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The Nahm transform of spatially periodic instantons

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We construct the Nahm transform from finite energy instantons on the product of a real line $\mathbb{R}$ and a three-dimensional torus T^3 to Dirac-type singular monopoles on the dual torus $\hat{T}^3$. Moreover, we show the correspondence between the data which handle the asymptotic behavior of instantons at infinity and one of monopoles at singular points.

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1. Introduction

Set $T^3 := \mathbb{R}^3/\Lambda_3$, where $\Lambda_3 \subset \mathbb{R}^3$ is a lattice of $\mathbb{R}^3$. Let (V, h) be a Hermitian vector bundle on $\mathbb{R} \times T^3$ and A be a connection on (V, h). The triple (V, h, A) is called an instanton on $\mathbb{R} \times T^3$ if its curvature $F(A)$ satisfies the ASD equation $F(A) = - * F(A)$. Additionally, an instanton (V, h, A) on $\mathbb{R} \times T^3$ is L^2-finite if it satisfies the finite energy condition $|F(A)| \in L^2(\mathbb{R} \times T^3)$. Let $\hat{T}^3$ be the dual torus of T^3 *i.e.* $\hat{T}^3 = \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})/\Lambda_3^*$, where $\Lambda_3^* = \{\xi \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}) \mid \xi(\Lambda_3) \subset \mathbb{Z}\}$ is the dual subgroup of Λ_3. Let $Z \subset \hat{T}^3$ be a finite subset. Let $(\hat{V}, \hat{h}, \hat{A})$ be a Hermitian vector bundle with a connection on $\hat{T}^3 \setminus Z$. Let $\hat{\Phi}$ be a skew-Hermitian section of $\mathrm{End}(V)$ on $\hat{T}^3 \setminus Z$. The tuple $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ is said to be a monopole on $\hat{T}^3 \setminus Z$ if it satisfies the Bogomolny equation $F(\hat{A}) = *\nabla_{\hat{A}}(\hat{\Phi})$. Moreover, Z is the Dirac-type singularities of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ if it has a certain type of the asymptotic behavior around each point of Z (Definition 2.6). Then we call $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ a Dirac-type singular monopole on $\hat{T}^3$. In this paper, we will construct the Nahm transform of an L^2-finite instanton (V, h, A) on the product of a real line $\mathbb{R} \times T^3$ to a Dirac-type singular monopole $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ on $\hat{T}^3$.

In general, for any closed subgroup $\Lambda \subset \mathbb{R}^4$ and the dual subgroup $\Lambda^* \subset \mathrm{Hom}(\mathbb{R}^4, \mathbb{R})$, it is believed that there exists a way to construct Λ^*-invariant instantons on $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R})$ from Λ-invariant instantons on $\mathbb{R}^4$. For example, if $\Lambda = \mathbb{R}^4$ and $\Lambda^* = \{0\}$, it was constructed by Atiyah, Drinfeld, Hitchin and Manin [2], and called the ADHM construction. The case $(\Lambda, \Lambda^*) \simeq (\mathbb{R}, \mathbb{R}^3)$ was studied by Nahm [21], Hitchin [12] and Nakajima [22]. See Jardim [13] for a list of many Nahm transformations.

Since $\mathbb{R}$-invariant instantons on $\mathbb{R} \times \hat{T}^3$ can be regarded as monopoles on $\hat{T}^3$, the construction in this paper corresponds to the case $(\Lambda, \Lambda^*) \simeq (\mathbb{Z}^3, \mathbb{R} \times \mathbb{Z}^3)$. This case was previously studied in Charbonneau [6] and Charbonneau-Hurtubise[8]. The difference between [6, 8] and this paper will be mentioned in detail after introducing our main results.

Next, let us consider relations between the Nahm transforms and the Kobayashi-Hitchin correspondences. On a connected compact Kähler surface (M, g) with the Kähler form ω, there exists a one-to-one correspondence between irreducible instantons and stable holomorphic vector bundles V with the condition $(c_1(V) \cup \omega)/[X] = 0$ up to their gauge transformations, which is called the Kobayashi-Hitchin correspondence (also called 'the Hitchin-Kobayashi correspondence' or 'the Donaldson-Uhlenbeck-Yau correspondence') and proved by Uhlenbeck and Yau [26]. In our case, there exist similar relations under the assumption that T^3 is isomorphic to $S^1 \times T^2$ as a Riemannian manifold. On one hand, in [7] Charbonneau and Hurtubise obtained the Kobayashi-Hitchin correspondence between Dirac-type singular monopoles on $\hat{T}^3$ and polystable singular mini-holomorphic bundles (Definition 2.8) on $\hat{T}^3$. On the other hand, we will give a construction of polystable parabolic bundles with parabolic degree 0 on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$ from L^2-finite instantons on $\mathbb{R} \times T^3$ (Theorem 5.11). However, it is only a half part of the Kobayashi-Hitchin correspondence and we have not yet proved the other part.

Next we will construct mini-holomorphic bundles on $\hat{T}^3$ from stable parabolic bundles on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$ of rank $r > 1$ in reference to [16]. We call this construction the *algebraic Nahm transform* as in [16] and it satisfies the following commutative diagram.

$$
\begin{tikzcd}
\boxed{\begin{array}{c} L^2\text{-finite instantons} \\ \text{on } \mathbb{R} \times T^3 \end{array}} \arrow[r, "\text{Nahm}"] \arrow[d, "\text{K-H corres.}"] & \boxed{\begin{array}{c} \text{Dirac-type singular} \\ \text{monopoles on } \hat{T}^3 \end{array}} \arrow[d, leftrightarrow, "\begin{array}{l}\text{K-H corres.} \\ \text{constructed in [7]}\end{array}"] \\
\boxed{\begin{array}{c} \text{stable parabolic bundles} \\ \text{on } (\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2) \end{array}} \arrow[r, "\text{Alg. Nahm}"] & \boxed{\begin{array}{c} \text{stable singular} \\ \text{mini-hol. bundles on } \hat{T}^3 \end{array}}
\end{tikzcd}
$$

1.1. Main result

The main results of this paper are summarized as follow.

(I) For any L^2-finite instanton (V, h, A) on $\mathbb{R} \times T^3$, there exist model solutions of the Nahm equation $(\Gamma_\pm, N_\pm) = (\Gamma_{i,\pm}, N_{i,\pm})_{i=1,2,3}$ such that (V, h, A) is approximated by the T^3-invariant instantons associated to $(\Gamma_\pm, N_\pm)$ at $t \to \pm\infty$ (Corollary 3.4).

(II) We construct a monopole $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ on $\hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$ from an L^2-finite instanton (V, h, A) on $\mathbb{R} \times T^3$, where $\mathrm{Sing}(V, h, A) \subset \hat{T}^3$ is a finite subset determined by $(\Gamma_\pm)$ (Proposition 4.2). Moreover, each point of $\mathrm{Sing}(V, h, A)$ is a Dirac-type singularity of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ (Theorem 4.3).

(III) Assume that T^3 is isomorphic to $S^1 \times T^2$ as a Riemannian manifold. If (V, h, A) is irreducible and $\mathrm{rank}(V) > 1$, then the weight $\vec{k} \in \mathbb{Z}^{\mathrm{rank}(\hat{V})}$ of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ at each singular point $\xi \in \mathrm{Sing}(V, h, A)$ is determined by the weights of the $\mathfrak{su}(2)$ representation $\rho_{\pm,\xi}$ constructed from $(N_\pm)$ (Theorem 6.2).

The first result is an analytical preparation of the Nahm transform in (II). The second is the construction of the Nahm transform. The third is an application of the commutativity in the above figure (Theorem 5.30). Let us explain more details in the following.

1.1.1. Main result (I). Let $h_{\mathbb{C}^r}$ be the canonical Hermitian metric on $\mathbb{C}^r$. For a smooth manifold S, we denote by $(\underline{\mathbb{C}^r}_S, \underline{h}_S)$ the product bundle of $(\mathbb{C}^r, h_{\mathbb{C}^r})$ on S. If there is no risk of confusion, then we abbreviate $(\underline{\mathbb{C}^r}_S, \underline{h}_S)$ to $(\underline{\mathbb{C}^r}, \underline{h})$.

Let A be a connection on $(\underline{\mathbb{C}^r}, \underline{h})$ on $(0,\infty) \times T^3$, and assume that the connection form $\alpha dt + \sum_i A_i dx^i$ of A is invariant under T^3-action *i.e.* α and A_i are T^3-invariant functions on $(0,\infty) \times T^3$. Then the ASD equation for $(\underline{\mathbb{C}^r}, \underline{h}, A)$ is equivalent to the following Nahm equation:

$$
\left\{
\begin{aligned}
&\frac{\partial A_1}{\partial t} + [\alpha, A_1] = -[A_2, A_3] \\
&\frac{\partial A_2}{\partial t} + [\alpha, A_2] = -[A_3, A_1] \\
&\frac{\partial A_3}{\partial t} + [\alpha, A_3] = -[A_1, A_2].
\end{aligned}
\right. \tag{1}
$$

Let $\Gamma_i \in \mathfrak{u}(r)$ $(i = 1, 2, 3)$ be skew-Hermitian matrices which are commutative each other. For the tuple $\Gamma = (\Gamma_i)$, we set the centralizer $\mathrm{Center}(\Gamma) := \{a \in \mathfrak{u}(r) \mid [\Gamma_i, a] = 0\ (i = 1, 2, 3)\}$. Take $N_i \in \mathrm{Center}(\Gamma)$ $(i = 1, 2, 3)$ satisfying $N_i = [N_j, N_k]$ for any even permutation (ijk) of (123). Then the tuple $\alpha = 0, A_i = \Gamma_i + N_i/t$ forms a solution of (1) on $(0,\infty)$.

Definition 1.1 (Definition 3.1). A tuple $(\Gamma = (\Gamma_i), N = (N_i))$ is called a model solution of the Nahm equation if it satisfies the above conditions.

We obtain the following theorem as a consequence of results in [3] and [20].

Theorem 1.2 (Corollary 3.4). *Let (V, h, A) be an L^2-finite instanton on $(0,\infty) \times T^3$, i.e. its curvature $F(A)$ is L^2.*

If we fix a positive number $0 < \lambda < 1$ and take a sufficiently large $R > 0$, then there exist a trivialization of $C^{4,\lambda}$-class $\sigma : (V, h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$ and a model solution of the Nahm equation (Γ, N) such that the following holds for the connection form $\alpha dt + \sum_i A_i dx^i$ of A with respect to σ.

(1) The trivialization σ is a temporal gauge i.e. we have $\alpha = 0$.

(2) For any $1 \leq i \leq 3$, there exists a decomposition $A_i - (\Gamma_i + N_i/t) = \varepsilon_{1,i}(t) + \varepsilon_{2,i}(t) + \varepsilon_{3,i}(t, x)$ such that we have $\varepsilon_{1,i}(t) \in \mathrm{Center}(\Gamma)$, $\varepsilon_{2,i}(t) \in (\mathrm{Center}(\Gamma))^\perp$ and the following estimates for any $1 \leq j \leq 3$,

where $(\mathrm{Center}(\Gamma))^\perp$ *means the orthogonal complement of* $\mathrm{Center}(\Gamma)$ *in* $\mathfrak{u}(r)$ *with respect to the inner product* $\langle A, B\rangle := -\mathrm{tr}(AB)$.

$$\begin{aligned} |\partial_t^j \varepsilon_{1,i}| &= O(t^{-(1+j+\delta)}) \\ |\partial_t^j \varepsilon_{2,i}| &= O(\exp(-\delta t)) \\ ||\varepsilon_{3,i}||_{C^{3,\lambda}([t,t+1]\times T^3)} &= O(\exp(-\delta t)), \end{aligned}$$

where δ *is a positive number.*

Remark 1.3. Since the tuple $\alpha = 0, A_i = \Gamma_i + N_i/t$ also form a solution of the Nahm equation on $(-\infty, 0)$, a similar result holds for L^2-finite instantons on $(-\infty, 0) \times T^3$.

1.1.2. Main result (II). Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$ of rank r. Applying Theorem 1.2 to $(V, h, A)|_{(-\infty,0)\times T^3}$ and $(V, h, A)|_{(0,\infty)\times T^3}$, we obtain model solutions $(\Gamma_\pm, N_\pm)$ which approximate (V, h, A) at $t \to \pm\infty$. Since the simultaneous eigenvalues of $\sum_i \Gamma_{\pm,i} dx^i \in \Omega^1(T^3)$ are T^3-invariant pure imaginary 1-forms on T^3, they can be regarded as elements of $\mathrm{Hom}(\mathbb{R}^3, \sqrt{-1}\mathbb{R})$. Thus we take $\overline{\mathrm{Spec}(\Gamma_\pm)} \subset \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ as the set of $(2\pi\sqrt{-1})^{-1}$ times simultaneous eigenvalues of $\sum_i \Gamma_{\pm,i} dx^i$. We define the spectrum set $\mathrm{Spec}(\Gamma_\pm) \subset \hat{T}^3$ to be the image of $\overline{\mathrm{Spec}(\Gamma_\pm)}$ by the quotient map $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R}) \to \hat{T}^3$. We define the singularity set of (V, h, A) as $\mathrm{Sing}(V, h, A) := \mathrm{Spec}(\Gamma_+) \cup \mathrm{Spec}(\Gamma_-)$. For $\xi \in \hat{T}^3$ and the associated flat Hermitian line bundle $L_\xi := (\underline{\mathbb{C}}, \underline{h}, d + 2\pi\sqrt{-1}\langle \xi, x\rangle)$ on $\mathbb{R} \times T^3$, we set the twisted instanton $(V, h, A_\xi) := (V, h, A) \otimes L_{-\xi}$. Then we have

$$\mathrm{Sing}(V, h, A_\xi) = \mathrm{Sing}(V, h, A) - \xi = \{\mu - \xi \in \hat{T}^3 \mid \mu \in \mathrm{Sing}(V, h, A)\}.$$

We construct the Nahm transform of (V, h, A) as follows. Let $S^\pm$ be the spinor bundle on $\mathbb{R} \times T^3$ with respect to the trivial spin structure and $\not{\partial}_A^\pm : S^\pm \otimes V \to S^\mp \otimes V$ be the Dirac operator of the connection A. Let $\mathcal{V}$ be a Hermitian flat vector bundle on $\hat{T}^3$ which is the quotient of the product bundle $(\underline{L^2(\mathbb{R} \times T^3, V \otimes S^-)}, ||\cdot||_{L^2})$ on $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ by a Λ_3^*-action $v \cdot (\xi, f) := (\xi + v, \exp(2\pi\sqrt{-1}\langle x, v\rangle) f)$. For a family of L^2-finite instantons $\{(V, h, A_\xi)\}_{\xi \in \hat{T}^3}$, we set $(\hat{V}, \hat{h})$ as a subbundle of $\mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)}$ defined by $\hat{V}_\xi := \mathrm{Ker}\left(\not{\partial}_{A_\xi}^-\right) \cap L^2$. Then $(\hat{V}, \hat{h})$ is well-defined and of finite rank because $\not{\partial}_{A_\xi}^-$ is a continuous family of surjective Fredholm operators for $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$ (see Theorem 3.13 and Remark 3.15). By Theorem 3.14, we have $\mathrm{rank}(\hat{V}) = (8\pi^2)^{-1} ||F(A)||_{L^2}^2$.

We define a connection $\hat{A}$ on $\hat{V}$ to be the induced connection from the flat connection $d_{\mathcal{V}}$ of $\mathcal{V}$, *i.e.* we can write $\hat{A} = Pd_{\mathcal{V}}$ for the orthogonal projection $P : \mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)} \to \hat{V}$. We take a skew-Hermitian endomorphism $\hat{\Phi}$ as $\hat{\Phi}(f) := P(2\pi\sqrt{-1}tf)$. Then we have the next theorem.

Theorem 1.4 (Proposition 4.2 and Theorem 4.3). $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ *is a monopole on* $\hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$*, and each point of* $\mathrm{Sing}(V, h, A)$ *is a Dirac-type singularity of* $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$*.*

Here we recall the definition of Dirac-type singularity of monopole by following [7]. Let (X, g) be an oriented Riemannian 3-fold and $Z \subset X$ be a discrete subset. Let (V, h, A, Φ) be a monopole on $X \setminus Z$ of rank r. Each point $p \in Z$ is a Dirac-type singularity of (V, h, A, Φ) with weight $\vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ if the following conditions are satisfied.

- There exists a neighborhood $B \subset X$ of p such that $(V, h)|_{B\setminus\{p\}}$ is decomposed into a direct sum of Hermitian line bundles $\bigoplus_{i=1}^r L_i$ such that we have $\deg(L_i) = \int_{\partial B} c_1(L_i) = k_i$ for any $1 \leq i \leq k$.
- Under the above decomposition, we have the next estimates.

$$\begin{cases} \Phi = \dfrac{\sqrt{-1}}{2R} \displaystyle\sum_{i=1}^{r} k_i \cdot \mathrm{Id}_{L_i} + O(1) \\ \nabla_A(R\Phi) = O(1), \end{cases}$$

where R is the distance from p.

1.1.3. Main result (III). Since $N_\pm = (N_{\pm,i})$ satisfies $N_{\pm,i} = [N_{\pm,j}, N_{\pm,k}]$ for any even permutation (ijk) of (123), we can construct $\mathfrak{su}(2)$ representation $\rho_\pm$ from $N_\pm$. Then, $\rho_\pm$ can be decomposed into $\rho_\pm = \bigoplus_{\xi\in\mathrm{Sing}(V,h,A)} \rho_{\pm,\xi}$ because $N_{\pm,i} \in \mathrm{Center}(\Gamma_\pm)$. Now we define the weight of $\rho_{\pm,\xi}$ to be $w_{\pm,\xi} := (\mathrm{rank}(\rho_{\pm,\xi,i})) \in \mathbb{Z}^{m_{\pm,\xi}}$, where $\rho_{\pm,\xi} = \bigoplus_{i=1}^{m_{\pm,\xi}} \rho_{\pm,\xi,i}$ is the irreducible decomposition. Let $\vec{k}_\xi$ be the weight of the monopole $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ at $\xi \in \mathrm{Sing}(V, h, A)$ and $\vec{k}_{+,\xi}, \vec{k}_{-,\xi}$ the positive and negative part of $\vec{k}_\xi$. Here main result (III) can be described as follows.

Theorem 1.5 (Theorem 6.2). *Assume* T^3 *is isomorphic to* $S^1 \times T^2$ *as a Riemannian manifold. If the* L^2*-finite instanton* (V, h, A) *is irreducible and of* $\mathrm{rank}(V) > 1$*, then* $\vec{k}_\pm$ *agrees with* $\pm w_{\pm,\xi}$ *with a suitable permutation.*

Comparison with previous studies. In [5, 6], Charbonneau constructed the Nahm transform to singular monopoles on $\hat{T}^3$ from L^2-finite instantons of rank 2 with an assumption $|\mathrm{Spec}(\Gamma_+)| = |\mathrm{Spec}(\Gamma_-)| = 2$. In [8], Charbonneau and Hurtubise constructed the bijection of the equivalence classes of spatially periodic instantons of rank r and Dirac-type singular monopoles on the dual torus, under the genericity conditions $|\mathrm{Spec}(\Gamma_+)| = |\mathrm{Spec}(\Gamma_-)| = r$ and $\mathrm{Spec}(\Gamma_+) \cap \mathrm{Spec}(\Gamma_-) = \emptyset$. Then $A_{\pm,i} = 0$ holds for any $i = 1, 2, 3$, and the weights of singularities of constructed monopoles are confined to $(\pm 1, 0, \ldots, 0)$.

In our Theorem 1.4, we study the Nahm transform $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ of any L^2-finite spatially periodic instantons (V, h, A) without the genericity assumptions as a refinement of the construction in [5, 6], and we prove that the singularities of the monopoles $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ are of Dirac type, in a more direct way using a result in [19]. We also study the comparison of the weights of the singularities of the instantons and the monopoles (Theorem 1.5) in this generalized context.

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2. Preliminary

For the product bundle $(\underline{\mathbb{C}^r}_S, \underline{h}_S)$ on a smooth manifold S, we denote by d_S the trivial connection on $(\underline{\mathbb{C}^r}_S, \underline{h}_S)$. If there is no risk of confusion, then we abbreviate d_S to d.

2.1. Tori and dual tori

For a finite dimensional $\mathbb{R}$-vector space X and a lattice $\Lambda \subset X$, we set $T = X/\Lambda$. Let X^* be the dual space of X. Let Λ^* denote the dual lattice of Λ, *i.e.*, $\Lambda^* := \{v \in X^* \mid v(\Lambda) \subset Z\}$. We define the dual torus $\hat{T}$ of T by $\hat{T} := X^*/\Lambda^*$.

For any $\xi \in \hat{T}$, we define a flat Hermitian line bundle L_ξ on T as $L_\xi := (\underline{\mathbb{C}}_T, \underline{h}_T, d_T + 2\pi\sqrt{-1}\langle \xi, dx\rangle)$. By this correspondence, we can naturally regard $\hat{T}$ as the moduli space of flat Hermitian line bundles on T. The double dual of T is naturally isomorphic to T, and hence $x \in T$ also gives a flat Hermitian line bundle $L_x := (\underline{\mathbb{C}}_{\hat{T}}, \underline{h}_{\hat{T}}, d_{\hat{T}} + 2\pi\sqrt{-1}\langle x, d\xi\rangle)$ on $\hat{T}$.

We recall a differential-geometric construction of the Poincaré bundle on $T \times \hat{T}$ in [10]. On $T \times X^*$, we have the following Hermitian line bundle with a connection on $T \times \hat{T}$

$$\tilde{\mathcal{L}} = \left(\underline{\mathbb{C}}, \underline{h}, d - 2\pi\sqrt{-1}\langle \xi, dx\rangle\right).$$

The Λ^*-action on $T \times X^*$ is naturally lifted to the action on $\tilde{\mathcal{L}}$ given by

$$v \cdot (x, \xi, s) = \left(x, \xi + v, \exp(2\pi\sqrt{-1}\langle x, v\rangle)s\right).$$

The induced Hermitian line bundle with a connection is called the Poincaré bundle, and denoted by $\mathcal{L}$.

Lemma 2.1 (Lemma 3.2.14 in [10]). *The Poincaré bundle $\mathcal{L}$ has the following properties.*

- *For any $\xi \in \hat{T}$, $\mathcal{L}|_{T\times\{\xi\}}$ is isomorphic to $L_{-\xi}$.*
- *For any $x \in T$, $\mathcal{L}|_{\{x\}\times\hat{T}}$ is isomorphic to L_x.*

Proof. The first claim is clear by the construction of $\mathcal{L}$. For $x \in T$, the connection form of $\tilde{\mathcal{L}}$ on the slice $\{x\} \times X^*$ vanishes, and $\mathcal{L}|_{\{x\}\times\hat{T}}$ has the global section induced by the function $s(\xi) := \exp(2\pi\sqrt{-1}\langle x, \xi\rangle)$ on $\{x\} \times X^*$, which satisfies $ds = s(2\pi\sqrt{-1}\langle x, d\xi\rangle)$. Hence, $\mathcal{L}|_{\{x\}\times\hat{T}}$ is isomorphic to L_x. $\square$

Remark 2.2. If X is a complex vector space, then T and $\hat{T}$ are equipped with the induced complex structures and $\mathcal{L}$ becomes a holomorphic line bundle on $T \times \hat{T}$ by the holomorphic structure induced by the $(0,1)$-part of the connection on $\mathcal{L}$.

In this paper, we fix a lattice $\Lambda_3 \subset \mathbb{R}^3$. Thus $T^3 = \mathbb{R}^3/\Lambda_3$, $\Lambda_3^* \subset (\mathbb{R}^3)^*$ and $\hat{T}^3 = \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})/\Lambda_3^*$ are also fixed.

2.2. L^2-finite instantons

Let (X, g) be a connected oriented Riemannian 4-fold and $*$ be the Hodge operator on X. Let $L^2(X, g)$ denote the space of L^2-functions on X with respect to the measure induced by g.

Definition 2.3. Let (V, h) be a Hermitian vector bundle and A be a unitary connection on (V, h).

- The tuple (V, h, A) is an instanton on X if the ASD equation $F(A) = - * F(A)$ holds.

- An instanton (V, h, A) is an L^2-finite instanton on X if we have $|F(A)| \in L^2(X, g)$, where the norm is induced by g and h.

2.2.1. Some easy properties of L^2-finite instantons on $\mathbb{R} \times T^3$.

Lemma 2.4. *Let (L, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$ of rank 1. Then, (L, h, A) is a flat Hermitian line bundle. In particular, (L, h) is a topologically trivial Hermitian line bundle.*

Proof. Let t and (x^1, x^2, x^3) be the standard coordinates of $\mathbb{R}$ and $\mathbb{R}^3$ respectively. By abuse of notation, we use (t, x^1, x^2, x^3) to denote a local chart of $\mathbb{R} \times T^3$. Using this coordinate, we write $F(A) = \sum_i F_{ti} dt \wedge dx^i + \sum_{i<j} F_{ij} dx^i \wedge dx^j$. By Bianchi's identity $\nabla_A(F(A)) = 0$ and the ASD equation $F(A) = - * F(A)$, we have $\Delta(F(A)) = 0$, where $\Delta = \nabla_A^* \nabla_A + \nabla_A \nabla_A^*$ is the Laplacian. It implies that the functions F_{ti} and F_{ij} are harmonic and L^2 on $\mathbb{R} \times T^3$, and hence 0. Thus we obtain that (L, h, A) is a flat Hermitian line bundle. $\square$

Corollary 2.5. *Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$ of rank r. Then, (V, h) is a trivial Hermitian vector bundle.*

Proof. By Lemma 2.4, $(\det(V), \det(h))$ is a trivial Hermitian line bundle. Thus, it suffices to prove that any principal $SU(r)$-bundle P on T^3 is topologically trivial.

We may assume $T^3 = (\mathbb{R}/\mathbb{Z})^3$. Let $q : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the quotient map, and put $T^2 = (\mathbb{R}/\mathbb{Z})^2$. Take the open intervals $I_1 = (0, 1)$, $I_2 = (1/2, 3/2)$. We set $U_i := q(I_i) \times T^2$, and we obtain an open covering $\{U_1, U_2\}$ of T^3. Then, $P|_{U_i}$ is trivial because $SU(r)$ is simply-connected and U_i and T^2 are homotopy equivalent. Moreover, we can patch each trivialization of $P|_{U_i}$ and get a global one. Indeed, any smooth map $f : T^2 \to SU(r)$ is homotopic to a smooth map $f_1 : T^2 \to SU(r)$ such that $f_1(\{0\} \times S^1) \subset \{e\}$ and $f_1(S^1 \times \{0\}) \subset \{e\}$ because $SU(r)$ is simply connected, and f_1 is homotopic to a constant map because $\pi_2(SU(r)) = 0$. Therefore, P is a trivial $SU(r)$-bundle. $\square$

2.3. Monopoles with Dirac-type singularities

In this subsection, we recall the definition of monopoles with Dirac-type singularities by following [7].

Definition 2.6. Let (X, g) be an oriented Riemannian 3-fold and $*$ be the Hodge operator on X.

(1) Let (V, h, A) be a Hermitian vector bundle with a unitary connection, and Φ be a skew-Hermitian section of $\mathrm{End}(V)$. The tuple (V, h, A, Φ) is called a monopole on X if it satisfies the Bogomolny equation $F(A) = *\nabla_A(\Phi)$.

(2) Let $Z \subset X$ be a discrete subset. Let (V, h, A, Φ) be a monopole of rank $r \in \mathbb{N}$ on $X \setminus Z$. Each point $p \in Z$ is called a Dirac-type singularity of the monopole (V, h, A, Φ) with weight $\vec{k} = (k_i) \in \mathbb{Z}^r$ if the following holds.
   - There exists a small neighborhood B of p such that $(V, h)|_{B\setminus\{p\}}$ is decomposed into a sum of Hermitian line bundles $\bigoplus_{i=1}^r L_i$ with $\int_{\partial B} c_1(L_i) = k_i$.
   - In the above decomposition, we have the following estimates,

$$\begin{cases} \Phi = \dfrac{\sqrt{-1}}{2R} \displaystyle\sum_{i=1}^{r} k_i \cdot Id_{L_i} + O(1) \\ \nabla_A(R\Phi) = O(1), \end{cases}$$

   where R is the distance from p.

In [19], the following proposition is proved.

Proposition 2.7. *Let $U \subset \mathbb{R}^3$ be a neighborhood of $0 \in \mathbb{R}^3$. Let (V, h, A, Φ) be a monopole on $(U \setminus \{0\}, g_{\mathbb{R}^3})$. Then, the point 0 is a Dirac-type singularity of (V, h, A, Φ) if and only if $|\Phi(x)| = O(|x|^{-1})$ $(x \to 0)$.*

2.3.1. Monopoles and mini-holomorphic structure. We introduce a complex-geometric interpretation of monopole by following [7] and [19]. Let Σ be a Riemann surface with a Kähler metric g_Σ. Set $S^1 := \mathbb{R}/\mathbb{Z}$ and $X := S^1 \times \Sigma$. Let p_i be the projection from X to the i-th component. Let $q : \mathbb{R} \to S^1$ be the quotient map. Let us recall the mini-holomorphic structure on X in [19].

Definition 2.8.

(1) We define $\Omega^{0,1}(X) := {p_1}^*\Omega^1_{\mathbb{C}}(S^1) \oplus {p_2}^*\Omega^{0,1}(\Sigma)$ and $\Omega^{0,2}(X) := \bigwedge^2 \Omega^{0,1}(X)$. We define $\overline{\partial}_X : \Omega^{0,i}(X) \to \Omega^{0,i+1}(X)$ to be $\overline{\partial}_X = d_{S^1} + \overline{\partial}_\Sigma$. We call the tuple $(\Omega^{0,i}(X), \overline{\partial}_X)$ *mini-holomorphic structure* on X.

(2) Let V be a vector bundle on an open subset $U \subset X$. Let $\Omega^{0,i}(U,V)$ denote the space of V-valued differential forms on U of degree $(0,i)$. A differential operator $\overline{\partial}_V : \Omega^{0,0}(U,V) \to \Omega^{0,1}(U,V)$ is called mini-holomorphic structure of V if it satisfies the following conditions.
   - For any $f \in C^\infty(U)$ and $s \in \Omega^{0,i}(U,V)$, we have $\overline{\partial}_V(fs) = \overline{\partial}_X(f) \wedge s + f\overline{\partial}_V(s)$. Note that the differential operators $\overline{\partial}_V : \Omega^{0,i}(U,V) \to \Omega^{0,i+1}(U,V)$ are naturally induced.
   - The integrability condition $\overline{\partial}_V \circ \overline{\partial}_V = 0$ is satisfied.

Let $I = (a,b) \subset \mathbb{R}$ be an open interval with $|b-a| < 1$ and $W \subset \Sigma$ be a domain. Let $(V, \overline{\partial}_V)$ be a mini-holomorphic bundle on the open subset of the form $q(I) \times W \subset X$. We decompose the differential operator $\overline{\partial}_V(s)$ as $\overline{\partial}_V(s) = d_{V,S^1}(s) + \overline{\partial}_{V,\Sigma}(s) \in {p_1}^*\Omega^1(S^1) \oplus {p_2}^*\Omega^{0,1}(\Sigma)$ for a local section s of V. For $t \in I$, let V^t denote the holomorphic bundle $(V|_{q(t)\times W}, \overline{\partial}_{V,q(t)})$ on W, where $\overline{\partial}_{V,q(t)}$ is the restriction of the differential operator $\overline{\partial}_{V,\Sigma}$ on $\{q(t)\} \times W$. For any fixed $x \in W$, we obtain a connection on $V|_{q(I)\times\{x\}}$ as the restriction of d_{V,S^1} on $q(I) \times \{x\}$. Hence we have the parallel transport $\Psi_{t,t'} : V^t \to V^{t'}$ for any $t, t' \in I$. The isomorphism $\Psi_{t,t'}$ is called the scattering map in [7]. Recall that the scattering map $\Psi_{t,t'} : V^t \to V^{t'}$ is a holomorphic isomorphism, which follows from the integrability condition $\overline{\partial}_V \circ \overline{\partial}_V = [d_{V,S^1}, \overline{\partial}_{V,\Sigma}] = 0$. The next proposition shows that monopoles on X have the underlying mini-holomorphic structures.

Proposition 2.9. *Let (L,h,A,Φ) be a monopole on X. We decompose the covariant derivative ∇_A into $\nabla_A(f) = \nabla_{A,t}(f)dt + \nabla_A^{1,0}(f) + \nabla_A^{0,1}(f) \in {p_1}^*\Omega^1(S^1) \oplus {p_2}^*\Omega^{1,0}(\Sigma) \oplus {p_2}^*\Omega^{0,1}(\Sigma)$ for a local section f of L. Then, the differential operator $\overline{\partial}_L := (\nabla_{A,t} - \sqrt{-1}\Phi)dt + \nabla_A^{0,1}$ is a mini-holomorphic structure on L.*

Proof. It is standard that the integrability condition $\overline{\partial}_L \circ \overline{\partial}_L = 0$ follows from the Bogomolny equation $F(A) = *\nabla_A(\Phi)$. □

Let w be the standard coordinate of $\mathbb{C}$. Let $U \subset \mathbb{C}$ be a neighborhood of 0 and put $U^* := U \setminus \{0\}$. Let (V,h,A,Φ) be a monopole of rank r on $([-\varepsilon,\varepsilon] \times U) \setminus \{(0,0)\} \subset \mathbb{R} \times \mathbb{C}$, and let $(V, \overline{\partial}_V)$ denote the underlying mini-holomorphic bundle. The following proposition in [7] allows us to interpret the weights of Dirac-type singularities in terms of the scattering maps of the underlying mini-holomorphic bundles.

Proposition 2.10. *If $(0,0)$ is a Dirac-type singularity of weight $\vec{k} = (k_i) \in \mathbb{Z}^r$, then the scattering map $\Psi_{-\varepsilon,\varepsilon} : V^{-\varepsilon}|_{U^*} \to V^{\varepsilon}|_{U^*}$ is extended to a meromorphic isomorphism $\Psi_{-\varepsilon,\varepsilon} : V^{-\varepsilon}(*0) \to V^{\varepsilon}(*0)$ and there exists a holomorphic frame $\boldsymbol{v}^-$ (resp. $\boldsymbol{v}^+$) of $V^{-\varepsilon}$ (resp. V^{ε}) such that $\Phi^{-\varepsilon,\varepsilon}$ can be represented as $\Psi_{-\varepsilon,\varepsilon}(\boldsymbol{v}^-) = \boldsymbol{v}^+ \cdot \operatorname{diag}(w^{k_i})$, where $\operatorname{diag}(c_i)$ is the diagonal matrix whose (i,i)-th entries are c_i. Moreover, this type of diagonal matrix representation is unique up to permutations.*

2.4. Filtered sheaves and filtered bundles

We recall the definitions of parabolic sheaves and bundles by following [16].

2.4.1. Filtered sheaves. Let X be a complex manifold. Let D be a smooth hypersurface of X and $D = \coprod_{i=1}^{d} D_i$ be the decomposition into connected components. Let $\mathcal{E}$ be a coherent $\mathcal{O}_X(*D) = \bigcup_{n\in\mathbb{Z}} \mathcal{O}_X(nD)$-module.

A tuple $P_*\mathcal{E} = \{P_{\boldsymbol{a}}\mathcal{E}\}_{\boldsymbol{a}=(a_i)\in\mathbb{R}^d}$ of $\mathcal{O}_X$-submodules of $\mathcal{E}$ is called a filtered sheaf over $\mathcal{E}$ if it satisfies the following conditions:

- $P_{\boldsymbol{a}}\mathcal{E} \subset \mathcal{E}$ is a coherent $\mathcal{O}_X$-module and $P_{\boldsymbol{a}}\mathcal{E}|_{X\setminus D} = \mathcal{E}|_{X\setminus D}$ holds.
- For $\boldsymbol{a} = (a_i)$ and $\boldsymbol{a'} = (a'_i) \in \mathbb{R}^d$, we have $P_{\boldsymbol{a'}}\mathcal{E} \subset P_{\boldsymbol{a}}\mathcal{E}$ if $a'_i \leq a_i$ for any $i = 1,\dots,d$.
- On a small neighborhood U of D_i, $P_{\boldsymbol{a}}\mathcal{E}|_U$ depends only on a_i, which we denote by ${}^iP_{a_i}(\mathcal{E}|_U)$.
- For any $i = 1,\dots,d$ and $a \in \mathbb{R}$, there exists $\epsilon > 0$ such that we have ${}^iP_a(\mathcal{E}|_U) = {}^iP_{a+\epsilon}(\mathcal{E}|_U)$.
- For any $\boldsymbol{a} \in \mathbb{R}^d$ and $\boldsymbol{n} = (n_i) \in \mathbb{Z}^d$, we have $P_{\boldsymbol{a}+\boldsymbol{n}}\mathcal{E} = P_{\boldsymbol{a}}\mathcal{E}(\sum n_i D_i)$.

A filtered subsheaf $P_*\mathcal{E}' \subset P_*\mathcal{E}$ is a filtered sheaf over a subsheaf $\mathcal{E}' \subset \mathcal{E}$ such that $P_{\boldsymbol{a}}\mathcal{E}' \subset P_{\boldsymbol{a}}\mathcal{E}$ for any $\boldsymbol{a} \in \mathbb{R}^d$. If $P_{\boldsymbol{a}}\mathcal{E}' = \mathcal{E}' \cap P_{\boldsymbol{a}}\mathcal{E}$ holds for any $\boldsymbol{a} \in \mathbb{R}^d$, it is called strict.

For a small neighborhood U of D_i, we set ${}^iP_{<a}(\mathcal{E}|_U) := \sum_{a'<a} {}^iP_{a'}(\mathcal{E}|_U)$. We also define a coherent $\mathcal{O}_{D_i}$-module ${}^i\mathrm{Gr}_a(\mathcal{E})$ by

$$ {}^i\mathrm{Gr}_a(\mathcal{E}) = {}^iP_{<a}(\mathcal{E}|_U)/{}^iP_{<a}(\mathcal{E}|_U). $$

We set

$$ \mathcal{P}ar(P_*\mathcal{E}, i) := \{a \in \mathbb{R} \mid {}^i\mathrm{Gr}_a(\mathcal{E}) \neq 0\}. $$

Suppose that $\mathcal{E}$ is torsion free. The parabolic first Chern class par-$\mathrm{c}_1(P_*\mathcal{E})$ is defined as

$$\text{par-}\mathrm{c}_1(P_*\mathcal{E}) := c_1(P_{0,\dots,0}V) - \sum_{i=1}^{d} \sum_{-1<a_i\leq 0} a_i \cdot \mathrm{rank}_{D_i}({}^i\mathrm{Gr}_{a_i}(\mathcal{E}))[D_i].$$

Here rank_{D_i} denote the rank of coherent sheaves on D_i, and $[D_i]$ is the cohomology class of D_i on X.

2.4.2. Filtered bundles. A filtered sheaf $P_*\mathcal{E}$ on (X, D) is called a filtered bundle if it satisfies the following conditions:

- For any $\boldsymbol{a} \in \mathbb{R}^d$, $P_{\boldsymbol{a}}\mathcal{E}$ is a locally free $\mathcal{O}_X$-module.
- For any $i = 1, \dots, d$ and $a \in \mathbb{R}$, ${}^i\mathrm{Gr}_a(\mathcal{E})$ is a locally free $\mathcal{O}_{D_i}$-module.

For example, we define the trivial filtered bundle $\mathcal{O}_X(*D)$ as $P_{\boldsymbol{a}}\mathcal{O}_X(*D) = \mathcal{O}_X(\sum[a_i]D_i)$, where $[a_i] \in \mathbb{Z}$ is the greatest integer satisfying $[a_i] \leq a_i$.

The filtered bundle $P_*\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{E}_1, \mathcal{E}_2)$ over $\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{E}_1, \mathcal{E}_2)$ is defined as follows:

$$P_{\boldsymbol{a}}\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{E}_1, \mathcal{E}_2) = \{f \in \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) \mid f(P_{\boldsymbol{b}}\mathcal{E}_1) \subset P_{\boldsymbol{a}+\boldsymbol{b}}\mathcal{E}_2 \, (\forall \boldsymbol{b} \in \mathbb{R}^d)\}.$$

We denote $P_*\mathcal{H}om_{\mathcal{O}_X(*D)}(\mathcal{E}_1, \mathcal{E}_2)$ by $P_*\mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2)$ if there is no risk of confusion.

For any filtered bundle $P_*\mathcal{E}$, the dual filtered bundle $P_*(\mathcal{E}^\vee)$ is defined as $P_*\mathcal{H}om(\mathcal{E}, \mathcal{O}_X(*D))$, Then we have a natural isomorphism

$$P_{\boldsymbol{a}}(\mathcal{E}^\vee) \simeq (P_{<-\boldsymbol{a}+\boldsymbol{\delta}}\mathcal{E})^\vee = (\bigcup_{\boldsymbol{b}<-\boldsymbol{a}+\boldsymbol{\delta}} P_{\boldsymbol{b}}\mathcal{E})^\vee,$$

where $\boldsymbol{\delta} = (1, \dots, 1) \in \mathbb{R}^d$.

2.4.3. Stable filtered bundles on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$. We consider the case $X = \mathbb{P}^1 \times T^2$ and $D = (\{0\} \times T^2) \sqcup (\{\infty\} \times T^2)$, where T^2 is an elliptic curve. For a filtered bundle $P_*\mathcal{E}$, we write $P_{ab}\mathcal{E}$ instead of $P_{\boldsymbol{a}}\mathcal{E}$ for $\boldsymbol{a} = (a, b) \in \mathbb{R}^2$.

For a filtered bundle $P_{**}\mathcal{E}$, we define the parabolic degree of $P_{**}\mathcal{E}$ by

$$\text{par-deg}(P_{**}\mathcal{E}) = \frac{\sqrt{-1}}{2} \int_{\mathbb{P}^1 \times \{w_0\}} \text{par-}\mathrm{c}_1(P_{**}\mathcal{E}), \; (\forall w_0 \in T^2).$$

Definition 2.11. Let $P_{**}\mathcal{E}$ be a filtered bundle on (X, D).

- The filtered bundle $P_{**}\mathcal{E}$ is stable if it satisfies the following:
  (1) For any $a, b \in \mathbb{R}$, $P_{ab}\mathcal{E}|_{\{0\}\times T^2}$ and $P_{ab}\mathcal{E}|_{\{\infty\}\times T^2}$ are semistable and of degree 0.
  (2) For any filtered subsheaf $P_{**}\mathcal{E}' \subset P_{**}\mathcal{E}$ satisfying (1) and $0 < \mathrm{rank}(\mathcal{E}') < \mathrm{rank}(\mathcal{E})$, we have

$$\text{par-deg}(P_{**}\mathcal{E}')/\mathrm{rank}(\mathcal{E}') \ < \ \text{par-deg}(P_{**}\mathcal{E})/\mathrm{rank}(\mathcal{E}).$$

- The filtered bundle $P_{**}\mathcal{E}$ is polystable if there exists a decomposition $P_{**}\mathcal{E} = \bigoplus_{i\in I} P_{**}\mathcal{E}_i$ such that we have $P_{**}\mathcal{E}_i$ is stable and that $\text{par-deg}(P_{**}\mathcal{E}_i)/\mathrm{rank}(\mathcal{E}_i) = \text{par-deg}(P_{**}\mathcal{E}_j)/\mathrm{rank}(\mathcal{E}_j)$ holds for any $i, j \in I$.

We have the following cohomology vanishing for stable filtered bundles of degree 0.

Proposition 2.12. *Let $p : \mathbb{P}^1 \times T^2 \to T^2$ be the projection map. Let F be a holomorphic line bundle of degree* 0 *on T^2. For a stable filtered bundle $P_{**}E$ on $\mathbb{P}^1 \times T^2$ satisfying* $\text{par-deg}(P_{**}E) = 0$ *and* $\mathrm{rank}(E) > 1$*, we have $H^i(\mathbb{P}^1 \times T^2, P_{-\hat{t}\hat{t}}E \otimes p^*F) = H^i(\mathbb{P}^1 \times T^2, P_{<-\hat{t}<\hat{t}}E \otimes p^*F) = 0$ for any $\hat{t} \in \mathbb{R}$ and any $i \neq 1$.*

Proof. By replacing E with $E \otimes p^*F$, we may assume that F is trivial. By considering $P_{*-\hat{t},*+\hat{t}}E$ instead of $P_{**}E$, we may also assume $\hat{t} = 0$. If $i < 0$ or $i > 2$ holds, then the cohomologies vanish obviously. Thus we prove only the cases $i = 0$ and $i = 2$. If there exists a non-zero global section of $P_{00}E$, we have a filtered subsheaf $P_{**}\mathcal{O} \subset P_{**}E$ of rank 1 that satisfies (1) in Definition 2.11 and $\text{par-deg}(P_{**}\mathcal{O}) \geq 0$. However, it contradicts the stability of $P_{**}E$. Thus, we have $H^0(\mathbb{P}^1 \times T^2, P_{00}E) = 0$. By the natural inclusion $P_{<0<0}V \subset P_{00}V$, we have $H^0(\mathbb{P}^1 \times T^2, P_{<0<0}E) = 0$. By the natural isomorphism $P_{00}(E^\vee) \simeq (P_{<1<1}E)^\vee$, we have $H^0(\mathbb{P}^1 \times T^2, (P_{<1<1}E)^\vee) = 0$. Using the Serre duality theorem and isomorphisms $P_{<1<1}E \simeq P_{<0<0}E \otimes O_X(D) \simeq P_{<0<0}E \otimes (\Omega^2_{\mathbb{P}^1\times T^2})^{-1}$, we obtain $H^2(\mathbb{P}^1 \times T^2, P_{<1<1}E \otimes \Omega^2_{\mathbb{P}^1\times T^2}) = H^2(\mathbb{P}^1 \times T^2, P_{<0<0}E) = 0$. We also have $H^2(P_{00}E) = 0$ by the short exact sequence $0 \to P_{<0<0}V \to P_{00}V \to {}^1\mathrm{Gr}_0(P_{**}V) \oplus {}^2\mathrm{Gr}_0(P_{**}V) \to 0$ and trivial vanishing of cohomologies $H^2(T^2, {}^1\mathrm{Gr}_0(P_{**}V)) = H^2(T^2, {}^2\mathrm{Gr}_0(P_{**}V)) = 0$. □

2.5. The Fourier-Mukai transform of semistable bundles

We recall the Fourier-Mukai transform of semistable bundles of degree 0 on an elliptic curve $T^2 := \mathbb{C}/\Lambda_2$ by following [16, Subsubsection 2.1.2].

Let w be the standard coordinate of $\mathbb{C}$. We will denote by $\hat{T}^2$ the dual torus of T^2. Let V be a semistable bundle of degree 0 and of rank r on T^2. As a part of result in [16, Proposition 2.2], we have the following proposition.

Proposition 2.13. *There exist $k \in \mathbb{N}$, $F_i \in \mathrm{Pic}^0(T^2)$ and nilpotent matrices $N_i \in \mathrm{Mat}(\mathbb{C}, r_i)$ $(1 \leq i \leq k$ and $\sum_i r_i = r)$ such that we have $F_i \not\simeq F_j$ and an isomorphism $V \simeq \bigoplus_{i=1}^k F_i \otimes (\underline{\mathbb{C}^{r_i}}, \overline{\partial} + N_i d\bar{w})$. Moreover, if we take another isomorphism $V \simeq \bigoplus_{i=1}^{k'} F'_i \otimes (\underline{\mathbb{C}^{r'_i}}, \overline{\partial} + N'_i d\bar{w})$, then we have $k = k'$ and there exist a permutation σ of $\{1, \dots, k\}$ and linear transformations $g_i \in GL(r_i, \mathbb{C})$ such that we have $F_i \simeq F_{\sigma(i)}$, $r_i = r'_i$ and $N_i = \mathrm{Ad}(g_i) N'_i$ for any $1 \leq r \leq k$.*

Recall the spectrum of semistable bundle of degree 0 is defined as follows. (See [16].)

Definition 2.14. We take the decomposition $V \simeq \bigoplus_{i=1}^k F_i \otimes (\underline{\mathbb{C}^{r_i}}, \overline{\partial} + N_i d\bar{w})$ as in Proposition 2.13. We define the spectrum set $\mathrm{Spec}(V) \subset \hat{T}^2$ to be $\mathrm{Spec}(V) := \{F_1, \dots, F_k\}$ under the identification $\hat{T}^2 \simeq \mathrm{Pic}^0(T^2)$.

The following corollary is also obtained by [16, Proposition 2.2].

Corollary 2.15. *We take the decomposition $V \simeq \bigoplus_{i=1}^k F_i \otimes (\underline{\mathbb{C}^{r_i}}, \overline{\partial} + N_i d\bar{w})$ as in Proposition 2.13. Let $\mathcal{L} \to T^2 \times \hat{T}^2$ be the Poincaré bundle. Let p_i be the projection of $T^2 \times \hat{T}^2$ to the i-th component. For any $\alpha \in \mathrm{Spec}(V)$, we set a multi-index $I_\alpha = (i_{\alpha,1}, \dots, i_{\alpha,k_\alpha}) \in \mathbb{N}^{k_\alpha}$ as a tuple of the sizes of the Jordan blocks of N_i corresponding to α. The Fourier-Mukai transform $\mathrm{FM}(V) := Rp_{2*}(p_1^* V \otimes L) \in D^b(\mathrm{Coh}(\mathcal{O}_{\hat{T}^2}))$ is given as follows:*

$$
\begin{aligned}
&H^i(\mathrm{FM}(V)) = 0, \ (i \neq 1) \\
&H^1(\mathrm{FM}(V)) \simeq \bigoplus_{\alpha \in \mathrm{Spec}(V)} \bigoplus_{j=1}^{k_\alpha} \mathcal{O}_{\hat{T}^2,\alpha} / m_{\hat{T}^2,\alpha}^{i_{\alpha,j}},
\end{aligned}
$$

where $m_{\hat{T}^2,\alpha}$ is the maximal ideal of the stalk $\mathcal{O}_{\hat{T}^2,\alpha}$.

3. L^2-finite instantons on $\mathbb{R} \times T^3$

In this section, we fix a positive number $0 < \lambda < 1$. Recall that a function f is called of $C^{i,\lambda}$-class if f is of C^i-class and all derivatives of f of order i are locally λ-Hölder continuous.

3.1. Asymptotic behavior of solutions of the Nahm equation on $(0, \infty)$

For a T^3-invariant unitary connection A on $(\underline{\mathbb{C}^r}, \underline{h})$ on $(0, \infty) \times T^3$, the tuple $(\underline{\mathbb{C}^r}, \underline{h}, A)$ is an instanton if and only if the connection form of $A = \alpha\, dt + \sum_i A_i dx^i$ satisfies the Nahm equation:

$$\begin{cases} \dfrac{\partial A_1}{\partial t} + [\alpha, A_1] = -[A_2, A_3] \\[2ex] \dfrac{\partial A_2}{\partial t} + [\alpha, A_2] = -[A_3, A_1] \\[2ex] \dfrac{\partial A_3}{\partial t} + [\alpha, A_3] = -[A_1, A_2]. \end{cases} \tag{2}$$

For skew-Hermitian commuting matrices $\Gamma_i \in \mathfrak{u}(r)$ $(i = 1, 2, 3)$, let $\mathrm{Center}(\Gamma)$ denote the centralizer of $\Gamma := (\Gamma_i)$ in $\mathfrak{u}(r)$, i.e. $\mathrm{Center}(\Gamma) = \{a \in \mathfrak{u}(r) \mid [\Gamma_i, a] = 0\ (i = 1, 2, 3)\}$. If we take $N_i \in \mathrm{Center}(\Gamma)$ $(i = 1, 2, 3)$ satisfying the relations $N_i = [N_j, N_k]$ for any even permutation (ijk) of (123), then the tuple of $\alpha = 0$ and $A_i = \Gamma_i + N_i/t$ $(i = 1, 2, 3)$ forms a solution of (2) on $(0, \infty)$.

Definition 3.1. A pair of tuples $(\Gamma = (\Gamma_i), N = (N_i))$ as described above is called a model solution of the Nahm equation.

In [3, Corollary 2.2 and Proposition 3.1], Biquard proved the following theorem.

Theorem 3.2. *Let $(\alpha(t), A_i(t))$ be a solution of the Nahm equation of $C^{3,\lambda}$-class on $(0, \infty)$ of rank r. Then, there exist a model solution of the Nahm equation (Γ, N) and a $C^{4,\lambda}$-gauge transformation $g : (0, \infty) \to U(r)$ such that the following conditions are satisfied.*

(i) The gauge transformation g satisfies $g^{-1}\alpha g + g^{-1}\partial_t g = 0$.

(ii) We take the decomposition $g^{-1}A_i g - (\Gamma_i + N_i/t) = \varepsilon_{1,i}(t) + \varepsilon_{2,i}(t)$ satisfying $\varepsilon_{1,i}(t) \in \mathrm{Center}(\Gamma)$ and $\varepsilon_{2,i}(t) \in (\mathrm{Center}(\Gamma))^{\perp}$, where $(\mathrm{Center}(\Gamma))^{\perp}$ means the orthogonal complement of $\mathrm{Center}(\Gamma)$ in $\mathfrak{u}(r)$ with respect to the inner product $\langle A, B\rangle := -\mathrm{tr}(AB)$. Then, there exists $\delta > 0$ such that the following estimates hold for any $1 \le i \le 3$ and

$0 \leq j \leq 3$.

$$\begin{cases} |\partial_t^j \varepsilon_{1,i}(t)| & = O(t^{-(1+j+\delta)}) \\ |\partial_t^j \varepsilon_{2,i}(t)| & = O(\exp(-\delta t)) \end{cases}$$

Proof. By Corollary 2.2 and Proposition 3.1 in [3], there exist a positive number $\delta > 0$ and a gauge transformation $g_0 : (0, \infty) \to U(r)$ such that for the transformed solution $(\tilde{\alpha}, \tilde{A}_i) = (g_0^{-1} \alpha g_0 + g_0^{-1} \partial_t g_0, g_0^{-1} A_i g_0)$ we have the following estimates:

$$\begin{cases} |\tilde{\alpha}| & = O(\exp(-\delta t)) \\ |\tilde{\varepsilon}_{1,i}| & = O(t^{-(1+\delta)}) \\ |\tilde{\varepsilon}_{2,i}| & = O(\exp(-\delta t)), \end{cases} \tag{3}$$

where we set $\tilde{A}_i - (\Gamma_i + N_i/t) =: \tilde{\varepsilon}_{i,1} + \tilde{\varepsilon}_{i,2} \in \mathrm{Center}(\Gamma) \oplus (\mathrm{Center}(\Gamma))^{\perp}$. We take another gauge transform $g_1 : (0, \infty) \to U(r)$ satisfying the following conditions:

$$\begin{cases} g_1^{-1} \tilde{\alpha} g_1 + g_1^{-1} \partial_t g_1 = 0 \\ \lim_{t \to \infty} g_1(t) = \mathrm{Id}. \end{cases}$$

Then we have an estimate $|g_1(t) - \mathrm{Id}| = O(\exp(-\delta t))$. Hence the same estimate as (3) holds for the gauge transform $g := g_0 g_1$. Moreover, by definition of g_1 we have $g^{-1} \alpha g + g^{-1} \partial_t g = 0$, and this shows that g is of $C^{4,\lambda}$-class. For a permutation (ijk) of (123), the equation (2) can be written as follows:

$$\begin{aligned} \partial_t(\varepsilon_{1,i} + \varepsilon_{2,i}) =& [N_j/t, \varepsilon_{1,k}] + [\varepsilon_{1,j}, N_k/t] + [\varepsilon_{1,j} + \varepsilon_{2,j}, \varepsilon_{1,k} + \varepsilon_{2,k}] + \\ & [\Gamma_j + N_j/t, \varepsilon_{2,k}] + [\varepsilon_{2,j}, \Gamma_k + N_k/t]. \end{aligned} \tag{4}$$

Since we have $[\mathrm{Center}(\Gamma), \mathrm{Center}(\Gamma)] \subset \mathrm{Center}(\Gamma)$, by bootstrapping argument from (4) we obtain the desired estimates for derivatives of $\varepsilon_{1,i}(t)$ and $\varepsilon_{2,i}(t)$. □

3.2. Asymptotic behavior of L^2-finite instantons on $(0, \infty) \times T^3$

The following theorem is proved in [20, Lemma 3.3.2, Theorem 4.3.1, Corollary 4.3.3, Corollary 5.1.3 and Theorem 5.2.2].

Theorem 3.3. *Let (V, h, A) be an L^2-finite instanton on $(0, \infty) \times T^3$. If we take a sufficiently large $R > 0$, then there exist a positive number $\delta > 0$, a trivialization of $C^{4,\lambda}$-class $\sigma : (V, h)|_{(R,\infty) \times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$ and a T^3-invariant*

L^2-finite instanton $(\underline{\mathbb{C}^r}, \underline{h}, \tilde{A})$ on $(R,\infty)\times T^3$, such that we have the following estimates.

$$\begin{cases} ||A_\sigma||_{C^{3,\lambda}([t,t+1]\times T^3)} & = O(1) \\ ||A_\sigma - \tilde{A}||_{C^{3,\lambda}([t,t+1]\times T^3)} & = O(\exp(-\delta t)), \end{cases}$$

where A_σ is the connection form of A with respect to σ, and we identify $\tilde{A}$ with its connection form.

Proof. Since we have $\pi_1(T^3)\simeq \mathbb{Z}^3$, by considering parallel transport, it is proved that for any flat Hermitian vector bundle F on T^3 of rank r there exists a tuple of Hermitian commuting matrices $\Gamma = (\Gamma_i) \subset \mathfrak{u}(r)$ $(i=1,2,3)$ such that we have $F \simeq (\underline{\mathbb{C}^r}, \underline{h}, d+\sum_i \Gamma_i dx^i)$. By taking a suitable gauge transformation of $(\underline{\mathbb{C}^r}, \underline{h})$, we may assume the condition $\alpha_i - \beta_i \notin (2\pi\sqrt{-1})\mathbb{Z}$ for any two distinct simultaneous eigenvalues $\alpha = (\alpha_i), \beta = (\beta_i) \in (\sqrt{-1}\mathbb{R})^3$ and for any $i=1,2,3$. By Lemma 3.3.2, Theorem 4.3.1 and Corollary 4.3.3 in [20], there exist $R>0$, a commuting tuple of skew-Hermitian matrices $\Gamma = (\Gamma_i) \subset \mathfrak{u}(r)$ $(i=1,2,3)$ and a trivialization of $C^{4,\lambda}$-class $\tilde{\sigma} : (V,h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$ such that $||A_{\tilde{\sigma}} - \sum_i \Gamma_i dx^i||_{C^{3,\lambda}([t,t+1]\times T^3)} = o(1)$. In particular, we obtain $||A_{\tilde{\sigma}}||_{C^{3,\lambda}([t,t+1]\times T^3)} = O(1)$.

Let $\mathcal{A}_{L^2_3}$ be the space of L^2_3-connections of $(\underline{\mathbb{C}^r}, \underline{h})$ and $\mathcal{H} \subset \mathcal{A}_{L^2_3}$ be the Center manifold of the flat connection $\nabla_\Gamma = (d + \sum_i \Gamma_i dx^i)$ on $(\underline{\mathbb{C}^r}, \underline{h})$ in [20, Section 5.1]. By the definition of the Center manifold, we have $\nabla_\Gamma \in \mathcal{H}$. The T^3-action on T^3 itself induces the T^3-action on $\mathcal{A}_{L^2_3}$. Since ∇_Γ is T^3-invariant, by Corollary 5.1.3 in [20], $\mathcal{H}$ is a connected Riemannian manifold equipped with a T^3-action. By [20], we have the T^3-equivariant isometry $T_{\nabla_\Gamma}\mathcal{H} \simeq H^1\left(\Omega^*_{T^3}(\underline{\mathfrak{u}(r)}), \nabla_{ad(\Gamma)}\right)$, where $\nabla_{ad(\Gamma)}$ is the flat connection on $\underline{\mathfrak{u}(r)}$ induced by ∇_Γ with the adjoint representation of $\mathfrak{u}(r)$. However, any elements of $H^1(T^3, \Omega^*(\underline{\mathbb{C}^r}, \nabla_\Gamma))$ are T^3-invariant because of the assumption on simultaneous eigenvalues of Γ. Therefore, since $\mathcal{H}$ is connected, the T^3-action on $\mathcal{H}$ is trivial. Hence, by Lemma 3.3.2 and Theorem 5.2.2 in [20], there exist a positive number $\delta > 0$, a trivialization $\sigma : (V,h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$ and a T^3-invariant L^2-finite instanton $(\underline{\mathbb{C}^r}, \underline{h}, \tilde{A})$ on $(R,\infty)\times T^3$, such that we have $||A_\sigma - \tilde{A}||_{C^{3,\lambda}([t,t+1]\times T^3)} = O(\exp(-\delta t))$, which completes the proof. □

We obtain the following corollary as a consequence of Theorem 3.3 and Theorem 3.2.

Corollary 3.4. *Let (V,h,A) be an L^2-finite instanton on $(0,\infty)\times T^3$ of rank r. If we take a sufficiently large $R>0$, then there exist a positive number $\delta>0$, a trivialization of $C^{4,\lambda}$-class $\sigma:(V,h)|_{(R,\infty)\times T^3}\simeq(\underline{\mathbb{C}^r},\underline{h})$ and a model solution of the Nahm equation (Γ,N) such that the following holds.*

(i) Let $\alpha dt+\sum_i A_i dx^i$ denote the connection form of A with respect to σ. Then we have $\alpha=0$.

(ii) There exist decompositions $A_i-(\Gamma_i+N_i/t)=\varepsilon_{1,i}(t)+\varepsilon_{2,i}(t)+\varepsilon_{3,i}(t,x)$ such that we have $\varepsilon_{1,i}(t)\in\mathrm{Center}(\Gamma)$, $\varepsilon_{2,i}(t)\in(\mathrm{Center}(\Gamma))^{\perp}$ and that the following estimates for $1\le i\le 3$ and $0\le j\le 3$,

$$\begin{cases} |\partial_t^j\varepsilon_{1,i}| & =O(t^{-(1+j+\delta)})\\ |\partial_t^j\varepsilon_{2,i}| & =O(\exp(-\delta t))\\ ||\varepsilon_{3,i}||_{C^{3,\lambda}([t,t+1]\times T^3)} & =O(\exp(-\delta t)).\end{cases}$$

Remark 3.5. For any model solution (Γ,N), the tuple of $\alpha=0$ and $A_i:=\Gamma_i+N_i/t$ also forms a solution of the Nahm equation on $(-\infty,0)$. Hence, a similar result holds for L^2-finite instantons on $(-\infty,0)\times T^3$.

Let (V,h,A) be an L^2-finite instanton on $\mathbb{R}\times T^3$. Applying Corollary 3.4 to $(V,h,A)|_{(0,\infty)\times T^3}$ and $(V,h,A)|_{(-\infty,0)\times T^3}$, we obtain model solutions $(\Gamma_\pm,N_\pm)$ which approximate (V,h,A) at $t\to\pm\infty$.

Since the simultaneous eigenvalues of $\sum_i\Gamma_{\pm,i}dx^i\in\Omega^1_{T^3}(\mathfrak{u}(r))$ are T^3-invariant pure imaginary 1-forms on T^3, they can be regarded as elements of $\mathrm{Hom}(\mathbb{R}^3,\sqrt{-1}\mathbb{R})$. Thus we take $\widetilde{\mathrm{Spec}(\Gamma_\pm)}\subset\mathrm{Hom}(\mathbb{R}^3,\mathbb{R})$ as the set of $(2\pi\sqrt{-1})^{-1}$ times simultaneous eigenvalues of $\sum_i\Gamma_{\pm,i}dx^i$. We take unitary representations $\rho_\pm:\mathfrak{su}(2)\to\mathfrak{u}(r)$ induced by $N_\pm$ to be $\rho(\sum_i a_ie_i):=\sum_i a_iN_{\pm,i}$, where $(e_i)_{i=1,2,3}$ is a basis of $\mathfrak{su}(2)$ satisfying $e_i=[e_j,e_k]$ for any even permutation (ijk) of (123). Because $N_{\pm,i}\in\mathrm{Center}(\Gamma_\pm)$, we have the decomposition $\rho_\pm=\bigoplus_{\xi\in\widetilde{\mathrm{Spec}(\Gamma_\pm)}}\rho_{\pm,\xi}$ which is induced by the simultaneous eigen decomposition of $\Gamma_\pm$.

Definition 3.6.

- We define the spectrum set $\mathrm{Spec}(\Gamma_\pm)\subset\hat{T}^3$ to be the image of $\widetilde{\mathrm{Spec}(\Gamma_\pm)}$ by the quotient map $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R})\to\hat{T}^3$.
- We define the singularity set of (V,h,A) as $\mathrm{Sing}(V,h,A):=\mathrm{Spec}(\Gamma_+)\cup\mathrm{Spec}(\Gamma_-)$.

- We assume that the quotient map $\widetilde{\mathrm{Spec}(\Gamma_\pm)} \to \mathrm{Spec}(\Gamma_\pm)$ is bijective, and identify $\widetilde{\mathrm{Spec}(\Gamma_\pm)}$ with $\mathrm{Spec}(\Gamma_\pm)$. (See below Remark 3.7 (i).) For $\xi \in \mathrm{Sing}(V,h,A)$, we define the unitary representations $\rho_{\pm,\xi}$ of $\mathfrak{su}(2)$ by putting $\rho_{\pm,\xi} := 0$ for $\xi \in \mathrm{Sing}(V,h,A) \setminus \mathrm{Spec}(\Gamma_\pm)$.

Remark 3.7.

(i) We assume that there exists a fundamental domain $H \subset \mathrm{Hom}(\mathbb{R}^3,\mathbb{R})$ of $\hat{T}^3$ such that $0 \in H$ and $\widetilde{\mathrm{Spec}(\Gamma_\pm)} \subset H$. Indeed, we can always take a suitable $\mathbb{R}$-invariant gauge transformation to satisfy this assumption.

(ii) For $\xi \in \hat{T}^3$, we take a flat Hermitian line bundle L_ξ on $\mathbb{R} \times T^3$ as in subsection 2.1. For any $\xi \in \hat{T}^3$ and any L^2-finite instanton (V,h,A), we define $(V,h,A_\xi) := (V,h,A) \otimes L_{-\xi}$. Then we have

$$\mathrm{Sing}(V,h,A_\xi) = \mathrm{Sing}(V,h,A) - \xi = \{\mu - \xi \mid \mu \in \mathrm{Sing}(V,h,A)\}.$$

3.3. Fredholmness of Dirac operators

Let (V,h,A) be an L^2-finite instanton on $\mathbb{R} \times T^3$. Take $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V,h,A)$, and we set $(V,h,A_\xi) := (V,h,A) \otimes L_{-\xi}$. We shall study the Dirac operators associated to (V,h,A_ξ) by following [6].

Let σ_0 be a global trivialization (V,h) (see Corollary 2.5). Let $R_\pm > 0$ be constants as in Corollary 3.4 with the L^2-finite instantons $(V,h,A)|_{(0,\infty)\times T^3}$ and $(V,h,A)|_{(-\infty,0)\times T^3}$ respectively. We set $R := \max(R_+, R_-)$. We also denote by $\sigma_\pm$ trivializations of (V,h) on $(R,\infty) \times T^3$ and $(-\infty,-R) \times T^3$ in Corollary 3.4 respectively. Let σ denote the triple $(\sigma_-, \sigma_0, \sigma_+)$. Let $S_{\mathbb{R}\times T^3} = S^+ \oplus S^-$ denote the spinor bundle of $\mathbb{R} \times T^3$ with respect to the trivial spin structure.

Definition 3.8. For $0 \le k \le 4$, we define a norm $||\cdot||_{L^2_{k,\sigma}} : L^2_{k,\mathrm{loc}}(\mathbb{R} \times T^3, V \otimes S^\pm) \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ as follows:

$$\begin{aligned} ||f||^2_{L^2_{k,\sigma}} &= ||\sigma_+(f)||^2_{L^2_k([R+1,\infty))} + ||\sigma_0(f)||^2_{L^2_k([-(R+2),R+2])} \\ &\quad + ||\sigma_-(f)||^2_{L^2_k((-\infty,-(R+1)])}. \end{aligned}$$

We set $L^2_{k,\sigma}(\mathbb{R}\times T^3, V\otimes S^\pm) := \left\{ f \in L^2_{k,\mathrm{loc}}(\mathbb{R}\times T^3, V\otimes S^\pm) \,\middle|\, ||f||_{L^2_{k,\sigma}} < \infty \right\}.$

Remark 3.9. Since $||f||_{L^2} \le ||f||_{L^2_{0,\sigma}} \le 3||f||_{L^2}$, the ordinary L^2-norm and $L^2_{0,\sigma}$-norm are equivalent.

Let S_{T^3} be the spinor bundle on T^3 with respect to the trivial spin structure. Let $p : \mathbb{R} \times T^3 \to T^3$ is the projection map. There exists the isomorphism $S^{\pm} \simeq p^* S_{T^3}$ such that the Clifford product can be written as follows.

$$\begin{aligned} \mathrm{clif}(dt) &= \begin{pmatrix} 0 & -\mathrm{Id}_{S_{T^3}} \\ \mathrm{Id}_{S_{T^3}} & 0 \end{pmatrix}, \\ \mathrm{clif}(dx^i) &= \begin{pmatrix} 0 & \mathrm{clif}_{T^3}(dx^i) \\ \mathrm{clif}_{T^3}(dx^i) & 0 \end{pmatrix}. \end{aligned}$$

Hence we obtain the following lemma.

Lemma 3.10. *Under the identification between S^+, S^- and $p_2^* S_{T^3}$, the Dirac operators $\partial\!\!\!/_A^{\pm} : V \otimes S^{\pm} \to V \otimes S^{\mp}$ with respect to the trivialization $\sigma_{\pm}$ can be written as follows:*

$$\partial\!\!\!/_A^{\pm} = \pm \frac{\partial}{\partial t} + D\!\!\!\!/_{A|_{\{t\}\times T^3}},$$

where $D\!\!\!\!/_{A|_{\{t\}\times T^3}}$ is the Dirac operator of $(V, A)|_{\{t\}\times T^3}$ on T^3.

Proof. The connection forms of A with respect to $\sigma_{\pm}$ are temporal *i.e.* they have no dt terms. □

Proposition 3.11. *For any $1 \le k \le 4$, there exist $K_k, C_k > 0$ such that the following estimates hold for any $f \in L^2_{k,\sigma}(\mathbb{R} \times T^3, V \otimes S^{\pm})$.*

$$||f||_{L^2_{k,\sigma}} \le C_k \left(||f||_{L^2(|t|<K_k)} + ||\partial\!\!\!/_{A_\xi}^{\pm}(f)||_{L^2_{k-1,\sigma}} \right)$$

Proof. By considering (V, h, A_ξ) instead of (V, h, A) we may assume $\xi = 0$. We consider the case $k = 1$. By the assumption $\xi = 0 \notin \mathrm{Sing}(V, h, A)$ we have $\mathrm{Ker}(D\!\!\!\!/_{\Gamma_\pm}) = \{0\}$, where $\Gamma_\pm = d + \sum_i (\Gamma_\pm)_i dx^i$ are flat unitary connections on the product bundle $(\underline{\mathbb{C}^r}, \underline{h})$ on T^3. Thus we can easily prove by using the Fourier series expansion that there exists $B_1 > 0$ such that for any section $g \in L^2_1(T^3, \underline{\mathbb{C}^r} \otimes S_{T^3})$ we have

$$B_1 ||g||_{L^2_1(T^3)} \le ||D\!\!\!\!/_{\Gamma_\pm}(g)||_{L^2(T^3)}.$$

Take $U_1 > R + 1$ such that

$$(5) \quad ||A_{\sigma_+} - \Gamma_+||_{C^0(t > U_1 - 1)} < B_1/4, \quad ||A_{\sigma_-} - \Gamma_-||_{C^0(t < -(U_1 - 1))} < B_1/4.$$

We take a smooth function $\varphi^+_{U_1} : \mathbb{R} \to [0,1]$ which satisfies the following:

$$\varphi^+_{U_1}(t) = \begin{cases} 1 & (t > U_1) \\ 0 & (t < U_1 - 1). \end{cases}$$

We also set $\varphi^-_{U_1}(t) := \varphi^+_{U_1}(-t)$ and $\varphi_{U_1}(t) := \varphi^+_{U_1}(t) + \varphi^-_{U_1}(t)$. For any $f \in L^2_{1,\sigma}(\mathbb{R} \times T^3, V \otimes S^\pm)$, we have

$$\begin{aligned} ||\partial\!\!\!/^{\pm}_A(\varphi_{U_1} f)||_{L^2} &= ||\partial_t(\varphi_{U_1} f) \pm D\!\!\!\!/_A(\varphi_{U_1} f)||_{L^2} \\ &\geq ||\partial_t(\varphi_{U_1} f) \pm \left(D\!\!\!\!/_{\Gamma_+}(\varphi^+_{U_1} f) + D\!\!\!\!/_{\Gamma_-}(\varphi^-_{U_1} f)\right)||_{L^2} \\ &\quad - ||\mathrm{clif}(A_{\sigma_+} - \Gamma_+) \cdot (\varphi^+_{U_1} f)||_{L^2} \\ &\quad - ||\mathrm{clif}(A_{\sigma_-} - \Gamma_-) \cdot (\varphi^-_{U_1} f)||_{L^2}. \end{aligned}$$

Here we use an equality $\langle \partial_t(\varphi^\pm_{U_1} f), D\!\!\!\!/_{\Gamma_\pm}(\varphi^\pm_{U_1} f)\rangle_{L^2} = -\langle D\!\!\!\!/_{\Gamma_\pm}(\varphi^\pm_{U_1} f), \partial_t(\varphi^\pm_{U_1} f)\rangle_{L^2}$, then we have

$$\begin{aligned} ||\partial\!\!\!/^{\pm}_A(\varphi_{U_1} f)||_{L^2} \geq \frac{1}{3}&\left(||\partial_t(\varphi_{U_1} f)||_{L^2} + ||D\!\!\!\!/_{\Gamma_+}(\varphi^+_{U_1} f)||_{L^2} + ||D\!\!\!\!/_{\Gamma_-}(\varphi^-_{U_1} f)||_{L^2}\right) \\ &- ||\mathrm{clif}(A_{\sigma_+} - \Gamma_+) \cdot (\varphi^+_{U_1} f)||_{L^2} - ||\mathrm{clif}(A_{\sigma_-} - \Gamma_-) \cdot (\varphi^-_{U_1} f)||_{L^2}. \end{aligned}$$

By this inequality and the inequalities (5), there exists $B_2 > 0$ such that we have

$$||\partial\!\!\!/^{\pm}_A(\varphi_{U_1} f)||_{L^2} \geq B_2 ||\varphi_{U_1} f||_{L^2_{1,\sigma}}.$$

Therefore, the following inequalities hold.

$$\begin{aligned} ||f||_{L^2_{1,\sigma}} &\leq ||\varphi_{U_1} f||_{L^2_{1,\sigma}} + ||f||_{L^2_{1,\sigma}(|t|<U_1)} \\ &\leq {B_2}^{-1} ||\partial\!\!\!/^{\pm}_A(\varphi_{U_1} f)||_{L^2} + ||f||_{L^2_{1,\sigma}(|t|<U_1)} \\ &\leq {B_2}^{-1}(||\varphi_{U_1} \partial\!\!\!/^{\pm}_A(f)||_{L^2} + ||\mathrm{clif}(d\varphi_{U_1}) f||_{L^2}) + ||f||_{L^2_{1,\sigma}(|t|<U_1)} \\ &\leq {B_2}^{-1}(||\partial\!\!\!/^{\pm}_A(f)||_{L^2}) + (1 + {B_2}^{-1} ||\partial_t \varphi_{U_1}||_{L^\infty}) ||f||_{L^2_{1,\sigma}(|t|<U_1)} \end{aligned}$$

Applying the interior estimate for elliptic operators to the above inequality, we obtain an inequality

$$||f||_{L^2_{1,\sigma}} \leq C_1 \left(||f||_{L^2([-K_1,K_1]\times T^3)} + ||\partial\!\!\!/^{\pm}_A(f)||_{L^2} \right),$$

where $C_1 > 0$ and $K_1 > U_1$ are constants independent of f. This is the desired inequality for $k = 1$.

We use an induction on k. Suppose that we have already obtained the desired inequality in the case $k = k_0$. We take a constant $U_2 > K_{k_0}$ and take a function φ_{U_2} as above. Then, for any $f \in L^2_{k_0+1,\sigma}$ we have

$$
\begin{aligned}
||f||_{L^2_{k_0+1,\sigma}} &\leq ||f||_{L^2_{k_0+1,\sigma}(|t|<K_2)} + ||\varphi_{U_2} f||_{L^2_{k_0+1,\sigma}} \\
&\leq ||f||_{L^2_{k_0+1,\sigma}(|t|<U_2)} + ||\varphi_{U_2} f||_{L^2_{k_0,\sigma}} + ||\partial_t(\varphi_{U_2} f)||_{L^2_{k_0,\sigma}} \\
&\quad + \sum_{i=1}^{3} ||\partial_i(\varphi_{U_2} f)||_{L^2_{k_0,\sigma}},
\end{aligned} \tag{6}
$$

where $\partial_t(\varphi_{U_2} f)$ and $\partial_i(\varphi_{U_2} f)$ are taken under the trivializations $\sigma_{\pm}$. Since we can apply the interior estimate of elliptic operators and the assumption of the induction to the first and second terms of (6), there exist $B_3 > 0$ and $U_3 > U_2$ such that we have

$$
||f||_{L^2_{k_0+1,\sigma}(|t|<U_2)} + ||\varphi_{U_2} f||_{L^2_{k_0,\sigma}} \leq B_3 \left(||f||_{L^2(|t|<U_3)} + ||\not{\partial}^{\pm}_A(f)||_{L^2_{k_0,\sigma}} \right). \tag{7}
$$

We also make an estimate of the third and fourth terms of (6) as follows:

$$
\begin{aligned}
&||\partial_t(\varphi_{U_2} f)||_{L^2_{k_0,\sigma}} + \sum_{i=1}^{3} ||\partial_i(\varphi_{U_2} f)||_{L^2_{k_0,\sigma}} \\
&\leq ||\not{\partial}^{\pm}_A(\partial_t(\varphi_{U_2} f))||_{L^2_{k_0-1,\sigma}} + \sum_{i=1}^{3} ||\not{\partial}^{\pm}_A(\partial_i(\varphi_{U_2} f))||_{L^2_{k_0-1,\sigma}} \\
&\leq ||\partial_t(\not{\partial}^{\pm}_A(\varphi_{U_2} f))||_{L^2_{k_0-1,\sigma}} + ||[\not{\partial}^{\pm}_A, \partial_t](\varphi_{U_2} f)||_{L^2_{k_0-1,\sigma}} \\
&\quad + \sum_{i=1}^{3} \left\{ ||\partial_i(\not{\partial}^{\pm}_A(\varphi_{U_2} f))||_{L^2_{k_0-1,\sigma}} + ||[\not{\partial}^{\pm}_A, \partial_i](\varphi_{U_2} f)||_{L^2_{k_0-1,\sigma}} \right\} \\
&\leq 4||\not{\partial}^{\pm}_A(\varphi_{U_2} f)||_{L^2_{k_0,\sigma}} + ||[\not{\partial}^{\pm}_A, \partial_t](\varphi_{K_2} f)||_{L^2_{k_0-1,\sigma}} \\
&\quad + \sum_{i=1}^{3} ||[\not{\partial}^{\pm}_A, \partial_i](\varphi_{U_2} f)||_{L^2_{k_0-1,\sigma}} \\
&\leq B_4(||f||_{L^2([|t|<U_4)} + ||\not{\partial}^{\pm}_A(f)||_{L^2_{k_0,\sigma}}).
\end{aligned} \tag{8}
$$

Here $B_4 > 0$ and $U_4 > U_2$ is a constant independent of f. As a consequence of (6), (7) and (8), we obtain the desired inequality for $k = k_0 + 1$, and the proof is complete. $\square$

Corollary 3.12. *For* $0 \leq k \leq 3$, *if* $f \in L^2_k(\mathbb{R} \times T^3, V \times S^\pm)$ *satisfies* $\partial\!\!\!/^{\pm}_{A_\xi}(f) = g \in L^2_k(\mathbb{R} \times T^3, V \times S^\mp)$ *as a distribution, then* $f \in L^2_{k+1,\sigma}(\mathbb{R} \times T^3, V \times S^\pm)$.

Proof. By the regularity of elliptic operators, we have $f \in L^2_{k+1,\mathrm{loc}}(\mathbb{R} \times T^3, V \times S^\pm)$. For $n \in \mathbb{N}$, we take bump functions $\varphi_n : \mathbb{R} \to [0,1]$ satisfying

$$\varphi_n(t) = \begin{cases} 1 & (|t| < n) \\ 0 & (|t| > n+1). \end{cases}$$

From Proposition 3.11, there exist $C, K > 0$ such that we have

$$\begin{aligned} &||\varphi_n f - \varphi_m f||_{L^2_{k+1,\sigma}} \\ &\quad \leq C \left(||\mathrm{clif}(d\varphi_n) f||_{L^2_{k,\sigma}} + ||\mathrm{clif}(d\varphi_m) f||_{L^2_{k,\sigma}} + ||(\varphi_n - \varphi_m) g||_{L^2_{k,\sigma}} \right) \end{aligned}$$

for any natural numbers $n, m > K$. Hence $\{\varphi_n f\}$ is a Cauchy sequence in $L^2_{k+1,\sigma}$. Moreover, this sequence converges pointwise to f. Therefore $f \in L^2_{k+1,\sigma}(\mathbb{R} \times T^3, V \times S^\pm)$. $\square$

Theorem 3.13. *For* $1 \leq k \leq 4$, *the operators*

$$\partial\!\!\!/^{\pm}_{A_\xi} : L^2_{k,\sigma}(\mathbb{R} \times T^3, V \times S^\pm) \to L^2_{k-1,\sigma}(\mathbb{R} \times T^3, V \times S^\mp)$$

are Fredholm, and $\mathrm{Ker}(\partial\!\!\!/^{\pm}_{A_\xi})$ *are independent of* k.

Proof. By Corollary 3.12, it suffices to prove the case $k = 1$. Thus we prove the following assertions.

(i) $\dim(\mathrm{Ker}(\partial\!\!\!/^{\pm}_{A_\xi})) < \infty$

(ii) $\dim(\mathrm{Cok}(\partial\!\!\!/^{\pm}_{A_\xi})) < \infty$

(iii) $R(\partial\!\!\!/^{\pm}_{A_\xi}) \subset L^2(V \times S^\mp)$ is closed

If a normed space has a relatively compact neighborhood of the origin, then it is finite dimensional. Hence (i) is an easy consequence of Proposition 3.11 and the compactness of the restriction map $L^2_{1,\sigma}(\mathbb{R} \times T^3, V \otimes S^\pm) \to L^2([-K,K] \times T^3, V \otimes S^\pm)$. By Corollary 3.12, we have $\mathrm{Cok}(\partial\!\!\!/^{\pm}_{A_\xi}) = \mathrm{Ker}(\partial\!\!\!/^{\mp}_{A_\xi})$. Therefore, (ii) is deduced from (i).

To prove (iii), it is enough to prove that there exists $C > 0$ such that for any $f \in (\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi}))^{\perp_{L^2}}$ we have

$$||f||_{L^2_{1,\sigma}} \leq C||\not{\partial}^{\pm}_{A_\xi}(f)||_{L^2}, \tag{9}$$

where $(\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi}))^{\perp_{L^2}}$ means the orthogonal complement of $\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi})$ in $L^2_{1,\sigma}$ with respect to the ordinary L^2 inner product. Suppose that there is no constant $C > 0$ satisfying the inequality (9) for any $f \in (\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi}))^{\perp_{L^2}}$. Take $f_n \in (\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi}))^{\perp_{L^2}}$ satisfying $||f_n||_{L^2_{1,\sigma}} = 1 > n||\not{\partial}^{\pm}_{A_\xi}(f_n)||_{L^2}$ for any $n \in \mathbb{N}$. Since the restriction map $L^2_{1,\sigma}(\mathbb{R} \times T^3, V \otimes S^{\pm}) \to L^2([-K, K] \times T^3, V \otimes S^{\pm})$ is compact, we may assume that $\{f_n|_{[-K,K]\times T^3}\}$ converges in $L^2([-K, K] \times T^3, V \otimes S^{\pm})$. We have $||\not{\partial}^{\pm}_{A_\xi}(f_n)||_{L^2} < 1/n \to 0$ $(n \to \infty)$, and hence $\{f_n\}$ also converges to some $f_\infty \in L^2_{1,\sigma}(\mathbb{R} \times T^3, V \otimes S^{\pm})$ by Proposition 3.11. Then we have $f_\infty \in \mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi})$ and $f_\infty \neq 0$. This contradicts $f_n \in (\mathrm{Ker}(\not{\partial}^{\pm}_{A_\xi}))^{\perp_{L^2}}$. Therefore the inequality (9) holds for some $C > 0$. $\square$

3.4. Index of Dirac operators

We calculate the index of Dirac operators by following Charbonneau [6].

Theorem 3.14. *Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$ of rank r and take $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$. The index of $\not{\partial}^{+}_{A_\xi}$ is given by*

$$\mathrm{index}(\not{\partial}^{+}_{A_\xi}) = -\frac{1}{8\pi^2}||F(A_\xi)||^2_{L^2} = -\frac{1}{8\pi^2}||F(A)||^2_{L^2}.$$

Proof. Replacing (V, h, A) with (V, h, A_ξ), and we may assume $\xi = 0$. We may also assume that any $\Gamma_{\pm,i}$ are diagonal matrices because $\Gamma_{\pm} = (\Gamma_{\pm,i})$ are commuting Hermitian matrices. Take a positive constant $K > R$, and a partition of unity $\{\phi_-, \phi_0, \phi_+\}$ on $\mathbb{R}$ which is subordinate to the open cover $\{(-\infty, -K), (-(K + 1), K + 1), (K + 1, \infty)\}$. We set a connection $a_K := \phi_-\Gamma_- + \phi_0 A + \phi_+\Gamma_+$, where $\Gamma_{\pm}$ are the connections given by $d + \sum_i \Gamma_{\pm,i}dx^i$ with respect to the trivialization $\sigma_{\pm}$. Then the difference $\not{\partial}^{+}_{A} - \not{\partial}^{+}_{a_K} = \mathrm{clif}\,(\phi_-(A - \Gamma_-) + \phi_+(A - \Gamma_+))$ is a compact operator, hence we have

$$\mathrm{index}(\not{\partial}^{+}_{A}) = \mathrm{index}(\not{\partial}^{+}_{a_K}).$$

We take a continuous family of flat unitary connections $\{\Gamma_s\}_{s\in[0,1]}$ on $(V, h)|_{(R,\infty)\times T^3}$ satisfying the following conditions:

- $\Gamma_0 = \Gamma_+$.

- The connection form of Γ_1 with respect to σ_+ is given by $\sum_i \Gamma_{-,i} dx^i$.
- For any $s \in [0,1]$, $0 \notin \mathrm{Spec}(\Gamma_s)$.

We set connections $\{a_K^s\}$ on (V,h) as $a_K^s = \phi_- \Gamma_- + \phi_0 A_{xi} + \phi_+ \Gamma_s$. Then, $\{\partial\!\!\!/^+_{a_K^s}\}$ forms a continuous family of Fredholm operators. Hence we have

$$\mathrm{index}(\partial\!\!\!/^+_A) = \mathrm{index}(\partial\!\!\!/^+_{a_K}) = \mathrm{index}(\partial\!\!\!/^+_{a_K^1}).$$

We construct a Hermitian vector bundle $(\tilde{V}, \tilde{h})$ on a four-dimensional torus T^4 by gluing (V,h) on $t < -(K+1)$ and $t > K+1$ with trivializations $\sigma_\pm$ respectively. Since the connection forms of a_K^1 on $|t| > K+1$ with respect to σ_+ and σ_- are equal, we also construct a connection $\widetilde{a_K^1}$ on $(\tilde{V}, \tilde{h})$ from a_K^1. Then the relative index theorem in [11] tells us

$$\mathrm{index}(\partial\!\!\!/^+_{a_K^1}) - \mathrm{index}(\partial\!\!\!/^+_{\gamma_-}) = \mathrm{index}(\partial\!\!\!/^+_{\widetilde{a_K^1}}) - \mathrm{index}(\partial\!\!\!/^+_{\widetilde{\gamma_-}}), \tag{10}$$

where γ_- (resp. $\widetilde{\gamma_-}$) is a flat connection on the product bundle $(\underline{\mathbb{C}^r}, \underline{h})$ on $\mathbb{R} \times T^3$ (resp. T^4) whose connection form is given by $\sum_i \Gamma_{-,i} dx^i$. By the assumption $\xi = 0 \notin \mathrm{Sing}(V,h,A)$, we have $\mathrm{index}(\partial\!\!\!/^+_{\gamma_-}) = \mathrm{index}(\partial\!\!\!/^+_{\widetilde{\gamma_-}}) = 0$. Hence we obtain $\mathrm{index}(\partial\!\!\!/^+_{a_K^1}) = \mathrm{index}(\partial\!\!\!/^+_{\widetilde{a_K^1}})$. By the Atiyah-Singer index theorem, we obtain

$$\mathrm{index}(\partial\!\!\!/^+_{\widetilde{a_K^1}}) = \mathrm{ch}_2(\widetilde{a_K^1})/[T^4].$$

Hence we have

$$\begin{aligned}\mathrm{index}(\partial\!\!\!/^+_A) &= \frac{1}{8\pi^2} \int_{T^4} \mathrm{Tr}\left(F(\widetilde{a_K^1}) \wedge F(\widetilde{a_K^1})\right) \\ &= \frac{1}{8\pi^2} \int_{[-(K+1), K+1] \times T^3} \mathrm{Tr}\left(F(a_K^1) \wedge F(a_K^1)\right).\end{aligned}$$

Since any $\Gamma_{\pm,i}$ are assumed to be diagonal matrices, by Corollary 3.4 we have

$$\begin{aligned}&\left| \int_{[-(K+1),K+1]\times T^3} \mathrm{Tr}\left(F(a_K^1) \wedge F(a_K^1)\right) - \int_{\mathbb{R}\times T^3} \mathrm{Tr}\left(F(A) \wedge F(A)\right) \right| \\ &\quad = O(K^{-2}).\end{aligned}$$

Taking the limit of $K \to \infty$, we obtain

$$\mathrm{index}(\partial\!\!\!/^+_A) = \frac{1}{8\pi^2} \int_{\mathbb{R}\times T^3} \mathrm{Tr}\left(F(A) \wedge F(A)\right) = -\frac{||F_A||^2_{L^2}}{8\pi^2},$$

which proves the theorem. □

Remark 3.15. Let $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$. Since $\mathbb{R} \times T^3$ has infinite volume, the Weitzenböck formula $\partial\!\!\!/^{-}_{A_\xi} \partial\!\!\!/^{+}_{A_\xi} = \nabla^*_{A_\xi} \nabla_{A_\xi}$ tells us

$$\dim\left(\mathrm{Ker}(\partial\!\!\!/^{-}_{A_\xi})\right) = \dim\left(\mathrm{Cok}(\partial\!\!\!/^{+}_{A_\xi})\right) = \frac{||F_{A_\xi}||^2_{L^2}}{8\pi^2},$$
$$\dim\left(\mathrm{Ker}(\partial\!\!\!/^{+}_{A_\xi})\right) = \dim\left(\mathrm{Cok}(\partial\!\!\!/^{-}_{A_\xi})\right) = 0.$$

3.5. Asymptotic behavior of Harmonic spinors

Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$.

Proposition 3.16. *There exist $K, \kappa : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that conditions are satisfied.*

- *$K(d), \kappa(d)^{-1} = O(d^{-1})$ as $d \to 0$.*
- *Let $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$ and $f \in \mathrm{Ker}(\partial\!\!\!/^{-}_{A_\xi}) \cap L^2$. We set the function $F(t) := \int_{\{t\} \times T^3} |f(t, x)|^2 dx$, where dx means the volume form of T^3. For any $t > K(d)$ (resp. $t < -K(d)$) we have $F'(t) \leq -\kappa(d)F(t)$ (resp. $F'(t) \geq \kappa(d)F(t)$). Here we abbreviate* $\mathrm{dist}(\xi, \mathrm{Sing}(V, h, A))$ *to d.*

Proof. We may assume $f \neq 0$. By the interior regularity of elliptic operators, we have $F \in C^4(\mathbb{R}) \cap L^1(\mathbb{R})$. Hence we can calculate derivatives of F.

$$F'(t) = 2 \int_{T^3} \langle \partial_t f, f \rangle dx.$$
$$F''(t) = 2 \left\{ \int_{T^3} \langle \partial_t^2 f, f \rangle dx + 2 \int_{T^3} |\partial_t f|^2 dx \right\}.$$

By Lemma 3.10, Dirac operators with respect to σ_+ can be written as $\partial\!\!\!/^{-}_{A_\xi} = -\partial_t + D\!\!\!\!/_{A_\xi|_{\{t\} \times T^3}}$. Thus, for $t > R$ we have

$$F'(t) = 2 \int_{T^3} \langle D\!\!\!\!/_{A_z} f, f \rangle dx.$$

$$\begin{aligned}
F''(t) =&2\int_{T^3}\langle\partial_t(\not{D}_{A_\xi}(f)), f\rangle dx + 2\int_{T^3}|\not{D}_{A_\xi} f|^2 dx \\
=&2\int_{T^3}\langle\not{D}_{A_\xi}(\partial_t(f)), f\rangle dx + 2\int_{T^3}\langle[\partial_t, \not{D}_{A_\xi}](f), f\rangle dx \\
&+ 2\int_{T^3}|\not{D}_{A_\xi} f|^2 dx \\
=&2\int_{T^3}\langle\not{D}_{A_\xi}(\not{D}_{A_\xi}(f)), f\rangle dx + 2\int_{T^3}\langle[\partial_t, \not{D}_{A_\xi}](f), f\rangle dx \\
&+ 2\int_{T^3}|\not{D}_{A_\xi} f|^2 dx \\
=&4\int_{T^3}|\not{D}_{A_\xi} f|^2 dx + 2\int_{T^3}\langle[\partial_t, \not{D}_{A_\xi}](f), f\rangle dx \\
=&4\int_{T^3}|\not{D}_{\Gamma_{+,\xi}}(f) + \mathrm{clif}(A-\Gamma_+)(f)|^2 dx + 2\int_{T^3}\langle[\partial_t, \not{D}_{A_\xi}](f), f\rangle dx. \qquad (11)
\end{aligned}$$

Now we use the Fourier series expansion on T^3 and get the following estimate: there exists $C_1 > 0$ such that we have

$$\int_{T^3}|\not{D}_{\Gamma_{+,z}}(f)|^2 dx \geq d^2 C_1 F(t). \qquad (12)$$

Moreover, Corollary 3.4 tells us

$$\left|\int_{T^3}|\mathrm{clif}(A-\Gamma_+)(f)|^2\right| = O(t^{-2})\cdot F(t), \qquad (13)$$

$$\left|\int_{T^3}\langle[\partial_t, \not{D}_{A_\xi}](f), f\rangle\right| = O(t^{-2})\cdot F(t). \qquad (14)$$

By applying (12), (13) and (14) to (11), we can take positive functions $K(d) = O(d^{-1})$ and $\kappa(d)^{-1} = O(d^{-1})$ such that if $t > K(d)$, then $F''(t) > \kappa(d)^2 F(t)$.

We set $\tilde{F}(t) := \exp(\kappa(d)t)F(t)$. Then, the inequality $F''(t) > \kappa(d)^2 F(t)$ is equivalent to $\tilde{F}''(t) > 2\kappa(d)\tilde{F}'(t)$. If we suppose there exists $t_0 > K(d)$ such that $\tilde{F}'(t_0) > 0$, then $F(t)\exp(-\kappa(d)t/2) \to \infty$ ($t \to \infty$) and this contradicts $F \in L^1(\mathbb{R})$. Therefore, for any $t > K$ we have $\tilde{F}'(t_0) \leq 0$ i.e. $F'(t) \leq -\kappa(d)F(t)$.

The same proof works for $t < 0$ mutatis mutandis. □

Corollary 3.17. *Let $\xi \in \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$. There exists $C > 0$ such that the following estimate holds for any $f \in \mathrm{Ker}(\not{\partial}^-_{A_\xi}) \cap L^2$:*

$$||f||_{C^{4,\lambda}([t,t+1]\times T^3)} = O\left(\exp(-C|t|)\right).$$

4. Construction of the Nahm transform

4.1. Construction of monopoles

We construct the Nahm transform by following Charbonneau [6]. Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$. Let $(\mathcal{V}, ||\cdot||_{L^2}, d)$ be a flat Hermitian vector bundle on $\hat{T}^3$ which is the quotient of the product vector bundle $\underline{\left(L^2(\mathbb{R} \times T^3, V \otimes S^-), ||\cdot||_{L^2}\right)}$ on $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ by the Λ_3^*-action $v \cdot (\xi, f) := (\xi + v, \exp(2\pi\sqrt{-1}\langle x, v\rangle) f)$. We set $(\hat{V}, \hat{h})$ as the finite-dimensional subbundle of $\mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)}$ defined by $\hat{V}_\xi := \mathrm{Ker}\left(\eth^-_{A_\xi}\right) \cap L^2$. Indeed, as mentioned in Remark 3.15, $\eth^-_{A_\xi} : L^2_{1,\sigma} \to L^2$ is a continuous family of surjective Fredholm operators, hence $(\hat{V}, \hat{h})$ is a finite-dimensional subbundle of $\mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)}$ by the implicit function theorem. Moreover, Theorem 3.14 tells us $\mathrm{rank}(\hat{V}) = (8\pi^2)^{-1}||F(A)||^2_{L^2}$.

Let $\hat{A}$ be the connection on $(\hat{V}, \hat{h})$ induced by the flat connection $d_{\mathcal{V}}$ on $\mathcal{V}$, namely $\hat{A} = P d_{\mathcal{V}}$, where $P : \mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)} \to \hat{V}$ is the orthogonal projection.

Let $\hat{\Phi}$ denote a skew-Hermitian section of $\mathrm{End}(\hat{V})$ given by $\Phi_\xi(f) := P_\xi(2\pi\sqrt{-1}tf)$. Since any $f \in \mathrm{Ker}\left(\eth^-_{A_\xi}\right) \cap L^2$ decays exponentially in $t \to \pm\infty$, $2\pi\sqrt{-1}tf$ is an L^2 section.

Definition 4.1. $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ is called the Nahm transform of (V, h, A).

Proposition 4.2. $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ *is a monopole on* $\hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$.

Proof. According to Charbonneau [6, Subsection 3.1 and 3.2], $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ is a monopole if for any open subset $U \subset \hat{T}^3 \setminus \mathrm{Sing}(V, h, A)$ and any local section $f \in \Gamma(U, \hat{V})$, $(d_{\mathcal{V}} f)_\xi \in L^2(\mathbb{R} \times T^3, V \otimes S^-) \otimes \Omega^1_{\hat{T}^3,\xi}$ decays exponentially at $t \to \pm\infty$ for any $\xi \in U$. Since $d_{\mathcal{V}} f$ satisfies the partial differential equation $\partial_t(d_{\mathcal{V}} f) = \not{D}_{A_\xi}(d_{\mathcal{V}} f) + \mathrm{clif}_{\mathbb{R}\times T^3}(\langle d\xi, dx\rangle) f$ and Corollary 3.17, the decay condition of $d_{\mathcal{V}} f$ can be proved by a similar way with the proof of Proposition 3.16. □

4.2. Singularities of the Nahm transform

Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$ and $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ be the Nahm transform of (V, h, A). In this subsection, we prove the following theorem.

Theorem 4.3. *Each point of* $\mathrm{Sing}(V,h,A)$ *is a Dirac-type singularity of* $(\hat{V},\hat{h},\hat{A},\hat{\Phi})$.

Proof. By Proposition 2.7, it suffices to show $|\hat{\Phi}(\xi)| = O(d(\xi,p)^{-1})$ $(\xi \to p)$ for any $p \in \mathrm{Sing}(V,h,A)$. Since $\hat{\Phi}$ is skew-Hermitian with respect to $\hat{h}$, we have $|\hat{\Phi}(\xi)| \leq \mathrm{rank}(\hat{V}) \cdot \max\{|\langle\hat{\Phi}(f),f\rangle| \mid f \in \hat{V}_\xi \text{ and } |f|=1\}$. Hence we have only to show $|\langle\hat{\Phi}(f),f\rangle| = O(d(\xi,p)^{-1})||f||^2_{L^2}$ for $f \in \mathrm{Ker}(\not{\partial}^-_{A_\xi}) \cap L^2$. We set $F(t) := \int_{\{t\}\times T^3} |f(t,x)|^2 dx$. By Proposition 3.16, we take functions $K,\kappa : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. We abbreviate $K(d(\xi,p))$ and $\kappa(d(\xi,p))$ to K and κ respectively. Then we have

$$\begin{aligned}|\langle\hat{\Phi}f,f\rangle| \leq& \int_{-K}^{K} |t|F(t)dt + \int_{\{|t|>K\}} |t|F(t)dt\\ \leq& K||f||^2_{L^2} + \int_{\{|t|>K\}} |t|F(t)dt.\end{aligned}$$

Here we use integration by parts, then

$$\begin{aligned}|\langle\hat{\Phi}f,f\rangle| \leq& K||f||^2_{L^2} + K\int_{\{|t|>K\}} F(t)dt + \int_{\{t>K\}} \left\{\int_{|s|>t} F(s)ds\right\} dt\\ \leq& 2K||f||^2_{L^2} + \int_{\{t>K\}} \left\{\int_{|s|>t} F(s)ds\right\} dt.\end{aligned}$$

Since the inequality $F'(t) \leq -\kappa(d)F(t)$ (resp. $F'(t) \geq \kappa(d)F(t)$) holds for any $t > K(d)$ (resp. $t < -K(d)$), we have $\int_{|s|>t} F(s)ds \leq \kappa^{-1}(F(t)+F(-t))$. Therefore we obtain

$$|\langle\hat{\Phi}f,f\rangle| \leq 2K||f||^2_{L^2} + \kappa^{-1}\int_{\{|t|>K\}} F(t)dt \leq (2K+\kappa^{-1})||f||^2_{L^2}.$$

Since $K,\kappa^{-1} = O(d(\xi,p)^{-1})$, we have $|\langle\hat{\Phi}(f),f\rangle| = O(d(\xi,p)^{-1})||f||^2_{L^2}$. $\square$

5. Algebraic Nahm transform

In this section, we assume that T^3 is isomorphic to the product of a circle $S^1 = \mathbb{R}/\mathbb{Z}$ and a 2-dimensional torus $T^2 = \mathbb{R}^2/\Lambda_2$ as a Riemannian manifold. Then, we have $\hat{T}^3 = S^1 \times \hat{T}^2$, where $\hat{T}^2 = \mathrm{Hom}(\mathbb{R}^2,\mathbb{R})/\Lambda_2^*$ is the dual torus of T^2. Under this assumption, we can regard $\mathbb{R}\times T^3$ as a Kähler manifold by setting holomorphic coordinates $\tau = t+\sqrt{-1}x^1 \in \mathbb{R}\times S^1$ and $w = x^2 + \sqrt{-1}x^3 \in T^2$. Since the map $\mathbb{R}\times S^1 \ni \tau = t+\sqrt{-1}x^1 \to \exp(2\pi\tau) = z \in \mathbb{C}^*$

is biholomorphic, we have a biholomorphic and isometric map $\mathbb{R}\times T^3\simeq(\mathbb{C}^*, dz d\bar{z}/|2\pi z|^2)\times T^2$.

We will construct a stable filtered bundle on $(\mathbb{P}^1\times T^2, \{0,\infty\}\times T^2)$ from an L^2-finite instanton on $\mathbb{R}\times T^3\simeq\mathbb{C}^*\times T^2$ as a prolongation of holomorphic vector bundles. Next, from a stable filtered bundle on $(\mathbb{P}^1\times T^2, \{0,\infty\}\times T^2)$ of rank $r>1$ we construct a mini-holomorphic bundle on $\hat{T}^3=S^1\times\hat{T}^2$ outside a finite subset, and we call this construction *the algebraic Nahm transform.* Finally, for an irreducible L^2-finite instanton (V,h,A) on $\mathbb{R}\times T^3$ of rank $r>1$ and the associated stable filtered bundle $P_{**}V$ on $(\mathbb{P}^1\times T^2, \{0,\infty\}\times T^2)$, we show that the algebraic Nahm transform of $P_{**}V$ is isomorphic to the underlying mini-holomorphic bundle of the Nahm transform of (V,h,A).

5.1. Asymptotic behavior of L^2-finite instantons as holomorphic bundles

We refine Corollary 3.4 in order to make it compatible with the complex structure of $\mathbb{R}\times T^3$.

Proposition 5.1. *Let (V,h,A) be an L^2-finite instanton on $(0,\infty)\times T^3$ of rank r. If we take a sufficiently large $R>0$, then there exist a C^2-frame $\boldsymbol{v}=(v_i)$ of V on $(R,\infty)\times T^3$, a model solution (Γ,N) of the Nahm equation and a positive number $\delta>0$ such that the following holds.*

(i) If we write the $(0,1)$-part of connection form of A with respect to $\boldsymbol{v}$ as $\nabla_A^{0,1}(\boldsymbol{v})=\boldsymbol{v}\,(A_{\bar\tau}d\bar\tau+A_{\bar w}d\bar w)$, then $A_{\bar\tau}$ and $A_{\bar w}$ are T^2-invariant, and we have $[A_{\bar\tau},\Gamma_{\bar w}]=[A_{\bar w},\Gamma_{\bar w}]=0$, where $\Gamma_{\bar\tau}d\bar\tau+\Gamma_{\bar w}d\bar w:=(\sum_i\Gamma_i dx^i)^{(0,1)}$.

(ii) We also take $N_{\bar w}, N_{\bar\tau}$ to be $N_{\bar\tau}d\bar\tau+N_{\bar w}d\bar w:=(\sum_i N_i dx^i)^{(0,1)}$. We set $\tilde\varepsilon_{\bar w}:=A_{\bar w}-(\Gamma_{\bar w}+N_{\bar w}/t)$ and $\tilde\varepsilon_{\bar\tau}:=A_{\bar\tau}-(\Gamma_{\bar\tau}+N_{\bar\tau}/t)$. Then, the following estimates hold:

$$\begin{cases} |\tilde\varepsilon_{\bar\tau}|, |\tilde\varepsilon_{\bar w}| & = O(t^{-(1+\delta)}) \\ |\partial_t\tilde\varepsilon_{\bar\tau}|, |\partial_t\tilde\varepsilon_{\bar w}| & = O(t^{-(2+\delta)}) \\ |\partial_1\tilde\varepsilon_{\bar\tau}|, |\partial_1\tilde\varepsilon_{\bar w}| & = O(\exp(-\delta t)), \end{cases}$$

where ∂_1 means the partial derivative with respect to x^1.

(iii) For any $1\le i,j\le r$, we have $||\langle v_i,v_j\rangle-\delta_{ij}||_{C^2([t,t+1]\times T^3)}=O(\exp(-\delta t))$.

Remark 5.2. As Corollary 3.4 and Remark 3.5, we obtain a similar result for L^2-finite instantons on $(-\infty,0)\times T^3$.

Proof. Applying Corollary 3.4 to (V,h,A) gives a trivialization $(V,h)|_{(R,\infty)\times T^3} \simeq (\underline{\mathbb{C}^r}, \underline{h})$ and a model solution (Γ, N) of the Nahm equation. Take an orthonormal frame $\boldsymbol{u}$ on $(R,\infty)\times T^3$ such that $\Gamma = (\Gamma_i)$ are diagonal. We take a Hermitian vector space $E = (\mathbb{C}^r, h_{\mathbb{C}^r})$ and the eigen decomposition of $\Gamma_{\bar{w}}$ *i.e.* $E = \bigoplus_{\alpha\in\mathbb{C}} E_\alpha$ and $\Gamma_{\bar{w}} = \sum_\alpha \alpha \mathrm{Id}_{E_\alpha}$.

We take Banach spaces X_1 and X_2 as

$$X_i := \Big(\bigoplus_\alpha C^{i,\lambda}(T^2, \underline{\mathrm{End}(E_\alpha)})^\perp\Big) \oplus \Big(\bigoplus_{\alpha\neq\beta} C^{i,\lambda}(T^2, \underline{\mathrm{Hom}(E_\alpha, E_\beta)})\Big),$$

where $C^{i,\lambda}(T^2, \underline{\mathrm{End}(E_\alpha)})^\perp$ is the kernel of the linear map $f \mapsto \int_{T^2} f$. We define a smooth map

$$F : X_2 \times \left(\bigoplus_\alpha \mathrm{End}(E_\alpha)\right) \times X_1 \to X_1$$

as

$$F(f,a,\varepsilon) := \pi\Big(\mathrm{Ad}(\exp(f))(\Gamma_{\bar{w}} + a + \varepsilon) + \exp(-f)\overline{\partial}_w \exp(f)\Big),$$

where $\pi : C^{1,\lambda}(T^2, \underline{\mathrm{End}(E)}) \to X_1$ is the projection. Then, $dF_{(0,0,0)}|_{X_2} : X_2 \to X_1$ is an isomorphism because we have $dF_{(0,0,0)}(f,0,0) = \pi(\overline{\partial}_w(f))$. Therefore, the implicit function theorem shows that there exist a small neighborhood U of $(0,0) \in \big(\bigoplus_\alpha \mathrm{End}(E_\alpha)\big) \times X_1$ and a smooth map $G : U \to X_2$ such that for any $(f,a,\varepsilon) \in \mathrm{Dom}(F)$ sufficiently close to $(0,0,0)$, $F(f,a,\varepsilon) = 0$ and $f = G(a,\varepsilon)$ are equivalent.

Lemma 5.3. *There exists $M > 0$ such that for any $(a,\varepsilon) \in U$ we have*

$$\begin{aligned}
&||G(a,\varepsilon)||_{C^{2,\lambda}(T^2)} \leq M||\varepsilon||_{C^{1,\lambda}(T^2)}\\
&||\partial_a G(a,\varepsilon)||_{\mathcal{B}(\bigoplus_\alpha \mathrm{End}(E_\alpha), C^{2,\lambda}(T^2))} \leq M||\varepsilon||_{C^{1,\lambda}(T^2)}\\
&||\partial_a^2 G(a,\varepsilon)||_{\mathcal{B}(\bigoplus_\alpha \mathrm{End}(E_\alpha), \mathcal{B}(\bigoplus_\alpha \mathrm{End}(E_\alpha), C^{2,\lambda}(T^2)))} \leq M||\varepsilon||_{C^{1,\lambda}(T^2)},
\end{aligned}$$

where $\mathcal{B}(X,Y)$ is the Banach space consisting of bounded linear maps from a Banach spaces X to a Banach space Y.

Proof. By $F(0,a,0) = 0$ and the definition of G, we have $G(a,0) = 0$. Thus, we have $\partial_a G(a,0) = 0$ and $\partial_a^2 G(a,0) = 0$, and this proves the lemma. □

For the error terms $\varepsilon_{i,j}$ $(1 \leq i,j \leq 3)$ in Corollary 3.4, we set $\varepsilon_{i,\bar{\tau}} d\bar{\tau} + \varepsilon_{i,\bar{w}} d\bar{w} := (\sum_j \varepsilon_{i,j} dx^j)^{(0,1)}$. Set a gauge transformation $g := \exp\big(G(\varepsilon_{1,\bar{w}}, \varepsilon_{2,\bar{w}} + \pi(\varepsilon_{3,\bar{w}}))\big)$

and take a frame $\boldsymbol{v} = \boldsymbol{u}g$. By Lemma 5.3 and the estimates of $\varepsilon_{2,j}, \varepsilon_{3,j}$ in Corollary 3.4, we have $||g - \mathrm{Id}||_{C^2([t,t+1]\times T^3)} = O(\exp(-\delta t))$. This proves (ii) and (iii).

We prove (i). By the definition of F, $A_{\bar{w}}$ is T^2-invariant and $[A_{\bar{w}}, \Gamma_{\bar{w}}] = 0$. We will prove that $A_{\bar{\tau}}$ is also T^2-invariant and $[A_{\bar{\tau}}, \Gamma_{\bar{w}}] = 0$. For an eigenvalue $\alpha \in \mathbb{C}$ of $\Gamma_{\bar{w}}$, let us denote by $\boldsymbol{v}_\alpha$ the subset of $\boldsymbol{v}$ corresponding to α. Then we write $\nabla_A^{0,1}$ as

$$\nabla_A^{0,1}(\boldsymbol{v}_\alpha) = (\sum_\beta \boldsymbol{v}_\beta A^\beta_{\bar{\tau},\alpha})d\bar{\tau} + \boldsymbol{v}_\alpha A_{\bar{w},\alpha}d\bar{w}.$$

Then, rewriting $\nabla_A^{0,1} \circ \nabla_A^{0,1} = 0$ gives

$$\overline{\partial}_w(A^\beta_{\bar{\tau},\alpha}) = A^\beta_{\bar{\tau},\alpha}A_{\bar{w},\alpha} - A_{\bar{w},\beta}A^\beta_{\bar{\tau},\alpha}$$

for any eigenvalues $\alpha \neq \beta$ of $\Gamma_{\bar{w}}$. We have $\lim_{t\to\infty} A_{\bar{w},\alpha} = \alpha\mathrm{Id}$, and by Remark 3.7 (i) we also have $\alpha - \beta \notin 2\pi\mathbb{Z}$ for any eigenvalues $\alpha \neq \beta$ of $\Gamma_{\bar{w}}$. Therefore, if $R > 0$ is sufficiently large, then we obtain $A^\beta_{\bar{\tau},\alpha} = 0$ on $(R,\infty) \times T^3$ by the Fourier series expansion. This is equivalent to $[A_{\bar{\tau}}, \Gamma_{\bar{w}}] = 0$. Here we use $\nabla_A^{0,1} \circ \nabla_A^{0,1} = 0$ again, and we have

$$\overline{\partial}_\tau(A_{\bar{w},\alpha}) - \overline{\partial}_w(A^\alpha_{\bar{\tau},\alpha}) + [A^\alpha_{\bar{\tau},\alpha}, A_{\bar{w},\alpha}] = 0. \tag{15}$$

Removing the T^2-invariant part of (15), and we obtain

$$\overline{\partial}_w\left((A^\alpha_{\bar{\tau},\alpha})^\perp\right) + \left[A_{\bar{w},\alpha}, (A^\alpha_{\bar{\tau},\alpha})^\perp\right] = 0,$$

where $(A^\alpha_{\bar{\tau},\alpha})^\perp$ means the non-constant part of the Fourier series expansion of $A^\alpha_{\bar{\tau},\alpha}$. This equation implies that $(A^\alpha_{\bar{\tau},\alpha})^\perp$ is a constant function on T^2, and hence 0. Thus, $A_{\bar{\tau}}$ is T^2-invariant. $\square$

Corollary 5.4. *Set $\widetilde{\Gamma_{\bar{w}}} \in \Gamma((R,\infty) \times T^3, \mathrm{End}(V))$ as $\widetilde{\Gamma_{\bar{w}}}(\boldsymbol{v}) := \boldsymbol{v}\Gamma_{\bar{w}}$. For an eigenvalue $\alpha \in \mathbb{C}$ of $\Gamma_{\bar{w}}$, we take a subbundle V_α of $V|_{(R,\infty)\times T^3}$ to be $V_\alpha := \mathrm{Ker}(\widetilde{\Gamma_{\bar{w}}} - \alpha\mathrm{Id}_V)$. Then the decomposition $V|_{(R,\infty)\times T^3} = \bigoplus_\alpha V_\alpha$ is a holomorphic decomposition. Moreover, there exist $C, \delta > 0$ such that for any $(t,x) \in \mathbb{R} \times T^3$, for any eigenvalues $\alpha \neq \beta$, for any $v_\alpha \in (V_\alpha)_{(t,x)}$ and for any $v_\beta \in (V_\beta)_{(t,x)}$, we have $|\langle v_\alpha, v_\beta\rangle| < C\exp(-\delta t)|v_\alpha| \cdot |v_\beta|$.*

Corollary 5.5. *Let $\pi : (0,\infty) \times T^3 \to (0,\infty) \times S^1$ be the projection. There exist a holomorphic Hermitian vector bundle $(E, \overline{\partial}_E, h_E)$ on $(R,\infty) \times S^1$*

and a holomorphic endomorphism $f \in \Gamma((R,\infty)\times S^1, \mathrm{End}(E))$ *such that the following holds:*

(i) $(V, \overline{\partial}_A)$ *and* $(\pi^*E, \pi^*(\overline{\partial}_E) + f d\bar{w})$ *are isomorphic.*

(ii) *Under the isomorphism between* $(V, \overline{\partial}_A)$ *and* $(\pi^*E, \pi^*(\overline{\partial}_E) + f d\bar{w})$*, we have the estimates* $||h - \pi^* h_E||_{C^2([t,t+1]\times T^3)} = O(\exp(-\delta t))$*, where* δ *is a positive number, and the norm is induced by* h*.*

(iii) *For the Chern connection* ∇_E *of* $(E, h_E, \overline{\partial}_E)$*, we have* $|F(\nabla_E)|_{h_E} = O(t^{-2})$*.*

(iv) $(E, \overline{\partial}_E, f)$ *has a holomorphic and orthogonal decomposition*

$$(E, \overline{\partial}_E, h_E, f) = \bigoplus_{\alpha} (E_\alpha, \overline{\partial}_{E_\alpha}, h_{E_\alpha}, f_\alpha),$$

which is compatible with the decomposition $V = \bigoplus_\alpha V_\alpha$ *in Corollary 5.4.*

Proof. Let (E, h_E) be a trivial Hermitian vector bundle on $(R,\infty)\times S^1$ and $\boldsymbol{e}$ be a orthonormal frame of (E, h_E). We set $\overline{\partial}_E(\boldsymbol{e}) = \boldsymbol{e} A_\tau$ and $f(\boldsymbol{e}) = \boldsymbol{e} A_{\bar{w}}$. Then all conditions are satisfied by Proposition 5.1. □

5.2. Prolongation of L^2-finite instantons

By following [16] and [18], we construct a polystable filtered bundles on $(\mathbb{P}^1 \times T^2, \{0,\infty\}\times T^2)$ from L^2-finite instantons on $\mathbb{R}\times T^3 \simeq \mathbb{C}^* \times T^2$.

Definition 5.6. Let (X, g) be a Kähler manifold. Let (V, h, A) be a holomorphic Hermitian bundle on $\Delta^* \times X$, where $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let $i : \Delta^* \times X \to \Delta \times X$ be the inclusion.

(i) (V, h, A) is *acceptable* if $|F(A)|$ is bounded on a neighborhood of $\{0\} \times X$, where the norm is induced by h and the Poincaré-like metric $g + |z|^{-2}(\log|z|)^{-2} dz d\bar{z}$.

(ii) For any $a \in \mathbb{R}$, we define a (possibly non-coherent) $\mathcal{O}_{\Delta\times X}$-submodule $P_a V$ of $i_* V$ as follows: For any open subset $U \subset \Delta \times X$, a section $s \in \Gamma(U, i_* V)$ belongs to $\Gamma(U, P_a V)$ if and only if for any $p \in (\{0\} \times X) \cap U$ and for any $\varepsilon > 0$, an estimate $|s| = O(|z|^{-(a+\varepsilon)})$ holds around p. We call $P_* V := \{P_a V\}_{a\in\mathbb{R}}$ the prolongation of V.

(iii) For a section $s \in \Gamma(X \times \Delta, P_a V)$, we define $\operatorname{ord}(s) \in \mathbb{R}$ to be

$$\operatorname{ord}(s) := \min\{a' \in \mathbb{R} \mid s \in \Gamma(X \times \Delta, P_{a'} V)\}.$$

Mochizuki [18] proved the following theorem.

Theorem 5.7. *Let $(X, g), (V, h, A)$ be as in Definition 5.6. If (V, h, A) is acceptable, then $P_* V = \{P_a V\}_{a \in \mathbb{R}}$ forms a filtered bundle on $(\Delta \times X, \{0\} \times X)$.*

Let (V, h, A) be an L^2-finite instanton on $(-\infty, 0) \times T^3$. Applying Corollary 5.5 to (V, h, A), we take a positive number $R > 0$, a holomorphic vector bundle $(E, \overline{\partial}_E, h_E)$ on $(-\infty, -R) \times S^1$ and a holomorphic endomorphism $f \in \Gamma((-\infty, -R) \times S^1, \operatorname{End}(E))$. For $r > 0$, we set $\Delta(r) := \{z \in \mathbb{C} \mid |z| < \exp(-2\pi r)\}$ and $\Delta(r)^* := \Delta(r) \setminus \{0\}$. Under the isomorphism $\Delta(0)^* \times T^2 \simeq (\infty, 0) \times T^3$ and $\Delta(R)^* \simeq (-\infty, R) \times S^1$, we obtain the prolongation $P_* E$ and $P_* V$ from $(E, h_E, \overline{\partial}_E)$ and (V, h, A) respectively.

Corollary 5.8. *Both (V, h, A) and $(E, h_E, \overline{\partial}_E)$ are acceptable on $\Delta(0)^* \times T^2$ and $\Delta(R)^*$ respectively. In particular, the prolongations $P_* V$ and $P_* E$ are filtered bundles on $(\Delta(0) \times T^2, \{0\} \times T^2)$ and $(\Delta(R), \{0\})$ respectively.*

Proof. Under the coordinate change $z = \exp(2\pi(t + \sqrt{-1}x^1))$, the Poincaré metric $|z|^{-2}(\log |z|)^{-2} dz d\bar{z}$ is written as $(dt^2 + (dx^1)^2)/t^2$. By Corollary 3.4, we have $|F(A)| = O(t^{-2})$, where the norm is induced by h and $g_{\mathbb{R} \times T^3}$. Thus, (V, h, A) is acceptable on $\Delta(0)^* \times T^2$. By Corollary 5.5 (iii), we also have $|F(\nabla_E)| = O(t^{-2})$, and $(E, h_E, \overline{\partial}_E)$ is acceptable on $\Delta(R)^*$. Therefore, by Theorem 5.7, $P_* V$ and $P_* E$ are filtered bundles on $(\Delta(0) \times T^2, \{0\} \times T^2)$ and $(\Delta(R), \{0\})$ respectively. $\square$

Let us prove that the isomorphism $(V, \overline{\partial}_A) \simeq (\pi^* E, \pi^* \overline{\partial}_E + \pi^* f d\bar{w})$ in Corollary 5.5 is extended over $\Delta(R) \times T^2$.

Proposition 5.9. *For any $a \in \mathbb{R}$, we have the holomorphic isomorphism $P_a(V|_{\Delta(R)^* \times T^2}) \cong (\pi^* P_a E, \pi^*(\overline{\partial}_{P_a E}) + \pi^*(P_a f) d\bar{w})$ constructed from the isomorphism in Corollary 5.5. In particular, the vector bundle $P_a V|_{\{0\} \times T^2}$ is semistable of degree 0.*

Proof. Let $\widetilde{P_a E}$ denote $(\pi^* P_a E, \pi^*(\overline{\partial}_{P_a E}) + \pi^*(P_a f) d\bar{w})$. Let $W \subset \Delta(R) \times T^2$ be an open subset. We take a local section $s \in \Gamma(W, \widetilde{P_a E})$. Then

$s|_{W\setminus(\{0\}\times T^2)}$ is a holomorphic section of $(V, \overline{\partial}_A)$. Since h and $\pi^* h_E$ are mutually bounded, s is a holomorphic section of $P_a V$.

Let s' be a holomorphic section of $P_a V$ on W. We obtain a holomorphic section $s'|_{W\setminus(\{0\}\times T^2)}$ of $(\pi^* E, \pi^* \overline{\partial}_E + \pi^* f d\bar{w})$. Because of the definition of $P_a E$, $s'|_{W\cap(\mathbb{P}^1\times\{w\})}$ is a holomorphic section of $P_a E$. Hence s' is a holomorphic section of $\widetilde{P_a E}$.

Under this isomorphism, $P_a V|_{\{0\}\times T^2}$ is naturally isomorphic to $\big(\pi^*(P_a E|_{\{0\}}), \overline{\partial}_{T^2} + \pi^*(P_a f|_{\{0\}}) d\bar{w}\big)$. Thus this vector bundle is semistable and of degree 0. $\square$

Corollary 5.10. *For any $a \in \mathbb{R}$, we have an isomorphism of vector bundles $\mathrm{Gr}_a(V) \simeq (\mathrm{Gr}_a(E) \times T^2, \overline{\partial}_{T^2} + \mathrm{Gr}_a(f) d\bar{w})$ on T^2, where we regard the skyscraper sheaf $\mathrm{Gr}_a(E)$ and the endomorphism $\mathrm{Gr}_a(f)$ as a vector space with an endomorphism.*

Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3$. We will denote by $P_{**}V$ the associated filtered bundle on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$. We prove that $P_{**}V$ is polystable.

Theorem 5.11. *The prolongation $P_{**}V$ is polystable and* $\text{par-deg}(P_{**}V) = 0$. *In particular, if (V, h, A) is irreducible, then $P_{**}V$ is stable.*

Once we admit this theorem, we obtain the next corollary from Proposition 2.12.

Corollary 5.12. *Let $p : \mathbb{P}^1 \times T^2 \to T^2$ be the projection. Let (V, h, A) be an irreducible L^2-finite instanton of rank $r > 1$ on $\mathbb{R} \times T^3$. Then, we have $H^0(\mathbb{P}^1 \times T^2, P_{00}V \otimes p^* F) = H^2(\mathbb{P}^1 \times T^2, P_{<0<0}V \otimes p^* F) = 0$ for any $F \in \mathrm{Pic}^0(T^2)$.*

5.2.1. Norm estimate. As a preparation of the proof of Theorem 5.11, we show the following norm estimates.

Let (V, h, A) be an L^2-finite instanton on $(-\infty, 0) \times T^3$ of rank r. Applying Corollary 5.5 to (V, h, A), we take a positive number $R > 0$, a holomorphic Hermitian vector bundle $(E, h_E, \overline{\partial}_E)$ on $(-\infty, -R) \times S^1 \simeq \Delta(R)^*$ and a holomorphic endomorphism $f \in \Gamma(\Delta(R)^*, \mathrm{End}(E))$. Let $P_* E$ denote the prolongation of E on $\Delta(R)$. On the fiber $(P_a E)|_0$, the parabolic filtration $\{F_c(P_a E|_0)\}_{a-1<c\leq a}$ is induced by the natural inclusion $P_c E \hookrightarrow P_a E$. Then an endomorphism $\mathrm{Gr}^F_c(f|_0)$ on $\mathrm{Gr}^F_c(P_a E|_0)$ is induced by f, and the nilpotent part of $\mathrm{Gr}^F_c(f|_0)$ induces the weight filtration $\{W_k \mathrm{Gr}^F_c(P_a E|_0)\}_{k\in\mathbb{Z}}$. For

$b \in \mathrm{Gr}^F_c(P_aE|_0)$, we denote by $\deg^W(b_i) \in \mathbb{Z}$ the degree of b with respect to the weight filtration. For a holomorphic frame $\boldsymbol{b} = (b_1, \dots, b_r)$ of P_aE on $\Delta(R)$, $\boldsymbol{b} = (b_i)$ is compatible with the parabolic filtration and the weight filtration if the following conditions are satisfied:

- For any $c \in (a-1, a]$, $\{b_i|_0 \mid \mathrm{ord}(b_i) \leq c\}$ forms a basis of $F_c(P_aE)$.
- For any $c \in (a-1, a]$ and any $k \in \mathbb{Z}$, $\{[b_i] \mid \mathrm{ord}(b_i) = c,\ \deg^W([b_i]) \leq k\}$ forms a basis of $W_k\mathrm{Gr}^F_c(P_aE)$, where $[b_i]$ is the image of b_i in $\mathrm{Gr}^F_c(P_aE|_0)$.

Proposition 5.13. *Let $a \in \mathbb{R}$ and $\boldsymbol{b} = (b_1, \dots, b_r)$ be a holomorphic frame of P_aE on $\Delta(R)$ that is compatible with the parabolic filtration and the weight filtration. We take a Hermitian metric h_1 on $E|_{\Delta(R)^*}$ given by $h_1(b_i, b_j) := \delta_{ij}|z|^{-2\mathrm{ord}(b_i)}(-\log|z|^2)^{\deg^W(b_i)}$. Then h_1 and h_E are mutually bounded.*

Proof. According to [24, Corollary 4.3], we only need to prove that there exists $p > 1$ such that we have $\left|F(E, h_E) + [f dz/z, f^\dagger d\bar{z}/\bar{z}]\right| \in L^p(\Delta(R)^*)$, where $f^\dagger$ is the adjoint of f with respect to the metric h_E, and the norm is induced by h_E and $dzd\bar{z}$. We set a holomorphic Hermitian vector bundle $(\tilde{E}, \overline{\partial}_{\tilde{E}}, h_{\tilde{E}})$ on $\Delta(R)^* \times T^2$ by $(\tilde{E}, \overline{\partial}_{\tilde{E}}, h_{\tilde{E}}) := (\pi^*E, \pi^*(\overline{\partial}_E) + f d\bar{w}, \pi^*h_E)$. We write the curvature of the Chern connection of $(\tilde{E}, \overline{\partial}_{\tilde{E}}, h_{\tilde{E}})$ as $F(\tilde{E}, h_{\tilde{E}}) = \tilde{F}_{z\bar{z}} dz \wedge d\bar{z} + \tilde{F}_{w\bar{w}} dw \wedge d\bar{w} + \tilde{F}_{w\bar{z}} dw \wedge d\bar{z} + \tilde{F}_{z\bar{w}} dz \wedge d\bar{w}$. Then we have

$$\pi^*\left(F(E, h_E) + [f dz/z, f^\dagger d\bar{z}/\bar{z}]\right) = \left(\tilde{F}_{z\bar{z}} + |z|^{-2}\tilde{F}_{w\bar{w}}\right) dz \wedge d\bar{z}.$$

Since (V, h, A) is an instanton, the ASD equation $F(A)_{z\bar{z}} = -|z|^{-2}F(A)_{w\bar{w}}$ holds, where $F(A)_{z\bar{z}}$ and $F(A)_{w\bar{w}}$ are the components of the curvature $F(A)$. Hence we have

$$\begin{aligned} &\pi^*\left(F(E, h_E) + [f dz/z, f^\dagger d\bar{z}/\bar{z}]\right) \\ &= \left((F_{z\bar{z}} - F(A)_{z\bar{z}}) + |z|^{-2}(F_{w\bar{w}} - F(A)_{w\bar{w}})\right) dz \wedge d\bar{z}. \end{aligned}$$

Applying Corollary 5.5, we obtain

$$\left|\pi^*\left(F(E, h_E) + [f dz/z, f^\dagger d\bar{z}/\bar{z}]\right)\right| = O(|z|^{-2+\delta}),$$

where δ is a positive number. Hence we have $\left|F(E, h_E) + [f dz/z, f^\dagger d\bar{z}/\bar{z}]\right| \in L^p(\Delta(R)^*)$ for some $p > 1$. $\square$

5.2.2. Analytic degree for acceptable bundles on $(\mathbb{P}^1,\{0,\infty\})$. As a preparation to prove Theorem 5.11, by following [16], we introduce the notion of analytic degree of acceptable bundles on $(\mathbb{P}^1,\{0,\infty\})$, and consider the relation between the parabolic degree and the analytic degree of acceptable bundles on $(\mathbb{P}^1,\{0,\infty\})$.

Let $(E,\overline{\partial}_E,h_E)$ be an acceptable holomorphic Hermitian vector bundle on $\mathbb{C}^*$. We assume that for any non-zero holomorphic section f of E on a neighborhood of $0\in\mathbb{P}^1$ there exist $C_1>0$ and $k_0(f)\in\mathbb{R}$ such that we have an estimate

$$C_1^{-1}|z|^{-\mathrm{ord}_0(f)}(-\log|z|^2)^{k_0(f)}\le|f|_{h_E}\le C_1|z|^{-\mathrm{ord}_0(f)}(-\log|z|^2)^{k_0(f)}.$$

We also assume a similar condition at $\infty\in\mathbb{P}^1$.

For any holomorphic subbundle $P_{00}\mathcal{L}$ of $P_{00}E$, we set $\mathcal{L}:=P_{00}\mathcal{L}|_{\mathbb{C}^*}$ and $h_{\mathcal{L}}$ be the Hermitian metric induced by h. We define the analytic degree and the parabolic degree of $\mathcal{L}$ by

$$\deg(\mathcal{L},h_E):=\sqrt{-1}\int_{\mathbb{C}^*}\mathrm{Tr}(\Lambda F(h_{\mathcal{L}}))d\mathrm{vol}_{\mathbb{C}^*}$$

and

$$\text{par-deg}(P_{**}\mathcal{L}):=\int_{P^1}\text{par-c}_1(P_{**}\mathcal{L}),$$

where $P_{**}\mathcal{L}$ is the strict filtered subbundle given by $P_{ab}\mathcal{L}:=\mathcal{L}\cap P_{ab}E$.

Proposition 5.14. *We have the equality* $\deg(\mathcal{L},h_E)=2\pi\ \text{par-deg}(P_{**}\mathcal{L})$.

Proof. By considering $\bigwedge^{\mathrm{rank}(\mathcal{L})}E$ and $\det(\mathcal{L})$ instead of E and $\mathcal{L}$, we may assume $\mathrm{rank}(\mathcal{L})=1$. Let e_0,e_∞ be holomorphic frames of $P_{00}\mathcal{L}$ on a neighborhood of $0,\infty\in\mathbb{P}^1$. We set a smooth function $\psi:\mathbb{C}^*\to\mathbb{R}$ satisfying

$$\psi(z)=\begin{cases}\mathrm{ord}(e_0)\log|z|^2 & (\text{on a small neighborhood of } 0\in\mathbb{P}^1)\\ -\mathrm{ord}(e_\infty)\log|z|^2 & (\text{on a small neighborhood of } \infty\in\mathbb{P}^1)\end{cases}$$

and set a metric h'_E on $(E,\overline{\partial}_E)$ by $h'_E:=h_Ee^{\psi}$. We will denote by $P'_{**}E, P'_{**}\mathcal{L}$ the prolongation of E and $\mathcal{L}$ with respect to the metric h'_E respectively. Then we have

$$\begin{aligned}\text{par-deg}(P_{**}\mathcal{L})&=\text{par-deg}(P'_{**}\mathcal{L})-\mathrm{ord}(e_0)-\mathrm{ord}(e_\infty)\\ \deg(\mathcal{L},h_E)&=\deg(\mathcal{L},h'_E)-2\pi\ \mathrm{ord}(e_0)+2\pi\ \mathrm{ord}(e_\infty).\end{aligned}$$

Therefore, by replacing h_E with h'_E, we may assume $\text{par-deg}(P_{**}\mathcal{L})=\deg(P_{00}\mathcal{L})$ and $\mathrm{ord}(e_0)=\mathrm{ord}(e_\infty)=0$. We take another metric $h_{1,\mathcal{L}}$ of $P_{00}\mathcal{L}$

satisfying $h_{1,\mathcal{L}}(e_0,e_0)=h_{1,\mathcal{L}}(e_\infty,e_\infty)=1$. We take a smooth function $\varphi:\mathbb{C}^*\to\mathbb{R}$ as $h_{1,\mathcal{L}}=e^{\varphi}h_{\mathcal{L}}$. By the definition of $h_{1,\mathcal{L}}$, we have $\deg(P_{00}L)=(2\pi)^{-1}\sqrt{-1}\int_{\mathbb{P}^1}F(h_{1,\mathcal{L}})$. We consider the following lemma.

Lemma 5.15. *The 2-form $\overline{\partial}\partial\varphi$ is integrable on $\mathbb{C}^*$, and we have $\int_{\mathbb{C}^*}\overline{\partial}\partial\varphi=0$.*

Once we admit this lemma, then we obtain the desired equality $\deg(L,h_E)=2\pi\text{par-deg}(P_{**}\mathcal{L})$ because we have $F(h_{\mathcal{L}})=F(h_{1,\mathcal{L}})+\overline{\partial}\partial\varphi$. □

proof of Lemma 5.15. We set $\eta:=z^{-1}$ and take a metric g_1 on $\mathbb{P}^1$ satisfying $g_1=dzd\bar{z}$ on a small neighborhood of $z=0$ and $g_1=d\eta d\bar{\eta}$ on a small neighborhood of $\eta=0$. We take a smooth function $\rho:\mathbb{R}\to[0,1]$ satisfying $\rho(t)=1$ ($|t|<1/2$) and $\rho(t)=0$ ($|t|>1$) and set $\chi_N:\mathbb{C}^*\to[0,1]$ as $\chi_N(z):=\rho(N^{-1}\log|z|^2)$ for $N\in\mathbb{N}$. Since χ_N is a compact supported function, we have

$$\begin{aligned}0=\int_{\mathbb{C}^*}\overline{\partial}\partial(\chi_N\varphi)=\int_{\mathbb{C}^*}\overline{\partial}\partial(\chi_N)\varphi+\int_{\mathbb{C}^*}\overline{\partial}\chi_N\cdot\partial\varphi\\ -\int_{\mathbb{C}^*}\partial\chi_N\cdot\overline{\partial}\varphi+\int_{\mathbb{C}^*}\chi_N\overline{\partial}\partial(\varphi).\end{aligned}\tag{16}$$

Here we consider the following lemma.

Lemma 5.16. *The integrands of the first to third term of rhs of (16) are dominated by an integrable functions independent of N. In particular, the dominated convergence theorem shows*

$$\lim_{N\to\infty}\left(\int_{\mathbb{C}^*}\overline{\partial}\partial(\chi_N)\varphi+\int_{\mathbb{C}^*}\overline{\partial}\chi_N\cdot\partial\varphi-\int_{\mathbb{C}^*}\partial\chi_N\cdot\overline{\partial}\varphi\right)=0.$$

If we admit this lemma, then we have $\lim_{N\to\infty}\int_{\mathbb{C}^*}\chi_N\overline{\partial}\partial(\varphi)=0$. Therefore, $\overline{\partial}\partial(\varphi)$ is integrable, and we have $\int_{\mathbb{C}^*}\overline{\partial}\partial\varphi=0$. □

proof of Lemma 5.16. We calculate the derivatives of χ_N, then

$$\begin{aligned}\partial\chi_N&=N^{-1}\rho'(N^{-1}\log|z|^2)z^{-1}dz\\ \overline{\partial}\chi_N&=N^{-1}\rho'(N^{-1}\log|z|^2)\bar{z}^{-1}d\bar{z}\\ \overline{\partial}\partial\chi_N&=-N^{-2}\rho''(N^{-1}\log|z|^2)|z|^{-2}dz\wedge d\bar{z}.\end{aligned}$$

Thus, there exists $C_2 > 0$ independent from N such that on a small neighborhood of $z = 0$ we have

$$|\partial \chi_N|_{g_1} = |\overline{\partial} \chi_N|_{g_1} \;\leq\; C_2 |z|^{-1}(-\log|z|)^{-1} \tag{17}$$

$$|\overline{\partial}\partial \chi_N|_{g_1} \;\leq\; C_2 |z|^{-2}(-\log|z|)^{-2}, \tag{18}$$

and on a small neighborhood of $\eta = 0$ we also have

$$|\partial \chi_N|_{g_1} = |\overline{\partial} \chi_N|_{g_1} \;\leq\; C_2 |\eta|^{-1}(-\log|\eta|)^{-1} \tag{19}$$

$$|\overline{\partial}\partial \chi_N|_{g_1} \;\leq\; C_2 |\eta|^{-2}(-\log|\eta|)^{-2}. \tag{20}$$

Hence by (18), (20) and the assumption of the norm of e_0, e_∞, there exist an integrable function dominating the first term of (16) and independent of N. To estimate the other terms, we prove the next lemma.

Lemma 5.17. $|\partial(\varphi)|_{g_1} = |\overline{\partial}(\varphi)|_{g_1}$ *is an* L^2*-function on* $(\mathbb{P}^1, g_1)$.

If we suppose that the lemma is true, by (17) and (19) we obtain an integrable function dominating the second and third terms of (16) independent of N. Hence the proof of Proposition 5.20 is complete. □

proof of Lemma 5.17. If we prove $|\partial(\varphi)|_{g_1}$ is an L^2-function on a neighborhood of $z = 0$, then the same proof works for a neighborhood of $\eta = 0$. Thus, we only needs to show $|\partial(\varphi)|_{g_1} \in L^2(\Delta^*, g_1)$, where $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

Let ∇ be the Chern connection of $(E, h_E, \overline{\partial}_E)$. By definition of φ we have $\exp(-2\varphi) = |e_0|^2_{h_E}$. Therefore we obtain $|\partial \exp(-2\varphi)| = |2\partial\varphi| \exp(-2\varphi) = |h_E(\nabla_z e_0, e_0)| \leq |e_0|_{h_E} \cdot |\nabla_z e_0|_{h_E}$, where we set $\nabla =: \nabla_z dz + \nabla_{\bar{z}} d\bar{z}$. Hence we have $|2\partial\varphi| \leq |e_0|^{-1}_{h_E} \cdot |\nabla_z e_0|_{h_E}$. By the norm estimate of e_0, it suffices to prove that $|(-\log|z|^2)^{-k} |\nabla_z(e_0)|_{h_E} \in L^2(\Delta)$, where $k = k_0(e_0)$. We take a smooth function $a : \mathbb{R} \to [0, 1]$ satisfying $a(t) = 0$ $(t > 1)$, $a(t) = 1$ $(t < 1/2)$ and the condition that $a^{1/2}$ and $a' \cdot a^{-1/2}$ are also smooth. We set a function $b_N : \Delta^* \to \mathbb{R}$ as

$$b_N(z) := \big(1 - a(-\log|z|^2)\big) \cdot a(-N^{-1}\log|z|^2)$$

for $N \in \mathbb{N}$. Then $\partial(b_N) \cdot b_N^{-1/2}$ is a smooth function on Δ^* because of the definition of a. Moreover, there exists $C_3 > 0$ such that we have $|\partial(b_N) b_N^{-1/2}|_{g_1} \leq$

$C_3|z|^{-1}(-\log|z|^2)^{-1}$. We consider the following integral.

$$\begin{aligned}
&\int_{\Delta^*} b_N \cdot h_E(\nabla_z e_0, \nabla_z e_0)(-\log|z|^2)^{-2k} d\text{vol} = \\
&\quad -\int_{\Delta^*} \partial(b_N)\cdot h_E(e_0, \nabla_z e_0)(-\log|z|^2)^{-2k} d\text{vol} \\
&\quad -\int_{\Delta^*} b_N \cdot h_E(e_0, F(h_E)e_0)(-\log|z|^2)^{-2k} d\text{vol} \\
&\quad +\int_{\Delta^*} b_N \cdot h_E(e_0, \nabla_z e_0)\cdot(-2k)(-\log|z|^2)^{-2k-1} z^{-1} d\text{vol},
\end{aligned}$$

where $d\text{vol} = \sqrt{-1}dz \wedge d\bar{z}$. We have the following estimate on the first term of rhs.

$$\begin{aligned}
&\left|\partial(b_N)\cdot h_E(e_0, \nabla_z e_0)(-\log|z|^2)^{-2k}\right| \\
&\le \left(C_3 C_1 |z|^{-1}(\log|z|^2)^{-1}\right)\left(b^{1/2}(z)\cdot|\nabla_z e_0|_{h_E}(-\log|z|^2)^{-k}\right).
\end{aligned}$$

For the second term, because of $(E, \overline{\partial}_E, h_E)$ is acceptable, there exists $C_4 > 0$ such that we have

$$\left|b_N \cdot h_E(e_0, F(h_E)e_0)(-\log|z|^2)^{-2k}\right| \le C_4|z|^{-2}(-\log|z|^2)^{-2}.$$

For the third term, we also have

$$\begin{aligned}
&\left|b_N \cdot h_E(e_0, \nabla_z e_0)\cdot(-2k)(-\log|z|^2)^{-2k-1} z^{-1}\right| \\
&\quad \le \left(C_1 b_N^{1/2}\cdot|z|^{-1}(-\log|z|^2)^{-1}\right)\left(b_N^{1/2}|\nabla_z e_0|_{h_E}(-\log|z|^2)^{-k}\right).
\end{aligned}$$

Therefore, there exist $C_5, C_6 > 0$ such that

$$\begin{aligned}
&\int_{\Delta^*} b_N \cdot h_E(\nabla_z e_0, \nabla_z e_0)(-\log|z|^2)^{-2k} d\text{vol} \\
\le & C_5 + C_6\left(\int_{\Delta^*} b_N \cdot h_E(\nabla_z e_0, \nabla_z e_0)(-\log|z|^2)^{-2k} d\text{vol}\right)^{1/2} < \infty.
\end{aligned}$$

Thus, we obtain

$$\int_{\Delta^*} b_N \cdot h_E(\nabla_z e_0, \nabla_z e_0)(-\log|z|^2)^{-2k} d\text{vol} < C_7,$$

where C_7 is a constant independent of N. Therefore, we conclude $|\partial(\varphi)|_{g_1} \in L^2(\mathbb{C}^*, g_1)$. $\square$

5.2.3. Analytic degree and parabolic degree on $\mathbb{R} \times T^3$. In order to prove Theorem 5.11, we also consider the analytic degree of holomorphic subsheaves of L^2-finite instantons on $\mathbb{R} \times T^3$ by following [16].

Definition 5.18. Let (V, h, A) be an L^2-finite instanton on $\mathbb{R} \times T^3 = \mathbb{C}^* \times T^2$. Let $\mathcal{F}$ be a saturated $\mathcal{O}_{\mathbb{C}^* \times T^2}$-submodule of $(V, \overline{\partial}_A)$ and $h_{\mathcal{F}}$ the induced metric of smooth part of $\mathcal{F}$. We define the analytic degree of $\mathcal{F}$ by

$$\deg(\mathcal{F}, h) := \sqrt{-1} \int_{\mathbb{C}^* \times T^2} \mathrm{Tr}(\Lambda F(h_{\mathcal{F}})) d\mathrm{vol}_{\mathbb{C}^* \times T^2},$$

where $d\mathrm{vol}_{\mathbb{C}^* \times T^2}$ is the volume form with respect to the Riemannian metric $|z|^{-2} dz d\bar{z} + dw d\bar{w}$. By the Chern-Weil formula in [23], this can be written as

$$\deg(\mathcal{F}, h) = -\int_{\mathbb{C}^* \times T^2} |\overline{\partial}\pi|_h^2 d\mathrm{vol}_{\mathbb{C}^* \times T^2},$$

where $\pi : V \to \mathcal{F}$ is the orthogonal projection.

Lemma 5.19. *Let $\mathcal{F}$ be a saturated subsheaf of an L^2-finite instanton (V, h, A) on $\mathbb{R} \times T^3$. Then, $\deg(\mathcal{F}, h)$ is finite if and only if the following conditions are satisfied.*

(i) $\mathcal{F}$ can be extended to a saturated subsheaf $P_{00}\mathcal{F}$ of $P_{00}V$.

(ii) For any $z \in \mathbb{C}^*$, we have $\deg(\mathcal{F}|_{\{z\} \times T^2}) = 0$.

Proof. Assume that $\deg(\mathcal{F}, h)$ is finite. We write $\overline{\partial} = \overline{\partial}_{\mathbb{C}^*} + \overline{\partial}_{T^2}$. On one hand we have

$$\int_{T^2} d\mathrm{vol}_{T^2} \int_{\mathbb{C}^*} |\overline{\partial}_{\mathbb{C}^*}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*} \leq \int_{T^2} d\mathrm{vol}_{T^2} \int_{\mathbb{C}^*} |\overline{\partial}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*} = -\deg(\mathcal{F}, h) < \infty.$$

Therefore, there exists a measurable subset $A \subset T^2$ such that $|A| = |T^2|$ and $\int_{\mathbb{C}^* \times \{w\}} |\overline{\partial}_{\mathbb{C}^*}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*} < \infty$ holds for any $w \in A$. According to [23, Lemma 10.5, Lemma 10.6], this is equivalent to the condition that $\mathcal{F}|_{\mathbb{C}^* \times \{w\}}$ can be extended to a saturated subsheaf of $P_{00}V|_{\mathbb{P}^1 \times \{w\}}$. Moreover, since A is a thick subset of T^2, by [25, Theorem 4.5] $\mathcal{F}$ can be extended to a saturated subsheaf of $P_{00}V$. On the other hand, we have

$$\int_{\mathbb{C}^*} d\mathrm{vol}_{\mathbb{C}^*} \int_{T^2} |\overline{\partial}_{T^2}\pi|^2 d\mathrm{vol}_{T^2} \leq -\deg(\mathcal{F}, h) < \infty.$$

Hence there exists a sequence $\{z_i\}$ on $\mathbb{C}^*$ such that we have $z_i \to \infty$ and $\int_{\{z_i\} \times T^2} |\overline{\partial}_{T^2}\pi|^2 d\mathrm{vol}_{T^2} \to 0$. We also have $|F(A)|_{\{z_i\} \times T^2} \to 0$ by Corollary

3.4. Therefore, we have $\deg(\mathcal{F}|_{\{z_i\}\times T^2}) \to 0$ because of the Chern-Weil formula

$$\begin{aligned}\deg(\mathcal{F}|_{\{z\}\times T^2}) &= \sqrt{-1}\int_{\{z\}\times T^2} \mathrm{Tr}\,(\Lambda_{T^2} F(h_{\mathcal{F}}))\, d\mathrm{vol}_{T^2} \\ &= \sqrt{-1}\int_{\{z\}\times T^2} \mathrm{Tr}\,(\pi\Lambda_{T^2} F(A))\, d\mathrm{vol}_{T^2} - \int_{\{z\}\times T^2} |\overline{\partial}_{T^2}\pi|^2 d\mathrm{vol}_{T^2}. \end{aligned} \tag{21}$$

Since $\deg(\mathcal{F}|_{\{z\}\times T^2})$ is a continuous and $2\pi\mathbb{Z}$-valued function, we conclude $\deg(\mathcal{F}|_{\{z\}\times T^2}) = 0$ for any $z \in \mathbb{C}^*$.

Conversely, We assume (i) and (ii). We have

$$\begin{aligned}-\deg(\mathcal{F}, h) &= \int_{\mathbb{C}^*\times T^2} (|\overline{\partial}_{\mathbb{C}^*}\pi|^2 + |\overline{\partial}_{T^2}\pi|^2) d\mathrm{vol}_{\mathbb{C}^*\times T^2} \\ &= \int_{\mathbb{C}^*\times T^2} |\overline{\partial}_{\mathbb{C}^*}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*\times T^2} + \int_{\mathbb{C}^*\times T^2} |\overline{\partial}_{T^2}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*\times T^2}.\end{aligned}$$

Here we use (ii) and (21), then we obtain

$$\begin{aligned}-\deg(\mathcal{F}, h) &= \int_{\mathbb{C}^*\times T^2} |\overline{\partial}_{\mathbb{C}^*}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*\times T^2} \\ &\quad + \sqrt{-1}\int_{\mathbb{C}^*\times T^2} \mathrm{Tr}\,(\pi\Lambda_{T^2} F(A))\, d\mathrm{vol}_{\mathbb{C}^*\times T^2} \\ &\leq \int_{\mathbb{C}^*\times T^2} |\overline{\partial}_{\mathbb{C}^*}\pi|^2 d\mathrm{vol}_{\mathbb{C}^*\times T^2} + ||F(A)||_{L^1(\mathbb{R}\times T^3)}.\end{aligned}$$

By Proposition 5.13, $(V, \overline{\partial}_A, h)|_{\mathbb{C}^*\times\{w\}}$ satisfies the assumption in Proposition 5.14 for any $w \in T^2$. Hence we have

$$\begin{aligned}-\deg(\mathcal{F}, h) &= -2\pi\int_{T^2} \text{par-deg}((P_{**}\mathcal{F})|_{\mathbb{C}^*\times\{w\}}) d\mathrm{vol}_{T^2} \\ &\quad + \sqrt{-1}\int_{\mathbb{C}^*\times T^2} \mathrm{Tr}\,(\pi\Lambda_{\mathbb{C}^*} F(A))\, d\mathrm{vol}_{\mathbb{C}^*\times T^2} + ||F(A)||_{L^1(\mathbb{R}\times T^3)} \\ &\leq -2\pi\int_{T^2} \text{par-deg}((P_{**}\mathcal{F})|_{\mathbb{C}^*\times\{w\}}) d\mathrm{vol}_{T^2} + 2||F(A)||_{L^1(\mathbb{R}\times T^3)}.\end{aligned}$$

Then $\text{par-deg}((P_{**}\mathcal{F})|_{\mathbb{C}^*\times\{w\}})$ is a constant on T^2, and we have the estimate $||F(A)||_{L^1(\mathbb{R}\times T^3)} < \infty$ by Corollary 3.4. Hence we obtain $0 \leq -\deg(\mathcal{F}, h) < \infty$. □

Proposition 5.20. *Let $\mathcal{F}$ be a saturated subsheaf of an L^2-finite instanton (V, h, A) on $\mathbb{R}\times T^3$. If $\deg(\mathcal{F}, h)$ is finite, then we have* $\deg(\mathcal{F}, h) =$

$2\pi\mathrm{Vol}(T^2)\cdot\text{par-deg}(P_{**}\mathcal{F})$, *where* $P_{**}\mathcal{F}$ *is a filtered subsheaf defined by* $P_{ab}\mathcal{F} := P_{ab}V \cap \mathcal{F}$.

Proof. Since $\deg(\mathcal{F}, h)$ is finite, $\mathcal{F}$ can be extended to the saturated subsheaf $P_{00}\mathcal{F}$ of $P_{00}V$ by Lemma 5.19. We denote by $B \subset T^2$ the set of all $w \in T^2$ that $P_{00}\mathcal{F}|_{\mathbb{P}^1\times\{w\}}$ is a subbundle of $P_{00}V|_{\mathbb{P}^1\times\{w\}}$. Since T^2 is compact and $P_{00}\mathcal{F}$ is saturated, $T^2 \setminus B$ is a finite subset. By applying (ii) of Lemma 5.19, we have

$$\begin{aligned}\deg(\mathcal{F}, h) &= \sqrt{-1}\int_{\mathbb{C}^*\times T^2} \Lambda_{\mathbb{C}^*}F(h_{\mathcal{F}})d\mathrm{vol}_{\mathbb{C}^*\times T^2}\\ &= \int_B \deg(\mathcal{F}|_{\mathbb{C}^*\times\{w\}}, h|_{\mathbb{C}^*\times\{w\}})d\mathrm{vol}_{T^2}\\ &= \int_B \deg(\mathcal{F}|_{\mathbb{C}^*\times\{w\}}, h|_{\mathbb{C}^*\times\{w\}})d\mathrm{vol}_{T^2}.\end{aligned}$$

By Proposition 5.13, $(V, \overline{\partial}_A, h)|_{\mathbb{P}^1\times\{w\}}$ satisfy the assumption in Proposition 5.14. Hence we have

$$\deg(\mathcal{F}, h) = \int_B \text{par-deg}(P_{**}(F)|_{\mathbb{P}^1\times\{w\}})d\mathrm{vol}_{T^2} = 2\pi|T^2|\text{par-deg}(P_{**}\mathcal{F}).$$

This is the desired equality. □

5.2.4. Proof of Theorem 5.11. By Lemma 2.4, $(\det(V), \det(h), \mathrm{Tr}(A))$ is a flat Hermitian line bundle. Hence we have the equality $\text{par-deg}(P_{**}V) = \text{par-deg}(P_{**}(\det(V))) = 0$. It is proved in Proposition 5.9 that $P_{ab}V|_{\{0\}\times T^2}$ and $P_{ab}V|_{\{\infty\}\times T^2}$ are semistable vector bundles of degree 0 for any $a, b \in \mathbb{R}$. Let $P_{**}\mathcal{F}$ be a filtered subsheaf of $P_{**}V$ satisfying $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(V)$ and (1) in Definition 2.11. We may assume that $P_{00}\mathcal{F}$ is saturated. We set $\mathcal{F} := P_{**}\mathcal{F}|_{\mathbb{C}^*\times T^2}$. Let $U \subset \mathbb{R}\times T^3$ be the maximal open subset such that $\mathcal{F}|_U$ is a subbundle of $V|_U$. Since $P_{00}\mathcal{F}$ is a saturated subsheaf of $P_{00}V$, $(\mathbb{R}\times T^3)\setminus U$ is a finite subset. By Lemma 5.19 and Proposition 5.20, we have $\deg(\mathcal{F}, h) = \text{par-deg}(P_{**}\mathcal{F})$. Therefore, $\text{par-deg}(P_{**}\mathcal{F}) \le 0$ holds by the Chern-Weil formula. Moreover, if $\text{par-deg}(P_{**}\mathcal{F}) = 0$ holds, then we have $\overline{\partial}\pi = 0$. Hence $\mathcal{F}|_U$ and $(\mathcal{F}|_U)^\perp$ become instantons by the induced metric from h, and $(V, h, A)|_U = (\mathcal{F}|_U) \oplus (\mathcal{F}|_U)^\perp$ is a decomposition as instantons. Moreover, this decomposition is invariant under parallel transports. Thus it can be extended to the decomposition on whole $\mathbb{R}\times T^3$. By repeating these arguments, we can prove that $P_{**}V$ is polystable. In particular, if (V, h, A) is irreducible, then $P_{**}V$ is stable. □

5.3. Some properties of $\mathrm{Gr}_a(P_*V)$

Let (V,h,A) be an L^2-finite instanton on $(-\infty,0)\times T^3$. Applying Proposition 5.1 to (V,h,A), we take a positive number $R>0$, a C^2-frame $\boldsymbol{v}=(v_i)$ of V on $(-\infty,-R)\times T^3$ and a model solution (Γ,N) of the Nahm equation. Assume that $\Gamma=(\Gamma_i)$ are diagonal, and we will denote by $\boldsymbol{v}_\alpha$ the subset of $\boldsymbol{v}$ corresponding to an eigenvalue $\alpha\in\mathbb{C}$ of $\Gamma_{\bar{w}}$. Applying Corollary 5.5 to (V,h,A), we take a holomorphic vector bundle $(E,\overline{\partial}_E,h_E)=\bigoplus_\alpha(E_\alpha,\overline{\partial}_{E_\alpha},h_{E_\alpha})$ on $(-\infty,-R)\times S^1$ and a holomorphic endomorphism $f=\bigoplus_\alpha f_\alpha\in\Gamma((-\infty,-R)\times S^1,\mathrm{End}(E))$. Let $\boldsymbol{e}$ (resp. $\boldsymbol{e}_\alpha$) be the C^∞-frame of E (resp. E_α) which corresponds to $\boldsymbol{v}$ (resp. $\boldsymbol{v}_\alpha$). We will denote by P_*V and P_*E the prolongations of V over $\Delta(0)\times T^2$ and E over $\Delta(R)$ respectively, where $\Delta(s):=\{z\in\mathbb{C}\mid |z|<\exp(-2\pi s)\}$ and $\Delta(s)^*:=\Delta(s)\setminus\{0\}$. For $a\in\mathcal{P}ar(P_*E_\alpha)$, we have the weight filtration $\{W_i\mathrm{Gr}_a(P_*E_\alpha)\}_{i\in\mathbb{Z}}$ on $\mathrm{Gr}_a(P_*E_\alpha)$ which is induced by the nilpotent part of $\mathrm{Gr}_a(f_\alpha)$ on $\mathrm{Gr}_a(E_\alpha)$.

We set a holomorphic Hermitian vector bundle $(E',\overline{\partial}_{E'},h_{E'})$ on $\Delta(R)^*$ and a holomorphic endomorphism $f'\in\Gamma(\Delta(R)^*,E')$ as

$$\begin{cases} \overline{\partial}_{E'}(\boldsymbol{e'}) & =\boldsymbol{e'}(\Gamma_{\bar{\tau}}+((2\pi)^{-1}\log|z|)^{-1}N_{\bar{\tau}})d\bar{z}/2\pi\bar{z} \\ h_{E'}(e'_i,e'_j) & =\delta_{ij} \\ f'(\boldsymbol{e'}) & =\boldsymbol{e'}(\Gamma_{\bar{w}}+((2\pi)^{-1}\log|z|)^{-1}N_{\bar{w}}), \end{cases}$$

where $\boldsymbol{e'}=(e'_i)$ is a C^∞-frame of E' on $\Delta(R)^*$. Let $E'=\bigoplus E'_\alpha$ be the holomorphic decomposition induced by the eigen decomposition of $\Gamma_{\bar{w}}$. We have $f'(E'_\alpha)\subset E'_\alpha$, hence we write $f'=\bigoplus_\alpha f'_\alpha$. Then $(E'_\alpha,h_{E'_\alpha},\overline{\partial}_{E'_\alpha})$ is also acceptable as $(E_\alpha,h_\alpha,\overline{\partial}_{E_\alpha})$. Moreover, for $a\in\mathcal{P}ar(P_*E'_\alpha)$ we also have the weight filtration $\{W_i\mathrm{Gr}_a(P_*E'_\alpha)\}_{i\in\mathbb{Z}}$ on $\mathrm{Gr}_a(P_*E'_\alpha)$ which is induced by the nilpotent part of $\mathrm{Gr}_a(f'_\alpha)$ on $\mathrm{Gr}_a(E'_\alpha)$.

Proposition 5.21. *There exists a holomorphic isomorphism $\Psi:(E,h_E,\overline{\partial}_E)\to(E',h_{E'},\overline{\partial}_{E'})$ such that Ψ and Ψ^{-1} are bounded, and Ψ preserves the decompositions $E=\bigoplus E_\alpha$ and $E'=\bigoplus E'_\alpha$. In particular, we have $\mathcal{P}ar(P_*E'_\alpha)=\mathcal{P}ar(P_*E_\alpha)$ and the induced isomorphisms*

$$\Psi:\mathrm{Gr}^W(\mathrm{Gr}_a(E_\alpha))\to\mathrm{Gr}^W(\mathrm{Gr}_a(E'_\alpha)).$$

Proof. For an eigenvalue $\alpha\in\mathbb{C}$ of $\Gamma_{\bar{w}}$, let $\boldsymbol{e'}_\alpha$ be the subset of $\boldsymbol{e'}$ which spans E'_α. We define $\Psi_{1,\alpha}:E_\alpha\to E'_\alpha$ as $\Psi_{1,\alpha}(\boldsymbol{e}_\alpha):=\boldsymbol{e'}_\alpha$. Then we have $\overline{\partial}\Psi_{1,\alpha}(\boldsymbol{e}_\alpha)=\boldsymbol{e'}_\alpha\tilde{\varepsilon}_{\bar{\tau},\alpha}d\bar{z}/\bar{z}$, where $\tilde{\varepsilon}_{\bar{\tau}}=\sum_\alpha\tilde{\varepsilon}_{\bar{\tau},\alpha}$ is the decomposition of $\tilde{\varepsilon}_{\bar{\tau}}$ in Proposition 5.1 induced by the decomposition $V|_{(-\infty,-R)\times T^3}=\bigoplus_\alpha V_\alpha$ in Corollary 5.4. Let $\boldsymbol{b}_\alpha=(b_{\alpha,i})$ and $\boldsymbol{b'}_\alpha=(b'_{\alpha,j})$ be holomorphic frames of

P_aE_α and $P_aE'_\alpha$ respectively that they have the norm estimates in Proposition 5.13. We set a function $K_\alpha = (K_{\alpha,ij}) : \Delta^*(R) \to \mathrm{Mat}(r_\alpha, \mathbb{C})$ as the change of basis of $\bar{z}^{-1}\tilde{\varepsilon}_{\overline{\tau},\alpha}$ from the frames $\boldsymbol{e}_\alpha$ and $\boldsymbol{e'}_\alpha$ to the frames $\boldsymbol{b}_\alpha$ and $\boldsymbol{b'}_\alpha$. Because of the estimate of $\tilde{\varepsilon}_{\overline{\tau}}$ in Proposition 5.1 and the norm estimates of $\boldsymbol{b}_\alpha$ and $\boldsymbol{b'}_\alpha$ in Proposition 5.13, we have the following estimate

$$|K_{\alpha,ij}| = O\left(|z|^{\mathrm{ord}(b_{\alpha,i})-\mathrm{ord}(b'_{\alpha,j})-1}(-\log|z|)^{-\deg^W(b_{\alpha,i})+\deg^W(b'_{\alpha,j})-1-\delta}\right).$$

Therefore, according to [24, Lemma 7.1] there exist functions $S_{\alpha,ij}$ such that we have $\overline{\partial}(S_{\alpha,ij}) = K_{\alpha,ij}d\bar{z}$, and it satisfies the following estimate

$$|S_{\alpha,ij}| = O\left(|z|^{\mathrm{ord}(b_{\alpha,i})-\mathrm{ord}(b'_{\alpha,j})}(-\log|z|)^{-\deg^W(b_{\alpha,i})+\deg^W(b'_{\alpha,j})-\delta}\right).$$

We set $S_\alpha : E_\alpha \to E'_\alpha$ as $S_\alpha(b_i) := \sum_j S_{\alpha,ij} \cdot b'_j$. Then we have $\overline{\partial}S_\alpha = \tilde{\varepsilon}_{\overline{\tau},\alpha}d\bar{z}/\bar{z}$ and $|S_\alpha| = O((-\log|z|)^{-\delta})$. Therefore, for some $R' > R$, $\Psi_\alpha := \Psi_{1,\alpha} - S_\alpha$ is a holomorphic isomorphism on $\Delta^*(R')$ such that Ψ_α and Ψ_α^{-1} are bounded. Finally, we set $\Psi := \oplus_\alpha \Psi_\alpha$. Then Ψ is holomorphic isomorphism, and both Ψ and Ψ^{-1} are bounded. We write $\Psi_\alpha(b_{\alpha,i}) = \sum_j \Psi_{\alpha,ij} \cdot b'_{\alpha,j}$. Then if $\mathrm{ord}(b_{\alpha,i}) < \mathrm{ord}(b'_{\alpha,j})$, or $\mathrm{ord}(b_{\alpha,i}) = \mathrm{ord}(b'_{\alpha,j})$ and $\deg^W(b_{\alpha,i}) < \deg^W(b'_{\alpha,j})$, then we have $\Psi_{ij}(0) = 0$ because of the norm estimates of $\boldsymbol{b}_\alpha$ and $\boldsymbol{b}'_\alpha$. Hence remains follow from it. □

For $a \in \mathcal{P}ar(P_*E) = \mathcal{P}ar(P_*E')$, the gradations $\mathrm{Gr}_a(E)$ and $\mathrm{Gr}_a(E')$ are skyscraper sheaves with the supports $\{0\} \subset \Delta(R)$, and f and f' induces the endomorphisms $\mathrm{Gr}_a(f)$ and $\mathrm{Gr}_a(f')$ on $\mathrm{Gr}_a(E)$ and $\mathrm{Gr}_a(E')$ respectively. We regard $(\mathrm{Gr}_a(E^{(\prime)}), \mathrm{Gr}_a(f^{(\prime)}))$ as a vector space with an endomorphism.

Corollary 5.22. *From the isomorphism Ψ, we can construct an isomorphism $(\mathrm{Gr}_a(E) \times T^2, \overline{\partial}_{T^2} + \mathrm{Gr}_a(f)d\bar{w}) \simeq (\mathrm{Gr}_a(E') \times T^2, \overline{\partial}_{T^2} + \mathrm{Gr}_a(f')d\bar{w})$ for $a \in \mathcal{P}ar(P_*E)$.*

Proof. In Proposition 5.21, we proved that Ψ induces an isomorphism between $\mathrm{Gr}_a(E_\alpha)$ and $\mathrm{Gr}_a(E'_\alpha)$, and it also induces an isomorphism between their gradation of the weight filtrations. Therefore, the Jordan normal forms of $\mathrm{Gr}_a(f)$ and $\mathrm{Gr}_a(f')$ are equivalent. Hence the proof is complete. □

5.4. Algebraic Nahm transform

We set a hypersurface $D = D_1 \sqcup D_2 := (\{0\} \times T^2) \sqcup (\{\infty\} \times T^2) \subset \mathbb{P}^1 \times T^2$. Let $P_{**}V$ be a stable filtered bundle of degree 0 and rank $r > 1$ on $(\mathbb{P}^1 \times$

$T^2, D)$ unless otherwise noted. We set $\mathrm{Sing}_{\mathbb{R}}(P_{**}V)$ to be $-\mathcal{P}ar(P_{**}V,1)\cup\mathcal{P}ar(P_{**}V,2)\subset\mathbb{R}$, where $-\mathcal{P}ar(P_{**}V,1):=\{a\in\mathbb{R}\mid -a\in\mathcal{P}ar(P_{**}V,1)\}$. We also set $\mathrm{Sing}_{S^1}(P_{**}V):=\pi(\mathrm{Sing}_{\mathbb{R}}(P_{**}V))\subset S^1$, where $\pi:\mathbb{R}\to S^1$ is the quotient map.

Let $\mathcal{L}\to T^2\times\hat{T}^2$ be the Poincaré bundle of T^2. For $I\subset\{1,2,3\}$, let p_I be the projection of $\mathbb{P}^1\times T^2\times\hat{T}^2$ onto the product of the i-th components ($i\in I$). We define a functor $F^i:\mathrm{Coh}(\mathcal{O}_{\mathbb{P}^1\times T^2})\to\mathrm{Coh}(\mathcal{O}_{\hat{T}^2})$ to be $F^i(\mathcal{F}):=R^ip_{3*}(p_{12}^*\mathcal{F}\otimes p_{23}^*\mathcal{L})$.

Proposition 5.23. *The sheaves $F^1(P_{<-\hat{t}<\hat{t}}V)$ and $F^1(P_{-\hat{t}\hat{t}}V)$ are locally free for any $\hat{t}\in\mathbb{R}$.*

Proof. Let $p:\mathbb{P}^1\times T^2\to T^2$ be the projection. By Proposition 2.12, we have $H^0(\mathbb{P}^1\times T^2, P_{-\hat{t}\hat{t}}V\otimes p^*F)=H^2(\mathbb{P}^1\times T^2,P_{-\hat{t}\hat{t}}V\otimes p^*F)=0$ for any $\hat{t}\in\mathbb{R}$ and any $F\in\mathrm{Pic}^0(T^2)$. Therefore $h^1(\mathbb{P}^1\times T^2,P_{-\hat{t}\hat{t}}V\otimes p^*F)$ is a constant for any $F\in\mathrm{Pic}^0(T^2)$ by the Riemann-Roch-Hirzebruch theorem. Hence $F^1(P_{-\hat{t}\hat{t}}V)$ is a locally free sheaves on $\hat{T}^2$. By the same way we can prove that $F^1(P_{<-\hat{t}<\hat{t}}V)$ is also locally free. □

We will denote by $\mathrm{AN}(P_{**}V)_{\hat{t}}$ the locally free sheaf $F^1(P_{-\hat{t}\hat{t}}V)$ for $\hat{t}\in\mathbb{R}$. Since we have $P_{-(\hat{t}+1)\ (\hat{t}+1)}V\simeq P_{-\hat{t}\hat{t}}V$, we can regard $\{\mathrm{AN}(P_{**}V)_{\hat{t}}\}$ as a family on $\hat{t}\in S^1$.

Definition 5.24. We define the algebraic singularity set $\mathrm{Sing}(P_{**}V)\subset\hat{T}^3=S^1\times\hat{T}^2$ as

$$\mathrm{Sing}(P_{**}V):=\left(\bigcup_{\hat{t}\in\mathcal{P}ar(P_{**}V,1)}\{-\pi(\hat{t})\}\times\mathrm{Spec}({}^1\mathrm{Gr}_{\hat{t}}(P_{**}V))\right)\cup\left(\bigcup_{\hat{t}\in\mathcal{P}ar(P_{**}V,2)}\{\pi(\hat{t})\}\times\mathrm{Spec}({}^2\mathrm{Gr}_{\hat{t}}(P_{**}V))\right),$$

where $\mathrm{Spec}(\cdot)$ is the spectrum set of a semistable bundle of degree 0 on T^2 (See Definition 2.14).

By Corollary 5.22, the notions of singularity set and algebraic singularity set are compatible.

Proposition 5.25. *Let (V,h,A) be an irreducible L^2-finite instanton on $\mathbb{R}\times T^3$ of rank $r>1$, and $P_{**}V$ be the associated stable filtered bundle. Then we have* $\mathrm{Sing}(V,h,A)=\mathrm{Sing}(P_{**}V)$. □

Proposition 5.26. *Let $I \subset \mathbb{R}$ be a closed interval with $|I| < 1$ and $\pi : \mathbb{R} \to S^1$ be the quotient map. Let $U \subset \hat{T}^2$ be the complement of the image of $\left(\pi(I) \times \hat{T}^2\right) \cap \mathrm{Sing}(P_{**}V)$ under the projection $\hat{T}^3 \to \hat{T}^2$. Then, for any $\hat{t}, \hat{t}' \in I$, we have a natural isomorphism $\mathrm{AN}(P_{**}V)_{\hat{t}}|_U \simeq \mathrm{AN}(P_{**}V)_{\hat{t}'}|_U$.*

Proof. We only need to prove the claim under the assumption $I = [-\varepsilon, \varepsilon]$ for a positive number $\varepsilon > 0$. By extending I, we may assume that any points in $\mathrm{Sing}_{\mathbb{R}}(P_{**}V) \cap I$ are interior points of I. Hence we may assume either $I \cap \mathrm{Sing}_{\mathbb{R}}(P_{**}V) = \emptyset$ or $I \cap \mathrm{Sing}_{\mathbb{R}}(P_{**}V) = \{0\}$. Under this assumption, we will show that there exists the natural isomorphism

$$\mathrm{AN}(P_{**}V)_{\hat{t}}|_U \simeq F^1(P_{-\varepsilon-\varepsilon}V)|_U$$

for any $\hat{t} \in I$. This isomorphism is obvious for $\hat{t} = 0$, and a proof for $\hat{t} > 0$ is also valid for $\hat{t} < 0$. Hence we assume $\hat{t} > 0$. Then, the short exact sequence $0 \to P_{-\varepsilon-\varepsilon}V \to P_{-\hat{t}\hat{t}}V \to {}^2\mathrm{Gr}_1(P_{**}V) \to 0$, we obtain the exact sequence $F^0({}^2\mathrm{Gr}_0(P_{**}V)) \to F^1(P_{-\varepsilon-\varepsilon}V) \to \mathrm{AN}(P_{**}V)_{\hat{t}} \to F^1({}^2\mathrm{Gr}_0(P_{**}V))$. Here we have $F^0({}^2\mathrm{Gr}_0(P_{**}V)) = 0$ and $F^1({}^2\mathrm{Gr}_0(P_{**}V))|_U = 0$ from Corollary 2.15. Hence we obtain $\mathrm{AN}(P_{**}V)_{\hat{t}}|_U \simeq F^1(P_{-\varepsilon-\varepsilon}V)|_U$. □

Definition 5.27. We call this isomorphism the *algebraic scattering map.*

Corollary 5.28. *The family $\{\mathrm{AN}(P_{**}V)_{\hat{t}}\}_{\hat{t}}$ forms a mini-holomorphic bundle $(\mathrm{AN}(P_{**}V), \overline{\partial}_{\mathrm{AN}}, \partial_{\mathrm{AN},\hat{t}})$ on $\hat{T}^3 \setminus \mathrm{Sing}(P_{**}V)$.*

Definition 5.29. We call this construction the *algebraic Nahm transform.*

We prove that the Nahm transform and the algebraic Nahm transform are compatible.

Theorem 5.30. *Let (V, h, A) be an irreducible L^2-finite instanton of rank $r > 1$ on $\mathbb{R} \times T^3$ and $P_{**}V$ the associated stable filtered bundle on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$. Then, the underlying mini-holomorphic bundle $(\hat{V}, \overline{\partial}_{\hat{A}}, \partial_{\hat{V},\hat{t}})$ of the Nahm transform $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ of (V, h, A) is isomorphic to the algebraic Nahm transform $(\mathrm{AN}(P_{**}V), \overline{\partial}_{\mathrm{AN}}, \partial_{\mathrm{AN},\hat{t}})$.*

Proof. For a smooth manifold M and a vector bundle F on M, let $C^\infty(F)$ denote the sheaf of C^∞-sections of F. If M is a complex manifold and F is a holomorphic bundle, then we also denote by $\mathcal{O}(F)$ the sheaf of holomorphic sections of F.

Let $U_{\hat{t}} \subset \hat{T}^2$ be the complement of the image of $\mathrm{Sing}(V, h, A) \cap (\{\hat{t}\} \times \hat{T}^2)$ under the projection map $\hat{T}^3 \to \hat{T}^2$. We first construct an isomorphism $(\hat{V}, \overline{\partial}_{\hat{A}})|_{\{\hat{t}\}\times U_{\hat{t}}} \simeq \mathrm{AN}(P_{**}V)_{\hat{t}}|_{U_{\hat{t}}}$ for any $\hat{t} \in S^1$. By replacing (V, h, A) with $(V, h, A + 2\pi\sqrt{-1}\hat{t}dt)$, we may assume $\hat{t} = 0$. From the short exact sequence $0 \to P_{<0<0}V \to P_{00}V \to {}^1\mathrm{Gr}_0(P_{**}V) \oplus {}^2\mathrm{Gr}_0(P_{**}V) \to 0$, we obtain the exact sequence

$$0 \to F^1(P_{<0<0}V) \to \mathrm{AN}(P_{**}V)_0 \to F^1({}^1\mathrm{Gr}_0(P_{**}V) \oplus {}^2\mathrm{Gr}_0(P_{**}V)).$$

Hence by the definition of U_0, the isomorphism

$$F^1(P_{<0<0}V)|_{U_0} \simeq \mathrm{AN}(P_{**}V)_0|_{U_0}$$

holds. Hence it suffices to prove $(\hat{V}, \overline{\partial}_{\hat{A}})|_{\{0\}\times U_0} \simeq F^1(P_{<0<0}V)|_{U_0}$. On one hand, since the Dolbeault resolution $(C^\infty(\Omega^{0,*}(p_{12}^*P_{<0<0}V \otimes p_{23}^*\mathcal{L})), \overline{\partial})$ of the holomorphic vector bundle $p_{12}^*P_{<0<0}V \otimes p_{23}^*\mathcal{L}$ is acyclic for the functor p_{3*}, we have an isomorphism $F^1(P_{<0<0}V) \simeq H^1(p_{3*}C^\infty(\Omega^{0,*}(p_{12}^*P_{<0<0}V \otimes p_{23}^*\mathcal{L})), \overline{\partial})$. Let $\mathcal{V}^i$ be the flat bundle on $\hat{T}^2$ which is the quotient of the product bundle $\underline{C^\infty(\mathbb{P}^1 \times T^2, \Omega^{0,i}(P_{<0<0}V))}$ on $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$ by Λ_2^*-action as in section 4. Then we have $F^1(P_{<0<0}V) \simeq H^1(p_{3*}C^\infty(\Omega^{0,*}(p_{12}^*P_{<0<0}V \otimes p_{23}^*\mathcal{L})), \overline{\partial}) \simeq H^1(\mathcal{O}(\mathcal{V}^*), \overline{\partial}_{P_{<0<0}V_\xi})$ on $\hat{T}^2$. On the other hand, for the normed vector space $X^i := \{f \in L^2(\mathbb{R} \times T^3, \Omega^{0,i}(V)) \mid \overline{\partial}_A(f) \in L^2\}$, we construct the flat vector bundle $\mathcal{V}^i_{L^2}$ on $\hat{T}^2$ from X^i in a similar way. Then, we also have an isomorphism $(\hat{V}, \overline{\partial}_{\hat{A}})|_{\{0\}\times U_0} \simeq H^1(\mathcal{O}(\mathcal{V}^*_{L^2}|_{U_0}), \overline{\partial}_{A_\xi})$. By Proposition 5.13, we obtain the natural inclusions

$$C^\infty(\mathbb{P}^1 \times T^2, \Omega^{0,i}(P_{<0<0}V)) \hookrightarrow L^2(\mathbb{R} \times T^3, \Omega^{0,i}(V))$$

and the induced chain homomorphism

$$\varphi : (\mathcal{O}(\mathcal{V}^*|_{U_0}), \overline{\partial}_{P_{<0<0}V_\xi}) \to (\mathcal{O}(\mathcal{V}^*_{L^2}|_{U_0}), \overline{\partial}_{A_\xi}).$$

In order to prove that φ is a quasi-isomorphism, we only need to prove that the specialization $\varphi_\xi : (C^\infty(\mathbb{P}^1 \times T^2, \Omega^{0,*}(P_{<0<0}V)), \overline{\partial}_{P_{<0<0}V_\xi}) \to (X^*, \overline{\partial}_{A_\xi})$ is a quasi-isomorphism for any $\xi \in U_0$. We will show this below in Proposition 5.32.

It remains to prove that the scattering map of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ and the algebraic scattering map of $(\mathrm{AN}(P_{**}V), \overline{\partial}_{\mathrm{AN}}, \partial_{\mathrm{AN},t})$ are compatible under the isomorphism $H^1(\varphi)$. It suffices to show the following lemma.

Lemma 5.31. *Let $I = [-\varepsilon, \varepsilon] \subset \mathbb{R}$ be a closed interval and $\xi \in \hat{T}^2$ with the condition $(\pi(I) \times \{\xi\}) \cap \mathrm{Sing}(V, h, A) = \emptyset$. For $\hat{t} \in I$, we take*

$$f_{\hat{t}} \in \mathrm{AN}(P_{**}V)_{(\hat{t},\xi)} \simeq H^1(\mathbb{P}^1 \times T^2, P_{-\hat{t}\hat{t}}V_\xi)$$

satisfying the condition that $f_{\hat{t}}$ is a constant under the algebraic scattering map for any $\hat{t} \in I$. Then, $H^1(\varphi)_{(\hat{t},\xi)}(f_{\hat{t}})$ is a constant under the scattering map of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ for any $\hat{t} \in I$. □

(Proof of Lemma 5.31). We may assume $\xi = 0$. As in section 4, we set $\mathcal{V}$ the flat vector bundle on $\hat{T}^3$ which is the quotient of the product bundle $\underline{L^2(\mathbb{R} \times T^3, S^- \otimes V)}$ on $\mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ by Λ_3^*-action. By using the orthogonal projection $P : \mathcal{V}|_{\hat{T}^3 \setminus \mathrm{Sing}(V,h,A)} \to \hat{V}$, the equation of the scattering map can be written as follows:

$$\nabla_{\hat{A}_{\hat{t}}}(\cdot) - \sqrt{-1}\hat{\Phi}(\cdot) = P\left(\partial_{\hat{t}}(\cdot) + \log|z|(\cdot)\right) = 0. \tag{22}$$

By the assumption, there exists $f \in H^1(\mathbb{P}^1 \times T^2, P_{-\varepsilon-\varepsilon}V)$ such that $f_{\hat{t}}$ is the image of f under the natural isomorphism $H^1(\mathbb{P}^1 \times T^2, P_{-\varepsilon-\varepsilon}V) \simeq H^1(\mathbb{P}^1 \times T^2, P_{-\hat{t}\hat{t}}V)$. Here the image of $f_{\hat{t}}$ in $\hat{V}_{(\hat{t},0)}$ is $P(f \cdot |z|^{-\hat{t}})$. Hence the derivative is $\partial_{\hat{t}} P(f \cdot |z|^{-\hat{t}}) = P(\partial_{\hat{t}}(f \cdot |z|^{-\hat{t}})) = -P(f \cdot |z|^{-\hat{t}} \cdot \log|z|)$. Thus $H^1(\varphi)_{(0,\hat{t})}(f_{\hat{t}})$ satisfies the equation (22), and this completes the proof. □

5.4.1. L^2-Dolbeault lemma. Let (V, h, A) be an irreducible L^2-finite instanton and $P_{**}V$ its prolongation. We assume $(0, 0) \notin \mathrm{Sing}(V, h, A)$. Let $\mathcal{A}^{0,i}(P_{<0<0}V)$ denote the sheaf of smooth sections of $\Omega^{0,i}(P_{<0<0}V)$. Then we have the natural isomorphism $H^1(\mathbb{P}^1 \times T^2, (\mathcal{A}^{0,*}(P_{<0<0}V), \overline{\partial}_{P_{<0<0}V})) \simeq \mathrm{AN}(P_{**}V)_{(0,0)}$. Let ${}_p\mathcal{A}^{0,i}_{L^2}(V)$ be the presheaf on $\mathbb{P}^1 \times T^2$ that associates an open subset $W \subset \mathbb{P}^1 \times T^2$ to a $\mathbb{C}$-vector space

$$\left\{ s \in L^2\big(W \cap (\mathbb{R} \times T^3), \Omega^{0,i}(V)\big) \;\middle|\; \overline{\partial}_A(s) \in L^2\big(W \cap (\mathbb{R} \times T^3), \Omega^{0,i+1}(V)\big) \right\},$$

where $L^2(W \cap (\mathbb{R} \times T^3), \Omega^{0,i}(V))$ means the set of L^2-sections of $\Omega^{0,i}(V)$ on $W \cap (\mathbb{R} \times T^3)$ with respect to h and $g_{\mathbb{R}\times T^3}$. Let $\mathcal{A}^{0,i}_{L^2}(V)$ denote the sheafification of ${}_p\mathcal{A}^{0,i}_{L^2}(V)$. Then we have the natural isomorphism $H^1(\mathbb{P}^1 \times T^2, (\mathcal{A}^{0,*}_{L^2}, \overline{\partial}_A)) \simeq \hat{V}_{(0,0)}$. Let $K : \mathcal{A}^{0,i}(P_{<0<0}V) \to \mathcal{A}^{0,i}_{L^2}(V)$ be the sheaf homomorphism induced by the inclusion map $C^\infty(W, \Omega^{0,i}(P_{<0<0}V)) \subset L^2\big(W \cap (\mathbb{R} \times T^3), \Omega^{0,i}(V)\big)$ for an open subset $W \subset \mathbb{P}^1 \times T^2$. For simplicity, we use the same symbol K for the chain map $(\mathcal{A}^{0,*}(P_{<0<0}V), \overline{\partial}_{P_{<0<0}V}) \to (\mathcal{A}^{0,*}_{L^2}, \overline{\partial}_A)$.

Proposition 5.32. *The chain map K induces an isomorphism $\hat{V}_{(0,0)} \simeq \mathrm{AN}(P_{**}V)_{(0,0)}$.*

Proof. Let $q : \mathbb{P}^1 \times T^2 \to \mathbb{P}^1$ be the projection. To show that K induces an isomorphism $\hat{V}_{(0,0)} \simeq \mathrm{AN}(P_{**}V)_{(0,0)}$, we only need to prove that q_*K is a quasi-isomorphism. Thus we consider the following lemma.

Lemma 5.33. *For any $z \in \mathbb{P}^1$, $(q_*K)_z$ is a quasi-isomorphism, where $(q_*K)_z$ means the induced chain map between the stalks at z.*

Once we admit the lemma, then q_*K is a quasi-isomorphism and the proof is complete. □

(Proof of Lemma 5.33). This lemma is trivial unless $z = 0$ or $z = \infty$, and the same proof works for both the cases $z = 0$ and $z = \infty$. Thus it suffices to consider only the case $z = 0$.

From Proposition 5.1, we take a sufficiently small neighborhood U of $0 \in \mathbb{P}^1$, a frame $\boldsymbol{v} = (v_i)$ of V on $U^* \times T^2$, and a model solution (Γ, N) of the Nahm equation that they satisfy conditions in Proposition 5.1, where $U^* := U \setminus \{0\}$. Let $(E, h_E, \overline{\partial}_E, f) = \bigoplus_\alpha (E_\alpha, h_{E_\alpha}, \overline{\partial}_{E_\alpha}, f_\alpha)$ be the holomorphic Hermitian vector bundle with the endomorphism on U^* constructed from (V, h, A) in Corollary 5.5. Let $\boldsymbol{e}$ be the C^∞-frame of E which corresponds to $\boldsymbol{v}$. We take the sheaves $\mathcal{A}^{0,i}_{L^2}(E)$ and $\mathcal{A}^{0,i}(P_{<0}E)$ on U constructed from E and $P_{<0}E$ in a similar way to $\mathcal{A}^{0,i}_{L^2}(V)$ and $\mathcal{A}^{0,i}(P_{<0<0}V)$. Then we have $\mathcal{A}^{0,i}_{L^2}(E) \simeq \bigoplus_\alpha \mathcal{A}^{0,i}_{L^2}(E_\alpha)$ and $\mathcal{A}^{0,i}(P_{<0}E) \simeq \bigoplus_\alpha \mathcal{A}^{0,i}(P_{<0}E_\alpha)$ because E_α and E_β are orthogonal for any $\alpha \neq \beta$.

We write $\nabla^{0,1}_A = \overline{\partial}_{A_{\bar{z}}} d\bar{z} + \overline{\partial}_{A_{\bar{w}}} d\bar{w}$. For $s \in L^2(U^* \times T^2, V)$, we denote the Fourier series expansion of s with respect to $\boldsymbol{v}$ by

$$s = \boldsymbol{v} \cdot \sum_{n \in \Lambda_2^*} s_n(z) \exp(2\pi\sqrt{-1}\langle n, w\rangle).$$

By using the Fourier series expansion, we set the bounded operator $I : L^2(U^* \times T^2, V) \to L^2(U^*, E)$ by $I(s) := \boldsymbol{e} \cdot s_0(z)$, and set the closed subspace $L^2(U^* \times T^2, V)^\perp := \mathrm{Ker}(I)$. Because of Remark 3.7 (i), we can construct the inverse operator $G_{L^2} : L^2(U^* \times T^2, V)^\perp \to L^2(U^* \times T^2, V)^\perp$ of $\overline{\partial}_{A_{\bar{w}}}$ as

$$G_{L^2}(s) := \boldsymbol{v} \cdot \sum_{\substack{n=(n_2,n_3) \\ \in \Lambda_2^* \setminus \{(0,0)\}}} \left((\sqrt{-1}n_2 - n_3)\pi + A_{\bar{w}}\right)^{-1} s_n(z) \exp(2\pi\sqrt{-1}\langle n, w\rangle),$$

where $A_{\overline{w}}$ is the component of $\nabla_A^{0,1}(\boldsymbol{v}) = \boldsymbol{v}(A_{\overline{w}} d\overline{w} + A_{\overline{z}} d\overline{z})$. Therefore, the complex $(q_* \mathcal{A}_{L^2}^{0,*}(V)(U), \overline{\partial}_A)$ is quasi-isomorphic to the following complex:

$$\mathcal{A}_{L^2}^{0,0}(E)(U) \xrightarrow{f \oplus \overline{\partial}_E} \mathcal{A}_{L^2}^{0,0}(E)(U) \oplus \mathcal{A}_{L^2}^{0,1}(E)(U) \xrightarrow{\overline{\partial}_E - f} \mathcal{A}_{L^2}^{0,1}(E)(U).$$

Since $f_\alpha : \mathcal{A}_{L^2}^{0,0}(E_\alpha)(U) \to \mathcal{A}_{L^2}^{0,0}(E_\alpha)(U)$ is an isomorphism for $\alpha \neq 0$, the complex $(q_* \mathcal{A}_{L^2}^{0,*}(V)(U), \overline{\partial}_A)$ is quasi-isomorphic to

$$\mathcal{A}_{L^2}^{0,0}(E_0)(U) \xrightarrow{f_0 \oplus \overline{\partial}_{E_0}} \mathcal{A}_{L^2}^{0,0}(E_0)(U) \oplus \mathcal{A}_{L^2}^{0,1}(E_0)(U) \xrightarrow{\overline{\partial}_{E_0} - f_0} \mathcal{A}_{L^2}^{0,1}(E_0)(U).$$

By the assumption $(0,0) \notin \mathrm{Sing}(V, h, A)$, the same argument in the proof of [27, Proposition 11.5] shows that $\overline{\partial}_{E_0}$ is surjective. Thus $(q_* \mathcal{A}_{L^2}^{0,*}(V)(U), \overline{\partial}_A)$ is quasi-isomorphic to

$$\left(\mathrm{Ker}(\overline{\partial}_{E_0}) \cap \mathcal{A}_{L^2}^{0,0}(E_0)(U)\right) \xrightarrow{f_0} \left(\mathrm{Ker}(\overline{\partial}_{E_0}) \cap \mathcal{A}_{L^2}^{0,0}(E_0)(U)\right).$$

By a similar way we can prove that $(q_* \mathcal{A}^{0,*}(P_{<0<0}V)(U), \overline{\partial}_{P_{<0}V})$ is also quasi-isomorphic to

$$\Gamma(U, P_{<0}(E_0)) \xrightarrow{P_{<0} f_0} \Gamma(U, P_{<0}(E_0)).$$

By the assumption $(0,0) \notin \mathrm{Sing}(V, h, A)$, we can show any $s \in \mathrm{Ker}(\overline{\partial}_{E_0}) \cap \mathcal{A}_{L^2}^{0,0}(E_0)(U^*)$ decays exponentially in $t \to -\infty$. Hence we have $\mathrm{Ker}(\overline{\partial}_{E_0}) \cap \mathcal{A}_{L^2}^{0,0}(E_0)(U^*) = \Gamma(U, P_{<0}(E_0))$ and the proof is complete. □

6. Correspondence between weights

As in Section 5, we assume that T^3 is isomorphic to the product of a circle $S^1 = \mathbb{R}/\mathbb{Z}$ and a 2-dimensional torus $T^2 = \mathbb{R}^2/\Lambda_2$ as a Riemannian manifold. Let $P_{**}V$ be a stable filtered bundle on $(\mathbb{P}^1 \times T^2, \{0, \infty\} \times T^2)$ of $\deg(P_{**}V) = 0$ and of rank $r > 1$. Let $(\mathrm{AN}(P_{**}V), \overline{\partial}_{\mathrm{AN}}, \partial_{\mathrm{AN},t})$ be the algebraic Nahm transform of $P_{**}V$. Let $\mathrm{Sing}_0(P_{**}V) \subset \hat{T}^2$ be the image of $\mathrm{Sing}(P_{**}V) \cap \{0\} \times \hat{T}^2$ under the projection $\hat{T}^3 = S^1 \times \hat{T}^2 \to \hat{T}^2$. Let $\mathcal{L} \to T^2 \times \hat{T}^2$ be the Poincaré bundle of T^2. For $I \subset \{1, 2, 3\}$, let p_I be the projection of $\mathbb{P}^1 \times T^2 \times \hat{T}^2$ onto the product of the i-th components $(i \in I)$. We define a functor $F^i : \mathrm{Coh}(\mathcal{O}_{\mathbb{P}^1 \times T^2}) \to \mathrm{Coh}(\mathcal{O}_{\hat{T}^2})$ to be $F^i(\mathcal{F}) := R^i p_{3*}(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{L})$ as in subsection 5.4.

Proposition 6.1. *We take a positive number $\varepsilon > 0$ small enough to satisfy* $\mathrm{Sing}(P_{**}V) \cap \big(q([-\varepsilon,\varepsilon]) \times \hat{T}^2\big) = \{0\} \times \mathrm{Sing}_0(P_{**}V)$*, where* $q : \mathbb{R} \to S^1$ *be the quotient map. Then, we have the following.*

- *We have a sequence of injections*

$$F^1(P_{-\varepsilon-\varepsilon}V) \hookrightarrow \mathrm{AN}(P_{**}V)_{\pm\varepsilon} \hookrightarrow F^1(P_{\varepsilon\varepsilon}V)$$

  which is compatible with the algebraic scattering map.

- *Under the above sequence of injections, we have the following isomorphisms:*

$$\mathrm{AN}(P_{**}V)_{-\varepsilon} \cap \mathrm{AN}(P_{**}V)_{\varepsilon} \simeq F^1(P_{-\varepsilon-\varepsilon}V).$$
$$\mathrm{AN}(P_{**}V)_{-\varepsilon} + \mathrm{AN}(P_{**}V)_{\varepsilon} \simeq F^1(P_{\varepsilon\varepsilon}V).$$
$$F^1(P_{\varepsilon\varepsilon}V)/\mathrm{AN}(P_{**}V)_{\varepsilon} \simeq F^1({}^1\mathrm{Gr}_0(P_{**}V)) \simeq H^1(\mathrm{FM}({}^1\mathrm{Gr}_0(P_{**}V))).$$
$$F^1(P_{\varepsilon\varepsilon}V)/\mathrm{AN}(P_{**}V)_{-\varepsilon} \simeq F^1({}^2\mathrm{Gr}_0(P_{**}V)) \simeq H^1(\mathrm{FM}({}^2\mathrm{Gr}_0(P_{**}V))).$$

  Here ${}^i\mathrm{Gr}_0(P_{**}V))$ *is the gradation of* $P_{**}V$ *(See subsubsection 2.4.1), and* $\mathrm{FM}(\mathcal{E}) \in D^b(\mathrm{Coh}(\mathcal{O}_{\hat{T}^2}))$ *is the Fourier-Mukai transform of a coherent sheaf* $\mathcal{E}$ *on* T^2*.*

Proof. From the definition of stable filtered bundle on $(\mathbb{P}^1 \times T^2, \{0,\infty\} \times T^2)$, ${}^i\mathrm{Gr}_0(P_{**}V)$ $(i = 1, 2)$ are semistable locally free sheaves on $\{0\} \times T^2$ and $\{\infty\} \times T^2$ respectively. Thus by Corollary 2.15 we have

$$F^0({}^i\mathrm{Gr}_0(P_{**}V)) = H^0(\mathrm{FM}({}^i\mathrm{Gr}_0(P_{**}V))) = 0.$$

Hence by the short exact sequences $0 \to P_{-\varepsilon-\varepsilon}V \to P_{\varepsilon-\varepsilon}V \to {}^1\mathrm{Gr}_0(P_{**}V) \to 0$ and $0 \to P_{\varepsilon-\varepsilon}V \to P_{\varepsilon\varepsilon}V \to {}^2\mathrm{Gr}_0(P_{**}V) \to 0$, we have an inclusion of sheaves $F^1(P_{-\varepsilon-\varepsilon}V) \hookrightarrow \mathrm{AN}(P_{**}V)_{-\varepsilon} \hookrightarrow F^1(P_{\varepsilon\varepsilon}V)$. Since $F^2(P_{\varepsilon-\varepsilon}V) = 0$ is shown in Proposition 5.23, we have

$$F^1(P_{\varepsilon\varepsilon}V)/\mathrm{AN}(P_{**}V)_{-\varepsilon} = F^1({}^2\mathrm{Gr}_0(P_{**}V)) = H^1(\mathrm{FM}({}^2\mathrm{Gr}_0(P_{**}V))).$$

In a similar way, we obtain $F^1(P_{-\varepsilon-\varepsilon}V) \hookrightarrow \mathrm{AN}(P_{**}V)_{\varepsilon} \hookrightarrow F^1(P_{\varepsilon\varepsilon}V)$ and

$$F^1(P_{\varepsilon\varepsilon}V)/\mathrm{AN}(P_{**}V)_{\varepsilon} = F^1({}^1\mathrm{Gr}_0(P_{**}V)) = H^1(\mathrm{FM}({}^1\mathrm{Gr}_0(P_{**}V))).$$

The compatibility with the algebraic scattering map is trivial by the definition of the algebraic scattering map.

We consider the short exact sequence $0 \to P_{-\varepsilon-\varepsilon}V \to P_{\varepsilon-\varepsilon}V \oplus P_{-\varepsilon\varepsilon}V \to P_{\varepsilon\varepsilon}V \to 0$. Since Proposition 5.23 shows $F^0(P_{\varepsilon\varepsilon}V) = 0$ and $F^2(P_{-\varepsilon-\varepsilon}V) = 0$, we obtain the exact sequence

$$0 \to F^1(P_{-\varepsilon-\varepsilon}V) \to \mathrm{AN}(P_{**}V)_{-\varepsilon} \oplus \mathrm{AN}(P_{**}V)_{\varepsilon} \to F^1(P_{\varepsilon\varepsilon}V) \to 0.$$

Therefore we have

$$\begin{aligned} \mathrm{AN}(P_{**}V)_{-\varepsilon} \cap \mathrm{AN}(P_{**}V)_{\varepsilon} &\simeq F^1(P_{-\varepsilon-\varepsilon}V), \\ \mathrm{AN}(P_{**}V)_{-\varepsilon} + \mathrm{AN}(P_{**}V)_{\varepsilon} &\simeq F^1(P_{\varepsilon\varepsilon}V). \end{aligned}$$

□

Let (V, h, A) be an irreducible L^2-finite instanton of rank $r > 1$ and $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ the Nahm transform of (V, h, A). For $\xi \in \mathrm{Sing}(V, h, A)$, let $\rho_{\pm,\xi}$ be the representation of $\mathfrak{su}(2)$ defined in Definition 3.6. For the irreducible decomposition $\rho_{\pm,\xi} = \bigoplus_{i=1}^{m_{\pm,\xi}} \rho_{\pm,\xi,i}$, we set $w_{\pm,\xi} := (\mathrm{rank}(\rho_{\pm,\xi,i}))$. We decompose the weight $\vec{k} = (k_i) \in \mathbb{Z}^{\mathrm{rank}(\hat{V})}$ of $(\hat{V}, \hat{h}, \hat{A}, \hat{\Phi})$ at $\xi \in \mathrm{Sing}(V, h, A)$ into the positive part k_+ and the negative part k_-.

Theorem 6.2. *$k_\pm$ agrees with $\pm w_{\pm,\xi}$ under a suitable permutation.*

Proof. By considering $V_\xi = (V, h, A_\xi)$ instead of (V, h, A), we assume $\xi = 0$. By Proposition 6.1 and Proposition 2.10, k_+ (resp. k_-) is determined by the stalk at $0 \in \hat{T}^2$ of $H^1(\mathrm{FM}({}^2\mathrm{Gr}_0(P_{**}V)))$ (resp. $H^1(\mathrm{FM}({}^1\mathrm{Gr}_0(P_{**}V)))$). Applying Proposition 5.1 to $(V, h, A)|_{(0,\infty)\times T^3}$ and $(V, h, A)|_{(-\infty,0)\times T^3}$, we obtain the model solutions $(\Gamma_\pm, N_\pm)$ of the Nahm equation. We set $\Gamma_{\pm,\bar{\tau}} d\bar{\tau} + \Gamma_{\pm,\bar{w}} d\bar{w} := (\sum_i \Gamma_{\pm,i} dx^i)^{(0,1)}$ and $N_{\pm,\bar{\tau}} d\bar{\tau} + N_{\pm,\bar{w}} d\bar{w} := (\sum_i N_{\pm,i} dx^i)^{(0,1)}$. We consider the following lemma.

Lemma 6.3. *Let $X_{\pm,0}$ be the eigenspace of $\Gamma_{\pm,\bar{\tau}}$ of eigenvalue 0. The gradation ${}^i\mathrm{Gr}_0(P_{**}V)$ $(i = 1, 2)$ are isomorphic with the holomorphic bundles $\tilde{E}_\pm := \big(X_{\pm,0} \times T^2, \overline{\partial}_{T^2} + (\Gamma_{\pm,\bar{w}} + N_{\pm,\bar{w}})|_{X_{\pm,0}} d\bar{w}\big)$ respectively.*

If we admit this lemma, then by Corollary 2.15, the stalk $H^1(\mathrm{FM}(\tilde{E}_\pm))_0$ is determined by the size of Jordan blocks of $(\Gamma_{\pm,\bar{w}} + N_{\pm,\bar{w}})|_{X_{\pm,0}}$ whose eigenvalues are 0. Since the size of Jordan blocks of $(\Gamma_{\pm,\bar{w}} + N_{\pm,\bar{w}})|_{X_{\pm,0}}$ whose eigenvalues are 0 is the rank of irreducible representations of $\mathfrak{su}(2)$ contained in $\rho_{\pm,0}$, hence $k_\pm$ agrees with $\pm w_{\pm,\xi}$ with a suitable permutation. □

(proof of Lemma 6.3). We may assume that any Γ_i are diagonal. As in subsection 5.3, we take $R > 0$ and set the holomorphic Hermitian vector bundle with the endomorphism $(E'_\pm, h_{E'_\pm}, \overline{\partial}_{E'_\pm}, f'_\pm)$ on $\Delta^*(R) := \{z \in \mathbb{C} \mid 0 <$

$|z| < \exp(-2\pi R)\}$ as follows:

$$\begin{cases} \overline{\partial}_{E'_\pm}(\boldsymbol{e'}_\pm) & = \boldsymbol{e'}\left(\Gamma_{\pm,\bar{\tau}} + ((2\pi)^{-1}\log|z|)^{-1}N_{\pm,\bar{\tau}}\right) d\bar{z}/(2\pi\bar{z}) \\ h_{E'_\pm}(e'_{\pm,i}, e'_{\pm,j}) & = \delta_{ij} \\ f'_\pm(\boldsymbol{e'}_\pm) & = \boldsymbol{e'}_\pm\left(\Gamma_{\pm,\bar{w}} + ((2\pi)^{-1}\log|z|)^{-1}N_{\pm,\bar{w}}\right), \end{cases}$$

where $\boldsymbol{e'}_\pm = (e'_{\pm,i})$ is a C^∞-frame of $E'_\pm$ on $\Delta(R)^*$. We take a holomorphic frame $\boldsymbol{b}'_\pm = (b'_{\pm,i})$ of $E'_\pm$ by $\boldsymbol{b}'_\pm := \boldsymbol{e'}_\pm \exp(-\Gamma_{\pm,\bar{\tau}}\bar{\tau} - 2N_{\pm,\bar{\tau}}\log(t)) = \boldsymbol{e'}_\pm \exp(-(2\pi)^{-1}\Gamma_{\pm,\bar{\tau}}\log(\bar{z}) - 2N_{\pm,\bar{\tau}}\log((2\pi)^{-1}\log|z|))$. Then, we also set the holomorphic frame $\boldsymbol{b''}_\pm = (b''_{\pm,i})$ of P_0E' by $b''_{\pm,i} := z^{-\lceil \mathrm{ord}(b'_{\pm,i})\rceil} b'_{\pm,i}$, where $\lceil\alpha\rceil$ is the least integer satisfying $\lceil\alpha\rceil \geq \alpha$. Hence we obtain $P_0f'(\boldsymbol{b''}_\pm) = \boldsymbol{b''}_\pm(\Gamma_{\pm,\bar{w}} + N_{\pm,\bar{w}})$ by the change of basis. By Remark 3.7 (i), ${}^i\mathrm{Gr}_0(P_{**}V)$ ($i = 1,2$) is spanned by the subset of $\boldsymbol{b''}_\pm$ that correspond to eigenvectors of $\Gamma_{\pm,1}$ of eigenvalue 0. By Corollary 5.22, ${}^i\mathrm{Gr}_0(P_{**}V)$ ($i = 1,2$) is isomorphic to the holomorphic bundles $\left(\mathrm{Gr}_0(P_*E'_\pm) \times T^2, \overline{\partial}_{T^2} + \mathrm{Gr}_0(P_*f'_\pm)d\bar{w}\right) \simeq \left(X_{\pm,0} \times T^2, \overline{\partial}_{T^2} + (\Gamma_{\pm,\bar{w}} + N_{\pm,\bar{w}})|_{X_{\pm,0}}d\bar{w}\right)$ respectively. $\square$

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