Communications in
Analysis and Geometry
Volume 29, Number 7, 1559–1570, 2021

Ricci-flat cubic graphs with girth five

David Cushing, Riikka Kangaslampi, Yong Lin, Shiping Liu,
Linyuan Lu, and Shing-Tung Yau

We classify all connected, simple, 3-regular graphs with girth at least 5 that are Ricci-flat. We use the definition of Ricci curvature on graphs given in Lin-Lu-Yau, Tohoku Math. J., 2011, which is a variation of Ollivier, J. Funct. Anal., 2009. A graph is Ricci-flat, if it has vanishing Ricci curvature on all edges. We show, that the only Ricci-flat cubic graphs with girth at least 5 are the Petersen graph, the Triplex and the dodecahedral graph. This will correct the classification in [8] that misses the Triplex.

1. Introduction

Ollivier developed a notion of Ricci curvature of Markov chains valid on metric spaces including graphs in [12]. The Ollivier-Ricci curvature κ_p depends on an idleness parameter, $p \in [0, 1]$. For graphs, the Ollivier-Ricci curvature has been studied mainly for idleness 0, see [1, 5, 6, 9, 14], but for example in [13] Ollivier and Villani considered idleness $\frac{1}{d+1}$, where d is the degree of a regular graph, in order to investigate the curvature of the hypercube. In [7], Lin, Lu, and Yau modified the definition of Ollivier-Ricci curvature to compute the negative of the derivative of the curvature with respect to the idleness p at $p = 1$, which they denote by κ. This modified Ollivier-Ricci curvature has been further investigated in, e.g., [10, 15].

Throughout this article, let $G = (V, E)$ be a connected simple (no loops and no multiedges) graph with vertex set V and edge set E. For any two adjacent vertices x and y, we write $x \sim y$, and denote by xy the edge between them. Let d_x denote the degree of the vertex $x \in V$. Let $d(x, y)$ denote the length of the shortest path between two vertices x and y.

We define the following probability distributions μ_x^p for any $x \in V$, $p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p, & \text{if } z = x, \\ \frac{1-p}{d_x}, & \text{if } z \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1. *Let $G = (V, E)$ be a locally finite graph. Let μ_1, μ_2 be two probability measures on V. The Wasserstein distance $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as*

$$W_1(\mu_1, \mu_2) = \inf_{\pi} \sum_{y \in V} \sum_{x \in V} d(x, y)\pi(x, y), \tag{1}$$

where the infimum runs over all transportation plans $\pi : V \times V \to [0, 1]$ satisfying

$$\mu_1(x) = \sum_{y \in V} \pi(x, y), \quad \mu_2(y) = \sum_{x \in V} \pi(x, y).$$

The transportation plan π moves a distribution given by μ_1 to a distribution given by μ_2, and $W_1(\mu_1, \mu_2)$ is a measure for the minimal effort which is required for such a transition. If π attains the infimum in (1) we call it an *optimal transport plan* transporting μ_1 to μ_2. For more details about the optimal transport theory, we refer to [16].

Definition 2. *The $p-$Ollivier-Ricci curvature on an edge xy in $G = (V, E)$ is*

$$\kappa_p(x, y) = 1 - W_1(\mu_x^p, \mu_y^p),$$

where p is called the idleness.

The Ollivier-Ricci curvature introduced by Lin-Lu-Yau in [7], is defined as

$$\kappa(x, y) = \lim_{p \to 1} \frac{\kappa_p(x, y)}{1 - p}.$$

We call a graph Ricci-flat if $\kappa(x, y) = 0$ on all edges $xy \in G$. Note that for regular graphs we can calculate the curvature $\kappa(x, y)$ for an edge xy as

$$\kappa(x, y) = \frac{d + 1}{d} \kappa_{\frac{1}{d+1}}(x, y),$$

as shown in [2] by Bourne, Cushing, Liu, Münch and Peyerimhoff. We show, that the only Ricci-flat cubic graphs with at least girth 5 are the Petersen graph, the Triplex and the dodecahedral graph (see Figure 1).

2. Classification of Ricci-flat cubic graphs with girth 5

In [8, Theorem 1] the authors classify Ricci-flat graphs with girth at least 5, but there is one 3-regular graph missing from their classification, namely

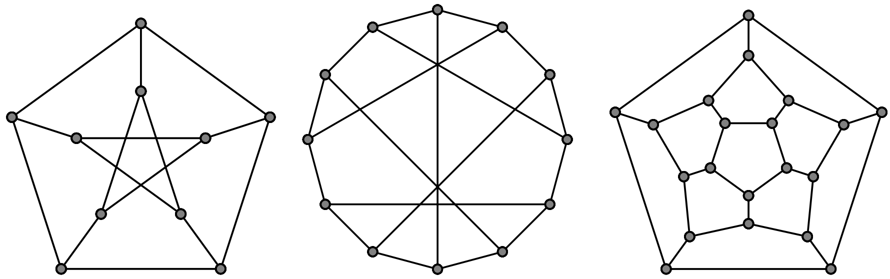


Figure 1. The three Ricci-flat cubic graphs with girth 5.

the Triplex (Figure 1b). With the following theorem we can complete the classification.

Theorem 1. *Let G be a 3-regular simple graph with girth $g(G) \geq 5$. If $\kappa(x, y) = 0$ on all edges $xy \in G$, then G is the Petersen graph, the Triplex, or the dodecahedral graph.*

Assuming that these graphs can be embedded to surfaces in such a way that they tile the surface with only pentagonal faces, the authors in [8] deduce that they only need to consider graphs with 10 or 20 vertices, obtaining the Petersen graph and the dodecahedral graph. However, the Triplex can be embedded to a torus with three pentagonal, two hexagonal and one 9-gonal face. A direct way to fix this problem in the proof of [8, Theorem 1] is given in [3].

Using Theorem 1 together with the results for non-cubic graphs from [8] we have the following classification:

Theorem 2. *Suppose that G is a Ricci-flat graph with girth $g(G) \geq 5$. Then G is one of the following graphs: the infinite path, cycle C_n with $n \geq 6$, the dodecahedral graph, the Triplex, the Petersen graph and the half-dodecahedral graph.*

Before proving Theorem 1, let us first consider the local structure of Ricci-flat cubic graphs. The following lemma shows that in order to have zero curvature on an edge xy, the edge must lie on two pentagons.

Lemma 1. *Let $G = (V, E)$ be a 3-regular graph of girth $g(G) \geq 5$. If $\kappa(x, y) = 0$, then the smallest cycle C_n supporting the edge xy has $n = 5$, and in addition xy belongs to two 5-cycles P_1 and P_2 such that $P_1 \cap P_2 = xy$.*

Proof. Denote the two other neighbours of x in addition to y by x_1 and x_2, and the neighbours of y by y_1 and y_2. We can assume that an optimal transport plan π only transports probability distribution from x_1 to y_1 and x_2 to y_2. Since there are no triangles or squares, $d(x_i, y_i) \geq 2$. The only possibility to have $\kappa(x, y) = 0$ i.e. $\kappa_{1/4}(x, y) = 0$ is if $d(x_i, y_i) = 2$, $i = 1, 2$. Since $d(x_1, y_1) = 2$, there exists a path $x_1 u y_1$, where the vertex u cannot be any of the vertices x_1, x_2, x, y, y_1 or y_2 without forming a triangle or a square. Similarly there exists another vertex v and a path $x_2 v y_2$. Thus we have two pentagons, $x x_1 u y_1 y x$ and $x x_2 v y_2 y x$, that intersect only on the edge xy, as in Figure 3.

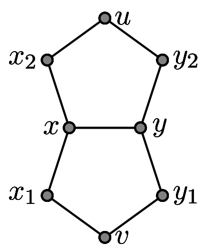


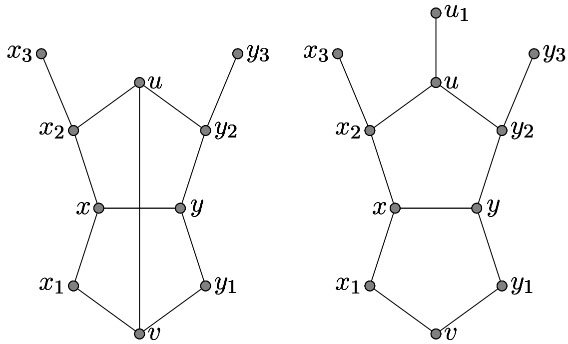
Figure 2. Two pentagons that intersect only on xy.

□

Remark. *Lemma 1 implies that Ricci-flat cubic graphs with girth at least 5 in fact have girth exactly 5.*

Let us now proceed to prove Theorem 1:

Proof of Theorem 1. Consider an edge xy in the graph. By lemma 1 we know that this edge is on two pentagons that intersect only on xy, as in Figure 2. We will now construct all possible 3-regular, girth 5 simple graphs with $\kappa = 0$ on all edges, starting from these pentagons. The vertices x_2 and y_2 cannot be adjacent to any other vertex in $\{x, y, x_1, y_1, u, v\}$ in order to have girth 5. Thus x_2 and y_2 are connected to two new vertices, x_3 and y_3, respectively. The vertex u is either connected to v or to a new vertex u_1, as in Figure 3. Let us consider these two cases separately.



(a) Case 1 (b) Case 2

Figure 3. Two possible graphs to start the construction.

Case 1:. Since the edge y_2y_3 has to lie on two pentagons, two of the following three possible ways for such pentagons to exist must be true (see Figure 4):

i) y_3 is adjacent to x_3,

ii) y_3 is adjacent to x_1 or

iii) there is a new vertex y_4, adjacent to both y_3 and y_1.

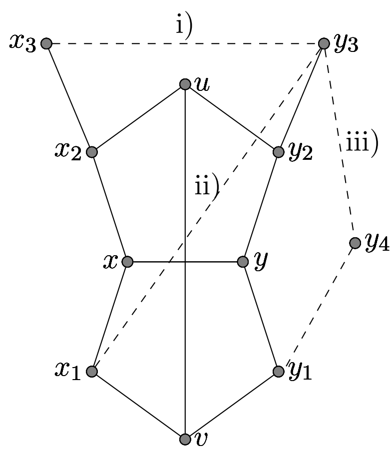


Figure 4. The three ways to construct a pentagon with y_2y_3 in Case 1.

The cases ii) and iii) cannot be true at the same time, since then it would not be possible to continue the construction in order to have pentagons with the edge x_2x_3: the only vertex with degree less than three, y_4, would be too far from x_3.

If i) and ii) are true, then the only vertices with degree less that 3 are x_3 and y_1, and $d(x_3, y_1) = 4$. Thus, the last edge from x_3 must go to y_1 for it to be on a pentagon. Now all edges lie on two pentagons, and we have constructed the Petersen graph with 10 vertices.

If i) and iii) are true, then there must be another vertex x_4 which is adjacent to x_1, since x_1 cannot be adjacent to either of the existing vertices with degree less than 3, that is, x_3 or y_4, without forming a square. Let us then construct pentagons with x_1x_4. There are two possibilities for the first pentagon: either x_4 is adjacent to x_3 or to y_4. Consider the first possibility. Then x_4 cannot be adjacent to y_4, since that would create a square $x_4x_3y_3y_4x_4$. So there must be yet another vertex x_5 to which x_4 is adjacent, and which is adjacent to y_4. But now all other vertices in the graph but x_5 have degree 3, and the construction cannot be continued. Consider then the second possibility, x_4 being adjacent to y_4. Similarly, now x_4 cannot be adjacent to x_3, we need a new vertex $x_5 \sim x_4$. Then the only other vertex with degree less than 3 is x_3, and $d(x_5, x_3) = 4$, so x_5 must be adjacent to x_3. But that leaves x_5 the only vertex with degree less than 3, and the construction cannot be continued. Thus, i) and iii) cannot be true at the same time, and the only graph with the edge uv is the Petersen graph.

Case 2:. Since the edge uu_1 must lie on two pentagons, one of them through x_2 and another through y_2, we have two isomorphically different possibilities:

a) there is one new vertex x_4 adjacent to u_1 and x_3 and u_1 is also adjacent to y_1 (Figure 5) or

b) there are two new vertices, x_4 adjacent to u_1 and x_3 and y_4 adjacent to u_1 and y_3 (Figure 6).

If u_1 were adjacent to both x_1 and y_1, that would create a square $u_1y_1vx_1u_1$. The case where u_1 would be adjacent to x_1 and to a vertex y_4 with $y_4 \sim y_3$ is isomorphic to the case a) above. Assume that a) is true. Then there are two isomorphically different possibilities to have a pentagon through x_3x_2x, illustrated in Figure 5:

i) Assume $x_4 \sim x_1$. Then in order to have two pentagons with y_2y_3, we must have $y_3 \sim x_3$ and $y_3 \sim v$, which gives us the Triplex.

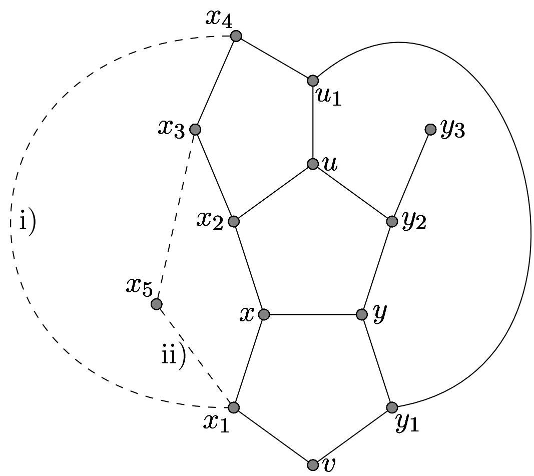


Figure 5. Case 2 a) with two non-isomorphic ways to continue.

ii) Assume that there is a new vertex x_5 adjacent to x_1 and x_3. Then there can exist two pentagons with y_2y_3 only if $y_3 \sim x_4$ and $y_3 \sim v$. But then the graph cannot be continued from x_5, since all other vertices already have degree 3.

Note that having $x_3 \sim v$ would also create a pentagon with x_3x_2x, but that is in fact isomorphic to i).

Assume then that b) is true. Then there are three possibilities to have a pentagon through x_2xx_1, illustrated in Figure 6 with labels i), ii) and iii). Symmetrically, there is three possibilities to have a pentagon with y_2yy_1, illustrated in Figure 6 with labels iv), v) and vi). Let us consider the possible non-isomorphic cases with pentagons on x_2xx_1 and y_2yy_1. There are four of them, since i) & v) is just a mirror image of ii) & iv), the four combinations i) & vi), its mirror iii) & iv), ii) & vi) and its mirror iii) & v) all give in fact the same graph, and the case ii) & v) would require $d(v) = 4$.

i) & iv) Assume that $x_4 \sim x_1$ and $y_4 \sim y_1$. Then in order to have a pentagon with $x_4u_1y_4$ the only possibility is that $x_3 \sim y_3$. But that leaves only the vertex v with degree less than three, and the construction cannot be continued.

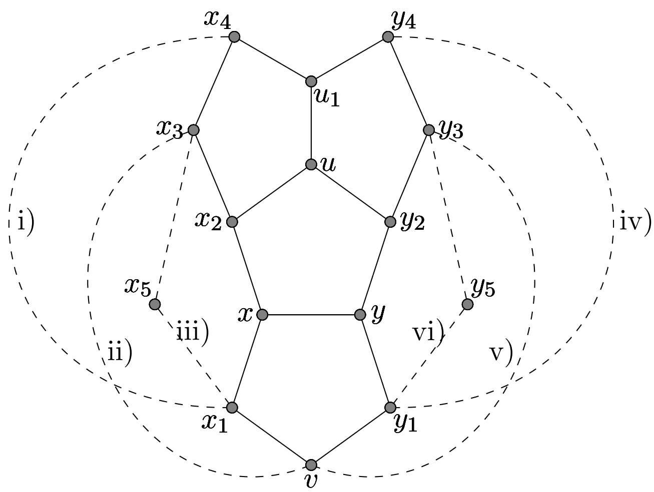


Figure 6. Case 2 b) with possible continuations.

i) & v) Assume that $x_4 \sim x_1$ and $y_3 \sim v$. Then to have a pentagon with x_4x_1v we must have $x_3 \sim y_1$. But that leaves only the vertex y_4 with degree less than three, and the construction cannot be continued.

i) & vi) Assume that $x_4 \sim x_1$ and that there exists a vertex y_5 such that $y_3 \sim y_5 \sim y_1$. Then to have a pentagon with $u_1x_4x_1$ we must have $y_4 \sim v$. But that leaves only the vertices x_3 and y_5 with degree less than three, and since now $d(x_3, y_5) = 5$, the construction cannot be continued.

iii) & vi) Assume that there exists a vertex x_5 such that $x_3 \sim x_5 \sim x_1$ and that a vertex y_5 such that $y_3 \sim y_5 \sim y_1$. Then there is two non-isomorphic ways to have a pentagon with vy_1y_5: either $x_5 \sim y_5$ or there exists two new verties v_1 and y_6 such that $v \sim v_1 \sim y_6 \sim y_5$. In the first case we must then have $v \sim y_4$ to have a pentagon with $y_4y_3y_5$. But then the only vertex with degree less than three is x_4, and the construction cannot be continued. In the latter case (Figure 7) there is two possibilities to have a pentagon with x_5x_1v: either $x_5 \sim y_6$, or there is a new vertex x_6 with $x_5 \sim x_6 \sim v_1$.

Assume first that $x_5 \sim y_6$. Then there are three vertices with degree less than three, v_1, x_4 and y_4. Since $d(v_1, x_4) = d(v_1, y_4) = 4$, v_1 must be adjacent to either x_4 or y_4. But that leaves only one vertex with

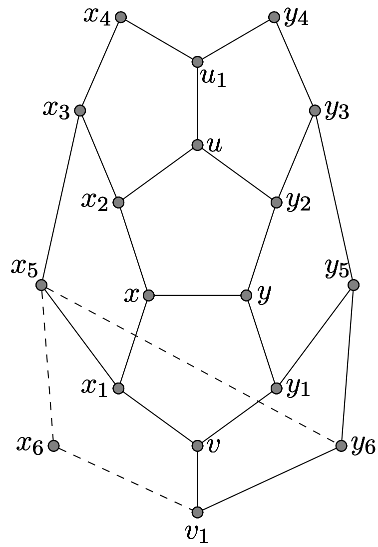


Figure 7. The graph for case 2 b) with iii), vi) and vertices v_1 and y_6 have two possible ways to continue.

degree less than three, and the construction cannot be continued. Thus we must have $x_5 \sim x_6 \sim v_1$. To have a pentagon with $x_4x_3x_5$ we need yet another vertex x_7 with $x_4 \sim x_7 \sim x_6$. Similarly for a pentagon with $y_4y_3y_5$ we need a vertex y_7 with $y_4 \sim y_7 \sim y_6$. Then there are two vertices with degree less than three in the graph, x_7 and y_7, and connecting them with an edge finalizes the construction by creating the dodecahedral graph.

We have now shown that the only 3-regular graphs with girth 5 and with every edge on two pentagons intersecting only at that edge are the Petersen graph, the Triplex and the dodecahedral graph. Therefore these are the only 3-regular Ricci-flat graphs with girth at least 5. □

We also calculated the curvatures of all cubic graphs with 20 vertices or less, and obtained the same three Ricci-flat graphs with girth 5. We used the graph generator package `nauty` by B. D. McKay and A. Piperno [11] to generate the graphs and the Graph Curvature Calculator [4] to calculate the curvatures.

References

[1] B. B. Bhattacharya and S. Mukherjee, *Exact and asymptotic results on coarse Ricci curvature of graphs*, Discrete Math. 338 (2015), no. 1, 23–42.

[2] D. Bourne, D. Cushing, S. Liu, F. Münch, and N. Peyerimhoff, *Ollivier-Ricci idleness functions of graphs*, SIAM J. Discrete Math. 32 (2018), no. 2, 1408–1424.

[3] D. Cushing, R. Kangaslampi, Y. Lin, S. Liu, L. Lu, and S.-T. Yau, *Erratum for Ricci-flat graphs with girth at least five*, Comm. Anal. Geom. 29 (2022), no. 8, 1775–1781.

[4] D. Cushing, R. Kangaslampi, V. Lipiäinen, S. Liu, and G. W. Stagg, *The Graph Curvature Calculator and the curvatures of cubic graphs*, Exp. Math. (2019), `https://doi.org/10.1080/10586458.2019.1660740`.

[5] S. Im, H. Kim, J. Maeng, J. Yu, Y. Cha and S.H. Paeng, *On the mean square displacement of a random walk on a graph*, European J. Combin. 51 (2016), 227–235.

[6] J. Jost and S. Liu, *Ollivier's Ricci curvature, local clustering and curvature-dimension inequalities on graphs*, Discrete. Comput. Geom. 51 (2014), no. 2, 300–322.

[7] Y. Lin, L. Lu, and S.-T. Yau, *Ricci curvature of graphs*, Tohoku Math. J. 63 (2011), 605–627.

[8] Y. Lin, L. Lu, and S.-T. Yau, *Ricci-flat graphs with girth at least five*, Comm. Anal. Geom. 22 (2014), no 4, 671–687.

[9] Y. Lin and S.-T. Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. 17 (2010), 343–356.

[10] B. Loisel and P. Romon, *Ricci curvature on polyhedral surfaces via optimal transportation.* Axioms 3 (2014), 119–139.

[11] B. D. McKay and A. Piperno, *Practical Graph Isomorphism, II*, J. Symbolic Computation 60 (2013), 94–112.

[12] Y. Ollivier, *Ricci curvature of Markov chains on metric spaces*, J. Funct. Anal. 256 (2009), 810–864.

[13] Y. Ollivier, C. Villani, *A curved Brunn-Minkowski inequality on the discrete hypercube, or: what is the Ricci curvature of the discrete hypercube?*, SIAM J. Discrete Math. 26 (2012), 983–996.

[14] S. H. Paeng, *Volume and diameter of a graph and Ollivier's Ricci curvature*, European J. Combin. 33 (2012), 1808–1819.

[15] J. D. H. Smith, *Ricci curvature, circulants, and a matching condition*, Discrete Math. 329 (2014), 88–98.

[16] C. Villani, *Topics in optimal transportation*, American Mathematical Society, 2003.

Department of Mathematical Sciences, Durham University
Durham DH1 3LE, UK
E-mail address: david.cushing@durham.ac.uk
Current address:
School of Mathematics, Statistics and Physics, Newcastle University
Newcastle upon Tyne NE1 7RU, UK
E-mail address: david.cushing1@newcastle.ac.uk

Computing sciences, Faculty of information technology and communication sciences, Tampere University, FI-33014 Tampere, Finland
E-mail address: riikka.kangaslampi@tuni.fi

Yau Mathematical Sciences Center and Department of Mathematics
Tsinghua University, Beijing 100084, China
E-mail address: yonglin@mail.tsinghua.edu.cn

School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026, China
E-mail address: spliu@ustc.edu.cn

Department of Mathematics, University of South Carolina
Columbia, SC 29208, USA
E-mail address: lu@math.sc.edu

Yau Mathematical Sciences Center, Tsinghua University
Beijing 100084, China
E-mail address: yau@math.harvard.edu

Received February 8, 2018
Accepted March 11, 2019