

Boundedness of anti-canonical volumes of singular log Fano threefolds

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We prove the Weak Borisov–Alexeev–Borisov Conjecture in dimension three which states that the anti-canonical volume of an ϵ -klt log Fano pair of dimension three is bounded from above.

1. Introduction

Throughout the article, we work over the field of complex numbers \mathbb{C} . We adopt the standard notations and definitions in [18] and [25], and will freely use them.

Definition 1.1. A pair (X, Δ) consists of a normal projective variety X and an effective \mathbb{Q} -divisor Δ on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. (X, Δ) is called a *log Fano pair* (resp. *weak log Fano pair*) if $-(K_X + \Delta)$ is ample (resp. nef and big). If $\dim X = 2$, we will use *del Pezzo* instead of Fano.

Definition 1.2. Let (X, Δ) be a pair. Let $f : Y \rightarrow X$ be a log resolution of (X, Δ) , write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i are distinct prime divisors. The coefficient a_i is called the *discrepancy* of F_i with respect to (X, Δ) , and denoted by $a_{F_i}(X, \Delta)$. For a real number $\epsilon \in [0, 1]$, the pair (X, Δ) is called

- (a) *ϵ -kawamata log terminal* (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i ;
- (b) *ϵ -log canonical* (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i ;
- (c) *terminal* if $a_i > 0$ for all f -exceptional divisors F_i .

Usually we write X instead of $(X, 0)$ in the case $\Delta = 0$.

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Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

Definition 1.3. A variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor Δ such that (X, Δ) is an ϵ -klt log Fano pair.

We are mainly interested in the boundedness of varieties of ϵ -Fano type.

Definition 1.4. A collection of varieties $\{X_\lambda\}_{\lambda \in \Lambda}$ is said to be *bounded* if there exists $h : \mathcal{X} \rightarrow S$ a projective morphism between schemes of finite type such that for each X_λ , $X_\lambda \simeq \mathcal{X}_s$ for some $s \in S$.

Our motivation is the following BAB Conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.5 (BAB Conjecture). *Fix $0 < \epsilon < 1$ and an integer $n > 0$. Then the set of all n -dimensional varieties of ϵ -Fano type is bounded.*

The BAB Conjecture is one of the most important conjectures in birational geometry and it is related to the termination of flips. The following weaker conjecture is an important step towards the proof of the BAB Conjecture.

Conjecture 1.6 (Weak BAB Conjecture). *Fix $0 < \epsilon < 1$ and an integer $n > 0$. Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property: if X is an n -dimensional variety of ϵ -Fano type, then*

$$\text{Vol}(-K_X) \leq M(n, \epsilon).$$

The BAB Conjecture was proved in dimension two by Alexeev [1] with a simplified argument by Alexeev and Mori [2]. In dimension three or higher, the BAB Conjecture is still open.¹ There are only some partial boundedness results. For example, we have boundedness of smooth Fano manifolds by Kollár, Miyaoka, and Mori [23], of terminal \mathbb{Q} -Fano \mathbb{Q} -factorial threefolds of Picard number one by Kawamata [17], of canonical \mathbb{Q} -Fano threefolds by Kollár, Miyaoka, Mori, and Takagi [24], and of toric varieties by Borisov and Borisov [9].

The Weak BAB Conjecture in dimension two was treated by Alexeev [1], Alexeev and Mori [2], and Lai [26]. Recently, the author [16] gave an

¹After this paper was posted on arXiv in 2014, Birkar [5, 6] gave a proof of the BAB Conjecture (along with the Weak BAB Conjecture and Ambro's conjecture) in arbitrary dimension in 2016 by different and much stronger methods.

optimal value for the number $M(2, \epsilon)$ (see also Corollary 4.5). For the Weak BAB Conjecture in dimension three assuming that the Picard number of X is one, an effective value of $M(3, \epsilon)$ was given by Lai [26]. For the general case in dimension three and higher, the Weak BAB Conjecture is still open.

As the main theorem of this paper, we prove the Weak BAB Conjecture in dimension three.

Theorem 1.7. *The Weak BAB Conjecture holds for $n = 3$.*

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2. Description of the proof

Firstly, we give an approach to the Weak BAB Conjecture via Mori fiber spaces.

Definition 2.1. A projective morphism $X \rightarrow T$ between normal varieties is called a *Mori fiber space* if the following conditions hold:

- (i) X is \mathbb{Q} -factorial with terminal singularities;
- (ii) f is a *contraction*, i.e., $f_*\mathcal{O}_X = \mathcal{O}_T$;
- (iii) $-K_X$ is ample over T ;
- (iv) $\rho(X/T) = 1$;
- (v) $\dim X > \dim T$.

In this case, we say that X is endowed with a *Mori fiber structure*.

We raise the following conjecture for Mori fiber spaces.

Conjecture 2.2 (Weak BAB Conjecture for Mori fiber spaces). *Fix $0 < \epsilon < 1$ and an integer $n > 0$. Then there exists a number $M(n, \epsilon)$ depending only on n and ϵ with the following property: if X is an n -dimensional*

variety of ϵ -Fano type with a Mori fiber structure, then

$$\mathrm{Vol}(-K_X) \leq M(n, \epsilon).$$

This is just a special case of the Weak BAB Conjecture. But we can prove the following theorem by using Minimal Model Program.

Theorem 2.3. *The Weak BAB Conjecture holds for fixed ϵ and n if and only if the Weak BAB Conjecture for Mori fiber spaces holds for fixed ϵ and n .*

By Theorem 2.3, to bound the anti-canonical volumes of varieties of Fano type, we only need to consider the ones with better singularities (\mathbb{Q} -factorial terminal singularities) and with additional structures (Mori fiber structures). This is the advantage of this theorem. In dimension two, this theorem appears as a crucial step to get the optimal value of $M(2, \epsilon)$ (c.f. [16]).

Restricting our interest to dimension three, we prove the following theorem.

Theorem 2.4. *The Weak BAB Conjecture for Mori fiber spaces holds for $n = 3$.*

Theorem 1.7 follows from Theorems 2.3 and 2.4 directly.

To prove Theorem 2.4, we need to consider 3-folds X of ϵ -Fano type with a Mori fiber structure $X \rightarrow T$. There are 3 cases:

- (1) $\dim T = 0$, X is a \mathbb{Q} -factorial terminal \mathbb{Q} -Fano 3-folds with $\rho = 1$;
- (2) $\dim T = 1$, $X \rightarrow T \simeq \mathbb{P}^1$ is a *del Pezzo fibration*, i.e. a general fiber of which is a smooth del Pezzo surface;
- (3) $\dim T = 2$, $X \rightarrow T$ is a *conic bundle*, i.e. a general fiber of which is a smooth rational curve.

The second statement is implied by the following fact: if (X, Δ) is a klt log Fano pair, then X is rationally connected (see [34, Theorem 1]), in particular, for any surjective morphism $X \rightarrow T$ to a normal curve, $T \simeq \mathbb{P}^1$.

In Case (1), X is bounded by Kawamata [17], and the optimal bound of $\mathrm{Vol}(-K_X) = (-K_X)^3$ is 64 due to the classification on smooth Fano 3-folds of Iskovskikh and Mori–Mukai and by Namikawa’s result [29] (Gorenstein case) and Prokhorov [30] (non-Gorenstein case).

We will mainly treat Cases (2) and (3).

One basic idea is to construct singular pairs which are not klt along fibers of $X \rightarrow T$. Then by Connectedness Lemma, we may find a non-klt center intersecting with the fibers. Then restricting on a general fiber, we get the bound after some arguments on lower dimensional varieties. But several difficulties arise here.

In Case (3), the difficulty arises in the construction of singular pairs because we need to avoid components which are vertical over T . To do this, we need a good understanding of the singularities and the boundedness of the surface T , which was done by several papers such as [1, 4].

In Case (2), the difficulty arises in the last step. After restricting to a general fiber, we need to bound the (unusual) log canonical thresholds on surfaces. So we are done by proving the (generalized) Ambro's conjecture in dimension two.

Definition 2.5. Let (X, B) be an lc pair and $D \geq 0$ a \mathbb{Q} -Cartier \mathbb{Q} -divisor. The *log canonical threshold* of D with respect to (X, B) is

$$\text{lct}(X, B; D) = \sup\{t \in \mathbb{Q} \mid (X, B + tD) \text{ is lc}\}.$$

For the purpose of this paper, we need to consider the case when D is not necessarily effective. Let G be a \mathbb{Q} -Cartier \mathbb{Q} -divisor satisfying $G + B \geq 0$, the *unusual log canonical threshold* of G with respect to (X, B) is

$$\text{ulct}(X, B; G) = \sup\{t \in [0, 1] \cap \mathbb{Q} \mid (X, B + tG) \text{ is lc}\}.$$

Note that the assumption $t \in [0, 1]$ guarantees that $B + tG \geq 0$.

Conjecture 2.6 (Ambro's conjecture). Fix $0 < \epsilon < 1$ and an integer $n > 0$. Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property: if (Y, B) is an ϵ -klt log Fano pair of dimension n , then

$$\inf\{\text{lct}(Y, B; D) \mid D \sim_{\mathbb{Q}} -(K_Y + B), D \geq 0\} \geq \mu(n, \epsilon).$$

Note that we do not assume any special conditions on the coefficients of B . The left-hand side of the inequality is called α -invariant of (Y, B) which generalizes the concept of α -invariant of Tian for Fano manifolds in differential geometry (see [10, 11, 33]). Recently Ambro [3] gave a proof of this conjecture assuming that (Y, B) is a toric pair where an explicit sharp number $\mu(n, \epsilon)$ was given. For the purpose of this paper, we need a stronger version of this conjecture where D may not be effective.

Conjecture 2.7 (generalized Ambro’s conjecture). *Fix $0 < \epsilon < 1$ and an integer $n > 0$. Then there exists a number $\mu(n, \epsilon) > 0$ depending only on n and ϵ with the following property: if (Y, B) is an ϵ -klt weak log Fano pair of dimension n and Y has at worst terminal singularities, then*

$$\inf\{\text{ulct}(Y, B; G) \mid G \sim_{\mathbb{Q}} -(K_Y + B), G + B \geq 0\} \geq \mu(n, \epsilon).$$

Note that Conjecture 2.6 follows from Conjecture 2.7 easily after taking a terminalization of (Y, B) .

We prove the conjecture in dimension two by following some ideas in the proof of the BAB Conjecture in dimension two ([1, 2]). But it seems that this conjecture does not follow from the BAB Conjecture trivially.

Theorem 2.8. *Conjecture 2.7 holds for $n = 2$.*

This paper is organized as follows. In Section 4, we prove the reduction step to Mori fiber spaces (Theorem 2.3). In Section 5, we prove the generalized Ambro’s conjecture in dimension two (Theorem 2.8). In Section 6, we prove the Weak BAB Conjecture for Mori fiber spaces in dimension three (Theorem 2.4).

3. Preliminaries

3.1. Volumes

Definition 3.1. Let X be an n -dimensional projective variety and D a Cartier divisor on X . The *volume* of D is the real number

$$\text{Vol}(D) = \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Note that the limsup is actually a limit (see, for example, [27, Example 11.4.7]). Moreover by the homogenous property of the volume, we can extend the definition to \mathbb{Q} -Cartier \mathbb{Q} -divisors. Note that if D is a nef \mathbb{Q} -divisor, then $\text{Vol}(D) = D^n$. If D is a non- \mathbb{Q} -Cartier \mathbb{Q} -divisors, we may take a \mathbb{Q} -factorialization of X , i.e., a birational morphism $\phi : Y \rightarrow X$ which is isomorphic in codimension one and Y is \mathbb{Q} -factorial, then $\text{Vol}(D) := \text{Vol}(\phi_*^{-1}D)$. Note that \mathbb{Q} -factorialization always exists for klt pairs (cf. [7, Theorem 1.4.3]).

For more background on volumes, see [27, 11.4.A].

3.2. Hirzebruch surfaces

We recall some basic properties of the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $n \geq 0$. Denote by h (resp. ℓ) the class in $\text{Pic } \mathbb{F}_n$ of the tautological bundle $\mathcal{O}_{\mathbb{F}_n}(1)$ (resp. of a fiber). Then $\text{Pic } \mathbb{F}_n = \mathbb{Z}h \oplus \mathbb{Z}\ell$ with $\ell^2 = 0$, $\ell \cdot h = 1$, $h^2 = n$. If $n > 0$, there is a unique irreducible curve $\sigma_n \subset \mathbb{F}_n$ such that $\sigma_n \sim h - n\ell$, $\sigma_n^2 = -n$. For $n = 0$, we can also choose one curve whose class in $\text{Pic } \mathbb{F}_0$ is h and denote it by σ_0 . Note that

$$-K_{\mathbb{F}_n} \sim 2h - (n - 2)\ell \sim 2\sigma_n + (n + 2)\ell.$$

Lemma 3.2. *For an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_{\mathbb{F}_n}$ and a fiber ℓ , $\text{mult}_{\ell}D \leq n + 2$.*

Proof. Since $D - (\text{mult}_{\ell}D)\ell$ is effective, $(D - (\text{mult}_{\ell}D)\ell) \cdot h \geq 0$. On the other hand, $(D - (\text{mult}_{\ell}D)\ell) \cdot h = n + 2 - \text{mult}_{\ell}D$. □

Lemma 3.3. *Let $T = \mathbb{P}^2$ or \mathbb{F}_n for some $n \geq 0$.*

- 1) *If $T = \mathbb{P}^2$, then for an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_T$ and a point Q , $\text{mult}_Q D \leq 3$ holds.*
- 2) *If $T = \mathbb{F}_n$, then for an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_T$ and a point Q , $\text{mult}_Q D \leq n + 4$ holds.*
- 3) *Moreover, if we write $D = \sum_j b_j D_j$ where D_j are distinct prime divisors and assume that $b_j \leq 1$ for all j , then $\sum_j b_j \leq 4$.*

Proof. If $T = \mathbb{P}^2$, taking a general line L through Q , we have

$$3 = (D \cdot L) \geq \text{mult}_Q(D).$$

If $T = \mathbb{F}_n$, take ℓ to be the fiber passing through Q , by Lemma 3.2 and intersection theory, we have

$$2 = D \cdot \ell \geq \text{mult}_Q D - \text{mult}_{\ell} D \geq \text{mult}_Q D - n - 2.$$

For the latter statement, if $T = \mathbb{F}_n$, then the conclusion follows from [2, Lemma 1.4]. If $T = \mathbb{P}^2$, then $\sum_j b_j \leq 3$ by degree computation. □

3.3. Non-klt centers and connectedness lemma

Definition 3.4. Let X be a normal projective variety and Δ a \mathbb{Q} -divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $f : Y \rightarrow X$ be a log resolution of

(X, Δ) , write

$$K_Y = f^*(K_X + \Delta) + \sum a_i F_i,$$

where F_i is a prime divisor. F_i is called a *non-klt place* of (X, Δ) if $a_i \leq -1$. A proper subvariety $V \subset X$ is called a *non-klt center* of (X, Δ) if it is the image of a non-klt place. The *non-klt locus* $\text{Nklt}(X, \Delta)$ is the union of all non-klt centers of (X, Δ) . A non-klt center is *maximal* if it is an irreducible component of $\text{Nklt}(X, \Delta)$.

The following lemma suggests a standard way to construct non-klt centers.

Lemma 3.5 (cf. [25, Lemma 2.29]). *Let (X, Δ) be a pair and $Z \subset X$ a closed subvariety of codimension k such that Z is not contained in the singular locus of X . If $\text{mult}_Z \Delta \geq k$, then Z is a non-klt center of (X, Δ) .*

Recall that the *multiplicity* $\text{mult}_Z F$ of a divisor F along a subvariety Z is defined by the multiplicity $\text{mult}_x F$ of F at a general point $x \in Z$.

Unfortunately, the converse of Lemma 3.5 is not true unless $k = 1$. Usually we do not have good estimates for the multiplicity along a non-klt center except the following lemma.

Lemma 3.6 (cf. [27, Theorem 9.5.13]). *Let (X, Δ) be a pair and $Z \subset X$ a non-klt center of (X, Δ) such that Z is not contained in the singular locus of X . Then $\text{mult}_Z \Delta \geq 1$.*

If we assume some simple normal crossing condition on the boundary, we can get more information on the multiplicity along a non-klt center. For simplicity, we just consider surfaces.

Lemma 3.7 (cf. [28, 4.1 Lemma]). *Fix $0 < e < 1$. Let S be a smooth surface, B an effective \mathbb{Q} -divisor, and D a (not necessarily effective) simple normal crossing supported \mathbb{Q} -divisor. Assume that the coefficients of D are at most e and $\text{mult}_P B \leq 1 - e$ for some point P , then for an arbitrary divisor E centered on P over S , $a_E(S, B + D) \geq -e$. In particular, if Z is a non-klt center of $(S, B + D)$ and the coefficients of D are at most e , then $\text{mult}_Z B > 1 - e$.*

Proof. By taking a sequence of point blow-ups, we can get the divisor E . Consider the blow-up at P , we have $f : S_1 \rightarrow S$ with

$$K_{S_1} + B_1 + D_1 + mE_1 = f^*(K_S + B + D)$$

where B_1 and D_1 are the strict transforms of B and D respectively, and E_1 is the exceptional divisor with

$$m = \text{mult}_P(B + D) - 1 \leq 1 - e + 2e - 1 = e.$$

Now $D_1 + mE_1$ is again simple normal crossing supported and $\text{mult}_Q B_1 \leq \text{mult}_P B$ for $Q \in E_1$. Hence, replacing (S, B, D, P) by $(S_1, B_1, D_1 + mE_1, Q)$, by induction on the minimal number of blow-ups, we conclude that the coefficient of E is at most e and hence $a_E(S, B + D) \geq -e$. \square

We have the following connectedness lemma of Kollár and Shokurov for non-klt locus (cf. Shokurov [32], Kollár [21, 17.4]).

Theorem 3.8 (Connectedness Lemma). *Let $f : X \rightarrow Z$ be a proper morphism of normal varieties with connected fibers and D is a \mathbb{Q} -divisor such that $-(K_X + D)$ is \mathbb{Q} -Cartier, f -nef, and f -big. Write $D = D^+ - D^-$ where D^+ and D^- are effective with no common components. If D^- is f -exceptional (i.e. all of its components have image of codimension at least 2), then $\text{Nklt}(X, D) \cap f^{-1}(z)$ is connected for any $z \in Z$.*

Remark 3.9. There are two main cases of interest in the Connectedness Lemma:

- (i) Z is a point and (X, D) is a weak log Fano pair. Then $\text{Nklt}(X, D)$ is connected.
- (ii) $f : X \rightarrow Z$ is birational, (Z, B) is a log pair and $K_X + D = f^*(K_Z + B)$.

4. Reduction to Mori fiber spaces

In this section, we prove the reduction step to Mori fiber spaces (Theorem 2.3). The “only if” direction is trivial, the “if” direction is a consequence of the following theorem.

Theorem 4.1. *Fix an integer $n > 0$ and $0 < \epsilon < 1$. Every n -dimensional variety X of ϵ -Fano type is birational to an n -dimensional variety X' of*

ϵ -Fano type with a Mori fibration structure such that

$$\mathrm{Vol}(-K_X) \leq \mathrm{Vol}(-K_{X'}).$$

We start with two lemmas. The first lemma is about equivalent definitions of ϵ -Fano type.

Lemma 4.2 (cf. [31, Lemma-Definition 2.6]). *Let Y be a projective normal variety, and $\epsilon \in [0, 1)$. The following are equivalent:*

- 1) Y is of ϵ -Fano type;
- 2) There exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$.

Proof. First we assume that Y is of ϵ -Fano type, that is, there exists an effective \mathbb{Q} -divisor B on Y such that (Y, B) is ϵ -klt log Fano pair. Then take a general effective ample \mathbb{Q} -divisor A on Y such that $(Y, B + A)$ is ϵ -klt and

$$K_Y + B + A \sim_{\mathbb{Q}} 0.$$

We may take $\Delta = A + B$.

Then we assume that there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$. Since Δ is big, we may write $\Delta = A + G$ where A is an ample \mathbb{Q} -divisor and G is an effective \mathbb{Q} -divisor. We may take a sufficiently small $\delta > 0$ such that $(Y, \Delta + \delta G)$ is again ϵ -klt. Hence $(Y, (1 - \delta)\Delta + \delta G)$ is ϵ -klt, and

$$-(K_Y + (1 - \delta)\Delta + \delta G) \equiv \delta A$$

is ample. Hence Y is of ϵ -Fano type. □

The minimal model program preserves ϵ -Fano type according to the following lemma.

Lemma 4.3 (cf. [13, Lemma 3.1]). *Let Y be a projective normal variety and $f : Y \rightarrow Z$ a projective birational morphism.*

- 1) If Y is of ϵ -Fano type, then so is Z ;
- 2) Assume that f is small, then Y is of ϵ -Fano type if and only if so is Z .

In particular, minimal model program preserves ϵ -Fano type.

Proof. First we assume that Y is of ϵ -Fano type, that is, by Lemma 4.2, there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (Y, Δ) is ϵ -klt, and $K_Y + \Delta \equiv 0$. Pushing forward by f , by negativity lemma,

$$K_Y + \Delta = f^*(K_Z + f_*\Delta) \equiv 0.$$

Hence $f_*\Delta$ is big, $(Z, f_*\Delta)$ is ϵ -klt, and $K_Z + f_*\Delta \equiv 0$, that is, Z is of ϵ -Fano type.

Next we assume that f is small and Z is of ϵ -Fano type. Let Γ be an effective big \mathbb{Q} -divisor on Z such that (Z, Γ) is ϵ -klt and $K_Z + \Gamma \equiv 0$. Let Δ be the strict transform of Γ on Y . Then Δ is big since f is small. Again by f is small,

$$K_Y + \Delta = f^*(K_Z + \Gamma).$$

Hence (Y, Δ) is ϵ -klt and $K_Y + \Delta \equiv 0$. Hence Y is of ϵ -Fano type. □

Proof of Theorem 4.1. Fix $0 < \epsilon < 1$ and an integer $n > 0$. Let X be a variety of ϵ -Fano type of dimension n , that is, by Lemma 4.2, there exists an effective \mathbb{Q} -divisor Δ such that Δ is big, (X, Δ) is ϵ -klt, and $K_X + \Delta \equiv 0$. By [7, Corollary 1.4.3], taking a \mathbb{Q} -factorialization of (X, Δ) , we have a birational morphism $\phi : X_0 \rightarrow X$ where $K_{X_0} + \phi_*^{-1}\Delta = \phi^*(K_X + \Delta)$, X_0 is \mathbb{Q} -factorial, and ϕ is isomorphic in codimension one.

Again by [7, Corollary 1.4.3], taking a terminalization of X_0 , we have a birational morphism $\pi : X_1 \rightarrow X_0$ where $K_{X_1} + \Delta_{X_1} = \pi^*(K_{X_0} + \phi_*^{-1}\Delta)$, $\Delta_{X_1} \geq \pi^*(\phi_*^{-1}\Delta)$ is an effective \mathbb{Q} -divisor, X_1 is \mathbb{Q} -factorial and terminal. Here $K_{X_1} + \Delta_{X_1} \equiv 0$ and (X_1, Δ_{X_1}) is ϵ -klt. Since Δ is big and ϕ is small, $\Delta_{X_1} \geq \pi^*(\phi_*^{-1}\Delta)$ is big. Therefore, X_1 is \mathbb{Q} -factorial terminal and of ϵ -Fano type.

Running the K -MMP on X_1 , we get a sequence of normal projective varieties:

$$X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_r \rightarrow T.$$

Since $-K_{X_1}$ is big, this sequence ends with a Mori fiber space $X_r \rightarrow T$ (cf. [7, Corollary 1.3.3]). Since we run the K -MMP, X_r is again \mathbb{Q} -factorial and terminal. By Lemma 4.3, for all i , X_i is of ϵ -Fano type. Now X_r is an n -dimensional variety of ϵ -Fano type with a Mori fiber structure by construction, which is birational to X .

The remaining thing is to compare the anti-canonical volumes. By definition,

$$\text{Vol}(-K_X) = \text{Vol}(-K_{X_0}).$$

Since X_1 is a terminalization of X_0 , we have $K_{X_1} + F = \pi^*K_{X_0}$ with F an effective \mathbb{Q} -divisor. Hence

$$\text{Vol}(-K_{X_0}) = \text{Vol}(-(K_{X_1} + F)) \leq \text{Vol}(-K_{X_1}).$$

By Lemma 4.4 below,

$$\text{Vol}(-K_{X_1}) \leq \text{Vol}(-K_{X_r}).$$

Hence we may take $X' = X_r$ and complete the proof of Theorem 4.1. \square

Lemma 4.4. *Let $X_i \dashrightarrow X_{i+1}$ be one step of the K -MMP. Then*

$$\text{Vol}(-K_{X_i}) \leq \text{Vol}(-K_{X_{i+1}}).$$

Proof. Take a common resolution $p : W \rightarrow X_i, q : W \rightarrow X_{i+1}$. Then

$$p^*(K_{X_i}) = q^*(K_{X_{i+1}}) + E,$$

where E is an effective q -exceptional \mathbb{Q} -divisor. Hence

$$\begin{aligned} \text{Vol}(-K_{X_i}) &= \text{Vol}(-p^*(K_{X_i})) \\ &= \text{Vol}(-q^*(K_{X_{i+1}}) - E) \\ &\leq \text{Vol}(-q^*(K_{X_{i+1}})) \\ &= \text{Vol}(-K_{X_{i+1}}). \end{aligned}$$

We proved the lemma. \square

As a direct corollary, we recover the main result in [16] on the Weak BAB Conjecture in dimension two.

Corollary 4.5. *Fix $0 < \epsilon < 1$. Then there exists a number*

$$M(2, \epsilon) := \max \left\{ 9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor} \right\}$$

with the following property: if X is a surface of ϵ -klt del Pezzo type, then

$$\text{Vol}(-K_X) \leq M(2, \epsilon).$$

Proof. By Theorem 4.1, we only need to consider the cases when $X = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$ (see [2, Lemma 1.4] or [16, Lemma 3.1]). And the result follows from an easy computation of volumes. \square

5. Generalized Ambro’s conjecture in dimension two

In this section, we prove the generalized Ambro’s conjecture in dimension two (Theorem 2.8).

Fix an ϵ -klt weak log del Pezzo pair (S, B) with S smooth and a \mathbb{Q} -divisor $G \sim_{\mathbb{Q}} -(K_S + B)$ such that $G + B \geq 0$. Set $a := \text{ulct}(S, B; G)$. Since we work with \mathbb{Q} -divisors, a is a positive rational number. The problem is to bound a from below. We may assume that $a < 1$. Set $D = G + B \geq 0$. Then $(S, B + aG) = (S, (1 - a)B + aD)$ is not klt. Note that $D \sim_{\mathbb{Q}} -K_S$.

By the Base Point Free Theorem (cf. [25, Theorem 3.3]), $-(K_S + B)$ is semi-ample. Hence there exists an effective \mathbb{Q} -divisor M such that $K_S + B + M \sim_{\mathbb{Q}} 0$ and $(S, B + M)$ is ϵ -klt. For any birational morphism $f : S \rightarrow T$ between smooth surfaces, we have

$$\begin{aligned} K_S + B + M &= f^*(K_T + f_*B + f_*M), \\ K_S + (1 - a)(B + M) + aD &= f^*(K_T + (1 - a)(f_*B + f_*M) + af_*D). \end{aligned}$$

Hence $(T, f_*B + f_*M)$ is ϵ -klt and $(T, (1 - a)(f_*B + f_*M) + af_*D)$ is not klt with

$$K_T + f_*B + f_*M \sim_{\mathbb{Q}} K_T + (1 - a)(f_*B + f_*M) + af_*D \sim_{\mathbb{Q}} 0.$$

Recall that either $S \simeq \mathbb{P}^2$ or there exists a birational morphism $g : S \rightarrow \mathbb{F}_n$ with $n \leq 2/\epsilon$ by [2, Lemma 1.4] or [16, Lemma 3.1].

Hence by replacing S by $T = \mathbb{P}^2$ or \mathbb{F}_n , we may assume that there exists a triple (T, B_T, D_T) satisfying the following conditions:

- (i) $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$;
- (ii) B_T, D_T are effective \mathbb{Q} -divisors on T ;
- (iii) (T, B_T) is ϵ -klt and $(T, (1 - a)B_T + aD_T)$ is not klt;
- (iv) $K_T + B_T \sim_{\mathbb{Q}} K_T + (1 - a)B_T + aD_T \sim_{\mathbb{Q}} 0$, equivalently, $B_T \sim_{\mathbb{Q}} D_T \sim_{\mathbb{Q}} -K_T$.

Since $(T, (1 - a)B_T + aD_T)$ is not klt, we may take a sequence of point blow-ups

$$T_{r+1} \rightarrow T_r \rightarrow \cdots \rightarrow T_2 \rightarrow T_1 = T$$

where $T_{i+1} \rightarrow T_i$ is the blow-up at a non-klt center $P_i \in \text{Nklt}(T_i, (1 - a)B_i + aD_i + E_i)$ where B_i and D_i are the strict transforms of B_T and D_T respectively and

$$K_{T_i} + (1 - a)B_i + aD_i + E_i = \pi_i^*(K_T + (1 - a)B_T + aD_T),$$

where $\pi_i : T_i \rightarrow T$ is the composition map and E_i is a π_i -exceptional \mathbb{Q} -divisor. We stop this process at T_{r+1} if

$$\dim \text{Nklt}(T_{r+1}, (1 - a)B_{r+1} + aD_{r+1} + E_{r+1}) > 0.$$

Since P_i is a non-klt center of $(T_i, (1 - a)B_i + aD_i + E_i)$, $\text{mult}_{P_i}((1 - a)B_i + aD_i + E_i) \geq 1$. Note that the coefficients of E_i are $(\text{mult}_{P_j}((1 - a)B_j + aD_j + E_j) - 1)$ for $j < i$, hence E_i is effective for all i . Furthermore, we may assume that $\text{mult}_{P_i}B_i$ is non-increasing. Take the integer $k \leq r$ such that $\text{mult}_{P_i}B_i \geq \epsilon/2$ for $i \leq k$ and $\text{mult}_{P_i}B_i < \epsilon/2$ for $i > k$. Write $B_T = \sum_j b_j B^j$ where B^j are distinct prime divisors and $B_i = \sum_j b_j B_i^j$ where B_i^j are the strict transforms of B^j . We have $b_j < 1 - \epsilon$ since (T, B_T) is ϵ -klt. Recall that $\sum_j b_j \leq 4$ by Lemma 3.3.

Claim 1. *If $\text{mult}_{B^j}(aD_T) > \epsilon/2$ for some j , then $a \geq \epsilon^2/(4 + 4\epsilon)$.*

Proof. Recall that $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$.

If $T = \mathbb{P}^2$, then $\text{mult}_{B^j}D_T \leq 3$ by degree counting. If $T = \mathbb{F}_n$ and B^j is a fiber, then $\text{mult}_{B^j}D_T \leq n + 2 \leq 2/\epsilon + 2$ by Lemma 3.2. If $T = \mathbb{F}_n$ and B^j is not a fiber, then $\text{mult}_{B^j}D_T \leq D_T \cdot \ell = 2$ where ℓ is a fiber. Hence

$$a \geq \frac{\epsilon}{2\text{mult}_{B^j}D_T} \geq \frac{\epsilon^2}{4 + 4\epsilon}.$$

We proved the claim. □

Since we need a lower bound of a , from now on, by Claim 1, we may assume that $\text{mult}_{B^j}(aD_T) \leq \epsilon/2$ for all j . In particular, $\text{mult}_{B_i^j}(aD_i) \leq \epsilon/2$ and

$$\text{mult}_{B_i^j}((1 - a)B_i + aD_i) < 1 - \epsilon/2$$

for all i and j .

Claim 2. $(B_{k+1}^j)^2 \geq -4/\epsilon$ for all j .

Proof. If $(B_{k+1}^j)^2 < 0$, then

$$\begin{aligned} -2 &\leq 2p_a(B_{k+1}^j) - 2 = (K_{T_{k+1}} + B_{k+1}^j) \cdot B_{k+1}^j \\ &= \frac{\epsilon}{2}(B_{k+1}^j)^2 + \left(K_{T_{k+1}} + \left(1 - \frac{\epsilon}{2}\right)B_{k+1}^j\right) \cdot B_{k+1}^j \\ &\leq \frac{\epsilon}{2}(B_{k+1}^j)^2 + (K_{T_{k+1}} + (1 - a)B_{k+1} + aD_{k+1} + E_{k+1}) \cdot B_{k+1}^j \\ &= \frac{\epsilon}{2}(B_{k+1}^j)^2 < 0, \end{aligned}$$

Hence we proved the claim. □

Now we can bound the number k . On T_{k+1} , we have

$$\begin{aligned} (B_{k+1})^2 &= \left(\sum_j b_j B_{k+1}^j\right)^2 \geq \sum_j b_j^2 (B_{k+1}^j)^2 \geq \left(\sum_j b_j^2\right)(-4/\epsilon) \\ &\geq \left(\sum_j b_j\right)(1 - \epsilon)(-4/\epsilon) \geq 16 - \frac{16}{\epsilon} \end{aligned}$$

and $(B_1)^2 = (K_T)^2 \leq 9$. On the other hand, after each blow-up, $(B_i)^2$ decreases by at least $\epsilon^2/4$ by the assumption $\text{mult}_{P_i} B_i \geq \epsilon/2$ for $i \leq k$. Hence

$$k \leq \frac{9 - (16 - 16/\epsilon)}{\epsilon^2/4} \leq \frac{64}{\epsilon^3}.$$

Now we consider $\pi_{k+1}^*(aD_T)$ on T_{k+1} .

Claim 3. *There exists a point Q on T_{k+1} such that $\text{mult}_Q \pi_{k+1}^*(aD_T) \geq \epsilon/4$.*

Proof. Consider the pair $(T_{k+1}, (1 - a)B_{k+1} + aD_{k+1} + E_{k+1})$. Note that E_{k+1} is simple normal crossing supported.

Suppose that there exists a curve E with coefficient at least $1 - 3\epsilon/4$ in E_{k+1} , that is,

$$\text{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1 - a)B_T + aD_T)) \leq -1 + 3\epsilon/4.$$

On the other hand, since $(T, (1 - a)B_T)$ is ϵ -klt,

$$\text{mult}_E(K_{T_{k+1}} - \pi_{k+1}^*(K_T + (1 - a)B_T)) > -1 + \epsilon.$$

Hence $\text{mult}_E \pi_{k+1}^*(aD_T) \geq \epsilon/4$.

Then we may assume that all coefficients of E_{k+1} are smaller than $1 - 3\epsilon/4$, then $k < r$ and P_{k+1} is a non-klt center of $(T_{k+1}, (1 - a)B_{k+1} +$

$aD_{k+1} + E_{k+1}$). By Lemma 3.7, $\text{mult}_{P_{k+1}}((1-a)B_{k+1} + aD_{k+1}) \geq 3\epsilon/4$. Then $\text{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$ since $\text{mult}_{P_{k+1}}B_{k+1} < \epsilon/2$ by definition of k . In particular, $\text{mult}_{P_{k+1}}\pi_{k+1}^*(aD_T) \geq \text{mult}_{P_{k+1}}(aD_{k+1}) \geq \epsilon/4$.

We proved the claim. □

Now we will estimate $\text{mult}_{Q_1}(aD_T)$ where Q_1 is the image of Q on T . By removing unnecessary blow-ups, we may assume that we have a sequence of blow-ups

$$T_{k+1} \rightarrow T_k \rightarrow \cdots \rightarrow T_2 \rightarrow T_1 = T$$

where $f_{i+1} : T_{i+1} \rightarrow T_i$ is the blow-up at Q_i which is the image of Q on T_i with $k \leq 64/\epsilon^3$. Recall that $\pi_i : T_i \rightarrow T$ is the composition map and D_i is the strict transform of D_T on T_i . Denote C_{i+1} to be the exceptional divisor of f_{i+1} and C_{i+1}^j its strict transform on T_j for $j \geq i + 1$. We can write

$$\pi_j^*(aD_T) = aD_j + \sum_{2 \leq i \leq j} c_i C_i^j,$$

with $c_i = \text{mult}_{Q_{i-1}}\pi_{i-1}^*(aD_T)$.

Claim 4. *If $\text{mult}_{Q_1}(aD_T) \leq \alpha$, then $\text{mult}_{Q_i}\pi_i^*(aD_T) \leq (F_{i+1} - 1)\alpha$ for $1 \leq i \leq k + 1$. Here F_n is the Fibonacci number with relation $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$ and $F_0 = F_1 = 1$.*

Proof. We run induction on i . The case $i = 1$ is trivial. Assume the conclusion holds for $i < j$, then noting that Q_j is contained in at most two exceptional curves, we have

$$\begin{aligned} \text{mult}_{Q_j}\pi_j^*(aD_T) &= \text{mult}_{Q_j}(aD_j + \sum_{2 \leq i \leq j} c_i C_i^j) \\ &\leq \text{mult}_{Q_j}(aD_j) + (F_j - 1)\alpha + (F_{j-1} - 1)\alpha \\ &\leq \text{mult}_{Q_1}(aD_1) + (F_j - 1)\alpha + (F_{j-1} - 1)\alpha \\ &\leq (F_{j+1} - 1)\alpha. \end{aligned}$$

We proved the claim. □

By Claims 3 and 4, $\text{mult}_{Q_1}(aD_T) \geq \epsilon/(4F_{k+2} - 4)$. Recall that $D_T \sim_{\mathbb{Q}} -K_T$ and $T = \mathbb{P}^2$ or \mathbb{F}_n with $n \leq 2/\epsilon$. By Lemma 3.3, $\text{mult}_{Q_1}(D_T) \leq n + 4$, combining with the inequality $k \leq 64/\epsilon^3$, we have

$$a \geq \frac{\epsilon^2}{(2 + 4\epsilon)(4F_{\lfloor 64/\epsilon^3 \rfloor + 2} - 4)},$$

and hence we may take this number to be $\mu(2, \epsilon)$.

We have proved Theorem 2.8.

6. Weak BAB Conjecture for Mori fiber spaces in dimension three

In this section, we prove the Weak BAB Conjecture for Mori fiber spaces in dimension 3 (Theorem 2.4). Recall that by a Mori fiber space we always mean a \mathbb{Q} -factorial terminal one.

Fix $0 < \epsilon < 1$ and consider an ϵ -klt log Fano pair (X, Δ) of dimension 3 with a Mori fiber structure. As explained, there are three cases:

- (1) X is a \mathbb{Q} -factorial terminal \mathbb{Q} -Fano 3-folds with $\rho = 1$;
- (2) $X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration;
- (3) $X \rightarrow S$ is a conic bundle.

As mentioned before, Case (1) is done by Kawamata [17]. We treat Cases (2) and (3) in the following two subsections, see Corollary 6.3 and Theorem 6.6.

6.1. Contractions to a curve

In this subsection, we treat the case under a more general setting when there is a contraction $f : X \rightarrow \mathbb{P}^1$ (e.g. X has a del Pezzo fibration structure).

Theorem 6.1. *Let (Y, B) be an ϵ -klt log Fano pair of dimension n with a contraction $g : Y \rightarrow \mathbb{P}^1$ and Y having terminal singularities. Assume that the Weak BAB Conjecture and the generalized Ambro's conjecture hold in dimension $n - 1$ with $M(n - 1, \epsilon)$ and $\mu(n - 1, \epsilon)$ the numbers defined in these conjectures. Then*

$$\text{Vol}(-K_Y) \leq \frac{2nM(n - 1, \epsilon)}{\mu(n - 1, \epsilon)}.$$

Proof. Note that Y is terminal by assumption. Hence a general fiber F of g is terminal and of ϵ -Fano type of dimension $n - 1$ by adjunction formula. It follows that $\text{Vol}(-K_F) \leq M(n - 1, \epsilon)$ by the Weak BAB Conjecture in dimension $n - 1$.

By contrary, assume that

$$\text{Vol}(-K_Y) > \frac{2nM(n - 1, \epsilon)}{\mu(n - 1, \epsilon)}.$$

Take a rational number s satisfying

$$\text{Vol}(-K_Y) > s \cdot nM(n - 1, \epsilon) > \frac{2nM(n - 1, \epsilon)}{\mu(n - 1, \epsilon)}.$$

Here we note that $s > 2$ since $\mu(n - 1, \epsilon) < 1$ (by considering, for example, $(\mathbb{P}^{n-1}, 0)$ in Ambro’s conjecture).

The following lemma allows us to construct non-klt centers.

Lemma 6.2. *For a general fiber F of g , $-K_Y - sF$ is \mathbb{Q} -effective.*

Proof. For a positive integer p and a sufficiently divisible positive integer m , we have an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_Y(-mK_Y - pF) &\rightarrow \mathcal{O}_Y(-mK_Y - (p - 1)F) \\ &\rightarrow \mathcal{O}_F(-mK_Y - (p - 1)F) \rightarrow 0. \end{aligned}$$

Note that $\mathcal{O}_F(-mK_Y - (p - 1)F) = \mathcal{O}_F(-mK_F)$. Hence

$$\begin{aligned} &h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \\ &\geq h^0(Y, \mathcal{O}_Y(-mK_Y - (p - 1)F)) - h^0(F, \mathcal{O}_F(-mK_F)). \end{aligned}$$

Inductively, we have

$$\begin{aligned} &h^0(Y, \mathcal{O}_Y(-mK_Y - pF)) \\ &\geq h^0(Y, \mathcal{O}_Y(-mK_Y)) - p \cdot h^0(F, \mathcal{O}_F(-mK_F)). \end{aligned}$$

Now we may take the integer $p = sm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{n!}{m^n} \left(h^0(Y, \mathcal{O}_Y(-mK_Y)) - sm \cdot h^0(F, \mathcal{O}_F(-mK_F)) \right) \\ &= \text{Vol}(-K_Y) - sn \text{Vol}(-K_F) \\ &\geq \text{Vol}(-K_Y) - snM(n - 1, \epsilon) > 0. \end{aligned}$$

Hence $h^0(Y, \mathcal{O}_Y(-mK_Y - smF)) > 0$ for m sufficiently divisible, that is, $-K_Y - sF$ is \mathbb{Q} -effective. □

By Lemma 6.2, there is an effective \mathbb{Q} -divisor $B' \sim_{\mathbb{Q}} -\frac{1}{s}K_Y - F$. Now for two general fibers F_1 and F_2 of g , consider the pair

$$\left(Y, \left(1 - \frac{2}{s}\right)B + 2B' + F_1 + F_2\right).$$

By construction, $F_1 \cup F_2 \subset \text{Nklt}(Y, (1 - \frac{2}{s})B + 2B' + F_1 + F_2)$. Note that

$$-\left(K_Y + \left(1 - \frac{2}{s}\right)B + 2B' + F_1 + F_2\right) \sim_{\mathbb{Q}} -\left(1 - \frac{2}{s}\right)(K_Y + B)$$

is ample, since $s > 2$. By Connectedness Lemma, $\text{Nklt}(Y, (1 - \frac{2}{s})B + 2B' + F_1 + F_2)$ is connected. Hence there is a non-klt center W connecting F_1 and F_2 . In particular, W dominates \mathbb{P}^1 . Restricting on a general fiber F , by adjunction, we have $(F, B|_F)$ is ϵ -klt log Fano with F terminal and $(F, (1 - \frac{2}{s})B|_F + 2B'|_F)$ is not klt (see [25, Lemma 5.17, Lemma 5.50]) with $B'|_F \sim_{\mathbb{Q}} -\frac{1}{s}K_F$. Hence

$$\frac{2}{s} \geq \text{ulct}(F, B|_F; sB'|_F - B|_F).$$

The generalized Ambro’s conjecture comes in to play when trying to bound s from above. By the generalized Ambro’s conjecture in dimension $n - 1$,

$$s \leq \frac{2}{\mu(n - 1, \epsilon)},$$

which contradicts the definition of s . □

In particular, by Corollary 4.5 and Theorem 2.8, the Weak BAB Conjecture and the generalized Ambro’s conjecture hold in dimension 2, and hence the following corollary holds.

Corollary 6.3. *Let X be a 3-fold of ϵ -Fano type with a contraction $f : X \rightarrow \mathbb{P}^1$ and X having terminal singularities. Then*

$$\text{Vol}(-K_X) \leq \frac{6M(2, \epsilon)}{\mu(2, \epsilon)}.$$

6.2. Conic bundles

In this subsection, we treat the case that X has a conic bundle structure $f : X \rightarrow S$. Firstly we collect some facts about singularities of the surface S .

Theorem 6.4. *Let (X, Δ) be an ϵ -klt log Fano pair of dimension 3 and $f : X \rightarrow S$ a Mori fiber space to a surface S , then*

- (i) *There exists an effective \mathbb{Q} -divisor Δ_S on S , such that (S, Δ_S) is klt log del Pezzo;*
- (ii) *There exists an effective \mathbb{Q} -divisor Δ_S on S , such that (S, Δ_S) is $\delta(\epsilon)$ -klt and $K_S + \Delta_S \sim_{\mathbb{Q}} 0$, where $\delta(\epsilon)$ is a positive real number depending only on ϵ ;*
- (iii) *The family of such S is bounded.*
- (iv) *There exists a positive integer $d(\epsilon)$ depending only on ϵ , such that on S there is a very ample divisor H satisfying $H^2 \leq d(\epsilon)$.*

Proof. (i) is by [12, Corollary 3.3]. (ii) is by [4, Corollary 1.7] since we may find a boundary $\Delta' \geq \Delta$ such that (X, Δ') is ϵ -klt and $K_X + \Delta' \sim_{\mathbb{Q}} 0$. (iii) is by (ii) and [1, Theorem 6.8]. (iv) is a direct consequence of (iii). \square

If taking H generally, we may assume that $G := f^{-1}(H)$ and H are smooth. Note that $f|_G : G \rightarrow H$ is a conic bundle from a smooth surface to a smooth curve. Note that for a general fiber F of f (or $f|_G$), $G|_G \sim_{\mathbb{Q}} dF$, where $d := H^2 \leq d(\epsilon)$.

Lemma 6.5. $\text{Vol}(-K_X|_G) \leq \frac{8(d(\epsilon)+2)}{\epsilon}$.

Proof. Assume by contrary that $\text{Vol}(-K_X|_G) > \frac{8(d(\epsilon)+2)}{\epsilon} \geq \frac{8(d+2)}{\epsilon}$. Choose a rational number r such that

$$\text{Vol}(-K_X|_G) > 4r > \frac{8(d+2)}{\epsilon}.$$

For a positive integer p and a sufficiently divisible positive integer m , for a general fiber F of $f|_G$, we have an exact sequence

$$0 \rightarrow \mathcal{O}_G(-mK_X|_G - pF) \rightarrow \mathcal{O}_G(-mK_X|_G - (p-1)F) \rightarrow \mathcal{O}_F(-mK_F) \rightarrow 0.$$

Hence

$$\begin{aligned} & h^0(G, \mathcal{O}_G(-mK_X|_G - pF)) \\ & \geq h^0(G, \mathcal{O}_G(-mK_X|_G - (p-1)F)) - h^0(F, -mK_F). \end{aligned}$$

Inductively, we have

$$h^0(G, \mathcal{O}_G(-mK_X|_G - pF)) \geq h^0(G, \mathcal{O}_G(-mK_X|_G)) - p \cdot h^0(F, -mK_F).$$

Now we may take the integer $p = rm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{2}{m^2} \left(h^0(G, \mathcal{O}_G(-mK_X|_G)) - rm \cdot h^0(F, -mK_F) \right) \\ &= \text{Vol}(-K_X|_G) - 2r \text{Vol}(-K_F) \\ &= \text{Vol}(-K_X|_G) - 4r > 0. \end{aligned}$$

Here note that $F \simeq \mathbb{P}^1$. Hence $h^0(G, \mathcal{O}_G(-mK_X|_G - rmF)) > 0$ for m sufficiently divisible, that is, $-K_X|_G - rF$ is \mathbb{Q} -effective. Take an effective \mathbb{Q} -divisor $D \sim_{\mathbb{Q}} -K_X|_G - rF$ on G . For two general fibers F_1 and F_2 of $f|_G$, consider the pair

$$\left(G, \left(1 - \frac{d+2}{r} \right) \Delta|_G + \frac{d+2}{r} D + F_1 + F_2 \right).$$

It is log Fano since

$$\begin{aligned} & - \left(K_G + \left(1 - \frac{d+2}{r} \right) \Delta|_G + \frac{d+2}{r} D + F_1 + F_2 \right) \\ & \sim_{\mathbb{Q}} - \left(1 - \frac{d+2}{r} \right) (K_X + \Delta)|_G \end{aligned}$$

is ample, where $r > d + 2$ by definition. Hence by Connectedness Lemma,

$$\text{Nklt} \left(G, \left(1 - \frac{d+2}{r} \right) \Delta|_G + \frac{d+2}{r} D + F_1 + F_2 \right)$$

is connected. Since it contains F_1 and F_2 , there exists a non-klt center connecting F_1 and F_2 , and hence dominating H . Restricting on a general fiber F of $f|_G : G \rightarrow H$, by adjunction,

$$\left(F, \left(1 - \frac{d+2}{r} \right) \Delta|_F + \frac{d+2}{r} D|_F \right)$$

is not klt. On the other hand, $(F, (1 - \frac{d+2}{r})\Delta|_F)$ is ϵ -klt since (X, Δ) is ϵ -klt. Hence

$$\text{deg} \left(\frac{d+2}{r} D|_F \right) \geq \epsilon.$$

Note that

$$\text{deg}(D|_F) = \text{deg}(-K_X|_F) = 2,$$

this implies that

$$\frac{2(d+2)}{r} \geq \epsilon,$$

which contradicts to the definition of r . □

Now we can prove the theorem of this subsection.

Theorem 6.6. *Let X be a 3-fold of ϵ -Fano type with a Mori fiber structure $f : X \rightarrow S$ to a surface. Then*

$$\text{Vol}(-K_X) \leq \frac{144(d(\epsilon) + 2)}{\epsilon^2}.$$

Proof. Assume to the contrary that $\text{Vol}(-K_X) > \frac{144(d(\epsilon)+2)}{\epsilon^2}$. Take a rational number t such that

$$\text{Vol}(-K_X) > t \cdot \frac{24(d(\epsilon) + 2)}{\epsilon} > \frac{144(d(\epsilon) + 2)}{\epsilon^2}.$$

Lemma 6.7. *$-K_X - tG$ is \mathbb{Q} -effective.*

Proof. For a positive integer p and a sufficiently divisible positive integer m , we have an exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(-mK_X - pG) &\rightarrow \mathcal{O}_X(-mK_X - (p - 1)G) \\ &\rightarrow \mathcal{O}_G(-mK_X - (p - 1)G) \rightarrow 0. \end{aligned}$$

Hence

$$\begin{aligned} &h^0(X, \mathcal{O}_X(-mK_X - pG)) \\ &\geq h^0(X, \mathcal{O}_X(-mK_X - (p - 1)G)) - h^0(G, -mK_X|_G - (p - 1)G|_G) \\ &\geq h^0(X, \mathcal{O}_X(-mK_X - (p - 1)G)) - h^0(G, -mK_X|_G). \end{aligned}$$

Here we use the fact that $G|_G \geq 0$. Inductively, we have

$$h^0(X, \mathcal{O}_X(-mK_X - pG)) \geq h^0(X, \mathcal{O}_X(-mK_X)) - p \cdot h^0(G, -mK_X|_G).$$

Now we may take the integer $p = tm$ since m is sufficiently divisible. By the definition of volume, we have

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{6}{m^3} \left(h^0(X, \mathcal{O}_X(-mK_X)) - tm \cdot h^0(G, \mathcal{O}_F(-mK_X|_G)) \right) \\ &= \text{Vol}(-K_X) - 3t \text{Vol}(-K_X|_G) \\ &\geq \text{Vol}(-K_X) - \frac{24t(d(\epsilon) + 2)}{\epsilon} > 0. \end{aligned}$$

Here we use Lemma 6.5. Hence $h^0(X, \mathcal{O}_X(-mK_X - tmG)) > 0$ for m sufficiently divisible, that is, $-K_X - tG$ is \mathbb{Q} -effective. \square

By Lemma 6.7, there exists an effective \mathbb{Q} -divisor $B \sim_{\mathbb{Q}} -K_X - tG$. For a general fiber F of X over $z \in S$, there exists a number $\eta > 0$ (cf. [20, 4.8]) such that for any general $H' \in |H|$ containing z ,

$$\text{Nklt}\left(X, \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B\right) = \text{Nklt}\left(X, \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B + \eta f^*(H')\right).$$

We may take general $H_j \in |H|$ containing z for $1 \leq j \leq J$ with $J > \frac{2}{\eta}$ and take $G_1 = \sum_{j=1}^J \frac{2}{j} f^*(H_j)$. Then $\text{mult}_F G_1 \geq 2$ and $G_1 \sim_{\mathbb{Q}} 2f^*(H) \sim_{\mathbb{Q}} 2G$. In particular, (X, G_1) is not klt at F and by construction, in a neighborhood of F ,

$$\begin{aligned} & \text{Nklt}\left(X, \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B\right) \cup F \\ &= \text{Nklt}\left(X, \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B + G_1\right). \end{aligned}$$

Take a general element $G_2 \in |f^*(H)|$ not containing F , consider the pair $(X, (1 - \frac{3}{t})\Delta + \frac{3}{t}B + G_1 + G_2)$ where $3/t < 1$. Then

$$-\left(K_X + \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B + G_1 + G_2\right) \sim_{\mathbb{Q}} -\left(1 - \frac{3}{t}\right)(K_X + \Delta)$$

is ample. Since

$$F \cup G_2 \subset \text{Nklt}\left(X, \left(1 - \frac{3}{t}\right)\Delta + \frac{3}{t}B + G_1 + G_2\right),$$

by Connectedness Lemma, there is a curve C contained in $\text{Nklt}(X, (1 - \frac{3}{t})\Delta + \frac{3}{t}B + G_1 + G_2)$, intersecting F and not contracted by f . Hence C is contained in $\text{Nklt}(X, (1 - \frac{3}{t})\Delta + \frac{3}{t}B)$ by the construction of G_1 and generality of G_2 . Since C intersects F , so does $\text{Nklt}(X, (1 - \frac{3}{t})\Delta + \frac{3}{t}B)$. By inversion of adjunction, $(F, (1 - \frac{3}{t})\Delta|_F + \frac{3}{t}B|_F)$ is not klt for a general fiber F . On the other hand, $(F, \Delta|_F)$ is ϵ -klt and $F \simeq \mathbb{P}^1$. Hence

$$\text{deg}\left(\frac{3}{t}B|_F\right) \geq \epsilon.$$

Note that

$$\text{deg}(B|_F) = \text{deg}(-K_X|_F) = 2,$$

Hence $t \leq \frac{6}{\epsilon}$, which contradicts the definition of t . Hence we proved Theorem 6.6. \square

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