

Standard planar double bubbles are dynamically stable under surface diffusion flow

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Although standard planar double bubbles are stable in the sense that the second variation of the perimeter functional is non-negative for all area-preserving perturbations the question arises whether they are dynamically stable. By presenting connections between these two concepts of stability for double bubbles, we prove that standard planar double bubbles are dynamically stable under the surface diffusion flow via the generalized principle of linearized stability in parabolic Hölder spaces.

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1. Introduction

The standard double bubble is stable in the sense that the second variation of the area functional is non-negative, i.e., it is variationally stable. This follows for example from the fact that it is a local minimum of the area functional under volume constraints. It is however an open problem whether the standard double bubble is dynamically stable for volume conserving geometric flows such as the surface diffusion flow.

The related problem for one bubble has been studied by Escher, Mayer and Simonett, see [8, 9], who showed that spheres are dynamically stable under the surface diffusion flow and the volume preserving mean curvature flow. In this paper we show that the standard double bubble in \mathbb{R}^2 is dynamically stable under the surface diffusion flow. In case of equal areas the result is illustrated in Figure 1.

Before moving on to define the problem more precisely, let us make one point clear: Consider a (cost) functional having local minimizers. Even though minimizers exist it is not clear that an associated gradient flow will converge to these minimizers, see [2] for ODE examples. In other words, if a stationary state of the associated gradient flow is a local minimum, this in general does not imply dynamical stability of this equilibrium under the flow.

As just mentioned, the surface diffusion flow is the volume preserving gradient flow of the area functional. Indeed, it is the fastest way to decrease area while preserving the volume w.r.t. the H^{-1} -inner product; see e.g. [11, 15, 21] and the discussion in [4, Section 2.2.3]. Let us now define the flow precisely. A surface is evolving in time under *the surface diffusion flow* if its normal velocity is equal to the negative surface Laplacian of its mean curvature at each point, that is, if a surface $\Gamma(t)$ satisfies

$$(1.1) \quad V(t) = -\Delta_{\Gamma(t)} H_{\Gamma(t)}.$$

Here V stands for the normal velocity, H is the mean curvature, and Δ is the Laplace-Beltrami operator, of the surface $\Gamma(t)$. Surfaces with constant mean curvature are stationary solutions of the flow (1.1). This flow leads to a fourth order parabolic partial differential equation (PDE). Thereby one may try to use PDE theories to answer the question on the dynamical stability of stationary solutions.

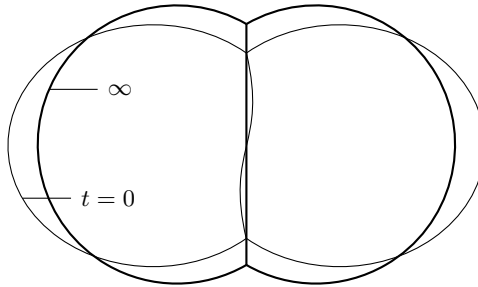


Figure 1: An illustration of the dynamical stability of standard planar double bubbles, possibly up to isometries. (cf. the cover page to G. Prokert's PhD thesis [17]).

Recently Prüss, Simonett and Zacher [18, 19] introduced a practical tool to show dynamical stability for evolution equations in infinite dimensional Banach spaces in cases where the linearization has a non-trivial kernel. It is called *the generalized principle of linearized stability*. This principle is extended in [1] to cover a more general setting. According to this principle, to prove dynamical stability, one needs to verify four assumptions known as the conditions of *normal stability*:

- (i) the set of stationary solutions creates locally a smooth manifold of finite dimension,
- (ii) the tangent space of the manifold of stationary solutions is given by the null space of the linearized operator,
- (iii) the eigenvalue 0 of the linearized operator is semi-simple,
- (iv) apart from zero, the spectrum of the linearized operator lies in \mathbb{C}_+ .

We will see that the non-negativity of the second variation of the area functional and having an appropriate inner product play an important role in verifying at least the assertion (iv) and the semi-simplicity condition (iii) for the double bubble problem.

We refer to the introduction of the PhD thesis of the second author [4, Section 1.1], where this subject is presented for a fundamental example from ODE theory with the Euclidean inner product. There general inner products are also considered and the appropriateness of these inner products with the Levi-Civita connection are discussed.

Let us note that the center manifold theory is used in [8, 9] to prove the dynamical stability of spheres under the surface diffusion flow and the volume preserving mean curvature flow. We remark that so far no center manifold theory exists in the case of non-homogenous boundary conditions. Due to the triple junctions, we indeed get nonlinear boundary conditions in the corresponding PDE.

Outline. In Section 2 we precisely define the problem which we summarize here: Let Γ^0 be an initial planar double bubble. We suppose that Γ^0 moves according to the surface diffusion flow including certain boundary conditions on the triple junctions. We continue then by observing that the set of stationary solutions consists precisely of all standard planar double bubbles.

Next we transfer, via suitable parameterization, this geometric problem to a system of fully nonlinear and nonlocal partial differential equations with nonlinear boundary conditions defined on fixed domains. We then linearize this nonlinear system. This is done in Section 3.

In Section 5.1 we rewrite this nonlinear system as a perturbation of the linearized problem. We then see how suitably the problem fits to the generalized principle of linearized stability setting which is summarized in Section 4.

It then remains to check the conditions of normal stability. Let us note here that understanding the geometric interpretations of the problem was of great help. Semi-simplicity is proved in Section 5.5. We use the fact that the inner product is appropriate. Lemma 5.13 proves assertion (iv). The non-negativity of the second variation is the main ingredient in the proof. We prove assertion (i) in Section 5.4 and Corollary 5.26 proves assertion (ii).

By applying the extended version of the generalized principle of linearized stability we then complete the proof of the dynamical stability under the surface diffusion flow, as summarized in Section 6. We continue in Section 6.1 to discuss general area preserving geometric flows. We then conjecture that the standard planar double bubbles are dynamically stable under sufficiently smooth appropriate gradient flows, see Conjecture 6.2. Finally, we discuss the higher dimensional cases.

In addition, Appendix A shows that the second variation is negative for two elements of the basis of the null space which correspond to non-area preserving perturbations.

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2. The geometric setting

A planar double bubble $\Gamma \subset \mathbb{R}^2$ consists of three curves $\Gamma_1, \Gamma_2, \Gamma_3$ meeting two common points p_+, p_- (triple junctions) at their boundaries such that Γ_1 and Γ_2 (resp. Γ_2 and Γ_3) enclose the connected region R_1 (resp. R_2). Hence the curve Γ_2 is the curve separating R_1 and R_2 , see Figure 2.

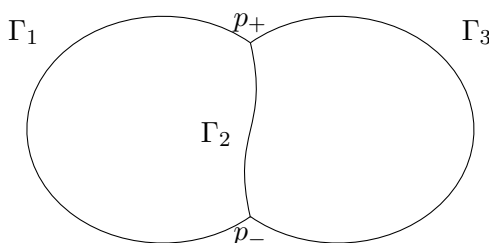


Figure 2: An example of a planar double bubble $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$.

We study the following problem introduced by Garcke and Novick-Cohen [13]: Find evolving planar double bubbles $\Gamma(t) = \{\Gamma_1(t), \Gamma_2(t), \Gamma_3(t)\}$ with the following properties:

$$(2.1) \quad \left. \begin{aligned} V_i &= -\Delta_{\Gamma_i} \kappa_i && \text{on } \Gamma_i(t), \\ \sphericalangle(\Gamma_1(t), \Gamma_2(t)) &= \sphericalangle(\Gamma_2(t), \Gamma_3(t)) = \sphericalangle(\Gamma_3(t), \Gamma_1(t)) = \frac{2\pi}{3} && \text{on } \Sigma(t), \\ \kappa_1 + \kappa_2 + \kappa_3 &= 0 && \text{on } \Sigma(t), \\ \nabla_{\Gamma_1} \kappa_1 \cdot n_{\partial\Gamma_1} &= \nabla_{\Gamma_2} \kappa_2 \cdot n_{\partial\Gamma_2} = \nabla_{\Gamma_3} \kappa_3 \cdot n_{\partial\Gamma_3} && \text{on } \Sigma(t), \\ \Gamma_i(t)|_{t=0} &= \Gamma_i^0, \end{aligned} \right\}$$

where $i = 1, 2, 3$, $\Gamma_i(t) \subset \mathbb{R}^2$, and

$$\partial\Gamma_1(t) = \partial\Gamma_2(t) = \partial\Gamma_3(t) (= \{p_+(t), p_-(t)\} =: \Sigma(t)).$$

Here V_i is the normal velocity, κ_i is the curvature, and Δ_{Γ_i} is the Laplace-Beltrami operator of the curve Γ_i ($i = 1, 2, 3$). Also ∇_{Γ_i} denotes the surface gradient and $n_{\partial\Gamma_i}$ denotes the outer unit conormal of Γ_i at $\partial\Gamma_i$ ($i = 1, 2, 3$).

Moreover $\Gamma^0 = \{\Gamma_1^0, \Gamma_2^0, \Gamma_3^0\}$ is a given initial planar double bubble, which fulfills the angle (2.1)₂, the curvature (2.1)₃ and the balance of flux condition

(2.1)₄ as above and satisfies the compatibility condition

$$(2.2) \quad \Delta_{\Gamma_1^0} \kappa_1^0 + \Delta_{\Gamma_2^0} \kappa_2^0 + \Delta_{\Gamma_3^0} \kappa_3^0 = 0 \quad \text{on } \Sigma(0).$$

Furthermore, the choice of unit normals $n_i(t)$ of $\Gamma_i(t)$ is illustrated in Figure 3, which in particular determines the sign of curvatures κ_1 , κ_2 and κ_3 . We say that the curve has positive curvature if it is curved in the direction of the normal. Therefore, according to this convention we always have

$$\kappa_1 < 0, \quad \kappa_3 > 0.$$

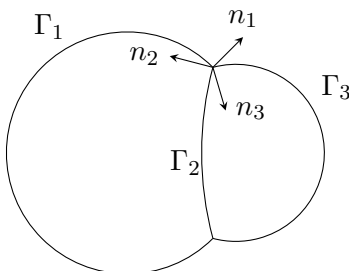


Figure 3: The choice of the normals.

To motivate the compatibility condition (2.2) note that for a classical solution of the surface diffusion flow (2.1) we have indeed

$$(2.3) \quad \sum_{i=1}^3 \Delta_{\Gamma_i} \kappa_i = 0 \quad \text{on } \Sigma(t).$$

Therefore if one seeks for a classical solution which is continuous up to the time $t = 0$, one should impose (2.3) on the initial data.

After introducing the problem, let us see its interesting geometric properties:

Lemma 2.1. *A classical solution to the surface diffusion flow (2.1) decreases the total length and preserves the enclosed areas.*

Proof. Assume $\Gamma(t)$ is a solution to the flow (2.1) and let

$$l(t) = \sum_{i=1}^3 \int_{\Gamma_i(t)} 1 \, ds$$

denote the total length. A transport theorem (see e.g. [5, Theorem 2.44]) gives:

$$\begin{aligned} \frac{d}{dt}l(t) &= - \sum_{i=1}^3 \int_{\Gamma_i(t)} V_i \kappa_i \, ds + \int_{\Sigma(t)} \overbrace{\sum_{i=1}^3 \nu_{\partial\Gamma_i}}^{=0} = \sum_{i=1}^3 \int_{\Gamma_i(t)} (\Delta_{\Gamma_i(t)} \kappa_i) \kappa_i \, ds \\ &= - \sum_{i=1}^3 \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)} \kappa_i|^2 ds + \int_{\Sigma(t)} \sum_{i=1}^3 (\nabla_{\Gamma_i(t)} \kappa_i \cdot n_{\partial\Gamma_i}) \kappa_i \\ &= - \sum_{i=1}^3 \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)} \kappa_i|^2 ds + \int_{\Sigma(t)} (\nabla_{\Gamma_1(t)} \kappa_1 \cdot n_{\partial\Gamma_1}) \underbrace{\sum_{i=1}^3 \kappa_i}_{=0} \\ (2.4) \quad &= - \sum_{i=1}^3 \int_{\Gamma_i(t)} |\nabla_{\Gamma_i(t)} \kappa_i|^2 ds \leq 0, \end{aligned}$$

where we used all the boundary conditions. Note that the sum of the normal boundary velocities $\nu_{\partial\Gamma_i}$ vanishes due to the angle condition, more precisely,

$$\sum_{i=1}^3 \nu_{\partial\Gamma_i}(t, p_{\pm}(t)) = \left(\frac{d}{d\tau} p_{\pm}(\tau) \Big|_{\tau=t} \right) \sum_{i=1}^3 n_{\partial\Gamma_i}(t, p_{\pm}(t)) = 0.$$

Moreover, the integral over $\Sigma(t) = \{p_+(t), p_-(t)\}$ should be understood as a sum over its elements.

Next, let us prove that the enclosed areas are preserved: It is a standard fact that (see e.g. [14, equation (3.1)])

$$\begin{aligned} \frac{d}{dt} \int_{R_1(t)} 1 \, dx &= \int_{\Gamma_1(t)} V_1 \, ds - \int_{\Gamma_2(t)} V_2 \, ds \\ &= - \int_{\Gamma_1(t)} \Delta_{\Gamma_1(t)} \kappa_1 \, ds + \int_{\Gamma_2(t)} \Delta_{\Gamma_2(t)} \kappa_2 \, ds \\ &= - \int_{\Sigma(t)} \nabla_{\Gamma_1(t)} \kappa_1 \cdot n_{\partial\Gamma_1(t)} + \int_{\Sigma(t)} \nabla_{\Gamma_2(t)} \kappa_2 \cdot n_{\partial\Gamma_2(t)} = 0. \end{aligned}$$

Similarly, we get $\frac{d}{dt} \int_{R_2(t)} 1 \, dx = 0$, which completes the proof. □

Let us mention that, via formally matched asymptotic expansions, the flow (2.1) is derived as an singular limit of a system of degenerate Cahn-Hilliard equations in [13], where in particular the boundary conditions at each triple junction are derived.

2.1. Equilibria

Let a planar double bubble $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ be a stationary solution of the flow (2.1), i.e., Γ satisfies (2.1) with $V_i = 0$ for $i = 1, 2, 3$. As a consequence

$$\Delta_{\Gamma_i} \kappa_i = 0 \quad (i = 1, 2, 3).$$

By the same arguments used in (2.4) we get

$$0 = \sum_{i=1}^3 \int_{\Gamma_i} (\Delta_{\Gamma_i} \kappa_i) \kappa_i \, ds = - \sum_{i=1}^3 \int_{\Gamma_i} |\nabla_{\Gamma_i} \kappa_i|^2 \, ds.$$

Thus $\nabla_{\Gamma_i} \kappa_i = 0$ on Γ_i . Therefore $\kappa_1, \kappa_2, \kappa_3$ are constant. Summing up, a planar double bubble Γ is a stationary solution of the flow (2.1) if and only if

- (i): the curvatures κ_i are constant, with $\kappa_1 + \kappa_2 + \kappa_3 = 0$, and
- (ii): $\sphericalangle(\Gamma_i, \Gamma_j) = \frac{2\pi}{3}$ on Σ or equivalently $\sum_{i=1}^3 n_{\partial\Gamma_i} = 0$ on Σ .

It will turn out that the set of stationary solutions consists precisely of all standard planar double bubbles:

Definition 2.2. A standard planar double bubble consists of three circular arcs meeting at their boundaries at 120 degree angles. (Here, we interpret a line segment as a circular arc too.)

We refer to Figure 3 for an example. Indeed, as circular arcs and line segments are the only curves with constant curvature, it just remains to verify the condition on curvatures. This is done in the following proposition given in [14, Proposition 2.1]:

Proposition 2.3. *There is a unique standard planar double bubble (up to rigid motions, i.e., translations and rotations) for given areas in \mathbb{R}^2 . The curvatures satisfy $\kappa_1 + \kappa_2 + \kappa_3 = 0$.*

Remark 2.4. As the choice of the normals in [14] differs from ours, some sign differences particularly for the curvature quantities can occur.

Therefore the set of all standard planar double bubbles $DB_{r,\gamma,\theta}(a_1, a_2)$ forms a 5-parameter family (see Figure 4), where

- (i) $r > 0$ is the radius of Γ_1 , corresponding to scaling,
- (ii) (a_1, a_2) is the center of Γ_1 , corresponding to translation,
- (iii) the angle θ corresponds to counterclockwise rotation around the center of Γ_1 ,
- (iv) the angle $0 < \gamma < \frac{2\pi}{3}$ corresponds to the curvature ratio.

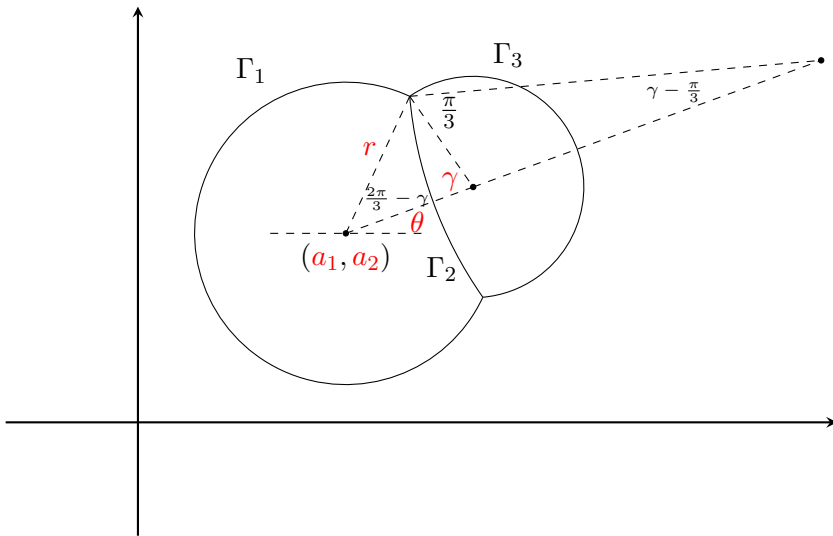


Figure 4: The standard planar double bubble $\Gamma = DB_{r,\gamma,\theta}(a_1, a_2)$.

Indeed, by the law of sines we have for $\gamma \neq \frac{\pi}{3}$

$$(2.5) \quad \frac{\kappa_1}{\sin(\gamma + \frac{\pi}{3})} = \frac{\kappa_2}{\sin(\gamma - \frac{\pi}{3})} = \frac{\kappa_3}{\sin(\gamma - \pi)}$$

and in case $\gamma = \frac{\pi}{3}$ we observe $\kappa_2 = 0$ and $\kappa_1 = -\kappa_3$. Note that due to our choice of normals we always have $\kappa_1 < 0$ and $\kappa_3 > 0$. Moreover,

$$\begin{cases} \kappa_2 > 0 & \text{for } \gamma < \frac{\pi}{3}, \\ \kappa_2 < 0 & \text{for } \gamma > \frac{\pi}{3}. \end{cases}$$

For later use we define the constants q_i as follows:

$$q_i := -\frac{1}{\sqrt{3}}(\kappa_j - \kappa_k)$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$. Then the following result is true.

Lemma 2.5. *We have*

$$q_1 = \cot\left(\gamma + \frac{\pi}{3}\right)\kappa_1, \quad q_2 = \begin{cases} \cot\left(\gamma - \frac{\pi}{3}\right)\kappa_2 & \gamma \neq \frac{\pi}{3}, \\ \frac{\kappa_1}{\sin\left(\frac{\pi}{3}\right)} & \gamma = \frac{\pi}{3}, \end{cases} \quad q_3 = \cot(\gamma - \pi)\kappa_3.$$

Proof. We calculate

$$\begin{aligned} q_2 &= -\frac{1}{\sqrt{3}}(\kappa_3 - \kappa_1) = -\frac{1}{\sqrt{3}} \left(\frac{-\sin(\gamma) - \sin\left(\gamma + \frac{\pi}{3}\right)}{\sin\left(\gamma - \frac{\pi}{3}\right)} \right) \kappa_2 \\ &= \frac{2}{\sqrt{3}} \left(\frac{\sin\left(\gamma + \frac{\pi}{6}\right) \cos\left(\frac{\pi}{6}\right)}{\sin\left(\gamma - \frac{\pi}{3}\right)} \right) \kappa_2 = \cot\left(\gamma - \frac{\pi}{3}\right)\kappa_2 \quad \text{for } \gamma \neq \frac{\pi}{3}, \end{aligned}$$

and obviously $q_2 = \frac{2}{\sqrt{3}}\kappa_1 = \frac{\kappa_1}{\sin\left(\frac{\pi}{3}\right)}$ for $\gamma = \frac{\pi}{3}$. The proof for q_1 and q_3 is similar. □

Moreover, using the sum-to-product trigonometric identity, we get

$$(2.6) \quad \begin{cases} \sin\left(\gamma + \frac{\pi}{3}\right) + \sin\left(\gamma - \frac{\pi}{3}\right) + \sin(\gamma - \pi) = 0, \\ \cos\left(\gamma + \frac{\pi}{3}\right) + \cos\left(\gamma - \frac{\pi}{3}\right) + \cos(\gamma - \pi) = 0. \end{cases}$$

One strategy to deal with geometric flows on hypersurfaces is to parameterize the evolving hypersurfaces with respect to a fixed reference hypersurface. This eventually leads to a PDE on a fixed domain allowing us to employ PDE theories.

3. PDE formulation and linearization

In this section we introduce the proper setting to reformulate the geometric flow (2.1) as a system of partial differential equations for unknown functions defined on fixed domains. For this, we employ a parameterization with two parameters. The parameters correspond to a movement in normal and tangential directions. This parameterization is adapted for two triple junctions from Depner and Garcke [6], see also [7].

3.1. Parameterization of planar double bubbles

Let us describe $\Gamma_i(t)$ as a graph over some fixed stationary solution Γ_i^* using functions

$$\rho_i : \Gamma_i^* \times [0, T) \rightarrow \mathbb{R} \quad (i = 1, 2, 3).$$

The precise way how ρ_i defines $\Gamma_i(t)$ will be derived in what follows.

Fix any stationary solution

$$\Gamma^* = DB_{r^*, \gamma^*, \theta^*}(a_1^*, a_2^*) \quad (r^* > 0, (a_1^*, a_2^*) \in \mathbb{R}^2, 0 < \gamma^* < \frac{2\pi}{3}, 0 \leq \theta^* < 2\pi).$$

Then we observe

$$\begin{aligned} l_1^* &= -\frac{1}{\kappa_1^*}(\gamma^* + \frac{\pi}{3}), \\ l_2^* &= \begin{cases} -\frac{1}{\kappa_2^*}(\gamma^* - \frac{\pi}{3}) = -\frac{1}{\kappa_1^*} \frac{(\gamma^* - \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \sin(\gamma^* + \frac{\pi}{3}) & \text{if } \gamma^* \neq \frac{\pi}{3}, \\ -\frac{1}{\kappa_1^*} \sin(\frac{\pi}{3}) & \text{if } \gamma^* = \frac{\pi}{3}, \end{cases} \\ l_3^* &= -\frac{1}{\kappa_3^*}(\gamma^* - \pi) = -\frac{1}{\kappa_1^*} \frac{(\gamma^* - \pi)}{\sin(\gamma^* - \pi)} \sin(\gamma^* + \frac{\pi}{3}), \end{aligned}$$

where $2l_i^*$ is the length of Γ_i^* ($i = 1, 2, 3$) and of course $\kappa_1^* = -\frac{1}{r^*}$.

Let $\Phi_i^* : [-l_i^*, l_i^*] \rightarrow \mathbb{R}^2$ be an arc-length parameterization of Γ_i^* . Hence

$$\Gamma_i^* = \{\Phi_i^*(x) : x \in [-l_i^*, l_i^*]\}.$$

Furthermore, set $(\Phi_i^*)^{-1}(\sigma) = x(\sigma) \in \mathbb{R}$, for $\sigma \in \Gamma_i^*$. To simplify the presentation, we hereafter set

$$(3.1) \quad \partial_\sigma w(\sigma) := \partial_x(w \circ \Phi_i^*)(x), \quad \sigma = \Phi_i^*(x),$$

that is, we do not state the parameterization explicitly. We also slightly abuse notation and write

$$(3.2) \quad w(\sigma) = w(x) \quad (\sigma \in \Gamma_i^*).$$

To parameterize a curve nearby Γ_i^* , define

$$(3.3) \quad \begin{aligned} \Psi_i : \Gamma_i^* \times (-\epsilon, \epsilon) \times (-\delta, \delta) &\longrightarrow \mathbb{R}^2, \\ (\sigma, w, r) &\mapsto \Psi_i(\sigma, w, r) := \sigma + w n_i^*(\sigma) + r \tau_i^*(\sigma). \end{aligned}$$

Here τ_i^* denotes a tangential vector field on Γ_i^* having support in a neighborhood of $\partial\Gamma_i^*$, which is equal to the outer unit conormal $n_{\partial\Gamma_i^*}$ at $\partial\Gamma_i^*$.

Define then $\Phi_i = (\Phi_i)_{\rho_i, \mu_i}$ (we often omit for shortness the subscript (ρ_i, μ_i)) by

$$(3.4) \quad \Phi_i : \Gamma_i^* \times [0, T] \rightarrow \mathbb{R}^2, \quad \Phi_i(\sigma, t) := \Psi_i(\sigma, \rho_i(\sigma, t), \mu_i(\text{pr}_i(\sigma), t)),$$

for the functions

$$(3.5) \quad \rho_i : \Gamma_i^* \times [0, T] \rightarrow (-\epsilon, \epsilon), \quad \mu_i : \Sigma^* \times [0, T] \rightarrow (-\delta, \delta),$$

where, similarly as before, $\Sigma^* = \partial\Gamma_i^* = \{p_+^*, p_-^*\}$.

The projection $\text{pr}_i : \Gamma_i^* \rightarrow \Sigma^*$ is defined by imposing the following condition: The point $\text{pr}_i(\sigma) \in \partial\Gamma_i^*$ has the shortest distance on Γ_i^* to σ . Clearly, in a small neighborhood of $\partial\Gamma_i^*$, the projection pr_i is well-defined and this is sufficient for us since this projection is just used in the product $\mu_i(\text{pr}_i(\sigma), t)\tau_i^*(\sigma)$, where the second term vanishes outside a (small) neighborhood of $\partial\Gamma_i^*$.

Now let us set, for small $\epsilon, \delta > 0$ and fixed t ,

$$(\Phi_i)_t : \Gamma_i^* \rightarrow \mathbb{R}^2, \quad (\Phi_i)_t(\sigma) := \Phi_i(\sigma, t) \quad \forall \sigma \in \Gamma_i^*$$

to finally define a new curve

$$(3.6) \quad \Gamma_{\rho_i, \mu_i}(t) := \text{image}((\Phi_i)_t).$$

Observe that for $\rho_i \equiv 0$ and $\mu_i \equiv 0$, the curve $\Gamma_{\rho_i, \mu_i}(t)$ coincides with Γ_i^* for all t .

At each triple junction, we have prepared for a movement in normal and tangential direction, allowing for an evolution of the triple junctions. Therefore, we can now formulate the condition, that the curves $\Gamma_i(t)$ meet at the triple junctions at their boundary by

$$(3.7) \quad \Phi_1(\sigma, t) = \Phi_2(\sigma, t) = \Phi_3(\sigma, t) \quad \text{for } \sigma \in \Sigma^*, t \geq 0.$$

Next we prove that this condition leads to a linear dependency at the boundary points. As a result, nonlocal terms will eventually enter into PDE-formulations of the geometric evolution problem.

Lemma 3.1. *Equivalent to the equations (3.7) are the following conditions*

$$(3.8) \quad \begin{cases} \text{(i)} & 0 = \rho_1 + \rho_2 + \rho_3 & \text{on } \Sigma^*, \\ \text{(ii)} & \mu_i = -\frac{1}{\sqrt{3}}(\rho_j - \rho_k) & \text{on } \Sigma^*, \end{cases}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$.

Here the linear dependency (ii) can be recast as the matrix equation

$$(3.9) \quad \mu = \mathcal{J}\rho \quad \text{on } \Sigma^*,$$

with the notations $\mu = (\mu_1, \mu_2, \mu_3)$, $\rho = (\rho_1, \rho_2, \rho_3)$ and the matrix

$$\mathcal{J} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Proof. First we prove that (3.7) implies (3.8). Using the definition of Φ_i , (3.7) can be rewritten as

$$(3.10) \quad \rho_i n_i^* + \mu_i n_{\partial\Gamma_i^*} = \rho_j n_j^* + \mu_j n_{\partial\Gamma_j^*} \quad \text{on } \Sigma^*$$

for $(i, j) = (1, 2), (2, 3)$. By setting

$$q := \rho_1 n_1^* + \mu_1 n_{\partial\Gamma_1^*} = \rho_2 n_2^* + \mu_2 n_{\partial\Gamma_2^*} = \rho_3 n_3^* + \mu_3 n_{\partial\Gamma_3^*} \quad \text{on } \Sigma^*$$

we obtain $\rho_i = \langle q, n_i^* \rangle$ for $i = 1, 2, 3$. Thus the angle condition for Γ^* gives

$$\sum_{i=1}^3 \rho_i = \sum_{i=1}^3 \langle q, n_i^* \rangle = \left\langle q, \sum_{i=1}^3 n_i^* \right\rangle = 0.$$

This proves (i). As a result of (3.10) we see further

$$\rho_i \langle n_i^*, n_j^* \rangle + \mu_i \langle n_{\partial\Gamma_i^*}, n_j^* \rangle = \rho_j \quad \text{on } \Sigma^*.$$

On the other hand the angle condition implies

$$\langle n_i^*, n_j^* \rangle = \cos\left(\frac{2\pi}{3}\right), \quad \langle n_{\partial\Gamma_i^*}, n_j^* \rangle = \cos\left(2\pi - \left(\frac{2\pi}{3} + \frac{\pi}{2}\right)\right) = -\sin\left(\frac{2\pi}{3}\right) \quad \text{on } \Sigma^*$$

for $(i, j) = (1, 2), (2, 3), (3, 1)$. Therefore using (i) we conclude

$$\mu_i = -\frac{1}{s}(\rho_j - c\rho_i) = -\frac{1}{s}((1+c)\rho_j + c\rho_k) = \frac{c}{s}(\rho_j - \rho_k),$$

where $s := \sin(\frac{2\pi}{3})$ and $c := \cos(\frac{2\pi}{3}) = -\frac{1}{2}$ and this yields assertion (ii). The proof of the converse statement is explicitly given in [6, Lemma 2.3]. \square

Note that we followed [12] in proving statement (i), while an easier proof is given here for assertion (ii). Notice further that (3.8) easily implies

$$(3.11) \quad \mu_1 + \mu_2 + \mu_3 = 0 \quad \text{on } \Sigma^*.$$

Remark 3.2. Let us now note that it is within this set, i.e., the set of all planar double bubbles which can be described as the graph over Γ^* , that we will seek a solution to the problem (2.1).

Naturally, we assume also that the initial double bubble Γ^0 from (2.1) is given as a graph over Γ^* , i.e.,

$$\Gamma_i^0 = \{\Psi_i(\sigma, \rho_i^0(\sigma), \mu_i^0(\text{pr}(\sigma))) : \sigma \in \Gamma_i^*\} \quad (i = 1, 2, 3)$$

for some function ρ^0 . Here $\mu^0 = \mathcal{J}\rho^0$ on Σ^* as Γ^0 is assumed to be a double bubble, i.e., the curves Γ_i^0 meet two triple junctions at their boundaries.

3.2. Nonlocal, nonlinear parabolic boundary-value PDE

The idea is to first derive evolution equations for ρ_i and μ_i which have to hold if the Γ_i ($i = 1, 2, 3$) in (3.6) satisfy the condition (3.7) and solve the surface diffusion flow (2.1) and then to make use of the linear dependency (3.9) in deriving evolution equations solely for the functions ρ_i .

As you may have noticed, nonlocal terms will appear in the formulations since this linear dependency (3.9) just holds at the boundary points.

We refer to the PhD thesis of the second author [4, Appendix B.1] for a detailed derivation. Indeed a similar derivation is derived in [1], see also [12], and [7]. Therefore, let us present the final system of fourth-order nonlinear, nonlocal PDEs for $t > 0$, $i = 1, 2, 3$ and $j = 1, 2, \dots, 6$:

$$(3.12) \quad \begin{cases} \partial_t \rho_i = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) \\ \quad + \mathfrak{B}_i(\rho_i, \rho|_{\Sigma^*}) (\{\mathcal{J}(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J})^{-1} \mathfrak{F}(\rho, \rho|_{\Sigma^*})\} \circ \text{pr}_i)_i & \text{on } \Gamma_i^*, \\ 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*, \end{cases}$$

with the initial conditions

$$\rho_i(\cdot, 0) = \rho_i^0 \text{ on } \Gamma_i^*,$$

where in particular $\mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*})$ is a fourth-order nonlinear equation in ρ_i .

Remark 3.3. Note that the price to pay for obtaining equations solely for functions ρ_i is the appearance of nonlocal terms, in particular the nonlocal terms of highest-order (fourth-order) $\mathfrak{F}(\rho, \rho|_{\Sigma^*}) \circ \text{pr}_i$, into the formulation.

The functions $\mathfrak{F}_i, \mathfrak{B}_i, \mathfrak{G}_j$ are rational functions in the ρ -dependent variables, with nonzero denominators in some neighborhood of $\rho \equiv 0$ in $C^1(\Gamma^*)$

(can be inside of square roots equalling to 1 in some neighborhood of $\rho \equiv 0$ in $C^1(\Gamma^*)$, see the term $\frac{1}{j_i}$). See [4, Appendix B.1].

3.3. Linearization around the stationary solution

The linearization of the surface diffusion equations and the angle conditions around the stationary solution $\rho \equiv 0$ is derived in [6, Lemma 3.2] and [6, Lemma 3.4] respectively.

Remark 3.4. Note that the situation in [6] is slightly different from ours, but nevertheless the results obtained there are applicable to our problem. More precisely, the authors in [6] consider the situation where, apart from the appearance of a triple junction, one has to deal with a fixed boundary. However, as they assume that the triple junction will not touch the outer fixed boundary, they can use an explicit parameterization, exactly as ours, around a triple junction and another parameterization near the fixed boundary and finally they compose them with the help of a cut-off function. Thus we can use their result for each triple junction.

Therefore, taking into account the linear dependency (ii) from Lemma 3.1, we get for the linearization of the nonlinear problem (3.12) around $\rho \equiv 0$ (that is, around the stationary solution Γ^*) the following linear system for $i = 1, 2, 3$

$$(3.13) \quad \partial_t \rho_i + \Delta_{\Gamma_i^*} (\Delta_{\Gamma_i^*} \rho_i + (\kappa_i^*)^2 \rho_i) = 0 \quad \text{in } \Gamma_i^*,$$

with the boundary conditions on Σ^*

$$(3.14) \quad \left\{ \begin{array}{ll} \rho_1 + \rho_2 + \rho_3 = 0, & \\ q_i^* \rho_i + \partial_{n_{\partial\Gamma_i^*}} \rho_i = q_j^* \rho_j + \partial_{n_{\partial\Gamma_j^*}} \rho_j & (i, j) = (1, 2), (2, 3), \\ \sum_{i=1}^3 \Delta_{\Gamma_i^*} \rho_i + (\kappa_i^*)^2 \rho_i = 0, & \\ \partial_{n_{\partial\Gamma_i^*}} (\Delta_{\Gamma_i^*} \rho_1 + (\kappa_i^*)^2 \rho_1) = \partial_{n_{\partial\Gamma_j^*}} (\Delta_{\Gamma_j^*} \rho_j + (\kappa_j^*)^2 \rho_j) & (i, j) = (1, 2), (2, 3), \end{array} \right.$$

where

$$q_i^* = -\frac{1}{\sqrt{3}}(\kappa_j^* - \kappa_k^*)$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$.

Let us recall the parameterization (remember our abuse of notation (3.2)) and employ the following facts

$$\begin{aligned} \Delta_{\Gamma_i^*} \rho_i &= \partial_x^2 \rho_i && \text{for } x \in [-l_i^*, l_i^*], \\ \partial_{n_{\partial\Gamma_i^*}} \rho_i &= \nabla_{\Gamma_i^*} \rho_i \cdot n_{\partial\Gamma_i^*} \\ &= \partial_x \rho_i (T_i^* \cdot n_{\partial\Gamma_i^*}) = \pm \partial_x \rho_i && \text{at } x = \pm l_i^*, \\ \kappa_{n_{\partial\Gamma_i^*}} &= \kappa_i^* && \text{at } x = \pm l_i^*, \end{aligned}$$

where x is the arc length parameter of Γ_i^* and denote by T_i^* the tangential vector of Γ_i^* . We can then rewrite the linearized problem in terms of functions $\rho_i : [-l_i^*, l_i^*] \times [0, T) \rightarrow \mathbb{R}$ as

$$\partial_t \rho_i + \partial_x^2 (\partial_x^2 + (\kappa_i^*)^2) \rho_i = 0 \quad \text{for } x \in [-l_i^*, l_i^*]$$

with the boundary conditions

$$(3.15) \quad \begin{cases} \rho_1 + \rho_2 + \rho_3 = 0, \\ q_1^* \rho_1 \pm \partial_x \rho_1 = q_2^* \rho_2 \pm \partial_x \rho_2 = q_3^* \rho_3 \pm \partial_x \rho_3, \\ \sum_{i=1}^3 (\partial_x^2 \rho_i + (\kappa_i^*)^2 \rho_i) = 0, \\ \partial_x (\partial_x^2 + (\kappa_1^*)^2) \rho_1 = \partial_x (\partial_x^2 + (\kappa_2^*)^2) \rho_2 = \partial_x (\partial_x^2 + (\kappa_3^*)^2) \rho_3. \end{cases}$$

In the boundary conditions (3.15) we have omitted the terms $\pm l_i^*$ in the functions ρ_i . That is, for instance the boundary condition $\rho_1 + \rho_2 + \rho_3 = 0$ should be read as

$$\rho_1(\pm l_1^*) + \rho_2(\pm l_2^*) + \rho_3(\pm l_3^*) = 0.$$

Furthermore, notice that the linearized problem is completely local as, in particular, we linearized around a stationary solution.

Now we are in a position to look for a suitable PDE theory in order to answer the question of the dynamical stability. The *generalized principle of linearized stability in parabolic Hölder spaces*, proved in [1], see also [18, 19], provides the tool.

4. The generalized principle of linearized stability in parabolic Hölder spaces

In this section we present the practical tool, proved in [1], for proving the dynamical stability of equilibria of fully nonlinear parabolic systems with

nonlinear boundary conditions in situations where the set of stationary solutions creates a C^2 -manifold of finite dimension which is normally stable. The parabolic Hölder spaces are used as function spaces.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $C^{2m+\alpha}$ for $m \in \mathbb{N}$, $0 < \alpha < 1$ with the boundary $\partial\Omega$. Consider the nonlinear boundary value problem

$$(4.1) \quad \begin{cases} \partial_t u(t, x) + A(u(t, \cdot))(x) = F(u(t, \cdot))(x), & x \in \bar{\Omega}, \quad t > 0, \\ B_j(u(t, \cdot))(x) = G_j(u(t, \cdot))(x), & x \in \partial\Omega, \quad j = 1, \dots, mN, \\ u(0, x) = u_0(x), & x \in \bar{\Omega}, \end{cases}$$

where $u : \bar{\Omega} \times [0, \infty) \rightarrow \mathbb{R}^N$. Here A denotes a linear $2m$ th-order partial differential operator having the form

$$(Au)(x) = \sum_{|\gamma| \leq 2m} a_\gamma(x) \nabla^\gamma u(x), \quad x \in \bar{\Omega},$$

and B_j denote linear partial differential operators of order m_j , i.e.,

$$(B_j u)(x) = \sum_{|\beta| \leq m_j} b_\beta^j(x) \nabla^\beta u(x), \quad x \in \partial\Omega, \quad j = 1, \dots, mN,$$

with $0 \leq m_1 \leq m_2 \leq \dots \leq m_{mN} \leq 2m - 1$.

The coefficients $a_\gamma(x) \in \mathbb{R}^{N \times N}$, $b_\beta^j(x) \in \mathbb{R}^N$. We assume that

- (H2) the elements of the matrix $a_\gamma(x)$ belong to $C^\alpha(\bar{\Omega})$ and the elements of the matrix $b_\beta^j(x)$ belong to $C^{2m+\alpha-m_j}(\partial\Omega)$.

Concerning the fully nonlinear terms F and G_j , let us suppose

- (H1) $F : B(0, R) \subset C^{2m}(\bar{\Omega}) \rightarrow C(\bar{\Omega})$ is C^1 with Lipschitz continuous derivative, $F(0) = 0, F'(0) = 0$, and the restriction of F to $B(0, R) \subset C^{2m+\alpha}(\bar{\Omega})$ has values in $C^\alpha(\bar{\Omega})$ and is continuously differentiable,
 $G_j : B(0, R) \subset C^{m_j}(\bar{\Omega}) \rightarrow C(\partial\Omega)$ is C^2 with Lipschitz continuous second-order derivative, $G_j(0) = 0, G'_j(0) = 0$, and the restriction of G_j to $B(0, R) \subset C^{2m+\alpha}(\bar{\Omega})$ has values in $C^{2m+\alpha-m_j}(\partial\Omega)$ and is continuously differentiable.

We set $B = (B_1, \dots, B_{mN})$ and $G = (G_1, \dots, G_{mN})$.

We denote by $\mathcal{E} \subset B_{X_1}(0, R)$ the set of stationary solutions of (4.1), i.e.,

$$(4.2) \quad u \in \mathcal{E} \iff u \in B_{X_1}(0, R), \quad Au = F(u) \quad \text{in } \Omega, \quad Bu = G(u) \quad \text{on } \partial\Omega,$$

where $X_1 = C^{2m+\alpha}(\overline{\Omega})$. The assumption (H1) in particular implies that

$$u^* \equiv 0 \text{ belongs to } \mathcal{E}.$$

Now the key assumption is that near $u^* \equiv 0$ the set of equilibria \mathcal{E} creates a finite dimensional C^2 -manifold. In other words we assume: There is a neighborhood $U \subset \mathbb{R}^k$ of $0 \in U$, and a C^2 -function $\Psi : U \rightarrow X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u^* \equiv 0$,
- the rank of $\Psi'(0)$ equals k .

Moreover, we at last require that there are no other stationary solutions near $u^* \equiv 0$ in X_1 than those given by $\Psi(U)$. That is we assume for some $r_1 > 0$,

$$\mathcal{E} \cap B_{X_1}(u^*, r_1) = \Psi(U).$$

The linearization of (4.1) at $u^* \equiv 0$ is given by the operator A_0 which is the realization of A with homogeneous boundary conditions in $X = C(\overline{\Omega})$, i.e., the operator with domain

$$(4.3) \quad D(A_0) = \left\{ u \in C(\overline{\Omega}) \cap \bigcap_{1 < p < +\infty} W^{2m,p}(\Omega) : Au \in X, Bu = 0 \text{ on } \partial\Omega \right\},$$

$$A_0u = Au, \quad u \in D(A_0).$$

Let $\nu(x)$ denote the outer normal of $\partial\Omega$ at $x \in \partial\Omega$. We assume further the *normality condition*:

$$(4.4) \quad \left\{ \begin{array}{l} \text{for each } x \in \partial\Omega, \text{ the matrix } \begin{pmatrix} \sum_{|\beta|=k} b_\beta^{j_1}(x)(\nu(x))^\beta \\ \vdots \\ \sum_{|\beta|=k} b_\beta^{j_{n_k}}(x)(\nu(x))^\beta \end{pmatrix} \text{ is surjective,} \\ \text{where } \{j_i : i = 1, \dots, n_k\} = \{j : m_j = k\}. \end{array} \right.$$

Suppose at last the following first-order compatibility conditions holds: For j such that $m_j = 0$ and $x \in \partial\Omega$

$$(4.5) \quad \begin{cases} Bu^0 = G(u^0), \\ B_j(Au^0 - F(u^0)) = G'_j(u^0)(Au^0 - F(u^0)). \end{cases}$$

Theorem 4.1. ([1, Theorem 3.1]) *Let $u^* \equiv 0$ be a stationary solution of (4.1). Assume that the regularity conditions (H1), (H2), the Lopatinskii-Shapiro condition, the strong parabolicity and finally the normality condition (4.4) hold. Moreover assume that u^* is normally stable, i.e., suppose that*

- (i) *near u^* the set of equilibria \mathcal{E} is a C^2 -manifold in X_1 of dimension $k \in \mathbb{N}$,*
- (ii) *the tangent space of \mathcal{E} at u^* is given by $N(A_0)$,*
- (iii) *the eigenvalue 0 of A_0 is semi-simple, i.e., $R(A_0) \oplus N(A_0) = X$,*
- (iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re } z > 0\}$.*

Then the stationary solution u^ is stable in X_1 . Moreover, if u^0 is sufficiently close to u^* in X_1 and satisfies the compatibility conditions (4.5), then the problem (4.1) has a unique solution in the parabolic Hölder spaces, i.e.,*

$$u \in C^{1+\frac{\alpha}{2m}, 2m+\alpha}([0, \infty) \times \overline{\Omega})$$

and approaches some $u^\infty \in \mathcal{E}$ exponentially fast in X_1 as $t \rightarrow \infty$. In other words, u^ is dynamically stable.*

Remark 4.2. We refer to [1, Section 2] for the definitions of Lopatinskii-Shapiro condition and the strong parabolicity as well as for a complete treatment.

In order to apply this theorem to prove dynamical stability, we must first show that our nonlinear, nonlocal problem (3.12) has the form (4.1). We then devote the rest of the paper to show that the problem (3.12) verifies all hypothesis of Theorem 4.1.

5. Verifying the hypotheses of Theorem 4.1

5.1. General setting

If we change the variables by setting for each $i = 2, 3$

$$x = \frac{\tilde{x} + l_1^*}{2l_1^*} l_i^* + \frac{\tilde{x} - l_1^*}{2l_1^*} l_i^* \quad \tilde{x} \in [-l_1^*, l_1^*],$$

then we easily can restate the nonlinear, nonlocal system (3.12) as a perturbation of a linearized problem, that is of the form (4.1), with $\Omega = [-l_1^*, l_1^*]$,

$$A\rho = \begin{bmatrix} (\mathbf{l}_1)^4 & 0 & 0 \\ 0 & (\mathbf{l}_2)^4 & 0 \\ 0 & 0 & (\mathbf{l}_3)^4 \end{bmatrix} \partial_x^4 \rho + \begin{bmatrix} (\mathbf{l}_1 \kappa_1^*)^2 & 0 & 0 \\ 0 & (\mathbf{l}_2 \kappa_2^*)^2 & 0 \\ 0 & 0 & (\mathbf{l}_3 \kappa_3^*)^2 \end{bmatrix} \partial_x^2 \rho,$$

and

$$\begin{aligned} B_1\rho &= [1 \quad 1 \quad 1] \rho, \\ B_2\rho &= \pm [\mathbf{l}_1 \quad -\mathbf{l}_2 \quad 0] \partial_x \rho + [q_1^* \quad -q_2^* \quad 0] \rho, \\ B_3\rho &= \pm [0 \quad \mathbf{l}_2 \quad -\mathbf{l}_3] \partial_x \rho + [0 \quad q_2^* \quad -q_3^*] \rho, \\ B_4\rho &= [(\mathbf{l}_1)^2 \quad (\mathbf{l}_2)^2 \quad (\mathbf{l}_3)^2] \partial_x^2 \rho + [(\kappa_1^*)^2 \quad (\kappa_2^*)^2 \quad (\kappa_3^*)^2] \rho, \\ B_5\rho &= [(\mathbf{l}_1)^3 \quad -(\mathbf{l}_2)^3 \quad 0] \partial_x^3 \rho + [\mathbf{l}_1(\kappa_1^*)^2 \quad -\mathbf{l}_2(\kappa_2^*)^2 \quad 0] \partial_x \rho, \\ B_6\rho &= [0 \quad (\mathbf{l}_2)^3 \quad -(\mathbf{l}_3)^3] \partial_x^3 \rho + [0 \quad \mathbf{l}_2(\kappa_2^*)^2 \quad -\mathbf{l}_3(\kappa_3^*)^2] \partial_x \rho. \end{aligned}$$

To simplify the presentation, we have dropped the tilde. Here

$$\rho : [-l_1^*, l_1^*] \times [0, \infty) \rightarrow \mathbb{R}^3, \quad \rho = (\rho_1, \rho_2, \rho_3)^T$$

and the constants are given as $\mathbf{l}_i := \frac{l_1^*}{l_i^*}$ ($i = 1, 2, 3$).

When we write (3.12) in the form of (4.1), the corresponding F is a smooth function defined in some neighborhood of 0 in $C^4(\overline{\Omega})$ having values in $C(\overline{\Omega})$. The reason is that, F is Fréchet-differentiable of arbitrary order in some neighborhood of 0 (using the differentiability of composition operators, see e.g. Theorem 1 and 2 of [20, Section 5.5.3]). The same argument works for the corresponding functions G_i . We have obtained that assumption (H1) is satisfied.

Obviously, the operators A and B_j satisfy the smoothness assumption (H2) and the operator A is strongly parabolic. Now Let us check that the Lopatinskii-Shapiro condition (LS) holds. To verify this, for $\lambda \in \overline{\mathbb{C}}_+, \lambda \neq 0$,

we consider the following ODE

$$(5.1) \quad \begin{cases} \lambda v_i(y) + (\mathbf{l}_i)^4 \partial_y^4 v_i(y) = 0, & (y > 0), \\ v_1(0) + v_2(0) + v_3(0) = 0, \\ \mathbf{l}_1 \partial_y v_1(0) = \mathbf{l}_2 \partial_y v_2(0) = \mathbf{l}_3 \partial_y v_3(0), \\ \sum_{i=1}^3 (\mathbf{l}_i)^2 \partial_y^2 v_i(0) = 0, \\ (\mathbf{l}_1)^3 \partial_y^3 v_1(0) = (\mathbf{l}_2)^3 \partial_y^3 v_2(0) = (\mathbf{l}_3)^3 \partial_y^3 v_3(0) \end{cases}$$

and we show that $v \equiv 0$ is the only classical solution that vanishes at infinity. The energy methods provide a simple proof: We test the first line of the equation (5.1) with the function $\frac{1}{\mathbf{l}_i} \bar{v}_i$ and sum for $i = 1, 2, 3$ to find

$$\begin{aligned} \sum_{i=1}^3 \frac{\lambda}{\mathbf{l}_i} \int_0^\infty |v_i|^2 dy &= - \sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty \bar{v}_i \partial_y^4 v_i dy \\ &= \sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty \partial_y \bar{v}_i \partial_y^3 v_i dy + \underbrace{\sum_{i=1}^3 \bar{v}_i \left[(\mathbf{l}_i)^3 \partial_y^3 v_i \right] \Big|_0^\infty}_{=0} \\ &= - \sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty |\partial_y^2 v_i|^2 dy + \underbrace{\sum_{i=1}^3 (\mathbf{l}_i)^2 \partial_y^2 v_i \left[\mathbf{l}_i \partial_y \bar{v}_i \right] \Big|_0^\infty}_{=0} \\ &= - \sum_{i=1}^3 (\mathbf{l}_i)^3 \int_0^\infty |\partial_y^2 v_i|^2 dy. \end{aligned}$$

Here we have used all boundary conditions at $y = 0$ and the fact that the functions v_i and consequently all their derivatives vanish exponentially at infinity. The latter holds due to the fact that the solutions of the above equations are linear combinations of exponential functions. The facts that $0 \neq \lambda \in \overline{\mathbb{C}_+}$ and $\mathbf{l}_i > 0$ enforce $v \equiv 0$. This verifies the claim.

Furthermore, the matrices

$$\begin{aligned} & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{l}_1 & -\mathbf{l}_2 & 0 \\ 0 & \mathbf{l}_2 & -\mathbf{l}_3 \end{bmatrix}, \\ & \begin{bmatrix} (\mathbf{l}_1)^2 & (\mathbf{l}_2)^2 & (\mathbf{l}_3)^2 \end{bmatrix}, \quad \begin{bmatrix} (\mathbf{l}_1)^3 & -(\mathbf{l}_2)^3 & 0 \\ 0 & (\mathbf{l}_2)^3 & -(\mathbf{l}_3)^3 \end{bmatrix} \end{aligned}$$

are surjective and hence the normality condition (4.4) is satisfied.

5.1.1. Compatibility condition. We next turn our attention to the corresponding compatibility condition (4.5). As we have assumed the initial planar double bubble Γ^0 fulfills the contact, angle, the curvature and the balance of flux condition, we see $\mu^0 = \mathcal{J}\rho^0$ and $\mathfrak{G}_j(\rho^0) = 0$ for $j = 1, 2, \dots, 6$. This is exactly the first condition in (4.5).

Concerning the second equation in the compatibility condition (4.5), the following lemma shows that it is equivalent to the geometric compatibility condition (2.2) if the existence of triple junctions and the angle condition for the initial data are already assumed.

Lemma 5.1. *Under the conditions $\mathfrak{G}_j(\rho^0) = 0$ ($j = 1, 2, 3$) and $\mu^0 = \mathcal{J}\rho^0$ on Σ^* , the second equation in the corresponding compatibility condition (4.5) and the geometric compatibility condition (2.2) are equivalent, provided ρ^0 is sufficiently small in the C^1 -norm.*

Proof. The second equation in the corresponding first-order compatibility condition (4.5) reads as

$$(5.2) \quad \sum_{i=1}^3 \mathfrak{F}_i(\rho_i^0, \rho^0) + \mathcal{B}_i(\rho_i, \rho^0) \left(\mathcal{J} \overbrace{(I - \mathcal{B}(\rho^0, \rho^0)\mathcal{J})^{-1} \mathfrak{F}(\rho^0, \rho^0)}^{z:=} \right)_i = 0$$

on Σ^* . Here we have used the facts that the zeroth-order boundary operator $B_1 u = \sum_{i=1}^3 u_i$ and $G_1 \equiv 0$. Let us remind that

$$\begin{aligned} \mathfrak{F}_i(\rho_i^0, \rho^0) &= \frac{1}{\langle n_i^*, n_i^0 \rangle} \Delta(\cdot, \rho_i^0, (\mathcal{J}\rho^0)_i) \kappa_i(\cdot, \rho_i^0, (\mathcal{J}\rho^0)_i), \\ \mathfrak{B}_i(\rho_i^0, \rho^0) &= \frac{\langle n_{\partial\Gamma_i^*}, n_i^0 \rangle}{\langle n_i^*, n_i^0 \rangle}, \end{aligned}$$

and $\tau_i^* = n_{\partial\Gamma_i^*}$ on Σ^* .

On the other hand, the angle condition implies

$$\langle n_i^*, n_i^0 \rangle = \langle n_j^*, n_j^0 \rangle, \quad \langle n_{\partial\Gamma_i^*}, n_i^0 \rangle = \langle n_{\partial\Gamma_j^*}, n_j^0 \rangle \quad \text{on } \Sigma^*.$$

Thus (5.2) can be rewritten as

$$\frac{1}{\langle n_1^*, n_1^0 \rangle} \sum_{i=1}^3 \Delta(\cdot, \rho_i^0, (\mathcal{J}\rho^0)_i) \kappa_i(\cdot, \rho_i^0, (\mathcal{J}\rho^0)_i) + \mathfrak{B}_1 \sum_{i=1}^3 (\mathcal{J}z)_i = 0 \quad \text{on } \Sigma^*,$$

where $\langle n_1^*, n_1^0 \rangle \neq 0$ if Γ^0 is close enough to Γ^* in C^1 -norm, that is if ρ^0 is sufficiently small in the C^1 -norm.

Moreover, due to the definition of the matrix \mathcal{J} , we have

$$\sum_{i=1}^3 (\mathcal{J}y)_i = 0 \quad \forall y \in \mathbb{R}^3.$$

Hence the compatibility condition (5.2) is equivalent to

$$\sum_{i=1}^3 \Delta(\sigma, \rho_i^0, (\mathcal{J}\rho^0)_i) \kappa_i(\sigma, \rho_i^0, (\mathcal{J}\rho^0)_i) = 0,$$

which is exactly the geometric compatibility condition (2.2) written in a parameterization. This finishes the proof. \square

5.2. The spectrum of the linearized problem

Since $\Omega = [-l_1^*, l_1^*] \subset \mathbb{R}$, the linearized operator A_0 (see (4.3)) is defined as $A_0u = Au$ with domain

$$D(A_0) = \left\{ u \in C^4(\overline{\Omega}) : Bu = 0 \text{ on } \partial\Omega \right\},$$

where A and B is defined in Section 5.1. Due to Remark 2.2 in [1], the spectrum of the linearized operator A_0 consists entirely of eigenvalues. As the analysis of the eigenvalue problem is invariant under the change of variables, we switch to the setting where the functions u_i ($i = 1, 2, 3$) have different domains.

Now, the eigenvalue problem for the linearized operator A_0 reads as follows: For $i = 1, 2, 3$,

$$(5.3) \quad \Delta_{\Gamma_i^*} (\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i) = \lambda u_i \quad \text{in } \Gamma_i^* \quad (i = 1, 2, 3),$$

subject to the boundary conditions on Σ^*

$$(5.4) \quad \begin{cases} u_1 + u_2 + u_3 = 0, \\ q_i^* u_i + \partial_{n_{\partial\Gamma_i^*}} u_i = q_j^* u_j + \partial_{n_{\partial\Gamma_j^*}} u_j, \\ \sum_{i=1}^3 \Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = 0, \\ \partial_{n_{\partial\Gamma_i^*}} (\Delta_{\Gamma_i^*} u_1 + (\kappa_i^*)^2 u_i) = \partial_{n_{\partial\Gamma_j^*}} (\Delta_{\Gamma_j^*} u_j + (\kappa_j^*)^2 u_j), \end{cases}$$

where $(i, j) = (1, 2), (2, 3)$.

To derive a bilinear form associated with this eigenvalue problem, let us multiply the equation (5.3) by $-(\Delta_{\Gamma_i^*} \bar{u}_i + (\kappa_i^*)^2 \bar{u}_i)$ and then integrate by parts and sum over $i = 1, 2, 3$ to find

$$\sum_{i=1}^3 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} (\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i)|^2 ds = -\lambda \sum_{i=1}^3 \int_{\Gamma_i^*} u_i (\Delta_{\Gamma_i^*} \bar{u}_i + (\kappa_i^*)^2 \bar{u}_i) ds.$$

Here, as usual, we have used the last two boundary conditions. We observe further

$$\begin{aligned} -\sum_{i=1}^3 \int_{\Gamma_i^*} u_i (\Delta_{\Gamma_i^*} \bar{u}_i + (\kappa_i^*)^2 \bar{u}_i) ds &= \sum_{i=1}^3 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} u_i|^2 - (\kappa_i^*)^2 |u_i|^2 ds \\ &\quad - \sum_{i=1}^3 \int_{\Sigma^*} u_i \partial_{n_{\partial\Gamma_i^*}} u_i \bar{u}_i. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{i=1}^3 \int_{\Sigma^*} u_i \partial_{n_{\partial\Gamma_i^*}} \bar{u}_i &= \sum_{i=1}^3 \int_{\Sigma^*} (u_i \partial_{n_{\partial\Gamma_i^*}} \bar{u}_i + q_i^* |u_i|^2 - q_i^* |u_i|^2) \\ &= \sum_{i=1}^3 \int_{\Sigma^*} (\partial_{n_{\partial\Gamma_i^*}} u_i + q_i^* u_i) \bar{u}_i - \sum_{i=1}^3 \int_{\Sigma^*} q_i^* |u_i|^2 \\ &= \int_{\Sigma^*} (\partial_{n_{\partial\Gamma_1^*}} u_1 + q_1^* u_1) \underbrace{\sum_{i=1}^3 \bar{u}_i}_{=0} - \sum_{i=1}^3 \int_{\Sigma^*} q_i^* |u_i|^2 \\ &= -\sum_{i=1}^3 \int_{\Sigma^*} q_i^* |u_i|^2. \end{aligned}$$

We now combine the three equalities above to discover

$$(5.5) \quad \sum_{i=1}^3 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} (\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i)|^2 ds = \lambda I(u, u),$$

where

$$\begin{aligned} I(u, u) &:= \sum_{i=1}^3 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} u_i|^2 - (\kappa_i^*)^2 |u_i|^2 ds + \sum_{i=1}^3 \int_{\Sigma^*} q_i^* |u_i|^2 \\ (5.6) \quad &= -\sum_{i=1}^3 \int_{\Gamma_i^*} \bar{u}_i (\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i) ds + \sum_{i=1}^3 \int_{\Sigma^*} (\partial_{n_{\partial\Gamma_i^*}} u_i + q_i^* u_i) \bar{u}_i. \end{aligned}$$

Note carefully that in (5.6) we just used integration by parts to obtain the second equality. It is interesting now to see that although (due to the linearized angle condition and the fact that on the boundary $u_1 + u_2 + u_3 = 0$) we have

$$(5.7) \quad \sum_{i=1}^3 \int_{\Sigma^*} (\partial_{n_{\partial\Gamma_i^*}} u_i + q_i^* u_i) \bar{u}_i = 0,$$

but nevertheless this does not effect the value of $I(u, u)$ (cf. [14, Remark 3.7]).

Remark 5.2. The identity (5.5) in particular shows that $\lambda \in \mathbb{R}$.

Remark 5.3. Indeed as one may have expected, the linearized problem (3.13), (3.14) is the gradient flow of the energy functional

$$E(u) = \frac{I(u, u)}{2},$$

with respect to the H^{-1} -inner product, see for instance [12].

5.2.1. Related problem: Double bubble conjecture. The goal of this section is to prove that, a part from zero, the spectrum of the linearized problem lies in \mathbb{R}_+ . We do this by considering the bilinear form $I(,)$.

In the following we state the second variation formula proved in general dimension by Morgan and co-authors:

Proposition 5.4. ([14, Proposition 3.3]). *Let Γ^* be a stationary planar double bubble and let φ_t be a one-parameter variation which preserves the areas of enclosed regions. Furthermore denote by $L(t)$ the length of $\varphi_t(\Gamma^*)$. Then*

$$\frac{d^2}{dt^2} L(t) \Big|_{t=0} = I(u, u),$$

where $u_i = \langle \frac{d}{dt} \varphi_t, n_i^* \rangle$.

Here and hereafter, by (one-parameter) variations $\{\varphi_t\}_{|t|<\epsilon} : \Gamma \rightarrow \mathbb{R}^2$ of a double bubble $\Gamma \subset \mathbb{R}^2$ we mean the variations which are smooth (up to the triple junctions) having equal values along triple junctions.

Remark 5.5. Notice that in (5.6) we have used outer unit conormals where inner unit conormals are used in [14]. In addition, the constants q_i^* and their

corresponding ones in [14] are also opposite in signs due to the different choice of normals. This explains the sign differences.

Remark 5.6. Of course, a double bubble is stationary for any variation preserving the area of the enclosed regions if and only if it is stationary for the surface diffusion flow (2.1), see Section 2.1 and [14, page 465].

Following [14], we denote by $\mathcal{F}(\Gamma)$ the space of functions $u \in H^1(\Gamma)$ satisfying

$$\begin{cases} u_1 + u_2 + u_3 = 0 & \text{on } \Sigma, \\ \int_{\Gamma_1} u_1 = \int_{\Gamma_2} u_2 = \int_{\Gamma_3} u_3. \end{cases}$$

Lemma 5.7. ([14, Lemma 3.2]). *Let Γ^* be a stationary double bubble. Then for any smooth $u \in \mathcal{F}(\Gamma^*)$ there is an area preserving variation $\{\varphi_t\}$ of Γ^* such that the normal components of the associated infinitesimal vector field are the functions u_i , i.e., $u_i = \langle \frac{d}{dt}\varphi_t, n_i^* \rangle$, $i = 1, 2, 3$.*

We are now ready to present:

Definition 5.8 (The concept of stability in differential geometry). A double bubble Γ^* is said to be variationally stable if it is stationary and

$$I(u, u) \geq 0 \quad \forall u \in \mathcal{F}(\Gamma^*).$$

We are forced here to name the concept of stability in differential geometry *variationally stable* instead of *stable*. Indeed it is an open problem whether for double bubbles this concept of stability in differential geometry is equivalent to the concept of stability in PDE theory. There are several evidences in this work which show how closely these two concepts are, starting from Lemma 5.13 below.

Remark 5.9. Note that the concept of stability in differential geometry was called *stable* in [14].

Corollary 5.10. *A perimeter-minimizing double bubble for prescribed areas is variationally stable.*

Proof. Let Γ be a perimeter-minimizing double bubble. As a minimizer, the second derivative of length is nonnegative along all variations which preserve the area, in other words by Proposition 5.4 $I(u, u) \geq 0$ for all functions u

given by normal components of area preserving variations. On the other hand, by Lemma 5.7 we know that every smooth element of $\mathcal{F}(\Gamma)$ is of this form. Therefore $I(u, u) \geq 0$ for all $u \in \mathcal{F}(\Gamma)$, which finishes the proof. \square

Theorem 5.11. ([10, Theorem 2.9]). *The standard planar double bubble is the unique perimeter-minimizing double bubble enclosing and separating two given regions of prescribed areas.*

Therefore as an important corollary one gets: (see also [16, Theorem 3.2])

Corollary 5.12. *The standard planar double bubble is variationally stable.*

We are now ready to see the first evidence.

Lemma 5.13. $\sigma(A_0) \setminus \{0\} \subset \mathbb{R}_+$.

Proof. Let $\lambda \in \sigma(A_0) \setminus \{0\}$. As mentioned before the spectrum consists entirely of eigenvalues. In addition, according to Remark 5.2, λ is real.

Therefore, let λ be an eigenvalue with a corresponding eigenvector $u \in C^{4+\alpha}(\Gamma^*)$. This means u solves the eigenvalue problem (5.3) subject to the boundary conditions (5.4) for λ . Since $\lambda \neq 0$, we deduce after integrating (5.3):

$$\int_{\Gamma_1^*} u_1 = \int_{\Gamma_2^*} u_2 = \int_{\Gamma_3^*} u_3,$$

where we employed the divergence theorem and the last boundary condition. This together with the first boundary condition implies that $u \in \mathcal{F}(\Gamma^*)$. Therefore $I(u, u) \geq 0$ by Corollary 5.12.

Now assume $I(u, u) = 0$. In view of the equation (5.5), we obtain

$$\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = c_i$$

for some constants c_i (cf. [14, Lemma 3.8]). This together with the equation (5.3) immediately implies $u \in N(A_0)$, i.e., $\lambda = 0$, a contradiction. Thus $I(u, u) > 0$ for the eigenvector u . Now $\lambda > 0$ by (5.3). This finishes the proof. \square

The bilinear form $I(\cdot, \cdot)$ is further discussed in Appendix A.

5.3. Null space of the linearized problem

We next determine the null space of the linearized operator A_0 . That is, we consider the case $\lambda = 0$ in the eigenvalue problem (5.3),(5.4).

Using the identity (5.5), we easily get $u \in N(A_0)$ if and only if there exists a constant vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ such that

$$(5.8) \quad \Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i = c_i \quad \text{on } \Gamma_i^* \quad (i = 1, 2, 3),$$

subject to the conditions

$$(5.9) \quad \begin{cases} u_1 + u_2 + u_3 = 0 & \text{on } \Sigma^*, \\ q_1^* u_1 + \partial_{n_{\partial\Gamma_1^*}} u_1 = q_2^* u_2 + \partial_{n_{\partial\Gamma_2^*}} u_2 = q_3^* u_3 + \partial_{n_{\partial\Gamma_3^*}} u_3 & \text{on } \Sigma^*, \\ c_1 + c_2 + c_3 = 0. \end{cases}$$

Notice that the constant vector $c = c(u)$ depends linearly on u by (5.8).

Definition 5.14. Following [14], we define the space of Jacobi functions

$$\mathcal{J}(\Gamma^*) := \{u \in N(A_0) : c = c(u) = 0\}.$$

We need, for later use, an identity that relates the null space $N(A_0)$ to the bilinear form $I(\cdot, \cdot)$.

Lemma 5.15. *Assume $u \in N(A_0)$. Then*

$$I(u, u) = - \sum_{i=1}^3 c_i \int_{\Gamma_i^*} u_i,$$

where the constants c_i , satisfying $\sum_{i=1}^3 c_i = 0$, depend linearly on u by (5.8).

Proof. By inserting (5.8) into the definition of the bilinear form (5.6) and taking into account the equation (5.7) coming from the first two boundary conditions in (5.9), we get the desired identity. □

As a corollary we get

Corollary 5.16. *If $u \in N(A_0) \cap \mathcal{F}(\Gamma^*)$, then $I(u, u) = 0$.*

Let us rewrite the linear equations (5.8) as a system of linear nonhomogeneous second order ordinary differential equations with constant coefficients

$$\partial_x^2 u_i + (\kappa_i^*)^2 u_i = c_i \quad \text{for } x \in [-l_i^*, l_i^*] \quad (i = 1, 2, 3),$$

with the conditions

$$\begin{cases} u_1 + u_2 + u_3 = 0 & \text{on } \Sigma^*, \\ q_1^* u_1 \pm \partial_x u_1 = q_2^* u_2 \pm \partial_x u_2 = q_3^* u_3 \pm \partial_x u_3 & \text{on } \Sigma^*, \\ c_1 + c_2 + c_3 = 0 \end{cases}$$

for the functions $u_i : [-l_i^*, l_i^*] \rightarrow \mathbb{R}$.

5.3.1. Determination of Jacobi functions . Let us first consider the case $\kappa_2^* \neq 0$. The general solution of the linearized problem is then

$$(5.10) \quad u_i(x) = a_i \sin(\kappa_i^* x) + b_i \cos(\kappa_i^* x) \quad (i = 1, 2, 3).$$

We calculate at $x = \pm l_1^*$

$$\begin{aligned} q_1^* u_1 &= \mp \cot(\gamma^* + \frac{\pi}{3}) \kappa_1^* a_1 \sin(\gamma^* + \frac{\pi}{3}) + \cot(\gamma^* + \frac{\pi}{3}) \kappa_1^* b_1 \cos(\gamma^* + \frac{\pi}{3}) \\ &= \mp a_1 \kappa_1^* \cos(\gamma^* + \frac{\pi}{3}) + b_1 \kappa_1^* \cot(\gamma^* + \frac{\pi}{3}) \cos(\gamma^* + \frac{\pi}{3}), \\ \pm \partial_x u_1 &= \pm a_1 \kappa_1^* \cos(\gamma^* + \frac{\pi}{3}) + b_1 \kappa_1^* \sin(\gamma^* + \frac{\pi}{3}). \end{aligned}$$

Therefore

$$q_1^* u_1 \pm \partial_x u_1 = b_1 \frac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} \quad \text{at } x = \pm l_1^*.$$

Similarly we get

$$\begin{aligned} q_2^* u_2 \pm \partial_x u_2 &= b_2 \frac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} \quad \text{at } x = \pm l_2^*, \\ q_3^* u_3 \pm \partial_x u_3 &= b_3 \frac{\kappa_3^*}{\sin(\gamma^* - \pi)} \quad \text{at } x = \pm l_3^*. \end{aligned}$$

Thus we conclude

$$b_1 \frac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} = b_2 \frac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} = b_3 \frac{\kappa_3^*}{\sin(\gamma^* - \pi)}.$$

Furthermore, $u_1(\pm l_1^*) + u_2(\pm l_2^*) + u_3(\pm l_3^*) = 0$ reads as

$$\begin{aligned} &\mp a_1 \sin\left(\gamma^* + \frac{\pi}{3}\right) + b_1 \cos\left(\gamma^* + \frac{\pi}{3}\right) \\ &\mp a_2 \sin\left(\gamma^* - \frac{\pi}{3}\right) + b_2 \cos\left(\gamma^* - \frac{\pi}{3}\right) \\ &\mp a_3 \sin(\gamma^* - \pi) + b_3 \cos(\gamma^* - \pi) = 0. \end{aligned}$$

Altogether, we have to find solutions to the following system

$$\begin{cases} a_1 \sin\left(\gamma^* + \frac{\pi}{3}\right) + a_2 \sin\left(\gamma^* - \frac{\pi}{3}\right) + a_3 \sin(\gamma^* - \pi) = 0, \\ b_1 \cos\left(\gamma^* + \frac{\pi}{3}\right) + b_2 \cos\left(\gamma^* - \frac{\pi}{3}\right) + b_3 \cos(\gamma^* - \pi) = 0, \\ b_1 \frac{\kappa_1^*}{\sin(\gamma^* + \frac{\pi}{3})} = b_2 \frac{\kappa_2^*}{\sin(\gamma^* - \frac{\pi}{3})} = b_3 \frac{\kappa_3^*}{\sin(\gamma^* - \pi)}. \end{cases}$$

Due to the identities (2.5) and (2.6) we get

$$\begin{cases} a_1 \sin\left(\gamma^* + \frac{\pi}{3}\right) + a_2 \sin\left(\gamma^* - \frac{\pi}{3}\right) + a_3 \sin(\gamma^* - \pi) = 0, \\ b_1 = b_2 = b_3. \end{cases}$$

Therefore, in view of the formula (2.6), we obtain

$$(a_1, a_2, a_3) \in \text{span}\left\{(1, 1, 1), \left(0, -\frac{\sin(\gamma^* - \pi)}{\sin(\gamma^* - \frac{\pi}{3})}, 1\right)\right\}, \quad (b_1, b_2, b_3) \in \text{span}\{(1, 1, 1)\}.$$

This shows the following lemma:

Lemma 5.17. *Assume $\kappa_2^* \neq 0$. Then the space of Jacobi functions is a three dimensional vector space whose basis consists of*

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ \cos(\kappa_2^* x) \\ \cos(\kappa_3^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^* - \frac{\pi}{3})} \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}.$$

We now consider the case $\kappa_2^* = 0$. The general solution of the linearized problem is then

$$\begin{aligned} u_1 &= a_1 \sin(\kappa_1^* x) + b_1 \cos(\kappa_1^* x), & u_2 &= a_2 x + b_2, \\ u_3 &= a_3 \sin(\kappa_3^* x) + b_3 \cos(\kappa_3^* x) \quad (= -a_3 \sin(\kappa_1^* x) + b_3 \cos(\kappa_1^* x)), \end{aligned}$$

where we used the fact that $\kappa_3^* = -\kappa_1^*$ in case $\gamma^* = \frac{\pi}{3}$. Let us also remind that for $\gamma^* = \frac{\pi}{3}$ we have

$$q_2^* = \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} \quad \text{and} \quad l_2^* = -\frac{\sin(\frac{\pi}{3})}{\kappa_1^*}$$

and so $q_2^* l_2^* = -1$. Therefore,

$$q_2^* u_2 \pm \partial_x u_2 = \mp a_2 + \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} b_2 \pm a_2 = \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} b_2 \quad \text{at } x = \pm l_2^*.$$

Taking into account the calculation done previously for u_1 and u_3 , the condition $q_1^* u_1 \pm \partial_x u_1 = q_2^* u_2 \pm \partial_x u_2 = q_3^* u_3 \pm \partial_x u_3$ reads as

$$b_1 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} = b_2 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} = b_3 \frac{\kappa_3^*}{\sin(-\frac{2\pi}{3})} \left(= b_3 \frac{-\kappa_1^*}{-\sin(\frac{\pi}{3})} = b_3 \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} \right).$$

Therefore, we conclude $b_1 = b_2 = b_3$. Furthermore, $u_1(\pm l_1^*) + u_2(\pm l_2^*) + u_3(\pm l_3^*) = 0$ reads as

$$\mp a_1 \sin(\frac{\pi}{3}) + b_1 \cos(\frac{2\pi}{3}) \mp a_2 \frac{\sin(\frac{\pi}{3})}{\kappa_1^*} + b_2 \pm a_3 \sin(\frac{\pi}{3}) + b_3 \cos(\frac{2\pi}{3}) = 0.$$

Moreover, using the facts that $b_1 = b_2 = b_3$ and $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$, we see that

$$b_1 \cos(\frac{2\pi}{3}) + b_2 + b_3 \cos(\frac{2\pi}{3}) = 0.$$

In summary, we have to find solutions to the following system

$$\begin{cases} a_1 + \frac{a_2}{\kappa_1^*} - a_3 = 0, \\ b_1 = b_2 = b_3. \end{cases}$$

Therefore,

$$(a_1, a_2, a_3) \in \text{span}\{(1, 0, 1), (0, \kappa_1^*, 1)\}, \quad (b_1, b_2, b_3) \in \text{span}\{(1, 1, 1)\}.$$

Lemma 5.18. *Assume $\kappa_2^* = 0$. Then the space of Jacobi functions is 3-dimensional and its basis is given by*

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ 1 \\ \cos(\kappa_1^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ 0 \\ -\sin(\kappa_1^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \kappa_1^* x \\ -\sin(\kappa_1^* x) \end{pmatrix}.$$

5.3.2. The null space $N(A_0)$ is at most five-dimensional. Next we try to get an upper bound on the dimension of the null space.

Lemma 5.19. *The null space $N(A_0)$ of the linearized operator A_0 is at most five-dimensional.*

Proof. We have already shown that the space of Jacobi functions is three-dimensional. Therefore it is enough to show that there exist at most two independent vectors in the null space $N(A_0)$ for which $c \neq 0$.

Take any three vector functions $u^{(1)}, u^{(2)}, u^{(3)} \in N(A_0)$ for which the vector constants $c^{(i)} = c^{(i)}(u^{(i)}) \neq 0, i = 1, 2, 3$. Then as

$$c^{(i)} \in \{ c = (c_1, c_2, c_3) \in \mathbb{R}^3 : c_1 + c_2 + c_3 = 0 \}$$

which is a two dimensional subspace of \mathbb{R}^3 , there exist scalars a_1, a_2, a_3 , not all zero, such that

$$0 = \sum_{j=1}^3 a_j c^{(j)} = \sum_i^3 a_i T u^{(i)} = T \left(\sum_{i=1}^3 a_i u^{(i)} \right).$$

Here T is the linear operator defined by the left hand side of (5.8). Thus we get $\sum_{i=1}^3 a_i u^{(i)} \in \mathcal{J}(\Gamma^*)$, in other words,

$$\sum_{i=1}^3 a_i u^{(i)} = \sum_{j=1}^3 b_j v^{(j)},$$

where $\{v^{(1)}, v^{(2)}, v^{(3)}\}$ is a basis of $J(\Gamma^*)$. This means that the vectors

$$u^{(1)}, u^{(2)}, u^{(3)}, v^{(1)}, v^{(2)}, v^{(3)}$$

are linearly dependent and completes the proof. □

Indeed, we will prove in Corollary 5.26 below that the dimension of the null space is exactly five.

5.4. Manifold of equilibria

Our goal in this section is to prove that near $\rho \equiv 0$, which corresponds to Γ^* , the set \mathcal{E} of equilibria of the nonlinear system (3.12) creates a smooth manifold of dimension 5.

5.4.1. Equilibria of the nonlinear system . Let us first identify the set of equilibria \mathcal{E} of the nonlinear system (3.12). According to (4.2), $\rho \in \mathcal{E}$ if and only if for $i = 1, 2, 3$ and $j = 1, 2, \dots, 6$,

$$\begin{cases} \rho \in B_{X_1}(0, R), \\ 0 = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) \\ \quad + \mathfrak{B}_i(\rho_i, \rho|_{\Sigma^*}) \left(\left\{ \mathcal{J}(I - \mathfrak{B}(\rho, \rho|_{\Sigma^*})\mathcal{J})^{-1} \mathfrak{F}(\rho, \rho|_{\Sigma^*}) \right\} \circ \text{pr}_i \right)_i & \text{on } \Gamma_i^*, \\ 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*. \end{cases}$$

Similarly as done in Section 3.2, we can write the first three equations as a vector identity on Σ^* and thereby obtain $\mathfrak{F}(\rho, \rho|_{\Sigma^*}) = 0$. Thus

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} \rho \in B_{X_1}(0, R), \\ 0 = \mathfrak{F}_i(\rho_i, \rho|_{\Sigma^*}) & \text{on } \Gamma_i^*, \quad i = 1, 2, 3, \\ 0 = \mathfrak{G}_j(\rho) & \text{on } \Sigma^*, \quad j = 1, \dots, 6. \end{cases}$$

Taking into account the definition of \mathfrak{F}_i , the balance of flux conditions $\mathfrak{G}_5, \mathfrak{G}_6$ and the condition on curvature \mathfrak{G}_4 (see [4, Appendix B.1]), by applying the Gauss theorem, we see

$$\rho \in \mathcal{E} \Leftrightarrow \begin{cases} \rho \in B_{X_1}(0, R), \\ \kappa_i(\rho_i, (\mathcal{J}\rho \circ \text{pr})_i) \text{ are constant,} & \text{on } \Gamma^*, \\ \mathfrak{G}_j(\rho) = 0 & \text{on } \Sigma^*, \quad j = 1, 2, 3, 4. \end{cases}$$

Therefore, using Lemma 3.1, we conclude:

$$\mathcal{E} = \left\{ \rho \in B_{X_1}(0, R) : \rho \text{ parameterizes a standard planar double bubble} \right\}.$$

5.4.2. Level set representation of standard double bubbles. Next we represent standard planar double bubbles as a subset of the zero level sets of some smooth functions. Let $S_{r_i}(O_i)$, $i = 1, 2, 3$, be the corresponding circles to standard planar double bubble $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$. In other words, $\Gamma_i \subset S_{r_i}(O_i)$, where r_i and O_i are the radius and the center of Γ_i respectively.

Lemma 5.20. *Let $\Gamma = DB_{r,\gamma,0}(0, 0)$. Then*

$$\left\{ \sigma \in \mathbb{R}^2 : G_i(\sigma, r, \gamma) = 0 \right\} = S_{r_i}(O_i) \supset \Gamma_i \quad (i = 1, 2, 3),$$

where $G_i : \mathbb{R}^2 \times (0, \infty) \times (0, \frac{2\pi}{3}) \rightarrow \mathbb{R}$ are smooth functions defined by

$$\begin{aligned} & r \sin(\gamma + \frac{\pi}{3})G_1(\sigma, r, \gamma) \\ &= \frac{1}{2} \sin(\gamma + \frac{\pi}{3})\left(|\sigma|^2 - r^2\right), \\ & r \sin(\gamma + \frac{\pi}{3})G_2(\sigma, r, \gamma) \\ &= \frac{1}{2} \left(\sin(\gamma - \frac{\pi}{3})|\sigma|^2 - 2r \sin(\frac{\pi}{3})\langle \sigma, (1, 0) \rangle - r^2 \sin(\gamma - \pi) \right), \\ & r \sin(\gamma + \frac{\pi}{3})G_3(\sigma, r, \gamma) \\ &= \frac{1}{2} \left(\sin(\gamma - \pi)|\sigma|^2 + 2r \sin(\frac{\pi}{3})\langle \sigma, (1, 0) \rangle - r^2 \sin(\gamma - \frac{\pi}{3}) \right), \end{aligned}$$

with the property that

$$(5.11) \quad G_1 + G_2 + G_3 = 0.$$

Proof. Consider the standard planar double bubble $\Gamma = DB_{r,\gamma,0}(0, 0)$. That is the left circular arc Γ_1 has radius $r_1 = r$ centered at $O_1 = (0, 0)$ and all the other centers also lie on the x -axis, for some $r > 0$, $0 < \gamma < \frac{2\pi}{3}$, see Figure 5.

It follows directly from the law of sines that in case $\gamma \neq \frac{\pi}{3}$

$$\begin{aligned} O_2 &= \left(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})} r, 0 \right), & r_2 &= \left| \frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma - \frac{\pi}{3})} \right| r, \\ O_3 &= \left(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma)} r, 0 \right), & r_3 &= \frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma)} r. \end{aligned}$$

Therefore, for $\sigma = (\sigma_1, \sigma_2) \in \Gamma_2$, $\gamma \neq \frac{\pi}{3}$, we have

$$\begin{aligned} 0 &= \left| \sigma - \left(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})} r, 0 \right) \right|^2 - \left(\frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma - \frac{\pi}{3})} r \right)^2 \\ &= |\sigma|^2 - 2\sigma \cdot \left(\frac{\sin(\frac{2\pi}{3})}{\sin(\gamma - \frac{\pi}{3})} r, 0 \right) + \frac{\sin^2(\frac{2\pi}{3}) - \sin^2(\frac{2\pi}{3} - \gamma)}{\sin^2(\gamma - \frac{\pi}{3})} r^2 \\ &= \frac{2}{\sin(\gamma - \frac{\pi}{3})} r \sin(\gamma + \frac{\pi}{3})G_2(\sigma, r, \gamma), \end{aligned}$$

where we applied the formula $\sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y)$.

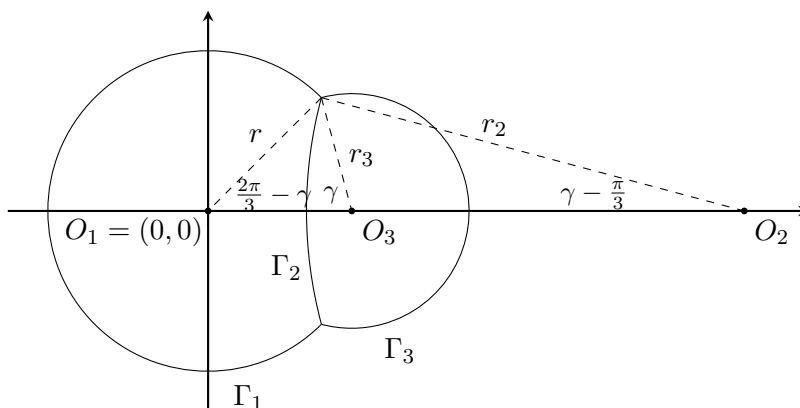


Figure 5: Standard planar double bubble $\Gamma = DB_{r,\gamma,0}(0,0)$.

Similarly, for $\sigma \in \Gamma_3$ we obtain

$$\begin{aligned} 0 &= \left| \sigma - \left(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma)} r, 0 \right) \right|^2 - \left(\frac{\sin(\frac{2\pi}{3} - \gamma)}{\sin(\gamma)} r \right)^2 \\ &= \frac{2}{\sin(\gamma - \pi)} r \sin(\gamma + \frac{\pi}{3}) G_3(\sigma, r, \gamma) \end{aligned}$$

and obviously for $\sigma \in \Gamma_1$ we have

$$0 = |\sigma|^2 - r^2 = \frac{2}{\sin(\gamma + \frac{\pi}{3})} r \sin(\gamma + \frac{\pi}{3}) G_1(\sigma, r, \gamma).$$

Furthermore, we see for $\sigma \in \Gamma_2$, $\gamma = \frac{\pi}{3}$ that $0 = \frac{r}{2} - \sigma_1 = G_2(\sigma, r, \frac{\pi}{3})$.

Finally, the identity (2.6) easily verifies (5.11). This finishes the proof as the coefficients appearing above are all nonzero and well-defined. \square

Next let us look at the gradient of the functions G_i .

Lemma 5.21. *Let $\Gamma = DB_{r,\gamma,0}(0,0)$. Then*

$$\nabla_\sigma G_i(\sigma, r, \gamma) = n_i(\sigma) \quad \text{for } \sigma \in \Gamma_i \quad (i = 1, 2, 3).$$

Proof. It is easy to see that

$$\sigma - O_i = -\frac{1}{\kappa_i} n_i(\sigma) \quad \text{for } \sigma \in \Gamma_i \quad (i = 1, 2, 3).$$

Using this, we calculate

$$\begin{aligned} r \sin(\gamma + \frac{\pi}{3}) \nabla_{\sigma} G_2(\sigma, r, \gamma) &= \sin(\gamma - \frac{\pi}{3}) \sigma - r \sin(\frac{\pi}{3})(1, 0) \\ &= \sin(\gamma - \frac{\pi}{3}) (\sigma - O_2) \\ &= -\frac{\sin(\gamma - \frac{\pi}{3})}{\kappa_2} n_2(\sigma) \quad \text{for } \sigma \in \Gamma_2. \end{aligned}$$

Similarly we get

$$\begin{aligned} r \sin(\gamma + \frac{\pi}{3}) \nabla_{\sigma} G_1(\sigma, r, \gamma) &= -\frac{\sin(\gamma + \frac{\pi}{3})}{\kappa_1} n_1(\sigma) \quad \text{for } \sigma \in \Gamma_1, \\ r \sin(\gamma + \frac{\pi}{3}) \nabla_{\sigma} G_3(x, r, \gamma) &= -\frac{\sin(\gamma - \pi)}{\kappa_3} n_3(\sigma) \quad \text{for } \sigma \in \Gamma_3. \end{aligned}$$

Since $r \sin(\gamma + \frac{\pi}{3}) = -\frac{\sin(\gamma + \frac{\pi}{3})}{\kappa_1}$, by the identity (2.5) we complete the proof. □

Furthermore, the following result holds.

Proposition 5.22. *Let $\Gamma = DB_{r,\gamma,0}(0, 0)$. Then*

$$\begin{cases} \partial_r G_1(\sigma, r, \gamma) = -1 & \text{for } \sigma \in \Gamma_1, \\ \partial_r G_2(\sigma, r, \gamma) = -\frac{1}{\sin(\gamma + \frac{\pi}{3})} \left(\frac{\sin(\frac{\pi}{3})}{r} \sigma_1 + \sin(\gamma - \pi) \right) & \text{for } \sigma \in \Gamma_2, \\ \partial_r G_3(\sigma, r, \gamma) = +\frac{1}{\sin(\gamma + \frac{\pi}{3})} \left(\frac{\sin(\frac{\pi}{3})}{r} \sigma_1 - \sin(\gamma - \frac{\pi}{3}) \right) & \text{for } \sigma \in \Gamma_3. \end{cases}$$

Proof. According to Lemma 5.20, we have

$$G_i(\sigma, r, \gamma) = 0 \quad \text{for } \sigma \in \Gamma_i.$$

Therefore, differentiating with respect to r in the definitions of functions G_i , we observe

$$\begin{aligned} -r \sin(\gamma + \frac{\pi}{3}) \partial_r G_1(\sigma, r, \gamma) &= \sin(\gamma + \frac{\pi}{3}) r & \text{for } \sigma \in \Gamma_1, \\ -r \sin(\gamma + \frac{\pi}{3}) \partial_r G_2(\sigma, r, \gamma) &= \sin(\frac{\pi}{3}) \langle \sigma, (1, 0) \rangle + \sin(\gamma - \pi) r & \text{for } \sigma \in \Gamma_2, \\ r \sin(\gamma + \frac{\pi}{3}) \partial_r G_3(\sigma, r, \gamma) &= \sin(\frac{\pi}{3}) \langle \sigma, (1, 0) \rangle - \sin(\gamma - \frac{\pi}{3}) r & \text{for } \sigma \in \Gamma_3, \end{aligned}$$

which finishes the proof. □

Similarly we get

Proposition 5.23. *Let $\Gamma = DB_{r,\gamma,0}(0, 0)$. Then*

$$\begin{cases} \partial_\gamma G_1(\sigma, r, \gamma) = 0 & \text{for } \sigma \in \Gamma_1, \\ \partial_\gamma G_2(\sigma, r, \gamma) = \frac{1}{2r \sin(\gamma + \frac{\pi}{3})} \left(\cos(\gamma - \frac{\pi}{3})|\sigma|^2 - r^2 \cos(\gamma - \pi) \right) & \text{for } \sigma \in \Gamma_2, \\ \partial_\gamma G_3(\sigma, r, \gamma) = \frac{1}{2r \sin(\gamma + \frac{\pi}{3})} \left(\cos(\gamma - \pi)|\sigma|^2 - r^2 \cos(\gamma - \frac{\pi}{3}) \right) & \text{for } \sigma \in \Gamma_3. \end{cases}$$

5.4.3. Five-dimensional smooth manifold. Throughout this section, without loss of generality, we may assume that the center of Γ_1^* is at the origin of \mathbb{R}^2 and that the angle $\theta^* = 0$, that is

$$\Gamma^* = DB_{r^*,\gamma^*,0}(0, 0).$$

Clearly, $\mathcal{E} \neq \emptyset$ as $\rho \equiv 0$ parameterizes $\Gamma^* = DB_{r^*,\gamma^*,0}(0, 0)$. First we demonstrate, by applying the implicit function theorem, that every standard planar double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$ sufficiently close to $\Gamma^* = DB_{r^*,\gamma^*,0}(0, 0)$ can be parameterized by some unique vector function $\rho = (\rho_1, \rho_2, \rho_3)$ depending smoothly on the parameters a_1, a_2, r, γ and θ . We continue then to verify that the set \mathcal{E} of equilibria is actually a smooth manifold of dimension five.

Theorem 5.24. *Any standard planar double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$ sufficiently close to Γ^* , i.e., $(a_1, a_2, r, \gamma, \theta) \in B_\epsilon(0, 0, r^*, \gamma^*, 0)$ for sufficiently small ϵ , can be parameterized by some unique smooth vector function $\rho = \rho(a_1, a_2, r, \gamma, \theta) \in B_{X_1}(0, R)$.*

Proof. We use the implicit function theorem of Hildebrandt and Graves, see Zeidler [22, Theorem 4.B], with $(x_0, y_0) = ((0, 0, r^*, \gamma^*, 0), 0)$,

$$\begin{aligned} X &= \mathbb{R}^2 \times B_{\delta_1}(r^*) \times B_{\delta_2}(\gamma^*) \times \mathbb{R}, \quad Z = Y, \\ Y &= \left\{ \rho \in C^{4+\alpha}(\Gamma_1^*) \times C^{4+\alpha}(\Gamma_2^*) \times C^{4+\alpha}(\Gamma_3^*) : \rho_1 + \rho_2 + \rho_3 = 0 \text{ on } \Sigma^* \right\}, \end{aligned}$$

$$\begin{aligned} F : X \times Y &\rightarrow Z, \\ ((a_1, a_2, r, \gamma, \theta), \rho) &\mapsto (F_1, F_2, F_3) \end{aligned}$$

with

$$F_i(a_1, a_2, r, \gamma, \theta, \rho) := G_i \left(Q_\theta T_{\vec{a}} \Psi_i(\cdot, \rho_i, \mu_i \circ \text{pr}_i), r, \gamma \right) \quad (i = 1, 2, 3).$$

Here G_i are the functions stated in Lemma 5.20 and

$$\Psi_i(\cdot, \rho_i, \mu_i \circ \text{pr}_i)(\sigma) = \sigma + \rho_i(\sigma)n_i^*(\sigma) + \mu_i(\text{pr}_i(\sigma))\tau_i^*(\sigma) \quad \text{for } \sigma \in \Gamma_i^*,$$

where $\mu = \mathcal{J}\rho$ on Σ^* . Furthermore,

$$Q_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad T_{\vec{a}}v = v - \vec{a}$$

are the clockwise rotation matrix and the translation operator respectively.

Indeed, the image of the function F lies in $Z = Y$, that is

$$(5.12) \quad F_1 + F_2 + F_3 = 0 \quad \text{on } \Sigma^*.$$

To see this, note that for $\sigma \in \Sigma^*$,

$$\Psi_1(\cdot, \rho_1, \mu_1 \circ \text{pr}_1)(\sigma) = \Psi_2(\cdot, \rho_2, \mu_2 \circ \text{pr}_2)(\sigma) = \Psi_3(\cdot, \rho_3, \mu_3 \circ \text{pr}_3)(\sigma),$$

by Lemma 3.1. This together with the identity (5.11) proves (5.12).

Moreover, since $\Psi_i|_{\rho=0} = I$, according to Lemma 5.20 we have

$$F_i(x_0, y_0)(\sigma) = F_i((0, 0, r^*, \gamma^*, 0), 0)(\sigma) = G_i(\sigma, r^*, \gamma^*) = 0 \quad \text{for } \sigma \in \Gamma_i^*.$$

Thus $F(x_0, y_0) = 0$. Now let us compute the derivative $\partial_\rho F(x_0, y_0)$:

$$\partial_\rho F_i(x_0, y_0)(v)(\sigma) = \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot (v_i n_i^*(\sigma) + (\mathcal{J}v(\text{pr}_i(\sigma)))_i \tau_i^*(\sigma)) = v_i,$$

where we used Lemma 5.21. Thus

$$(5.13) \quad \partial_\rho F(x_0, y_0) = I.$$

Furthermore, F is a smooth map on a neighborhood of (x_0, y_0) .

Therefore, according to the implicit function theorem, there exist neighborhoods $U = B_\epsilon(x_0)$ of x_0 and $V = B_{X_1}(0, R)$ of $y_0 = 0$ and a smooth function

$$\begin{aligned} \rho : U &\longrightarrow V \\ (a_1, a_2, r, \gamma, \theta) &\mapsto \rho(a_1, a_2, r, \gamma, \theta), \end{aligned}$$

such that $\rho(x_0) = 0$ and for every $(a_1, a_2, r, \gamma, \theta) \in B_\epsilon(0, 0, r^*, \frac{\pi}{3}, 0)$ we have

$$(5.14) \quad F((a_1, a_2, r, \gamma, \theta), \rho(a_1, a_2, r, \gamma, \theta)) = 0.$$

Moreover if $(x, y) \in U \times V$ and $F(x, y) = 0$ then $y = \rho(x)$.

We now claim that $\Gamma_\rho = \{\Gamma_{\rho_1}, \Gamma_{\rho_2}, \Gamma_{\rho_3}\}$ parameterized by the function $\rho = \rho(a_1, a_2, r, \gamma, \theta)$ is the standard planar double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$. To see this, note

$$\begin{aligned} F_i((a_1, a_2, r, \gamma, \theta), \rho(a_1, a_2, r, \gamma, \theta)) &= 0 \\ \iff G_i(Q_\theta T_{\bar{a}} \Psi_i(\cdot, \rho_i, \mu_i \circ \text{pr}_i), r, \gamma) &= 0 \\ \iff Q_\theta T_{\bar{a}} \Gamma_{\rho_i} \subset S_{r_i}(O_i) &\text{ by Lemma 5.20.} \end{aligned}$$

Therefore, since Lemma 3.1 guaranties that the curves $\Gamma_{\rho_1}, \Gamma_{\rho_2}, \Gamma_{\rho_3}$ meet at their boundaries, we end up with two choices: Either $\Gamma_{\rho_i} = \Gamma_i$, where $\Gamma = \{\Gamma_1, \Gamma_2, \Gamma_3\}$ is a standard double bubble $DB_{r,\gamma,\theta}(a_1, a_2)$ or Γ_{ρ_i} is the complementary part of Γ_i in $S_{r_i}(O_i)$. But the latter can not happen since the norm of ρ is small. Hence

$$\Gamma_{\rho(a_1, a_2, r, \gamma, \theta)} = DB_{r,\gamma,\theta}(a_1, a_2),$$

as required. □

Theorem 5.25. *The set of equilibria \mathcal{E} is in a neighborhood of zero a C^2 -manifold in X_1 of dimension 5.*

Proof. Remind that we have shown

$$\begin{aligned} \mathcal{E} \cap U &= \left\{ \rho \in B_{X_1}(0, R) : \rho \text{ parameterizes a standard planar double bubble} \right\} \\ &\quad \cap U \\ &= \left\{ \rho(a_1, a_2, r, \gamma, \theta) : (a_1, a_2, r, \gamma, \theta) \in U = B_\epsilon(0, 0, r^*, \gamma^*, 0) \right\}, \end{aligned}$$

where the function

$$\begin{aligned} \rho : U &\longrightarrow X_1 = C^{4+\alpha}(\Gamma_1^*) \times C^{4+\alpha}(\Gamma_2^*) \times C^{4+\alpha}(\Gamma_3^*) \\ (a_1, a_2, r, \gamma, \theta) &\mapsto \rho(a_1, a_2, r, \gamma, \theta) \end{aligned}$$

is smooth, in particular C^2 and $\rho(U) = \mathcal{E}$, $\rho(x_0) = \rho(0, 0, r^*, \gamma^*, 0) = 0$.

Therefore, it is left to check that the rank of $\rho'(x_0)$ equals five (see the definition of a manifold on page 1023). To do this, we differentiate (5.14)

with respect to $\iota \in \{a_1, a_2, r, \gamma, \theta\}$ and evaluate at x_0 to get

$$\partial_\iota F(x_0, 0) + \partial_\rho F(x_0, 0) \partial_\iota \rho(x_0) = 0.$$

Therefore, (5.13) gives

$$\partial_\iota \rho(x_0) = -\partial_\iota F(x_0, 0) \quad (\iota \in \{a_1, a_2, r, \gamma, \theta\}).$$

We now calculate

$$\partial_{a_1} F_i(x_0, 0) = \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot (-1, 0) = n_i^*(\sigma) \cdot (-1, 0) = \cos(\kappa_i^* x),$$

where we used the fact $n_i^*(\sigma) = -(\cos(\kappa_i^* x), \sin(\kappa_i^* x))$, $i = 1, 2, 3$. Thus

$$\partial_{a_1} \rho(x_0) = (\cos(\kappa_1^* x), \cos(\kappa_2^* x), \cos(\kappa_3^* x)).$$

Similarly, we get $\partial_{a_2} \rho(x_0) = (\sin(\kappa_1^* x), \sin(\kappa_2^* x), \sin(\kappa_3^* x))$.

Next we calculate

$$\begin{aligned} \partial_\theta F_i(x_0, 0) &= \nabla_\sigma G_i(\sigma, r^*, \gamma^*) \cdot \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \sigma \right) \\ &= n_i^*(\sigma) \cdot \sigma^\perp = n_i^*(\sigma) \cdot \left(-\frac{1}{\kappa_i^*} n_i^*(\sigma) + O_i^* \right)^\perp \\ &= n_i^*(\sigma) \cdot O_i^{*\perp} \end{aligned}$$

and so

$$\partial_\theta \rho(x_0) = \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^*)} r^* \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^* - \frac{\pi}{3})} \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}.$$

We now compute the derivative $\partial_r F(x_0, 0) = \partial_r G(\sigma, r^*, \gamma^*)$. According to Proposition 5.22

$$\partial_r G_2(\sigma, r^*, \gamma^*) = -\frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \left(\frac{\sin(\frac{\pi}{3})}{r^*} \sigma_1 + \sin(\gamma^* - \pi) \right).$$

First we consider the case $\kappa_2^* \neq 0$. Employing the arc-length parameterization of Γ_2^* derived in Proposition B.1 we obtain

$$\begin{aligned} \partial_r G_2(\sigma, r^*, \gamma^*) &= -\frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \left(\frac{\sin(\frac{\pi}{3})}{r^*} \sigma_1 + \sin(\gamma^* - \pi) \right) \\ &= \frac{\kappa_1^* \sin(\frac{\pi}{3})}{\kappa_2^* \sin(\gamma^* + \frac{\pi}{3})} \cos(\kappa_2^* x) \\ &\quad - \frac{1}{\sin(\gamma^* + \frac{\pi}{3})} \left(\frac{\sin^2(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} + \sin(\gamma^* - \pi) \right) \\ &= \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x) - \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})}, \end{aligned}$$

where we applied the formula $\sin^2(x) - \sin^2(y) = \sin(x + y) \sin(x - y)$.

A similar argument works for $\partial_r G_3(\sigma, r^*, \gamma^*)$. Altogether we derive in case $\gamma^* \neq \frac{\pi}{3}$,

$$\partial_r \rho(x_0) = \begin{pmatrix} 1 \\ -\frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x) + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \\ \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \pi)} \cos(\kappa_3^* x) + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \pi)} \end{pmatrix}.$$

Next we consider the case $\kappa_2^* = 0$: We calculate

$$\partial_r G_2(\sigma, r^*, \frac{\pi}{3}) = -\frac{1}{\sin(\frac{2\pi}{3})} \left(\frac{\sin(\frac{\pi}{3})}{r^*} r^* + \sin\left(-\frac{2\pi}{3}\right) \right) = \frac{1}{2}.$$

Therefore, we derive in case $\kappa_2^* = 0$, i.e., when $x_0 = (0, 0, r^*, \frac{\pi}{3}, 0)$,

$$\partial_r \rho(x_0) = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\cos(\kappa_1^* x) - 1 \end{pmatrix}.$$

Finally let us calculate $\partial_\gamma F(x_0, 0)$. We have $\partial_\gamma F(x_0, 0) = \partial_\gamma G(\sigma, r^*, \gamma^*)$. We first consider the case $\kappa_2^* \neq 0$: Employing the arc length parameterization of Γ^* to the formulas derived in Proposition 5.23 we derive in case $\kappa_2^* \neq 0$ that

$$\partial_\gamma \rho(x_0) = \begin{pmatrix} 0 \\ a_2 \cos(\kappa_2^* x) + b_2 \\ a_3 \cos(\kappa_3^* x) + b_3 \end{pmatrix}$$

for some constants a_i, b_i (see the Appendix for the explicit form of the constants). This immediately implies that $\partial_\gamma \rho(x_0)$ is independent from the other elements of $\rho'(x_0)$.

However, we give the explicit formula in case $\kappa_2^* = 0$. Using Proposition 5.23 we see

$$\begin{aligned} \partial_\gamma G_2(\sigma, r^*, \frac{\pi}{3}) &= \frac{1}{2r^* \sin(\frac{2\pi}{3})} \left(\frac{1}{4}(r^*)^2 + x^2 + \frac{1}{2}(r^*)^2 \right) \\ &= -\frac{\kappa_1^*}{\sin(\frac{2\pi}{3})} \left(\frac{1}{2}x^2 + \frac{3}{8} \frac{1}{(\kappa_1^*)^2} \right), \end{aligned}$$

$$\begin{aligned} \partial_\gamma G_3(\sigma, r^*, \frac{\pi}{3}) &= \frac{1}{2r^* \sin(\frac{2\pi}{3})} \left(\cos(-\frac{2\pi}{3})|\sigma|^2 - (r^*)^2 \right) \\ &= \frac{-1}{2r^* \sin(\frac{2\pi}{3})} \left(\frac{1}{2}|\sigma|^2 + (r^*)^2 \right) \\ &= \frac{1}{\kappa_1^* \sin(\frac{2\pi}{3})} \left(1 + \frac{1}{2} \cos(\kappa_1^* x) \right). \end{aligned}$$

In summary, we have proved that the rank of $\rho'(x_0)$ is equal to five and we have shown that the set of equilibria \mathcal{E} is a C^2 -manifold in X_1 of dimension five. Moreover

$$T_0\mathcal{E} = \text{span}\{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}\},$$

where

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ \cos(\kappa_2^* x) \\ \cos(\kappa_3^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \frac{\sin(\gamma^*)}{\sin(\gamma^* - \frac{\pi}{3})} \sin(\kappa_2^* x) \\ \sin(\kappa_3^* x) \end{pmatrix},$$

$$v^{(4)} = \begin{pmatrix} 1 \\ \frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \cos(\kappa_2^* x) + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} \\ \frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \pi)} \cos(\kappa_3^* x) + \frac{\sin(\gamma^* + \frac{\pi}{3})}{\sin(\gamma^* - \pi)} \end{pmatrix}, \quad v^{(5)} = \begin{pmatrix} 0 \\ a_2 \cos(\kappa_2^* x) + b_2 \\ a_3 \cos(\kappa_3^* x) + b_3 \end{pmatrix}.$$

Although $v^{(i)}$ are continuous in particular at $\kappa_2^* = 0$, for convenience we state them in case $\kappa_2^* = 0$:

$$v^{(1)} = \begin{pmatrix} \cos(\kappa_1^* x) \\ 1 \\ \cos(\kappa_1^* x) \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} \sin(\kappa_1^* x) \\ 0 \\ -\sin(\kappa_1^* x) \end{pmatrix}, \quad v^{(3)} = \begin{pmatrix} 0 \\ \kappa_1^* x \\ -\sin(\kappa_1^* x) \end{pmatrix},$$

$$v^{(4)} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\cos(\kappa_1^* x) - 1 \end{pmatrix}, \quad v^{(5)} = \begin{pmatrix} 0 \\ \frac{\kappa_1^*}{\sin(\frac{\pi}{3})} (\frac{1}{2} x^2 + \frac{3}{8} \frac{1}{(\kappa_1^*)^2}) \\ \frac{-1}{\kappa_1^* \sin(\frac{\pi}{3})} (\frac{1}{2} \cos(\kappa_1^* x) + 1) \end{pmatrix}.$$

□

5.4.4. Geometric interpretation of the null space. As an immediate corollary of Theorem 5.25 we get

Corollary 5.26. *The null space $N(A_0)$ is five dimensional. Furthermore,*

$$T_0\mathcal{E} = N(A_0).$$

Proof. It always holds

$$T_0\mathcal{E} \subseteq N(A_0),$$

see [1, equation (2.8)]. Thus, according to Theorem 5.25 and Lemma 5.19,

$$5 = \dim(T_0\mathcal{E}) \leq \dim(N(A_0)) \leq 5.$$

It follows that $\dim(N(A_0)) = 5$ and moreover $T_0\mathcal{E} = N(A_0)$. □

Variations preserving areas and curvatures. We easily see, using formula (2.5), that

$$\begin{cases} \int_{\Gamma_1^*} v^{(1)}_1 = \int_{\Gamma_2^*} v^{(1)}_2 = \int_{\Gamma_3^*} v^{(1)}_3 = -2 \frac{\sin(\gamma^* + \frac{\pi}{3})}{\kappa_1^*}, \\ \int_{\Gamma_1^*} v^{(2)}_1 = \int_{\Gamma_2^*} v^{(2)}_2 = \int_{\Gamma_3^*} v^{(2)}_3 = 0, \\ \int_{\Gamma_1^*} v^{(3)}_1 = \int_{\Gamma_2^*} v^{(3)}_2 = \int_{\Gamma_3^*} v^{(3)}_3 = 0. \end{cases}$$

In other words,

$$(5.15) \quad \mathcal{J}(\Gamma^*) \subseteq \mathcal{F}(\Gamma^*).$$

By Lemma 5.7, each of the $v^{(i)}$ ($i = 1, 2, 3$) corresponds to a first variation of Γ^* which preserves the areas, and the curvatures to first order. Indeed, we have demonstrated in the proof of Theorem 5.25 that $v^{(1)}$, $v^{(2)}$, $v^{(3)}$ correspond to the first variations of the double bubble Γ^* associated with translation along x -axis, translation along y -axis and rotation around the center of Γ_1^* , respectively.

Variations not preserving areas and curvatures. It is shown in the proof of Theorem 5.25 that $v^{(4)}$ corresponds to the first variations of the double bubble Γ^* associated with uniform scaling (with the scale factor $\frac{r}{r^*}$). Let $A_i(r)$ denote the area of the regions $R_i(r)$ corresponding to the double bubble $DB_{r,\gamma^*,\theta^*}(a_1^*, a_2^*)$. Then (see equation (3.1) in [14])

$$(5.16) \quad \partial_r A_1 = \int_{\Gamma_1^*} v^{(4)}_1 - \int_{\Gamma_2^*} v^{(4)}_2 > 0, \quad \partial_r A_2 = \int_{\Gamma_2^*} v^{(4)}_2 - \int_{\Gamma_3^*} v^{(4)}_3 > 0$$

according to Lemma C.1 (ii).

Again remember from the proof of Theorem 5.25 that $v^{(5)}$ corresponds to the first variation of Γ^* with respect to the angle γ , that is w.r.t. the curvature ratio. Similarly we denote by $A_i(\gamma)$ the area of the regions $R_i(\gamma)$ corresponding to the double bubble $DB_{r^*,\gamma,\theta^*}(a_1^*, a_2^*)$. Then

$$(5.17) \quad \partial_\gamma A_1 = \int_{\Gamma_1^*} v^{(5)}_1 - \int_{\Gamma_2^*} v^{(5)}_2 > 0, \quad \partial_\gamma A_2 = \int_{\Gamma_2^*} v^{(5)}_2 - \int_{\Gamma_3^*} v^{(5)}_3 < 0$$

according to Lemma C.2 (ii).

We now define the matrix

$$D := \begin{pmatrix} \partial_r A_1 & \partial_\gamma A_1 \\ \partial_r A_2 & \partial_\gamma A_2 \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_1^*} v^{(4)}_1 - \int_{\Gamma_2^*} v^{(4)}_2 & \int_{\Gamma_1^*} v^{(5)}_1 - \int_{\Gamma_2^*} v^{(5)}_2 \\ \int_{\Gamma_2^*} v^{(4)}_2 - \int_{\Gamma_3^*} v^{(4)}_3 & \int_{\Gamma_2^*} v^{(5)}_2 - \int_{\Gamma_3^*} v^{(5)}_3 \end{pmatrix}.$$

Lemma 5.27. *The matrix D is invertible for each $0 < \gamma^* < \frac{2\pi}{3}$.*

Proof. Let us calculate its determinant. Inequalities (5.16) and (5.17) imply

$$\det D = \partial_r A_1 \partial_\gamma A_2 - \partial_\gamma A_1 \partial_r A_2 < 0$$

Now as the determinant of the matrix D is strictly negative, we conclude that the matrix D is for each $0 < \gamma^* < \frac{2\pi}{3}$ invertible, □

As a further result of Lemma C.1 and C.2 (ii), we get $v^{(4)}, v^{(5)} \notin \mathcal{F}(\Gamma^*)$. Therefore, we conclude from Lemma 5.7 that the corresponding variations do not preserve areas to first order. Indeed we will show below in Lemma A.2 that

$$I(u, u) < 0 \quad \text{for } u = v^{(4)}, v^{(5)}.$$

In addition they do not preserve the curvatures to first order too as the constant vectors $c(v^{(4)})$ and $c(v^{(5)})$ are nonzero.

5.5. Semi-simplicity

We need to show two small propositions. The first one is stated and proved in the proof of Lemma 3.8 in [14].

Proposition 5.28. *If $u \in \mathcal{F}(\Gamma^*)$ satisfies $I(u, u) = 0$, then*

$$I(u, v) = 0 \quad \forall v \in \mathcal{F}(\Gamma^*).$$

Proof. According to Corollary 5.12, $I(v + tu, v + tu) \geq 0$ for all $v \in \mathcal{F}(\Gamma^*)$ and $t \in \mathbb{R}$. Therefore

$$\begin{aligned} I(v + tu, v + tu) &= I(v, v) + 2tI(u, v) + t^2I(u, u) \\ &= I(v, v) + 2tI(u, v). \end{aligned}$$

This forces $I(u, v) = 0$ as t can take arbitrary negative values. □

Proposition 5.29. *Let $z \in R(A_0)$. Then there exists $u \in \mathcal{F}(\Gamma^*) \cap D(A_0)$ such that $Au = z$.*

Proof. Clearly, there exists $\tilde{u} \in D(A_0)$ such that $A\tilde{u} = z$. The actual task is to find two constants $\alpha(\tilde{u}), \beta(\tilde{u})$ such that

$$u := \tilde{u} + \alpha(\tilde{u})v^{(4)} + \beta(\tilde{u})v^{(5)}$$

satisfies

$$\int_{\Gamma_1^*} u_1 = \int_{\Gamma_2^*} u_2 = \int_{\Gamma_3^*} u_3.$$

(This will finish the proof since $v^{(4)}, v^{(5)} \in N(A_0)$ implies $Au = A\tilde{u} = z$.) To do so, let us recast this integral constraint into the matrix form

$$D \begin{pmatrix} \alpha(\tilde{u}) \\ \beta(\tilde{u}) \end{pmatrix} = \begin{pmatrix} \int_{\Gamma_1^*} \tilde{u}_2 - \int_{\Gamma_2^*} \tilde{u}_1 \\ \int_{\Gamma_1^*} \tilde{u}_3 - \int_{\Gamma_3^*} \tilde{u}_2 \end{pmatrix},$$

where the matrix D is given above. The invertibility of this matrix proved in Lemma 5.27 finishes the proof. □

We are now ready to prove:

Lemma 5.30. *The eigenvalue 0 of A_0 is semi-simple.*

Proof. Since the operator A_0 has a compact resolvent, the semi-simplicity condition is equivalent to the condition that $N(A_0) = N(A_0^2)$ (use the spectral theory of compact operators, e.g. see [4, Section 9.9]). In other words, it suffices to check that

$$R(A_0) \cap N(A_0) = \{0\}.$$

To prove this, let $z \in R(A_0) \cap N(A_0)$ ($\subset D(A_0)$). According to Proposition 5.29 there exists $u \in \mathcal{D}(A_0) \cap \mathcal{F}(\Gamma^*)$ such that $Au = z$. From this, exactly as done in Section 5.2, we derive the identity

$$(5.18) \quad \sum_{i=1}^3 \int_{\Gamma_i^*} |\nabla_{\Gamma_i^*} (\Delta_{\Gamma_i^*} u_i + (\kappa_i^*)^2 u_i)|^2 ds = I(z, u),$$

where we used only the facts that $u, z \in D(A_0)$.

Moreover, similarly as before, an integration and application of the divergence theorem using the fact that $u \in D(A_0)$ gives

$$\int_{\Gamma_1^*} z_1 = \int_{\Gamma_2^*} z_2 = \int_{\Gamma_3^*} z_3,$$

and so $z \in \mathcal{F}(\Gamma^*)$.

Now since $z \in N(A_0) \cap \mathcal{F}(\Gamma^*)$, Corollary 5.16 tells us $I(z, z) = 0$. Therefore, according to Proposition 5.28,

$$I(z, u) = 0$$

as $u \in \mathcal{F}(\Gamma^*)$. In view of the identity (5.18), we obtain $u \in N(A_0)$. Consequently $z = Au = 0$, which finishes the proof. \square

Remark 5.31. The main ingredient in the proof is the identity 5.18. Here of course we slightly use the non-negativity of the bilinear form. However, one could write the identity 5.18 in terms of the linearized operator A_0 and the H^{-1} -inner product. This would give immediately, i.e., without assuming the non-negativity of the bilinear form, the semi-simplicity condition.

6. Standard planar double bubbles are dynamically stable under surface diffusion flow

Summing up, we have shown that all the hypotheses of Theorem 4.1 are satisfied. Thereby applying Theorem 4.1 we conclude:

Theorem 6.1. *Let Γ^* be a standard planar double bubble. Then $\rho^* \equiv 0$ is a stable equilibrium of (3.12) in $X_1 = C^{4+\alpha}(\bar{\Omega}, \mathbb{R}^3)$. Moreover, if ρ^0 is sufficiently close to $\rho^* \equiv 0$ in X_1 and satisfies the corresponding compatibility conditions (4.5), then the problem (3.12) has a unique solution*

$$\rho \in C^{1+\frac{\alpha}{4}, 4+\alpha}([0, \infty) \times \bar{\Omega}, \mathbb{R}^3)$$

and approaches some $\rho^\infty \in \mathcal{E}$, parameterizing some standard planar double bubble, exponentially fast in X_1 as $t \rightarrow \infty$.

In this sense, the standard planar double bubble Γ^* is stable under the surface diffusion flow. In addition, every planar double bubble that starts sufficiently close to Γ^* and satisfies the angle, curvature, balance of flux condition and the condition on the Laplace of the curvatures, see (2.2), at $t = 0$ exists globally and converges to some standard planar double bubble, enclosing the same areas as its initial data, at an exponential rate as $t \rightarrow \infty$. In other words, they are dynamically stable under the surface diffusion flow. We illustrate this result in Figure 1.

6.1. General area preserving gradient flows

It is to be expected that for any area preserving gradient flow

$$V = -\text{grad}_{\mathcal{H}(\Gamma)} \text{Length}$$

one obtains the following identity

$$(6.1) \quad \|z\|_{\mathcal{H}(\Gamma^*)}^2 = I(z, u),$$

where $z := \delta(\text{grad}_{\mathcal{H}(\Gamma)} \text{Length})(u)$. Here $\mathcal{H}(\Gamma)$ denotes a (pre-)Hilbert manifold with a Levi-Civita connection and with some area constraints.

In particular, if u is an eigenvector of the operator $\delta(\text{grad}_{\mathcal{H}(\Gamma)} \text{Length})$ with respect to the eigenvalue λ , then we get

$$(6.2) \quad \left\| \delta(\text{grad}_{\mathcal{H}(\Gamma)} \text{Length})(u) \right\|_{\mathcal{H}(\Gamma^*)}^2 = \lambda I(u, u).$$

Comparing the identities (6.2) and (6.1) with the identities (5.5) and (5.18) respectively, we expect that our approach can be used for other area preserving gradient flows. Therefore we conjecture that

Conjecture 6.2. *Standard planar double bubbles are dynamically stable under area preserving gradient flows with respect to the Riemannian metric with a Levi-Civita connection.*

It would be desirable to analyze the problem systematically. See Section 1.1, and Section 2.2 in [4] for some initial works. Let us point out that it is not known whether there exists an example of a gradient flow close to the surface diffusion flow for which standard planar double bubbles (even spheres) are unstable.

6.2. Higher dimensional case

The problem in higher dimension reads as follows:

$$(6.3) \quad \left. \begin{aligned} V_i &= -\Delta_{\Gamma_i} H_i && \text{on } \Gamma_i(t), \\ \angle(\Gamma_1(t), \Gamma_2(t)) &= \angle(\Gamma_2(t), \Gamma_3(t)) = \angle(\Gamma_3(t), \Gamma_1(t)) = \frac{2\pi}{3} && \text{on } \Sigma(t), \\ H_1 + H_2 + H_3 &= 0 && \text{on } \Sigma(t), \\ \nabla_{\Gamma_1} H_1 \cdot n_{\partial\Gamma_1} &= \nabla_{\Gamma_2} H_2 \cdot n_{\partial\Gamma_2} = \nabla_{\Gamma_3} H_3 \cdot n_{\partial\Gamma_3} && \text{on } \Sigma(t), \\ \Gamma_i(t)|_{t=0} &= \Gamma_i^0, \end{aligned} \right\}$$

where $\Gamma_i \subset \mathbb{R}^{n+1}$ is a hypersurface ($i = 1, 2, 3$), and $\partial\Gamma_1(t) = \partial\Gamma_2(t) = \partial\Gamma_3(t) =: \Sigma(t)$. Here H_i is the mean curvature.

As you could already see from above formulation, in this paper we have used a general terminology, which makes it easy to visualize the problem in higher dimension. If one now goes through the whole paper section by section one will soon realize that apart from two almost all the arguments and proofs are not restricted to one dimension and work also in higher dimension. In the following we make this point clear.

Starting from Section 2 one gets Lemma 2.1 for higher dimension with the same proof, just replace κ_i by H_i . We face the first difficulty in Section 2.1: verifying that the set of equilibria only consists of the standard double bubbles, as not only spherical caps have constant mean curvature. In the compact, embedded case this leaves only the spheres, see [3]. One may also try to use the implicit function theorem for the verification.

For the PDE formulation and linearization, i.e., Section 3, one could rely on [6, 7]. In Section 4, the generalized principle of linearized stability in parabolic Hölder spaces is presented in general dimensions. Concerning Section 5.2, i.e., the spectrum of the linearized problem, just replace κ_i by H_i to see it in higher dimension. Note that the double bubble conjecture has been proved in general dimension in [14].

The second difficulty appears in Section 5.3, determining the space of Jacobi functions. We expect the justification of the separation ansatz and consequently reduction of the problem to ODEs. Furthermore, it is well known that the spherical harmonics of degree 1 span the space of Jacobi functions in the case of spheres, see [8, Lemma 3.2]. It is to be expected that the space of Jacobi functions is $2n + 1$ -dimensional.

Similarly, as given in Section 5.4, one proves that the set of all standard double bubbles in \mathbb{R}^{n+1} creates a smooth manifold of dimension $2n + 3$, $n + 1$ translations, n rotations, and one for scaling and the other one for the curvature ratio. The main ingredient in the proof is the level set representation of the standard double bubble, which clearly works also in general dimensions.

Note that one could prove the invertibility of the matrix D in general dimension by observing that the areas A_1 and A_2 strictly increase by uniform scaling and increasing γ will strictly increase the area A_1 and strictly decrease the area A_2 . Finally, a functional analysis proof is given in Section 5.5 for showing semi-simplicity.

Appendix A. More about the bilinear form $I(,)$

Lemma A.1. *Within the class of functions u satisfying the linearized angle condition, we have*

$$\{ u : I(u, u) = 0 \} \cap \mathcal{F}(\Gamma^*) = N(A_0) \cap \mathcal{F}(\Gamma^*).$$

Proof. We first assume $u \in \mathcal{F}(\Gamma^*)$ such that $I(u, u) = 0$. Then by Lemma 3.8 in [14] and the fact that u satisfies the linearized angle condition, we conclude that $u \in N(A_0)$. The converse statement is Corollary 5.16. \square

Note that we have already shown in Section 5.4.4 that

$$N(A_0) \cap \mathcal{F}(\Gamma^*) = \mathcal{J}(\Gamma^*) = \text{span}\{v^{(1)}, v^{(2)}, v^{(3)}\}.$$

On the other hand we obtain:

Lemma A.2. *For the bilinear form I it holds*

$$I(u, u) < 0 \quad \text{for } u = v^{(4)}, v^{(5)} \quad (0 < \gamma^* < \frac{2\pi}{3}).$$

Proof. According to Lemma 5.15,

$$I(u, u) = - \sum_{i=1}^3 c_i(u) \int_{\Gamma_i^*} u_i \quad \text{for } v^{(4)}, v^{(5)} (\in N(A_0)).$$

Now assertion (iii) in Lemma C.1 and Lemma C.2 proves the lemma. \square

Appendix B. Arc-length parameterization of Γ^*

Proposition B.1. *An arc-length parameterization of Γ_i^* , $i = 1, 2, 3$ is given as follows: For $\gamma^* \neq \frac{\pi}{3}$,*

$$(\sigma_1, \sigma_2) = \sigma = \begin{cases} \frac{1}{\kappa_1^*} (\cos(\kappa_1^* x), \sin(\kappa_1^* x)) & \text{for } \sigma \in \Gamma_1^*, \\ \left(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} r^* + \frac{1}{\kappa_2^*} \cos(\kappa_2^* x), \frac{1}{\kappa_2^*} \sin(\kappa_2^* x) \right) & \text{for } \sigma \in \Gamma_2^*, \\ \left(\frac{-\sin(\frac{\pi}{3})}{\sin(\gamma^* - \pi)} r^* + \frac{1}{\kappa_3^*} \cos(\kappa_3^* x), \frac{1}{\kappa_3^*} \sin(\kappa_3^* x) \right) & \text{for } \sigma \in \Gamma_3^*, \end{cases}$$

Moreover this arc-length parameterization is continuous at $\gamma^* = \frac{\pi}{3}$ and in particular $\sigma = (\frac{r^*}{2}, x)$ for $\sigma \in \Gamma_2^*$, $\gamma^* = \frac{\pi}{3}$.

Proof. We give the proof for Γ_2^* . We observe

$$\begin{aligned} (\sigma_1, \sigma_2) = \sigma &= O_2^* - \frac{1}{\kappa_2^*} n_2^* = O_2^* + \frac{1}{\kappa_2^*} (\cos(\kappa_2^* x), \sin(\kappa_2^* x)) \\ &= \left(\frac{\sin(\frac{\pi}{3})}{\sin(\gamma^* - \frac{\pi}{3})} r^* + \frac{1}{\kappa_2^*} \cos(\kappa_2^* x), \frac{1}{\kappa_2^*} \sin(\kappa_2^* x) \right) \quad \text{for } \sigma \in \Gamma_2^*. \end{aligned}$$

The proof of the continuity can be done using the identity (2.5) and the L'Hôpital's rule. \square

Appendix C. The signs of the integrals

We refer to [4, Appendix A.4] for the proof of the following two lemmas.

Lemma C.1. *Let $0 < \gamma^* < \frac{2\pi}{3}$. Then*

$$\begin{aligned} \text{(i)} \quad & \int_{\Gamma_1^*} v^{(4)}_1 > 0, \quad \int_{\Gamma_2^*} v^{(4)}_2 < 0, \quad \int_{\Gamma_3^*} v^{(4)}_3 < 0, \\ \text{(ii)} \quad & \int_{\Gamma_1^*} v^{(4)}_1 - \int_{\Gamma_2^*} v^{(4)}_2 > 0, \quad \int_{\Gamma_2^*} v^{(4)}_2 - \int_{\Gamma_3^*} v^{(4)}_3 > 0, \\ \text{(iii)} \quad & \sum_{i=1}^3 c_i(v^4) \int_{\Gamma_i^*} v^{(4)}_i > 0. \end{aligned}$$

Lemma C.2. *Let $0 < \gamma^* < \frac{2\pi}{3}$. Then*

$$\begin{aligned} \text{(i)} \quad & \int_{\Gamma_1^*} v^{(5)}_1 = 0, \quad \int_{\Gamma_2^*} v^{(5)}_2 < 0, \quad \int_{\Gamma_3^*} v^{(5)}_3 > 0, \\ \text{(ii)} \quad & \int_{\Gamma_1^*} v^{(5)}_1 - \int_{\Gamma_2^*} v^{(5)}_2 > 0, \quad \int_{\Gamma_2^*} v^{(5)}_2 - \int_{\Gamma_3^*} v^{(5)}_3 < 0, \\ \text{(iii)} \quad & \sum_{i=1}^3 c_i(v^5) \int_{\Gamma_i^*} v^{(5)}_i > 0. \end{aligned}$$

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