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On an open problem of characterizing the birationality of 4K

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We answer an open problem raised by Chen-Zhang in 2008 and prove that, for any minimal projective 3-fold X of general type with the geometric genus $p_g(X) \geq 5$, the 4-canonical map $\varphi_{4,X}$ is non-birational if and only if X is birationally equivalent to a fibration, onto a curve, of which the general fiber is a minimal surface of general type with $(c_1^2, p_g) = (1, 2)$. The statement does not hold for those with the geometric genus $p_g(X) \leq 4$ according to our examples.

1. Introduction

Throughout we work over an algebraically closed field of characteristic 0.

In this note, a $(1, 2)$-surface means a nonsingular projective surface of general type whose minimal model has the invariants: $c_1^2 = 1$ and $p_g = 2$.

We mean a fibration by a surjective projective morphism with connected fibers. Let $f : Y \to T$ be a fibration from a smooth projective 3-fold Y onto a smooth projective curve T. Denote by F a general fiber of f. When the general fiber F is a $(1, 2)$-surface, we say that f is a pencil of $(1, 2)$-surfaces. For a projective 3-fold Z, we say that Z is *birationally fibred by a pencil of* $(1, 2)$*-surfaces* if Z is birationally equivalent to a nonsingular projective 3-fold which admits a pencil of $(1, 2)$-surfaces.

A famous theorem of Bombieri says that, for any nonsingular projective surface S of general type, $\varphi_{4,S}$ is non-birational if and only if S is a $(1, 2)$-surface. A direct corollary is that any nonsingular projective 3-fold of general type, admitting a pencil of $(1, 2)$-surfaces, necessarily has non-birational

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4-canonical map. A very natural question (raised in Chen-Zhang [CZ08, 6.4(1)]) is whether the converse is true!

The purpose of this paper is to prove the following theorem which answers the above question:

Theorem 1.1. *Let X be a minimal projective 3-fold of general type with $p_g(X) \geq 5$. Then $\varphi_{4,X}$ is non-birational if and only if X is birationally fibred by a pencil of $(1,2)$-surfaces.*

One has the following example:

Example 1.2. (see Fletcher [Flet]) The general hypersurface of degree 10:

$$X = X_{10} \subset \mathbb{P}(1,1,1,1,5)$$

is a smooth canonical 3-fold with $p_g = 4$ and non-birational 4-canonical map $\varphi_{4,X}$. Since X is a double cover onto $\mathbb{P}^3$, X admits no genus 2 curve class of canonical degree 1. Hence X admits no pencil of $(1,2)$-surfaces (by the same argument as in [CZ08, Example 6.3]).

Example 1.2, together with Example 3.1 and Example 3.2 in the last section, shows that the condition "$p_g(X) \geq 5$" in Theorem 1.1 is sharp.

Throughout we use the following symbols:

- ◇ "$\sim$" denotes linear equivalence or $\mathbb{Q}$-linear equivalence (subject to the context);
- ◇ "$\equiv$" denotes numerical equivalence;
- ◇ "$|D_1| \succcurlyeq |D_2|$" (or, equivalently, "$|D_2| \preccurlyeq |D_1|$") means, for linear systems $|D_1|$ and $|D_2|$ of divisors on a variety,

$$|D_1| \supseteq |D_2| + \text{certain fixed effective divisor}.$$

2. Proof of Theorem 1.1

Throughout this section, X denotes a minimal projective $\mathbb{Q}$FT 3-fold of general type with $p_g(X) \geq 5$. Let K_X be a canonical divisor of X and denote by $\mathrm{Sing}(X)$ the singular locus of X. Since 3-dimensional terminal singularities are isolated, $\mathrm{Sing}(X)$ consists of only finitely many points.

2.1. Fixed notation and the standard resolution for $\mathrm{Mov}|K_X|$

First of all, we take a resolution of singularities of X, say: $\alpha : X_0 \longrightarrow X$ where X_0 is projective. In particular, we may choose α such that α is an isomorphism over the smooth locus of X. As X is minimal, we have $p_g(X_0) = p_g(X) \geq 5$. We may write

$$\alpha^*(K_X) = M_0 + Z_0',$$

where $|M_0| = \mathrm{Mov}|K_{X_0}|$ and Z_0' is an effective $\mathbb{Q}$-divisor.

By Hironaka's big theorem, we may resolve the base locus $\mathrm{Bs}|M_0|$ by taking successive blowups, say:

$$\beta\colon X' = X_{n+1} \overset{\pi_n}{\to} X_n \to \cdots \to X_{i+1} \overset{\pi_i}{\to} X_i \to \cdots \to X_1 \overset{\pi_0}{\to} X_0$$

where each π_i is a blow-up along a nonsingular center W_i (W_i is contained in the base locus of the movable part $\mathrm{Mov}|(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1})^*(M_0)|$. Moreover, the morphism $\beta = \pi_n \circ \cdots \circ \pi_0$ satisfies the following properties:

1) The linear system $|M| = \mathrm{Mov}|\beta^*(M_0)|$ is base point free.

2) One may write

$$K_{X'} = \beta^*(K_{X_0}) + \sum_{i=0}^{n} a_i E_i, \tag{2.1}$$

$$\beta^*(M_0) = M + \sum_{i=0}^{n} b_i E_i, \tag{2.2}$$

where each E_i is the strict transform of the exceptional divisor of π_i for $0 \leq i \leq n$, a_i and b_i are positive integers.

For any positive integer m, denote by $|M_m|$ the moving part of $|mK_{X'}|$. By our notation, $M = M_1$.

Lemma 2.1. *(see [Ch04, Lemma 4.2]) In the above setting, the following properties hold:*

(i) *For any i, $a_i \leq 2b_i$.*

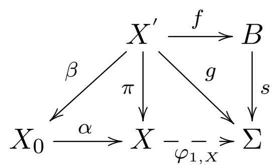
(ii) *If $a_k = b_k = 1$ for some k with $0 \leq k \leq n$, then W_k is a smooth curve contained in X_k.*

(iii) *If* $a_k = 2b_k$ *for some* k *such that* $0 \le k \le n$*, then* W_k *is a closed point of* X_k*.*

Let $\pi = \alpha \circ \beta : X' \longrightarrow X$ be the composition. We may write

$$K_{X'} \sim_{\mathbb{Q}} \pi^*(K_X) + E_\pi, \qquad \pi^*(K_X) \sim_{\mathbb{Q}} M + E'_\pi, \tag{2.3}$$

where E_π is an effective π-exceptional $\mathbb{Q}$-divisor and E'_π is an effective $\mathbb{Q}$-divisor. Let $g = \varphi_{1,X} \circ \pi$ and set $\Sigma = g(X')$. Take the Stein factorization of g, say $X' \xrightarrow{f} B \xrightarrow{s} \Sigma$. We have the following commutative diagram:



where B is a normal projective variety.

2.2. The case of $\dim(B) = 1$ and 3

This is a known case since we have the following theorem:

Theorem 2.2. *(see Chen–Zhang [CZ08, 4.2, 4.8, 4.9]) Let* X *be a minimal* 3*-fold of general type with* $p_g(X) \ge 5$*. Keep the notation in 2.1. The following statements hold:*

(i) *Assume* $\dim(B) = 1$*. Then* $\varphi_{4,X}$ *is non-birational if and only if the general fiber of* f *is a* $(1,2)$*-surface.*

(ii) *Assume* $\dim(B) = 3$*. Then* $\varphi_{4,X}$ *is birational onto its image.*

2.3. The case of $\dim(B) = 2$

Let C be a general fiber of f. The following result was proved by Chen–Zhang as well:

Theorem 2.3. *(see [CZ08, 4.3]) Let* X *be a minimal* 3*-fold of general type with* $p_g(X) \ge 5$*. Keep the notation in 2.1. Assume* $\dim(B) = 2$*. Then* $\varphi_{4,X}$ *is non-birational if and only if* $g(C) = 2$ *and* $(\pi^*(K_X) \cdot C) = 1$*.*

(‡) **From now on, we always assume:**

$$g(C) = 2 \text{ and } (\pi^*(K_X) \cdot C) = 1.$$

Pick a general member S in $|M|$. By Chen–Zhang [CZ16, Theorem 2.4], one has

$$|2nK_{X'}||_S \succcurlyeq |n(K_{X'} + S)||_S = |nK_S|$$

for any sufficiently large and divisible integer n. Noting that

$$2n\pi^*(K_X) \geq M_{2n}$$

and that $|n\sigma^*(K_{S_0})|$ is base point free, we have

$$\pi^*(K_X)|_S \sim_{\mathbb{Q}} \frac{1}{2}\sigma^*(K_{S_0}) + H_S, \tag{2.4}$$

where H_S is an effective $\mathbb{Q}$-divisor on S. We may write

$$\pi^*(K_X)|_S \sim_{\mathbb{Q}} S|_S + E'_\pi|_S \text{ and } S|_S \equiv aC, \tag{2.5}$$

where $a \geq p_g(X) - 2 \geq 3$.

Lemma 2.4. *Let X be a minimal 3-fold of general type with $p_g(X) \geq 5$. Keep the notation in 2.1. Assume that $\dim(B) = 2$ and that $\varphi_{4,X}$ is non-birational. Then there exists exactly one exceptional divisor $E \subset \mathrm{Supp}(E_\pi)$ such that $(E \cdot C) = 1$.*

Proof. By Theorem 2.3 and Relation (2.5), we have $(E'_\pi|_S \cdot C) = 1$. By (2.3) and the assumption, we have $(E_\pi|_S \cdot C) = 1$.

First we prove that the horizontal part of $\mathrm{Supp}(E'_\pi|_S)$ is an integral curve Γ. Take a general member K_1 in $|K_X|$, we have $\pi^*(K_1)|_C = E'_\pi|_C$. It is clear that one of the following cases occurs:

(a) $\mathrm{Supp}(E'_\pi|_C)$ consists of one single point P with $2P \sim K_C$;

(b) $\mathrm{Supp}(E'_\pi|_C)$ consists of two different points P and Q, where $P + Q \sim K_C$.

We will exclude the possibility of (b). Otherwise, we may write $E'_\pi|_C = \varepsilon P + (1 - \varepsilon)Q$, where $0 < \varepsilon < 1$.

By the argument in the proof of [CZ08, Proposition 4.6], we know that $\mathrm{Mov}|4K_{X'}||_C \succcurlyeq |2K_C|$. Noting that $\deg(4E'_\pi|_C) = 4$, we see $4E'_\pi|_C \sim 2K_C$.

Thus 4ε is a positive integer. If $4\varepsilon = 1$, then $4E'_\pi|_C \sim K_C + 2Q$, which implies that $2Q \sim K_C$, a contradiction. Similarly, we can conclude that $4\varepsilon \neq 3$. Thus we have $\varepsilon = \frac{1}{2}$ and

$$2P + 2Q = \llcorner 5E'_\pi|_C \lrcorner \geq M_5|_C,$$

for a general fiber C. This simply implies that $\varphi_{5,X}|_C$ is not birational and neither is $\varphi_{5,X}$, which contradicts to [Ch03, Theorem 1.2(2)]. Therefore the only possibility is case (a).

Since $E_\pi|_C + E'_\pi|_C \in |K_C|$ and $2P \in |K_C|$, we have $E_\pi|_C = P$, which implies that the horizontal part of $E_\pi|_S$ (with respect to the fibration $f|_S$) coincides with the horizontal part of $E'_\pi|_S$ (with respect to the fibration $f|_S$). Since $\mathrm{Supp}(E_\pi|_C)$ consists of exactly one point for a general C, there exists only one exceptional divisor E such that $(E \cdot C) = 1$. In particular, the coefficient of E in E_π (and hence in E'_π) is 1.

Furthermore, for any other π-exceptional divisor $E' \neq E$, $E'|_S$ is vertical with respect to $f|_S$ for a general member S. □

By Lemma 2.4, for a general member $S \in |M|$, we may write

$$E_\pi|_S = \Gamma + E_V, \quad E'_\pi|_S = \Gamma + E'_V, \tag{2.6}$$

where Γ is the horizontal part satisfying $(\Gamma \cdot C) = 1$ for a smooth fiber C contained in S, E_V and E'_V are both vertical parts with respect to $f|_S$.

Lemma 2.5. *Let X be a minimal 3-fold of general type with $p_g(X) \geq 5$. Keep the notation in 2.1. Assume that $\dim(B) = 2$ and that $\varphi_{4,X}$ is non-birational. We have $(\pi^*(K_X)|_S \cdot \Gamma) > 0$. In particular, we have $E = E_i$ for some i, $a_i = b_i = 1$ and $\pi(E)$ is an irreducible curve on X.*

Proof. It is clear that $p_g(S) = h^0((K_{X'} + S)|_S) \geq 3$, so S is not a $(1,2)$-surface and we have $(\sigma^*(K_{S_0}) \cdot C) \geq 2$ by the Hodge index theorem and the result of Bombieri [Bom] that a minimal $(1,1)$ surface is simply connected (see also [CC15, Lemma 2.4] for a direct reference).

By Relation (2.4), we have $(\sigma^*(K_{S_0}) \cdot C) = 2$ and $(H_S \cdot C) = 0$. Thus H_S is composed of vertical divisors with respect to $f|_S$. Since Γ is the section of the fibration $f|_S$, we have $(\pi^*(K_X)|_S \cdot \Gamma) \geq \frac{1}{2}(\sigma^*(K_{S_0}) \cdot \Gamma)$ by (2.4). Thus it is sufficient to prove $(\sigma^*(K_{S_0}) \cdot \Gamma) > 0$.

Suppose $(\sigma^*(K_{S_0}) \cdot \Gamma) = 0$. We consider the contraction $\sigma : S \to S_0$ onto the minimal model S_0. Since $g(C) = 2$, $(\sigma^*(K_{S_0}) \cdot C) = 2$ and $C^2 = 0$, we see that all exceptional divisors of σ are contained in special fibers of $f|_S$.

Thus $C = \sigma^*(\overline{C})$ where $\overline{C}$ comes from a free pencil of genus 2 on S_0. Let $\overline{\Gamma} = \sigma_*(\Gamma)$. Since $(\overline{\Gamma} \cdot \overline{C}) = (\Gamma \cdot C) = 1$, we conclude that $\overline{\Gamma}$ is a section of fibration induced from the free pencil generated by $\overline{C}$. In particular, $\overline{\Gamma} \neq 0$. Thus $\overline{\Gamma}$ is a (-2)-curve on S_0. By the adjunction formula and (2.6), we can write

$$K_S = (K_{X'} + S)|_S = \pi^*(K_X)|_S + S|_S + \Gamma + E_V.$$

Considering the Zariski decomposition of the above divisor, we can write

$$\pi^*(K_X)|_S + S|_S + \Gamma + E_V \equiv (\pi^*(K_X)|_S + S|_S + N^+) + N^-,$$

where

(z1) both N^+ and N^- are effective $\mathbb{Q}$-divisors and $N^+ + N^- = \Gamma + E_V$;

(z2) the $\mathbb{Q}$-divisor $\pi^*(K_X)|_S + S|_S + N^+$ is equal to $\sigma^*(K_{S_0})$;

(z3) $((\pi^*(K_X)|_S + S|_S + N^+) \cdot N^-) = 0$.

Since $(\sigma^*(K_{S_0}) \cdot C) = 2$ and $(\pi^*(K_X)|_S \cdot C) = 1$, we have $N^+ = \Gamma + A$, where A is an effective vertical divisor. Thus we can write

$$\begin{aligned} \sigma^*(K_{S_0}) &= \pi^*(K_X)|_S + S|_S + \Gamma + A \\ &\equiv 2aC + 2\Gamma + E'_V + A. \end{aligned}$$

Pushing forward to S_0, we have

$$K_{S_0} \equiv 2a\overline{C} + 2\overline{\Gamma} + \sigma_*(E'_V + A),$$

where $\sigma_*(E'_V + A)$ is clearly vertical. Then we get $(K_{S_0} \cdot \overline{\Gamma}) \geq 2a - 4 \geq 2$, which contradicts to our assumption. So our conclusion is that $(\sigma^*(K_{S_0}) \cdot \Gamma) > 0$.

Note that Γ comes from the exceptional divisor E. Since $(\pi^*(K_X)|_S \cdot E|_S) \geq (\pi^*(K_X) \cdot \Gamma) > 0$, we see that $E = E_i$ for some index i by the construction of π. In particular, by Lemma 2.5, we have $a_i = b_i = 1$. □

By Lemma 2.1, one sees that E comes from the blow-up of a smooth curve. Thus E carries a natural fibration whose general fiber is a smooth rational curve. Denote by l_E the general fiber of this fibration. We have the following observation:

Lemma 2.6. *Under the same assumption as that of Lemma 2.5, keep the above notation. We have* $(S \cdot l_E) = 1$*. In particular, we have* $S|_E \geq l_1 + l_2$ *for two distinct general elements in the same algebraic class of* l_E *on* E.

Proof. Denote by E_i^i the exceptional divisor of π_i so that E dominates E_i^i and by $l_{E_i^i}$ the corresponding general ruling. We have $(E_i^i \cdot l_{E_i^i}) = -1$. Denote by $\tilde{E}$ the total transform of E_i^i on X'. Then we have $(\tilde{E} \cdot l_E) = -1$ by the projection formula. For any exceptional divisor D not contained in $\tilde{E}$, we have $(D \cdot l_E) = 0$ by the choice of l_E. By (2.3), $(\pi^*(K_X) \cdot l_E) = 0$ and our construction of π, we have $(S \cdot l_E) \leq 1$. Since $f|_E$ is a birational morphism and f is induced by $|S|$, we have $(S \cdot l_E)_{X'} = (S|_E \cdot l_E)_E > 0$. Since E is a smooth projective surface and $S|_E$ is a Cartier divisor, we have $(S \cdot l_E) = 1$.

Take two distinct general fibers l_1 and l_2 in the ruling of E. Since l_E is a smooth rational curve, we have $h^0(l_E, S|_{l_E}) = 2$. Since $((S - E) \cdot C) = -1 < 0$, we have $h^0(X', S - E) = 0$. Thus we have $h^0(E, S|_E) \geq p_g(X) \geq 5$. Consider the natural exact sequence

$$0 \to H^0(E, S|_E - l_1 - l_2) \to H^0(E, S|_E) \to H^0(l_1, S|_{l_1}) \oplus H^0(l_2, S|_{l_2}).$$

We naturally get $h^0(E, S|_E - l_1 - l_2) \geq 1$, which implies that $S|_E \geq l_1 + l_2$. $\square$

Now we are ready to prove the main statement.

Theorem 2.7. *Let* X *be a minimal* 3*-fold of general type with* $p_g(X) \geq 5$*. Keep the notation in 2.1. Assume that* $\dim(B) = 2$ *and that* $\varphi_{4,X}$ *is non-birational. Then* X *is birationally fibred by a pencil of* $(1,2)$*-surfaces.*

Proof. First of all, we note that all our above arguments remain effective if we replace π by any further birational modification over π.

Since E is birational to B, we may take a common smooth projective birational modification W of both B and E. Take a birational modification $\pi' \colon X'' \to X'$ such that $f \circ \pi'$ factors through $W \to B$. Denote by $f'' \colon X'' \to W$ the corresponding fibration. The natural $\mathbb{P}^1$-fibration on E induces a fibration on W. Denote by l_W the general fiber of the fibration induced from the ruling. Set $\tilde{\pi} = \pi \circ \pi'$.

Now we work on the higher model X'', on which we have the base point free linear system $|M''| = |\pi^*(M)|$ and the general member S'' has the property: $S'' = \pi'^*(S) = f''^*(H)$ for a certain nef and big divisor H on W. By Lemma 2.6, we have $H \geq l_{1,W} + l_{2,W}$ for two general distinct fibers on W (in the same algebraic class as that of l_W). Set $F'' = f''^*(l_W)$

and $F''_i = f''^*(l_{i,W})$ for $i = 1, 2$. Clearly F'' induces a pencil on X'' and $\tilde{\pi}^*(K_X) \geq S'' \geq F''_1 + F''_2 \equiv 2F''$.

Since $S''|_{F''}$ is moving, we have $p_g(F'') \geq 2$. On the other hand, the canonical system $|K_{X''}|$ contains a free sub-pencil $|F''_1 + F''_2|$ with a generic irreducible element F'', which is smooth and projective. By [CC15, Lemma 2.1], we have

$$\tilde{\pi}^*(K_X)|_{F''} \geq \frac{2}{3}\sigma''^*(K_{F''_0}) \tag{2.7}$$

where $\sigma'' : F'' \to F''_0$ denotes the contraction onto the minimal model.

Denote by C'' a general fiber of f''. Pick a smooth such element $C''_F \subset F''$. Clearly we have

$$1 = (\tilde{\pi}^*(K_X)|_{F''} \cdot C''_F) \geq \frac{2}{3}(\sigma''^*(K_{F''_0}) \cdot C''_F),$$

which means that $(\sigma''^*(K_{F''_0}) \cdot C''_F) = 1$. Hence F''_0 must be a $(1, 2)$-surface by Bombieri (see also [CC15, Lemma 2.4] for a direct reference). We are done. □

Now it is clear that Theorem 1.1 follows directly from Theorem 2.2, Theorem 2.3 and Theorem 2.7. We have finished the proof of our main theorem.

3. Examples

It is interesting to know whether a pencil of $(1, 2)$-surfaces necessarily appears in those 3-folds of general type with $p_g \leq 4$ and with non-birational 4-canonical maps. We provide two more examples here.

Example 3.1. Consider the general hypersurface of degree 12 (canonical 3-fold) $X = X_{12} \subset \mathbb{P}(1, 1, 1, 2, 6)$. One knows that $K_X^3 = 1$, $p_g(X) = 3$ and X has 2 orbifold points $\frac{1}{2}(1, -1, 1)$. It is also clear that $\varphi_{4,X}$ is non-birational. We claim that X does not admit any pencil of $(1, 2)$-surfaces.

Assume, to the contrary, that X admits a pencil of $(1, 2)$-surfaces, say $\Lambda \subset |F_1|$ where $\dim \Lambda = 1$, F_1 is irreducible and is of $(1, 2)$-type. We keep the notation in 2.1 and modify π (for simplicity, still denoted by π), if necessary, so that $\mathrm{Mov}|\lfloor \pi^*(F_1) \rfloor|$ is base point free. Denote by F the generic irreducible element of $\mathrm{Mov}|\lfloor \pi^*(F_1) \rfloor|$. By assumption, F is a $(1, 2)$-surface. Since $|K_X|$ is not composed of a pencil and, in fact, $\varphi_{1,X}$ induces a genus 2 fibration (see [Ch07]), we see that the natural map

$$H^0(K_{X'}) \longrightarrow H^0(F, K_F)$$

is surjective for a general element F. In particular, $K_{X'} \geq F$ and $\pi^*(K_X)|_F \geq \mathrm{Mov}|K_F|$. Recall that we have $\rho(X) = 1$ by Dolgacgev [Dolg, 3.2.4]. Then we may write $K_X \equiv aF_1$ for some rational number $a \geq 1$. Since $r_X = 2$, we have $2(K_X^2 \cdot F_1) \in \mathbb{Z}_{>0}$. Hence $a = 1$ or 2.

First, we consider the case $a = 1$. We have $K_X \sim F_1$ and $(K_X^2 \cdot F_1) = 1$. In fact, we may take such a partial resolution $\hat{\pi} : \hat{X} \longrightarrow X$ that $\hat{\pi}$ is a composition of blow-ups along those centers over $\mathrm{Bs}(\Lambda)$ and that $\mathrm{Mov}(\hat{\pi}^*(\Lambda))$ is free of base points. By assumption, the generic irreducible element $\hat{F}$ in $\mathrm{Mov}(\hat{\pi}^*(\Lambda))$ is a nonsingular projective surface of $(1,2)$-type. Thus we may write

$$\hat{\pi}^*(F_1) = \hat{F} + E'_{\hat{\pi}},$$

$$K_{\hat{X}} = \hat{\pi}^*(K_X) + E_{\hat{\pi}},$$

where $\mathrm{Supp}(E'_{\hat{\pi}}) = \mathrm{Supp}(E_{\hat{\pi}})$ by the construction. Noting that $|\hat{F}|$ is a free pencil, we have $(\hat{\pi}^*(K_X)|_{\hat{F}})^2 = (K_X^2 \cdot F_1) = 1$. The uniqueness of Zariski decomposition implies that $\hat{\pi}^*(K_X)|_{\hat{F}}$ is the positive part of $K_{\hat{F}}$. Thus $(\hat{\pi}^*(K_X)|_{\hat{F}} \cdot E_{\hat{\pi}}|_{\hat{F}}) = 0$, which also means that

$$(K_X \cdot F_1^2) = (\hat{\pi}^*(K_X)|_{\hat{F}} \cdot E'_{\hat{\pi}}|_{\hat{F}}) = 0,$$

a contradiction.

We consider the case $a = 2$. Clearly we have $(K_X^2 \cdot F_1) = \frac{1}{2}$. On the other hand, we have $\pi^*(K_X)|_S \geq \frac{1}{2}\sigma^*(K_{S_0})$ by (2.4). Noting that $S|_F \equiv C \equiv \mathrm{Mov}|K_F|$, we have

$$\begin{aligned}(\pi^*(K_X)^2 \cdot F) &\geq (\pi^*(K_X)|_F \cdot S|_F)\\ &= (\pi^*(K_X)|_S \cdot F|_S) \geq \frac{1}{2}(\sigma^*(K_{S_0}) \cdot F|_S) \geq 1\end{aligned}$$

by [CC15, Lemma 2.4] since S is not a $(1,2)$-surface. This is also absurd.

Example 3.2. Consider the general complete intersection $X = X_{6,10} \subset \mathbb{P}(1,2,2,2,3,5)$, which has invariants: $K_X^3 = \frac{1}{2}$, $p_g(X) = 1$ and has 15 orbifold points of type $\frac{1}{2}(1,-1,1)$. We claim that X does not admit any pencil of $(1,2)$-surfaces. Assume, to the contrary, that X admits a pencil $|\overline{F}|$ of $(1,2)$-surfaces where $\overline{F}$ is irreducible. We aim at deducing a contradiction.

Since $\rho(X)=1$ (see [Dolg, 3.2.4]), we may write $K_X\equiv a\overline{F}$ for some positive rational number a. Noting that $(K_X^2\cdot\overline{F})\leq 1$ (since $\overline{F}$ is a $(1,2)$-surface), we have $a\geq\frac{1}{2}$.

First of all, let us fix the notation. Since $P_2(X)=4$ and the bicanonical map $\varphi_{2,X}$ gives a generically finite map, we set $|\overline{S}|=\mathrm{Mov}|2K_X|$. Take a birational modification $\mu:\tilde{X}\longrightarrow X$ such that the following properties hold:

(i) $\tilde{X}$ is nonsingular and projective;

(ii) both $\mathrm{Mov}|2K_{\tilde{X}}|$ and $\mathrm{Mov}|\lfloor\mu^*(\overline{F})\rfloor|$ are base point free.

Take general members $S\in\mathrm{Mov}|2K_{\tilde{X}}|$ and $F\in\mathrm{Mov}|\lfloor\mu^*(\overline{F})\rfloor|$. We may write

$$\mu^*(\overline{S})\sim_{\mathbb{Q}} S+E_2,$$

$$\mu^*(\overline{F})\sim_{\mathbb{Q}} F+E_1,$$

where E_1 and E_2 are effective $\mathbb{Q}$-divisors. By assumption we know that $p_g(S)\geq 3$, that $\Phi_{|S|}$ is generically finite and that F is a nonsingular $(1,2)$-surface.

Since $(K_X^2\cdot\overline{F})=(\pi^*(K_X)^2\cdot F)>0$ and $r_X(K_X^2\cdot\overline{F})\in\mathbb{Z}$ (by the intersection theory and the fact that X has isolated singularities), we see $(K_X^2\cdot\overline{F})=\frac{1}{2}$ or 1. In a word, either $a=\frac{1}{2}$ or $a=1$ is true.

If $a=\frac{1}{2}$, then $\overline{F}\equiv 2K_X$ and $(K_X^2\cdot\overline{F})=1$. Since $(\mu^*(K_X)|_F)^2=(K_X^2\cdot\overline{F})=1$ and $\mu^*(K_X)|_F\leq K_F$, the uniqueness of Zariski decomposition implies that $\mu^*(K_X)|_F\sim\sigma_0^*(K_{F_0})$ where $\sigma_0:F\to F_0$ is the contraction onto the minimal model. The similar argument to that in Example 3.1 (the case $a=1$) shows that $(K_X\cdot F_1^2)=0$, a contradiction.

If $a=1$, we have

$$2\geq(K_X\cdot\overline{S}^2)\geq(\mu^*(K_X)\cdot S^2)\geq(S|_F)^2\geq 2,$$

which implies $2K_X\equiv\overline{S}$ and $(\mu^*(K_X)\cdot S^2)=2$. By [CZ16, Lemma 2.4, Corollary 2.5], we have

$$\mu^*(K_X)|_F\geq\frac{1}{2}\sigma_0^*(K_{F_0}).$$

Hence it follows that

$$1\leq\frac{1}{2}(\sigma_0^*(K_{F_0})\cdot S|_F)\leq(\mu^*(K_X)\cdot F\cdot S)\leq\frac{1}{2}(\mu^*(K_X)\cdot S^2)=1,$$

which implies $(\mu^*(K_X)\cdot F\cdot S)=1$. Since, by the Hodge index theorem,

$$1=(\mu^*(K_X)|_F\cdot S|_F)\geq\sqrt{\mu^*(K_X)|_F^2\cdot S|_F^2}\geq 1,$$

one has $S|_F \equiv 2\mu^*(K_X)|_F$. Let $C \sim S|_F$ be a general curve. Since we have shown that $C^2 = 2$, C must be hyperelliptic and $C|_C$ gives a g_2^1 of C. Now we consider the linear system

$$|K_{X'} + \lfloor 4\mu^*(K_X)\rfloor| \succcurlyeq |\lfloor 5\mu^*(K_X)\rfloor|.$$

It is clear that, for a general member F of $|F|$,

$$|K_{X'} + \lfloor 4\mu^*(K_X)\rfloor||_F \preccurlyeq |K_F + 2C|.$$

Since $|K_F + 2C|$ does not give a birational map, neither do $|K_{X'} + \lfloor 4\mu^*(K_X)\rfloor||_F$, which contradicts to the fact that $\varphi_{5,X}$ is birational. The conclusion is that X does not admit any pencil of $(1,2)$-surfaces.

It might be interesting to know more such examples. However the difficulty is how to prove the non-existence of a pencil of $(1,2)$-surfaces on a 3-fold. For the case of $p_g = 4$, the reader may refer to [CZ16] for a complete characterization of the birationality of φ_4.

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