Suppose $\mathcal{A}_0$ is cobordant to $\mathcal{A}_1$ and $\mathcal{A}_1$ is cobordant to $\mathcal{A}_2$. Then $\mathcal{A}_0$ is cobordant to $\mathcal{A}_2$.*

Proof. The proof is a verbatim repetition of the proof of Lemmas 8.3 and 8.4 of [17]. □

Definition 4.4. Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are cobordant self-adjoint strongly Callias-type operators and let $(M, \mathcal{D})$ be an almost compact cobordism between them. Let $B_0 = H^{1/2}_{(-\infty,0)}(\mathcal{A}_0)$ and $B_1 = H^{1/2}_{(-\infty,0)}(-\mathcal{A}_1)$ be the APS boundary conditions for $\mathcal{D}^+$. The *relative η-invariant* is defined as

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = 2\left(\operatorname{ind} \mathcal{D}^+_{B_0 \oplus B_1} - \int_M \alpha_{\rm AS}(\mathcal{D}^+)\right) + \dim\ker \mathcal{A}_0 + \dim\ker \mathcal{A}_1. \tag{4.1}$$

Theorem 3.4 implies that $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is independent of the choice of the cobordism $(M, \mathcal{D})$.

Remark 4.5. Sometimes it is convenient to use the dual APS boundary conditions $\bar{B}_0 = H^{1/2}_{(-\infty,0]}(\mathcal{A}_0)$ and $\bar{B}_2 = H^{1/2}_{(-\infty,0]}(-\mathcal{A}_1)$ instead of B_0 and B_1. It follows from Corollary 2.22 that the relative η-invariant can be written as

(4.2)
$$\eta(\mathcal{A}_1, \mathcal{A}_0) = 2\left(\operatorname{ind}\mathcal{D}^+_{\bar{B}_0\oplus\bar{B}_1} - \int_M \alpha_{\rm AS}(\mathcal{D}^+)\right) - \dim\ker\mathcal{A}_0 - \dim\ker\mathcal{A}_1.$$

4.6. The case when the heat kernel has an asymptotic expansion

In [27], Fox and Haskell studied the index of a boundary value problem on manifolds of bounded geometry. They showed that under certain conditions (satisfied for natural operators on manifolds with conical or cylindrical ends) on M and $\mathcal{D}$, the heat kernel $e^{-t(\mathcal{D}_B)^*\mathcal{D}_B}$ is of trace class and its trace has an asymptotic expansion similar to the one on compact manifolds. In this case the η-function, defined by a usual formula

$$\eta(s;\mathcal{A}) := \sum_{\lambda\in\operatorname{spec}(\mathcal{A})} \operatorname{sign}(\lambda)|\lambda|^s, \qquad \operatorname{Re}s \ll 0,$$

is an analytic function of s, which has a meromorphic continuation to the whole complex plane and is regular at 0. So one can define the η-invariant of $\mathcal{A}$ by $\eta(\mathcal{A}) = \eta(0;\mathcal{A})$.

Proposition 4.7. *Suppose now that $\mathcal{D}$ is an operator on an essentially cylindrical manifold M which satisfies the conditions of [27]. We also assume that $\mathcal{D}$ is product near $\partial M = N_0 \sqcup N_1$ and that M is an almost compact essential support of $\mathcal{D}$. Let $\mathcal{A}_0$ and $-\mathcal{A}_1$ be the restrictions of $\mathcal{D}^+$ to N_0 and N_1 respectively. Let $\eta(\mathcal{A}_j)$ ($j = 0, 1$) be the η-invariant of $\mathcal{A}_j$. Then*

$$\eta(\mathcal{A}_1, \mathcal{A}_0) = \eta(\mathcal{A}_1) - \eta(\mathcal{A}_0). \tag{4.3}$$

Proof. An analogue of this proposition for the case when $\dim M$ is odd is proven in [17, Proposition 8.8]. This proof extends to the case when $\dim M$ is even without any changes. □

4.8. Basic properties of the relative η-invariant

Proposition 4.7 shows that under certain conditions the η-invariants of $\mathcal{A}_0$ and $\mathcal{A}_1$ are defined and $\eta(\mathcal{A}_1, \mathcal{A}_0)$ is their difference. We now show that in general case, when $\eta(\mathcal{A}_0)$ and $\eta(\mathcal{A}_1)$ do not necessarily exist, $\eta(\mathcal{A}_1, \mathcal{A}_0)$ behaves like it were a difference of an invariant of N_1 and an invariant of N_0.

Proposition 4.9 (Antisymmetry). *Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are cobordant self-adjoint strongly Callias-type operators. Then*

$$\eta(\mathcal{A}_0, \mathcal{A}_1) = -\eta(\mathcal{A}_1, \mathcal{A}_0). \tag{4.4}$$

Proof. Let $-M$ denote the manifold M with the opposite orientation and let $\tilde{M} := M \cup_{\partial M} (-M)$ denote the *double* of M. Let $\mathcal{D}$ be an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Using the construction of Section 3.6 (with $\mathcal{D}' = \mathcal{D}$) we obtain a graded self-adjoint strongly Callias-type operator $\tilde{\mathcal{D}}$ on $\tilde{M}$ whose restriction to M is isometric to $\mathcal{D}$. Let $\mathcal{D}''$ denote the restriction of $\tilde{\mathcal{D}}$ to $-M = \tilde{M} \setminus M$. Then the restriction of $(\mathcal{D}'')^+$ to N_1 is equal to $\mathcal{A}_1$ and the restriction of $(\mathcal{D}'')^+$ to N_0 is equal to $-\mathcal{A}_0$.

Let

$$\bar{B}_0 = H^{1/2}_{[0,\infty)}(\mathcal{A}_0) = H^{1/2}_{(-\infty,0]}(-\mathcal{A}_0),$$
$$\bar{B}_1 = H^{1/2}_{[0,\infty)}(-\mathcal{A}_1) = H^{1/2}_{(-\infty,0]}(\mathcal{A}_1)$$

be the dual APS boundary conditions for $(\mathcal{D}'')^+$. By (3.6),

$$\operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1} - \int_{M'} \alpha_{\mathrm{AS}}((\mathcal{D}'')^+) = -\operatorname{ind}\mathcal{D}^+_{B_0 \oplus B_1} + \int_M \alpha_{\mathrm{AS}}(\mathcal{D}^+). \tag{4.5}$$

Since $\mathcal{D}''$ is an almost compact cobordism between $\mathcal{A}_1$ and $\mathcal{A}_0$ we conclude from (4.2) that

$$\begin{aligned}\eta(\mathcal{A}_0, \mathcal{A}_1) = 2\left(\operatorname{ind}(\mathcal{D}'')^+_{\bar{B}_0 \oplus \bar{B}_1} - \int_{M'} \alpha_{\mathrm{AS}}((\mathcal{D}'')^+)\right) \\ - \dim\ker \mathcal{A}_0 - \dim\ker \mathcal{A}_1.\end{aligned} \tag{4.6}$$

Combining (4.6) and (4.5) we obtain (4.4). □

Note that (4.4) implies that

$$\eta(\mathcal{A}, \mathcal{A}) = 0 \tag{4.7}$$

for every self-adjoint strongly Callias-type operator $\mathcal{A}$.

Proposition 4.10 (The cocycle condition). *Let $\mathcal{A}_0, \mathcal{A}_1$ and $\mathcal{A}_2$ be self-adjoint strongly Callias-type operators which are cobordant to each other.*

Then

$$\eta(\mathcal{A}_2,\mathcal{A}_0)=\eta(\mathcal{A}_2,\mathcal{A}_1)+\eta(\mathcal{A}_1,\mathcal{A}_0). \tag{4.8}$$

Proof. Let M_1 and M_2 be essentially cylindrical manifolds such that $\partial M_1 = N_0 \sqcup N_1$ and $\partial M_2 = N_1 \sqcup N_2$. Let $\mathcal{D}_1$ be an operator on M_1 which is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Let $\mathcal{D}_2$ be an operator on M_2 which is an almost compact cobordism between $\mathcal{A}_1$ and $\mathcal{A}_2$. Then the operator $\mathcal{D}_3$ on $M_1 \cup_{N_1} M_2$ whose restriction to M_j $(j=1,2)$ is equal to $\mathcal{D}_j$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}_2$.

Let B_0 and B_1 be the APS boundary conditions for $\mathcal{D}_1^+$ at N_0 and N_1 respectively. Then $\bar{B}_1 = H^{1/2}_{[0,\infty)}(\mathcal{A}_1)$ is equal to the dual APS boundary condition for $\mathcal{D}_2^+$. Let B_2 be the APS boundary condition for $\mathcal{D}_2^+$ at N_2. From Corollary 2.22 we obtain

$$\begin{aligned}\eta(\mathcal{A}_2,\mathcal{A}_1)=2\left(\operatorname{ind}(\mathcal{D}_2^+)_{\bar{B}_1\oplus B_2}-\int_M \alpha_{\mathrm{AS}}(\mathcal{D}_2^+)\right)\\ -\dim\ker\mathcal{A}_1+\dim\ker\mathcal{A}_2.\end{aligned} \tag{4.9}$$

By the Splitting Theorem 2.24

$$\operatorname{ind}(\mathcal{D}_3^+)_{B_0\oplus B_2}=\operatorname{ind}(\mathcal{D}_1^+)_{B_0\oplus B_1}+\operatorname{ind}(\mathcal{D}_2^+)_{\bar{B}_1\oplus B_2}. \tag{4.10}$$

Clearly,

$$\int_{M_1\cup M_2}\alpha_{\mathrm{AS}}(\mathcal{D}_3^+)=\int_{M_1}\alpha_{\mathrm{AS}}(\mathcal{D}_1^+)+\int_{M_2}\alpha_{\mathrm{AS}}(\mathcal{D}_2^+). \tag{4.11}$$

Combining (4.9), (4.10), and (4.11) we obtain (4.8). □

5. The spectral flow

Suppose $\mathbb{A} := \{\mathcal{A}^s\}_{0\le s\le 1}$ is a smooth family of self-adjoint elliptic operators on a closed manifold N. Let $\bar{\eta}(\mathcal{A}^s)\in\mathbb{R}/\mathbb{Z}$ denote the mod $\mathbb{Z}$ reduction of the η-invariant $\eta(\mathcal{A}^s)$. Atiyah, Patodi, and Singer, [5], showed that $s\mapsto\bar{\eta}(\mathcal{A}^s)$ is a smooth function whose derivative $\frac{d}{ds}\bar{\eta}(\mathcal{A}^s)$ is given by an explicit local formula. Further, Atiyah, Patodi and Singer, [5], introduced a notion of spectral flow $\mathrm{sf}(\mathbb{A})$ and showed that it computes the net number of integer jumps of $\eta(\mathcal{A}^s)$, i.e.,

$$2\,\mathrm{sf}(\mathbb{A}) = \eta(\mathcal{A}^1) - \eta(\mathcal{A}^0) - \int_0^1 \left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s)\right) ds.$$

In this section we consider a family of self-adjoint strongly Callias-type operators $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ on a complete Riemannian manifold. Assuming that the restriction of $\mathcal{A}^s$ to a complement of a compact set $K \subset N$ is independent of s, we show that for any operator $\mathcal{A}_0$ cobordant to $\mathcal{A}^0$ the mod $\mathbb{Z}$ reduction $\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0)$ of the relative η-invariant depends smoothly on s and

$$2\,\mathrm{sf}(\mathbb{A}) = \eta(\mathcal{A}^1, \mathcal{A}_0) - \eta(\mathcal{A}^0, \mathcal{A}_0) - \int_0^1 \left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0)\right) ds.$$

5.1. A family of boundary operators

Let $E_N \to N$ be a Dirac bundle over a complete *odd-dimensional* Riemannian manifold N. We denote the Clifford multiplication of T^*N on E_N by $c_N : T^*N \to \mathrm{End}(E_N)$. Let $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ be a family of self-adjoint strongly Callias-type operators $\mathcal{A}^s : C^\infty(N, E_N) \to C^\infty(N, E_N)$.

Definition 5.2. The family $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ is called *almost constant* if there exists a compact set $K \subset N$ such that the restriction of $\mathcal{A}^s$ to $N \setminus K$ is independent of s.

Consider the cylinder $M := [0,1] \times N$ and denote by t the coordinate along $[0,1]$. Set

$$E^+ = E^- := [0,1] \times E_N.$$

Then $E = E^+ \oplus E^- \to M$ is naturally a $\mathbb{Z}_2$-graded Dirac bundle over M with

$$c(dt) := \begin{pmatrix} 0 & -\mathrm{id}_{E_N} \\ \mathrm{id}_{E_N} & 0 \end{pmatrix}$$

and

$$c(\xi) := \begin{pmatrix} 0 & c_N(\xi) \\ c_N(\xi) & 0 \end{pmatrix}, \qquad \text{for} \quad \xi \in T^*N.$$

Definition 5.3. The family $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ is called *smooth* if

$$\mathcal{D} := c(dt)\left(\partial_t + \begin{pmatrix} \mathcal{A}^t & 0 \\ 0 & -\mathcal{A}^t \end{pmatrix}\right) : C^\infty(M, E) \to C^\infty(M, E) \tag{5.1}$$

is a smooth differential operator on M.

Fix a smooth non-decreasing function $\kappa : [0,1] \to [0,1]$ such that $\kappa(t) = 0$ for $t \leq 1/3$ and $\kappa(t) = 1$ for $t \geq 2/3$ and consider the operator

$$\mathcal{D} := c(dt)\left(\partial_t + \begin{pmatrix} \mathcal{A}^{\kappa(t)} & 0 \\ 0 & -\mathcal{A}^{\kappa(t)} \end{pmatrix}\right) : C^\infty(M,E) \to C^\infty(M,E). \tag{5.2}$$

Then $\mathcal{D}$ is product near ∂M. If $\mathbb{A}$ is a smooth almost constant family of self-adjoint strongly Callias-type operators then (5.2) is a strongly Callias-type operator for which M is an almost compact essential support. Hence it is a non-compact cobordism (cf. Definition 4.2) between $\mathcal{A}^0$ and $\mathcal{A}^1$.

5.4. The spectral section

If $\mathbb{A} = \{\mathcal{A}^s\}_{0\leq s\leq 1}$ is a smooth almost constant family of self-adjoint strongly Callias-type operators then it satisfies the conditions of the Kato Selection Theorem [30, Theorems II.5.4 and II.6.8], [36, Theorem 3.2]. Thus there is a family of eigenvalues $\lambda_j(s)$ $(j \in \mathbb{Z})$ which depend continuously on s. We order the eigenvalues so that $\lambda_j(0) \leq \lambda_{j+1}(0)$ for all $j \in \mathbb{Z}$ and $\lambda_j(0) \leq 0$ for $j \leq 0$ while $\lambda_j(0) > 0$ for $j > 0$.

Atiyah, Patodi and Singer [5] defined the spectral flow $\mathrm{sf}(\mathbb{A})$ for a family of operators satisfying the conditions of the Kato Selection Theorem ([30, Theorems II.5.4 and II.6.8], [36, Theorem 3.2]) as an integer that counts the net number of eigenvalues that change sign when s changes from 0 to 1. Several other equivalent definitions of the spectral flow based on different assumptions on the family $\mathbb{A}$ exist in the literature. For our purposes the most convenient is the Dai and Zhang's definition [24] which is based on the notion of *spectral section* introduced by Melrose and Piazza [34].

Definition 5.5. A *spectral section* for $\mathbb{A}$ is a continuous family $\mathbb{P} = \{P^s\}_{0\leq s\leq 1}$ of self-adjoint projections such that there exists a constant $R > 0$ such that for all $0 \leq s \leq 1$, if $\mathcal{A}^s u = \lambda u$ then

$$P^s u = \begin{cases} 0, & \text{if } \lambda < -R; \\ u, & \text{if } \lambda > R. \end{cases}$$

If $\mathbb{A}$ satisfies the conditions of the Kato Selection Theorem, then the arguments of the proof of [34, Proposition 1] show that $\mathbb{A}$ admits a spectral section.

5.6. The spectral flow

Let $\mathbb{P} = \{P^s\}$ be a spectral section for $\mathbb{A}$. Set $B^s := \ker P^s$. Let $B_0^s := H^{1/2}_{(-\infty,0)}(\mathcal{A}^s)$ denote the APS boundary condition defined by the boundary operator $\mathcal{A}^s$. Since the spectrum of $\mathcal{A}^s$ is discrete, it follows immediately from the definition of the spectral section that for every $s \in [0,1]$ the space B^s is a finite rank perturbation of B_0^s, cf. Section 2.19. Recall that the relative index $[B^s, B_0^s]$ was defined in Definition 2.20. Following Dai and Zhang [24] (see also [17, §9.8]) we give the following definition.

Definition 5.7. Let $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ be a smooth almost constant family of self-adjoint strongly Callias-type operators which admits a spectral section $\mathbb{P} = \{P^s\}_{0\le s\le 1}$. Assume that the operators $\mathcal{A}^0$ and $\mathcal{A}^1$ are invertible. Let $B^s := \ker P^s$ and $B_0^s := H^{1/2}_{(-\infty,0)}(\mathcal{A}^s)$. The *spectral flow* $\mathrm{sf}(\mathbb{A})$ of the family $\mathbb{A}$ is defined by the formula

$$\mathrm{sf}(\mathbb{A}) := [B^1, B_0^1] - [B^0, B_0^0]. \tag{5.3}$$

By Theorem 1.4 of [24] the spectral flow is independent of the choice of the spectral section $\mathbb{P}$ and computes the net number of eigenvalues that change sign when s changes from 0 to 1.

Lemma 5.8. *Let $-\mathbb{A}$ denote the family $\{-\mathcal{A}^s\}_{0\le s\le 1}$. Then*

$$\mathrm{sf}(-\mathbb{A}) = -\,\mathrm{sf}(\mathbb{A}). \tag{5.4}$$

Proof. The lemma is an immediate consequence of Lemma 5.7 of [17]. □

5.9. Deformation of the relative η-invariant

Let $\mathbb{A} = \{\mathcal{A}^s\}_{0\le s\le 1}$ be a smooth almost constant family of self-adjoint strongly Callias-type operators on a complete odd-dimensional Riemannian manifold N_1. Let $\mathcal{A}_0$ be another self-adjoint strongly Callias-type operator, which is cobordant to $\mathcal{A}^0$. In Section 5.1 we showed that $\mathcal{A}^0$ is cobordant to $\mathcal{A}^s$ for all $s \in [0,1]$. Hence, by Lemma 4.3.(ii), $\mathcal{A}_0$ is cobordant to $\mathcal{A}^1$. In this situation we say the $\mathcal{A}_0$ is *cobordant to the family* $\mathbb{A}$.

The following theorem is the main result of this section.

Theorem 5.10. *Suppose $\mathbb{A} = \big\{\mathcal{A}^s : C^\infty(N_1, E_1) \to C^\infty(N_1, E_1)\big\}_{0\le s\le 1}$ is a smooth almost constant family of self-adjoint strongly Callias-type operators on a complete odd-dimensional Riemannian manifold N_1. Assume that*

$\mathcal{A}^0$ and $\mathcal{A}^1$ are invertible. Let $\mathcal{A}_0 : C^\infty(N_0, E_0) \to C^\infty(N_0, E_0)$ be an invertible self-adjoint strongly Callias-type operator on a complete Riemannian manifold N_0 which is cobordant to the family $\mathbb{A}$. Then the mod $\mathbb{Z}$ reduction $\bar{\eta}(\mathcal{A}^s, \mathcal{A}_0) \in \mathbb{R}/\mathbb{Z}$ of the relative η-invariant depends smoothly on $s \in [0,1]$ and

$$\eta(\mathcal{A}^1, \mathcal{A}_0) - \eta(\mathcal{A}^0, \mathcal{A}_0) - \int_0^1 \left(\frac{d}{ds} \bar{\eta}(\mathcal{A}^s, \mathcal{A}_0) \right) ds = 2\,\mathrm{sf}(\mathbb{A}). \tag{5.5}$$

The proof of this theorem occupies Sections 5.11–5.14.

5.11. A family of almost compact cobordisms

Let M be an essentially cylindrical manifold whose boundary is the disjoint union of N_0 and N_1. First, we construct a smooth family $\mathcal{D}^r$ $(0 \le r \le 1)$ of graded self-adjoint strongly Callias-type operators on the manifold

$$M' := M \cup_{N_1} \big([0,1] \times N_1 \big), \tag{5.6}$$

such that for each $r \in [0,1]$ the pair $(M', \mathcal{D}^r)$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^r$.

Let $\mathcal{D} : C^\infty(M, E) \to C^\infty(M, E)$ be an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^0$. Let E_0 and E_1 denote the restrictions of E to N_0 and N_1 respectively.

Let M' be given by (5.6) and let $E' \to M'$ be the bundle over M' whose restriction to M is equal to E and whose restriction to the cylinder $[0,1] \times N_1$ is equal to $[0,1] \times E_1$.

We fix a smooth function $\rho : [0,1] \times [0,1] \to [0,1]$ such that for each $r \in [0,1]$

- the function $s \mapsto \rho(r,s)$ is non-decreasing.
- $\rho(r,s) = 0$ for $s \le 1/3$ and $\rho(r,s) = r$ for $s \ge 2/3$.

Consider the family of strongly Callias-type operators $\mathcal{D}^r : C^\infty(M', E') \to C^\infty(M', E')$ whose restriction to M is equal to $\mathcal{D}$ and whose restriction to $[0,1] \times N_1$ is given by

$$\mathcal{D}^r := c(dt) \left(\partial_t + \begin{pmatrix} \mathcal{A}^{\rho(r,t)} & 0 \\ 0 & -\mathcal{A}^{\rho(r,t)} \end{pmatrix} \right).$$

Then $\mathcal{D}^r$ is an almost compact cobordism between $\mathcal{A}_0$ and $\mathcal{A}^r$. In particular, the restriction of $\mathcal{D}^r$ to N_1 is equal to $-\mathcal{A}^r$.

Recall that we denote by $-\mathbb{A}$ the family $\{-\mathcal{A}^s\}_{0\le s\le 1}$. Let $\mathbb{P}=\{P^s\}$ be a spectral section for $-\mathbb{A}$. Then for each $r\in[0,1]$ the space $B^r:=\ker P^r$ is a finite rank perturbation of the APS boundary condition for $\mathcal{D}^r$ at $\{1\}\times N_1$. Let $B_0:=H^{1/2}_{(-\infty,0)}(\mathcal{A}_0)$ be the APS boundary condition for $\mathcal{D}^r$ at N_0. Then, by Proposition 2.21, the operator $\mathcal{D}^r_{B_0\oplus B^r}$ is Fredholm. Recall that the domain $\operatorname{dom}\mathcal{D}^r_{B_0\oplus B^r}$ consists of sections u whose restriction to $\partial M'=N_0\sqcup N_1$ lies in $B_0\oplus B^r$.

Lemma 5.12. $\operatorname{ind}\mathcal{D}^r_{B_0\oplus B^r}=\operatorname{ind}\mathcal{D}^1_{B_0\oplus B^1}$ *for all* $r\in[0,1]$.

Proof. For $r_0,r\in[0,1]$, let $\pi_{r_0r}:B^{r_0}\to B^r$ denote the orthogonal projection. Then for every $r_0\in[0,1]$ there exists $\varepsilon>0$ such that if $|r-r_0|<\varepsilon$ then π_{r_0r} is an isomorphism. As in the proofs of [17, Theorem 5.11] and [6, Theorem 8.12], it induces an isomorphism

$$\Pi_{r_0r}:\operatorname{dom}\mathcal{D}^{r_0}_{B_0\oplus B^{r_0}}\to\operatorname{dom}\mathcal{D}^r_{B_0\oplus B^r}.$$

Hence

$$\operatorname{ind}\big(\mathcal{D}^r_{B_0\oplus B^r}\circ\Pi_{r_0r}\big)=\operatorname{ind}\mathcal{D}^r_{B_0\oplus B^r}. \tag{5.7}$$

Since for $|r-r_0|<\varepsilon$

$$\mathcal{D}^r_{B_0\oplus B^r}\circ\Pi_{r_0r}:\operatorname{dom}\mathcal{D}^{r_0}_{B_0\oplus B^{r_0}}\to L^2(M',E')$$

is a continuous family of bounded operators, $\operatorname{ind}\mathcal{D}^r_{B_0\oplus B^r}\circ\Pi_{r_0r}$ is independent of r. The lemma follows now from (5.7). □

5.13. Variation of the reduced relative η-invariant

By Definition 4.4, the mod $\mathbb{Z}$ reduction of the relative η-invariant is given by

$$\bar\eta(\mathcal{A}^r,\mathcal{A}_0):=-2\int_{M'}\alpha_{\rm AS}(\mathcal{D}^r). \tag{5.8}$$

It follows that $\bar\eta(\mathcal{A}^r,\mathcal{A}_0)$ depends smoothly on r and

$$\frac{d}{dr}\bar\eta(\mathcal{A}^r,\mathcal{A}_0)=-2\int_{M'}\frac{d}{dr}\alpha_{\rm AS}(\mathcal{D}^r). \tag{5.9}$$

A more explicit local expression for the right hand side of this equation is given in Section 5.15. For the moment we just note that (5.9) implies that

$$\int_0^1 \left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s,\mathcal{A}_0)\right) ds = -2\int_{M'} \left(\alpha_{\mathrm{AS}}(\mathcal{D}^1) - \alpha_{\mathrm{AS}}(\mathcal{D}^0)\right). \tag{5.10}$$

5.14. Proof of Theorem 5.10

Since the operators $\mathcal{A}_0, \mathcal{A}^0$, and $\mathcal{A}^1$ are invertible, we have

$$\eta(\mathcal{A}^j,\mathcal{A}_0) = 2\left(\operatorname{ind}\mathcal{D}^j_{B_0\oplus B_0^j} - \int_{M'}\alpha_{\mathrm{AS}}(\mathcal{D}^j)\right), \qquad j=0,1.$$

Thus, using (5.10), we obtain

$$\begin{aligned}\eta(\mathcal{A}^1,\mathcal{A}_0) - \eta(\mathcal{A}^0,\mathcal{A}_0) - \int_0^1 \left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s,\mathcal{A}_0)\right) ds \\ = 2\left(\operatorname{ind}\mathcal{D}^1_{B_0\oplus B_0^1} - \operatorname{ind}\mathcal{D}^0_{B_0\oplus B_0^0}\right).\end{aligned} \tag{5.11}$$

Recall that, by Proposition 2.21,

$$\operatorname{ind}\mathcal{D}^r_{B_0\oplus B^r} = \operatorname{ind}\mathcal{D}^r_{B_0\oplus B_0^r} + [B^r, B_0^r],$$

where $B^r = \ker P^r$ and $B_0^r = H^{1/2}_{(-\infty,0)}(\mathcal{A}^r)$ are defined in Subsection 5.6. Hence, from (5.11) we obtain

$$\begin{aligned}&\frac{1}{2}\left(\eta(\mathcal{A}^1,\mathcal{A}_0) - \eta(\mathcal{A}^0,\mathcal{A}_0) - \int_0^1\left(\frac{d}{ds}\bar{\eta}(\mathcal{A}^s,\mathcal{A}_0)\right)ds\right)\\ &= \left(\operatorname{ind}\mathcal{D}^1_{B_0\oplus B^1} - [B^1,B_0^1]\right) - \left(\operatorname{ind}\mathcal{D}^0_{B_0\oplus B^0} - [B^0,B_0^0]\right)\\ &\overset{\text{Lemma 5.12}}{=} -[B^1,B_0^1] + [B^0,B_0^0] = -\operatorname{sf}(-\mathbb{A}) \overset{\text{Lemma 5.8}}{=} \operatorname{sf}(\mathbb{A}).\end{aligned}$$

□

5.15. A local formula for variation of the reduced relative η-invariant

It is well known that there exists a family of differential forms β_r $(0 \le r \le 1)$, called the *transgression form* such that

$$d\beta_r = \frac{d}{dr}\alpha_{\mathrm{AS}}(\mathcal{D}^r). \tag{5.12}$$

The transgression form depends on the symbol of $\mathcal{D}^r$ and its derivatives with respect to r. For geometric Dirac operators one can write very explicit formulas for β_r. For example, if $\mathcal{D}^r$ is the signature operator (so that $\mathcal{A}^r$ is the odd signature operator) corresponding to a family ∇^r of flat connections on E, then $\beta_r = L(M) \wedge \frac{d}{dr}\nabla^r$, where $L(M)$ is the L-genus of M, cf, for example, [14, Theorem 2.3]. For general Dirac-type operators, a formula for β_r is more complicated, cf. [15, §6].

We note that since the family $\mathcal{A}^r$ is constant outside of the compact set K, the form β_r vanishes outside of K. Hence, $\int_{\partial M'} \beta_r$ is well defined and finite. Thus we obtain from (5.9) that

$$\begin{aligned} \frac{d}{dr}\bar{\eta}(\mathcal{A}^r, \mathcal{A}_0) &= -2\int_{M'} d\beta_r \\ &= -2\int_{\partial M'} \beta_r = 2\left(\int_{\{1\}\times N_1} \beta_r - \int_{N_0} \beta_r\right). \end{aligned} \tag{5.13}$$

Hence, (5.5) expresses $\eta(\mathcal{A}^1, \mathcal{A}_0) - \eta(\mathcal{A}^0, \mathcal{A}_0)$ as a sum of $2\,\mathrm{sf}(\mathbb{A})$ and a local differential geometric expression $2\int_{\partial M'}(\int_0^1 \beta_r)dr$.

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Received March 6, 2018
Accepted November 26, 2018