A maximum principle for free boundary minimal varieties of arbitrary codimension

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We establish a boundary maximum principle for free boundary minimal submanifolds in a Riemannian manifold with boundary, in any dimension and codimension. Our result holds more generally in the context of varifolds.

1. Introduction

Let N^* be a smooth $(n+1)$-dimensional Riemannian manifold with smooth boundary $\partial N^* \neq \emptyset$, whose inward unit normal (relative to N^*) is denoted by $\nu_{\partial N^*}$. The metric and the Levi-Civita connection on N^* will be denoted by $\langle \cdot, \cdot \rangle$ and ∇ respectively. We fix m to be any positive integer with $1 \leq m \leq n$.

Suppose $N \subset N^*$ is a compact domain whose topological boundary $S := \partial N$ is a smooth *properly* embedded hypersurface in N^*. In other words, S is a smooth embedded hypersurface with boundary $\partial S = S \cap \partial N^*$. Let ν_S denote the unit normal of $S = \partial N$ pointing into N. Recall that a point $p \in S$ is said to be *strongly m-convex* provided that

$$\kappa_1 + \kappa_2 + \cdots + \kappa_m > 0$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ are the principal curvatures [1] of S at p with respect to ν_S. We will often denote $T := N \cap \partial N^*$ with inward unit normal ν_T pointing into N. Note that any of the hypersurfaces S, T and their common boundary $S \cap T$ could be disconnected. See Figure 1.

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[1] The principal curvatures of a hypersurface S (possibly with smooth boundary) at a point $p \in S$ are defined to be the eigenvalues of the second fundamental form A^S as a self-adjoint operator on $T_p S$ given by $A^S(u) := -\nabla_u \nu_S$ where ν_S is a fixed unit normal to S.
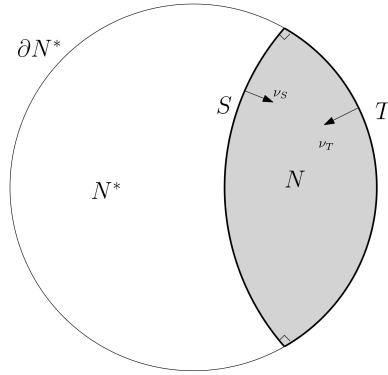
Figure 1: A compact domain $N \subset N^*$ whose topological boundary $S := \partial N$ is a properly embedded hypersurface meeting ∂N^* orthogonally.

Consider the following space of "tangential" vector fields

$$\mathfrak{X}(N^*) := \left\{ \begin{array}{c} \text{compactly supported } C^1 \text{ vector field } X \text{ on } N^* \\ \text{such that } \langle X, \nu_{\partial N^*} \rangle = 0 \text{ along } \partial N^* \end{array} \right\},$$

any $X \in \mathfrak{X}(N^*)$ generates a one-parameter family of diffeomorphisms $\{\phi_t\}_{t \in \mathbb{R}}$ of N^* such that ϕ_0 is the identity map of N^* and $\phi_t(\partial N^*) = \partial N^*$ for all t. If V is a C^1 submanifold of N^* with boundary $\partial V \subset \partial N^*$ such that V has locally finite area, then we denote the first variation of area of V with respect to X by (see [8, §9] for more details):

$$(1.1) \qquad \delta V(X) := \frac{d}{dt}\Big|_{t=0} \text{area}(\phi_t(V)) = \int_V \text{div}_V X.$$

If V is a C^2 submanifold, one can further express (1.1) as (see [8, (7.6)])

$$\delta V(X) = -\int_V \langle X, H \rangle - \int_{\partial V} \langle X, \eta \rangle$$

where H is the (inward pointing) mean curvature vector of V and η is the inward unit co-normal of ∂V. Note that (1.1) makes sense even when V has infinite total area as the vector field X (hence ϕ_t) is compactly supported. In fact, the same discussion holds for any varifold V. We refer the readers to the appendix of [9] for a quick introduction to varifolds. We will be following the notations in [9] closely. Readers who are not familiar with the notion of varifolds may simply replace any varifold V by a C^1 submanifold with boundary lying inside ∂N^*.

Definition 1.1. An m-dimensional varifold V is said to be *stationary with free boundary* if $\delta V(X) = 0$ for all $X \in \mathfrak{X}(N^*)$.

Note that any C^2 submanifold M of N^* with boundary $\partial M = M \cap \partial N^*$ is stationary with free boundary if and only if M is a minimal submanifold in N^* meeting ∂N^* orthogonally along ∂M. These are commonly called *properly embedded* [2] *free boundary minimal submanifolds*.

The goal of this paper is to prove the following result, which generalizes the main result of [10] to the free boundary setting.

Theorem 1.2 (Boundary maximum principle for stationary varifolds with free boundary). *Let $N \subset N^*$ be a compact domain whose topological boundary $S := \partial N$ is a properly embedded hypersurface meeting ∂N^* orthogonally. Suppose S is strongly m-convex at a point $p \in \partial S$. Then, p is not contained in the support of any m-dimensional varifold V which is supported in N and stationary with free boundary.*

Theorem 1 of [10] establishes the maximum principle at any *interior* point of S which is strongly m-convex. Our result above shows that any stationary varifold with free boundary cannot touch S from inside of N at a strongly m-convex point on the *boundary* of S either. In case the varifold V is a C^2 hypersurface (i.e. $m = n$) with free boundary lying inside T, our theorem follows from the classical boundary Hopf lemma [1, Lemma 3.4] as follows. Suppose p is a boundary point [3] of the C^2 hypersurface V. Using the Fermi coordinate system relative to T centered at p (see [2, Section 7] for example), one can locally express S and V as graphs of functions f_S and f_V respectively over an n-dimensional half-ball $B_{r_0}^+ = \{x_1^2 + \cdots + x_n^2 < r_0^2, x_1 \geq 0, x_{n+1} = 0\}$ such that $f_V \geq f_S$ because V lies completely on one side of S. Then, the difference $u := f_V - f_S$ is a C^2 function on $B_{r_0}^+$ satisfying $Lu \leq 0$ in the interior of $B_{r_0}^+$ for some uniformly elliptic second order differential operator L. Moreover, since S is orthogonal to T and V is a free boundary hypersurface, the function u satisfies the following homogeneous Neumann boundary condition along $\{x_1 = 0\}$:

$$(1.2) \qquad \frac{\partial u}{\partial x_1} = 0.$$

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[2]See [5] and [6] for a more detailed discussion on properness.
[3]Note that p cannot be an interior point. Otherwise, V would have non-empty support outside N by transversality.

Since $u \geq 0$ everywhere in $B_{r_0}^+$ and attains zero as its minimum value at the origin, (1.2) violates the boundary Hopf lemma [1, Lemma 3.4]. Our main theorem (Theorem 1.2) shows that the same result holds in any codimension and in the context of varifolds as well.

The interior maximum principle for minimal submanifolds *without boundary* has been proved in various context. The case for C^2 hypersurfaces follows directly from Hopf's classical interior maximum principle [1, Theorem 3.5]. Jorge and Tomi [3] generalized the result to C^2 submanifolds in any codimension. Later, White [10] proved that the maximum principle holds in the context of varifolds, which has important consequences as for example in the Almgren-Pitts min-max theory on the existence and regularity of minimal hypersurfaces in Riemannian manifolds (see [7, Proposition 2.5] for example). Similarly, our boundary maximum principle (Theorem 1.2) is a key ingredient in the regularity part of the min-max theory for free boundary minimal hypersurfaces in compact Riemannian manifolds with non-empty boundary, which is developed in [6] by the authors. We expect to see more applications of Theorem 1.2 to other situations related to the study of free boundary minimal submanifolds.

Our method of proof of Theorem 1.2 is mostly inspired by the arguments in [10] (also in [7, Proposition 2.5]). The key point is to construct a suitable test vector field X which is compactly supported locally near the point p and is universally area-decreasing for any varifold V contained inside N (see [10, Theorem 2]). However, the situation is somewhat trickier in the free boundary setting as the test vector field X constructed has to be *tangential*, i.e. $X \in \mathfrak{X}(N^*)$. In the interior setting of [10], the vector field X is constructed as the gradient of the distance function from a perturbed hypersurface which touches the boundary of N up to second order at p. Unfortunately, the distance function from a free boundary hypersurface does not behave well near the free boundary for at least two reasons. First of all, the distance function may fail to be C^2 near the boundary. Second, even if it is smooth, its gradient may not be tangential and thus cannot be used as a test vector field. We overcome these difficulties by constructing a pair of mutually orthogonal foliations near p, one of which consists of free boundary hypersurfaces for each leaf of the foliation. We then define our test vector field X to be the unit normal to the foliation consisting of free boundary hypersurfaces and show that it is universally area-decreasing as in [10]. We would like to point out that the same argument also applies to varifolds which only *minimize area to first order in N* in the sense of [10] and to free boundary varieties with bounded mean curvature in a weak sense.

The paper is organized as follows. In Section 2, we give a detailed local construction (Lemma 2.1) of orthogonal foliations near a boundary point $p \in \partial N^*$ where a hypersurface S meets ∂N^* orthogonally. We can then choose a local orthonormal frame adapted to such foliation which gives a nice decomposition of the second fundamental form (Lemma 2.2). We give the proof of our main result (Theorem 1.2) in Section 3. All functions and hypersurfaces are assumed to be smooth (i.e. C^∞) unless otherwise stated.

2. Orthogonal foliations

Throughout this section, let N^* be an $(n + 1)$-dimensional Riemannian manifold with boundary $\partial N^* \neq \emptyset$ as in Section 1. Let $p \in \partial N^*$ be a point on the boundary of N^*. Suppose S is a hypersurface in N^* which meets ∂N^* orthogonally along its boundary $\partial S = S \cap \partial N^*$ containing the point p. We first show that one can extend S and ∂N^* locally near p to foliations whose leaves are mutually orthogonal to each other.

Lemma 2.1. *There exists a constant $\delta > 0$, a neighborhood $U \subset N^*$ containing p and foliations* [4] *$\{S_s\}, \{T_t\}$, with $s \in (-\delta, \delta)$ and $t \in [0, \delta)$, of U such that $S_0 = S \cap U$, $T_0 = \partial N^* \cap U$; and S_s intersect T_t orthogonally for every s and t. In particular, each hypersurface S_s meets ∂N^* orthogonally. (See Figure 2.)*

Proof. We first extend S locally near p to a foliation $\{S_s\}$ such that each S_s meets ∂N^* orthogonally. This can be done in a rather straightforward

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[4]See for example [4] for a precise definition of a foliation. When U possess a boundary, one requires one of the following: (i) all the leaves are transversal to the boundary; or (ii) every leaf is either contained in the boundary or is completely disjoint from it.

manner as follows. Let $(x_1, \ldots, x_{n+1})$ be a local Fermi coordinate system of N^* centered at p such that $x_1 = \operatorname{dist}_{N^*}(\cdot, \partial N^*)$. Furthermore, we can assume that $(x_2, \ldots, x_{n+1})$ is a local Fermi coordinate system of ∂N^* relative to the hypersurface $S \cap \partial N^*$, i.e. x_{n+1} is the signed distance in ∂N^* from $S \cap \partial N^*$. As in Section 1 we can express S in such local coordinates as the graph $x_{n+1} = f(x_1, \ldots, x_n)$ of a function f defined on a half ball $B_{r_0}^+$ such that $f = 0 = \frac{\partial f}{\partial x_1}$ along $B_{r_0}^+ \cap \{x_1 = 0\}$. The translated graphs $x_{n+1} = f(x_1, \ldots, x_n) + s$ then gives a local foliation $\{S_s\}$ near p such that each leaf S_s is a hypersurface in N^* which meets ∂N^* orthogonally along its boundary $\partial S_s = S_s \cap \partial N^*$. Note that ∂S_s gives a local foliation of ∂N^* near p obtained from the equidistant hypersurfaces of $\partial S \subset \partial N^*$.

Next, we construct another foliation $\{T_t\}$ which is orthogonal to *every* leaf of the foliation $\{S_s\}$ defined above. Let $q \in N^*$ be a point near p which lies on the leaf S_s. We define $\nu(q)$ to be a unit vector normal to the hypersurface S_s. By a continuous choice of ν it gives a smooth unit vector field in a neighborhood of p such that $\nu(q') \in T_{q'} \partial N^*$ for each $q' \in \partial N^*$ since each S_s meets ∂N^* orthogonally. As ν is nowhere vanishing near p, the integral curves of ν gives a local 1-dimensional foliation of N^* near p. We can put together these integral curves to form our desired foliation $\{T_t\}$ as follows. Let $\Gamma_t \subset S$ be the parallel hypersurface in S which is of distance $t > 0$ away from $S \cap \partial N^*$ (measured with respect to the intrinsic distance in S). Define T_t to be the union of all the integral curves of ν which passes through Γ_t. It is clear that $\{T_t\}$ gives a local foliation near p. Since $\nu(q'')$ is tangent to the leaf T_t which contains q'', the leaves S_s and T_t must be orthogonal to each other for *every* s and t. By continuity, one can restrict the range of s and t such that $|s|$ and t are less than some $\delta > 0$ for which all the above discussions hold. This proves the lemma. $\qquad\square$
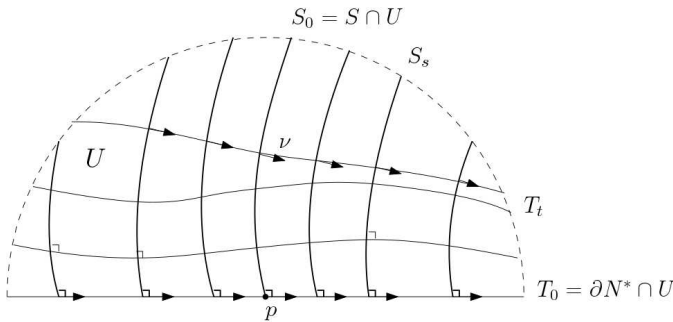

Figure 2: A local orthogonal foliation near a boundary point $p \in S \cap \partial N^*$.

Next, we make use of the local orthogonal foliation in Lemma 2.1 to give a decomposition of the second fundamental form of the leaves of $\{S_s\}$ under a suitable orthonormal frame.

Lemma 2.2. *Let $\{e_1, \ldots, e_{n+1}\}$ be a local orthonormal frame of N^* near p such that at each $q \in S_s \cap T_t$, $e_1(q)$ and $e_{n+1}(q)$ is normal to $S_s \cap T_t$ inside S_s and T_t respectively. Then, we have $\langle A^{S_s}(e_1), e_i \rangle = -\langle A^{T_t}(e_i), e_{n+1} \rangle$ for each $i = 2, \ldots, n$, where A^{S_s} and A^{T_t} are the second fundamental forms of the hypersurfaces S_s and T_t in N^* with respect to the unit normals e_{n+1} and e_1 respectively.*

Proof. By definition of A^{S_s} and A^{T_t} (see Section 1), we have

$$\langle A^{S_s}(e_1), e_i \rangle = \langle -\nabla_{e_1} e_{n+1}, e_i \rangle = \langle e_{n+1}, \nabla_{e_i} e_1 \rangle = -\langle A^{T_t}(e_i), e_{n+1} \rangle,$$

where we used the fact that $[e_1, e_i]$ is tangent to S_s in the second equality. $\quad\square$

3. Proof of Theorem 1.2

The proof is by a contradiction argument motivated by some of the ideas in [10]. Note that we will continue to adopt the notations in Section 1. Suppose on the contrary that there exists a point $p \in \partial S = S \cap T$ which lies in the support of an m-dimensional varifold V in N which is stationary with free boundary. Our goal is to construct a tangential vector field $X \in \mathfrak{X}(N^*)$ which is compactly supported near p such that $\delta V(X) < 0$ (recall (1.1)), which contradicts the stationarity of V.

For every $\epsilon > 0$ small, we can define

$$\Gamma := \left\{ x \in \partial N^* : \mathrm{dist}_{\partial N^*}(x, \partial S) = \epsilon \, \mathrm{dist}^4_{\partial N^*}(x, p) \right\},$$

which is an $(n-1)$-dimensional hypersurface in ∂N^* that is smooth in a neighborhood of p.

Claim 1: Γ touches ∂S from outside T up to second order at p.

Proof of Claim 1. Let $(y_1, \ldots, y_{n-1}, t)$ be the Fermi coordinate system of ∂N^* centered at p adapted to the hypersurface ∂S, i.e. $(y_1, \ldots, y_{n-1})$ is the geodesic normal coordinates of ∂S centered at p and t is the signed distance function from ∂S in ∂N^* (taken to be negative in T). In such a Fermi coordinate system, locally near p we have $\partial S = \{t = 0\}$, $T = \{t \leq 0\}$

and

$$\Gamma = \{t = \epsilon\varphi(y_1, \ldots, y_{n-1}, t)\}$$

for some function φ which is smooth defined near the origin. Using the definition of Fermi coordinates, we have φ vanishes up to second order at the origin since the metric components g_{ij} of ∂N^* in Fermi coordinates has C^2 bound only in terms of the geometry of ∂S and ∂N^*. This proves the claim. $\qquad\square$

Next we want to extend Γ to a hypersurface S' in N^* which meets ∂N^* orthogonally along Γ such that S' touches N from outside at p up to second order. The construction of such an S' can be done locally as follows. As in the proof of Lemma 2.1, let $(x_1, \ldots, x_{n+1})$ be a Fermi coordinate system around p such that

- $\{x_1 \geq 0\} \subset N^*$,
- $\{x_{n+1} = f(x_1, \ldots, x_n)\} \subset S$,
- $\{x_{n+1} \geq f(x_1, \ldots, x_n)\} \subset N$,
- $\{x_1 = x_{n+1} = 0\} \subset \Gamma$.

Note that this f could be slightly different from that as in the proof of Lemma 2.1 as we further require that x_{n+1} to agree with the signed distance function from Γ in ∂N^*. By Claim 1, we have $f(0, x_2, \ldots, x_n) \geq 0$ with equality holds only at the origin. Moreover, we have $\frac{\partial f}{\partial x_1} = 0$ along $\{x_1 = 0\}$ since S meets ∂N^* orthogonally. Take S' to be the graph $x_{n+1} = u(x_1, \ldots, x_n)$ of the smooth function

$$u(x_1, \ldots, x_n) := \frac{x_1^2}{2} \frac{\partial^2 f}{\partial^2 x_1}(0) + \frac{x_1^3}{6} \left(\frac{\partial^3 f}{\partial^3 x_1}(0) - \epsilon \right).$$

Since $u = \frac{\partial u}{\partial x_1} = 0$ along $\{x_1 = 0\}$, S' is indeed an extension of Γ meeting ∂N^* orthogonally. By Claim 1 we know that all the partial derivatives (with respect to the coordinates $x_1, \ldots, x_n$) of u and f at 0 agree up to second order, therefore the Hessian of u and f agrees at the origin. For ϵ sufficiently small, $f \geq u$ everywhere in a neighborhood of p with equality holds only at the origin where f and u agrees up to second order. In order words, S' touches N from outside up to second order at p.

Since S' meets ∂N^* orthogonally, we can apply all the results in Section 2 to S' to obtain local foliations $\{S'_s\}$ and $\{T_t\}$ as in Lemma 2.1. We will use the same notations as in the proof of Lemma 2.1 in what follows (with S

replaced by S'). Define a smooth function s in a neighborhood of p such that $s(q)$ is the unique s such that $q \in S'_s$. Recall that $s \geq 0$ on N.

Lemma 3.1. $\nabla s = \psi \nu$ for some function ψ which is smooth in a neighborhood of p such that $\psi = 1$ along ∂N^*.

Proof. Since s is constant on each leaf S'_s by definition, ∇s is normal to the hypersurface S'_s and thus $\nabla s = \psi \nu$ for some smooth function ψ in a neighborhood of p. The last assertion follows from our construction that $\partial S'_s$ are parallel hypersurfaces from $\partial S'$ in ∂N^*. $\qquad \square$

Now, we define a vector field X on N^* by

$$X(q) := \phi(s(q))\nu(q),$$

where $\phi(s)$ is the cutoff function defined by

$$\phi(s) = \begin{cases} e^{1/(s-\epsilon)} & \text{if } 0 \leq s < \epsilon, \\ 0 & \text{if } s \geq \epsilon. \end{cases}$$

As S' touches N at p from outside, we see that X is compactly supported in a neighborhood of p. Moreover, since $\nu(q) \in T_q \partial N^*$ at all points $q \in \partial N^*$, we have $X \in \mathfrak{X}(N^*)$. To finish the proof, we just have to show that X decreases the area of V up to first order, i.e. $\delta V(X) < 0$.

At each q in a neighborhood of p, we consider the bilinear form on $T_q N^*$ defined by

$$Q(u, v) := \langle \nabla_u X, v \rangle(q).$$

Let $\{e_1, \ldots, e_{n+1}\}$ be an orthonormal frame as in Lemma 2.2 (note that $e_{n+1} = \nu$). By Lemma 3.1, when $u = e_i$, $v = e_j$, $i, j = 1, \ldots, n$, we have

$$Q(e_i, e_j) = \langle \nabla_{e_i}(\phi\nu), e_j \rangle = -\phi \langle A^{S'_s}(e_i), e_j \rangle.$$

Moreover, since $\langle \nu, \nu \rangle \equiv 1$ and $\nabla_{e_i} s \equiv 0$, we have for $i = 1, \ldots, n$,

$$Q(e_i, e_{n+1}) = \langle \nabla_{e_i}(\phi\nu), e_{n+1} \rangle = \phi \langle \nabla_{e_i}\nu, \nu \rangle = 0.$$

On the other hand, when $u = e_{n+1} = \nu$, we have

$$Q(e_{n+1}, e_1) = \langle \nabla_{e_{n+1}}(\phi\nu), e_1 \rangle = \phi \langle \nabla_\nu \nu, e_1 \rangle = \phi \langle A^{T_t}(\nu), \nu \rangle.$$

Since $\langle \nu, e_j \rangle \equiv 0$, we have for $j = 2, \ldots, n$,

$$Q(e_{n+1}, e_j) = \langle \nabla_{e_{n+1}}(\phi\nu), e_j \rangle = \phi\langle \nabla_\nu \nu, e_j \rangle.$$

Finally, when $u = v = e_{n+1} = \nu$, using Lemma 3.1 and $\langle \nu, \nu \rangle \equiv 1$,

$$Q(e_{n+1}, e_{n+1}) = \langle (\nabla_\nu \phi)\nu, \nu \rangle + \phi\langle \nabla_\nu \nu, \nu \rangle = \phi'\psi.$$

Therefore, we can express Q in this frame as the following $n + 1$ by $n + 1$ matrix:

$$(3.1) \qquad Q = \begin{bmatrix} -\phi A_{11}^{S'_s} & \phi A_{n+1,j}^{T_t} & 0 \\ \phi A_{i,n+1}^{T_t} & -\phi A_{ij}^{S'_s} & 0 \\ \phi A_{n+1,n+1}^{T_t} & \phi\langle \nabla_\nu \nu, e_j \rangle & \phi'\psi \end{bmatrix}$$

where $i, j = 2, \ldots, n$, and $q \in S_s \cap T_t$.

Lemma 3.2. *When $\epsilon > 0$ is small enough, $\text{tr}_P Q < 0$ for all m-dimensional subspaces $P \subset T_q N^*$.*

Proof. If $P \subset T_q S'_s$, then $\text{tr}_P Q < 0$ since S'_s is strongly m-convex in a neighborhood of p. Therefore, we focus on the case $P \not\subset T_q S'_s$. In this case, one can fix an orthonormal basis $\{v_1, \ldots, v_m\}$ for P such that $\{v_1, \ldots, v_{m-1}\} \subset T_q(S'_s \cap T_t)$. As $P \not\subset T_q S'_s$, there exists some unit vector $v_0 \in T_q S'_s$ with $v_0 \perp v_i$ for $i = 1, \ldots, m-1$ and $\theta \in (0, \pi)$ such that

$$v_m = (\cos\theta)\, v_0 + (\sin\theta)\, e_{n+1}.$$

Denote $P' = \text{span}\{v_0, v_1, \ldots, v_{m-1}\} \subset T_q S'_s$. On the other hand, since $v_0 \in T_q S'_s$, one can write

$$v_0 = a_1 e_1 + \cdots + a_n e_n,$$

where $a_1^2 + \cdots + a_n^2 = 1$. Therefore, using (3.1) and that $\phi' \leq -\frac{1}{\epsilon^2}\phi$, by possibly shrinking the neighborhood of p we have $\psi \geq 1/2$, $|A^{T_t}| \leq K$, $|A^{S'_s}| \leq K$ and $|\langle \nabla_\nu \nu, e_j \rangle| \leq K$ for some constant $K > 0$ (depending on the chosen orthogonal foliation in Lemma 2.1 but independent of ϵ), one then obtains

$$\operatorname{tr}_P Q = \sum_{i=1}^{m-1} Q(v_i, v_i) + Q(v_m, v_m)$$

$$= \sum_{i=1}^{m-1} Q(v_i, v_i) + \cos^2\theta\, Q(v_0, v_0) + \sin\theta\cos\theta\, Q(e_{n+1}, v_0)$$

$$+ \sin^2\theta\, Q(e_{n+1}, e_{n+1})$$

$$= -\phi\operatorname{tr}_{P'} A^{S'_s} + \sin^2\theta\left(\phi'\psi + \phi A^{S'_s}(v_0, v_0)\right) + a_1\phi\sin\theta\cos\theta\, A^{T_t}_{n+1,n+1}$$

$$+ \sum_{j=2}^{n} a_j\phi\sin\theta\cos\theta\,\langle\nabla_\nu\nu, e_j\rangle$$

$$\leq -\phi\operatorname{tr}_{P'} A^{S'_s} + \phi\left(\left(K - \frac{1}{2\epsilon^2}\right)\sin^2\theta + \sqrt{n}K|\sin\theta\cos\theta|\right)$$

Lemma 3.3. *As $\epsilon \to 0$, we have*

$$\max_{\theta\in[0,\pi]}\left[\left(K - \frac{1}{2\epsilon^2}\right)\sin^2\theta + \sqrt{n}K|\sin\theta\cos\theta|\right] \to 0.$$

Proof. Define the function $F : [0,\pi] \to \mathbb{R}$ by

$$F(\theta) := \left(K - \frac{1}{2\epsilon^2}\right)\sin^2\theta + \sqrt{n}K|\sin\theta\cos\theta|.$$

Notice that $F(\theta) = F(\pi - \theta)$ for all $\theta \in [0, \pi/2]$ and that $F(0) = 0$, $F(\pi/2) = K - \frac{1}{2}\epsilon^{-2}$ which is negative as long as $\epsilon < 1/\sqrt{2K}$. Moreover, if $F'(\theta_0) = 0$ at some $\theta_0 \in (0, \pi/2)$, then we have

$$(3.2) \qquad\qquad \tan 2\theta_0 = \frac{\sqrt{n}K}{\frac{1}{2}\epsilon^{-2} - K}.$$

Note that such a θ_0 is unique and $\theta_0 \to 0$ as $\epsilon \to 0$. Using (3.2) and L'Hospital's rule, $F(\theta_0) \to 0$ as $\epsilon \to 0$. This proves Lemma 3.3. $\qquad\square$

Using Lemma 3.3 and that S'_s is strongly m-convex in a small neighborhood of p when ϵ is sufficiently small, we have $\operatorname{tr}_P Q < 0$. This finishes the proof of Lemma 3.2. $\qquad\square$

To finish the proof, recall that the first variation formula [8, 39.2] says

$$\delta V(X) = \int \operatorname{div}_P X(q) \, dV(q, P)$$
$$= \int \operatorname{tr}_P Q(q) \, dV(q, P) < 0,$$

where the last inequality follows from Lemma 3.2 and that the support of the vector field X inside N is contained in a very small neighborhood of p. This gives a contradiction to the assumption that V is stationary with free boundary. This finishes the proof of Theorem 1.2.

References

[1] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin (2001), ISBN 3-540-41160-7. Reprint of the 1998 edition.

[2] Q. Guang, M. M.-c. Li, and X. Zhou, *Curvature estimates for stable minimal hypersurfaces with free boundary*, J. Reine Angew. Math. **759** (2020), 245–264.

[3] L. P. Jorge and F. Tomi, *The barrier principle for minimal submanifolds of arbitrary codimension*, Ann. Global Anal. Geom. **24** (2003), no. 3, 261–267.

[4] H. B. Lawson, Jr., *Foliations*, Bull. Amer. Math. Soc. **80** (1974), 369–418.

[5] M. M.-c. Li, *A general existence theorem for embedded minimal surfaces with free boundary*, Comm. Pure Appl. Math. **68** (2015), no. 2, 286–331.

[6] M. M.-c. Li and X. Zhou, *Min-max theory for free boundary minimal hypersurfaces I: regularity theory*, J. Differential Geom. **118** (2021), no. 3, 487–553.

[7] J. T. Pitts, Existence and Regularity of Minimal Surfaces on Riemannian Manifolds, Vol. 27 of Mathematical Notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1981), ISBN 0-691-08290-1.

[8] L. Simon, Lectures on Geometric Measure Theory, Vol. 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University, Australian National University, Centre for Mathematical Analysis, Canberra (1983), ISBN 0-86784-429-9.

[9] B. White, *Which ambient spaces admit isoperimetric inequalities for submanifolds?* J. Differential Geom. **83** (2009), no. 1, 213–228.

[10] B. White, *The maximum principle for minimal varieties of arbitrary codimension*, Comm. Anal. Geom. **18** (2010), no. 3, 421–432.

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