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Erratum for Ricci-flat graphs with girth at least five

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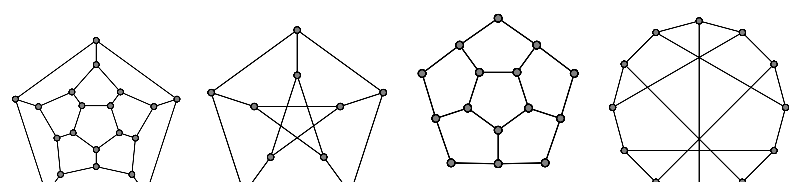
This erratum will correct the classification of Theorem 1 in [1] that misses the Triplex graph.

In Theorem 1 of [1], the classification of Ricci-flat graph with girth $g(G) \geq 5$ missed one graph – the Triplex graph, as discovered by three authors: Cushing, Kangaslampi, and Liu. Here is the correct theorem.

Theorem 1. *Suppose that* G *is a Ricci-flat graph with girth* $g(G) \geq 5$*. Then* G *is one of the following graphs,*

1) *the infinite path,*
2) *cycle* C_n *with* $n \geq 6$*,*
3) *the dodecahedral graph,*
4) *the Petersen graph,*
5) *the half-dodecahedral graph.*
6) *the Triplex graph.*



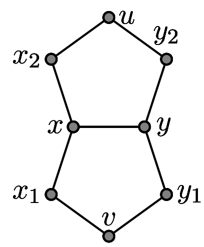
(a) Dodecahedral (b) Petersen (c) Half-Dodecahedral (d) Triplex

Figure 1. The four Ricci-flat graphs with girth 5

This error was caused by an incorrect implicit statement (in [1]) that any 3-regular Ricci-flat graph G has a surface embedding whose faces are all pentagons. In this erratum, we analyze the case that G does not have a surface embedding whose faces are all pentagons. We will show that this case leads a unique missing graph — the Triplex graph. An alternative method to correct Theorem 1 in [1] is given in [2].

Recall that Lemma 3 item 2 in [1] states:

Lemma 1. *For any edge xy of a graph of girth at least 5, if $d_x = d_y = 3$ and $\kappa(x, y) = 0$, then xy belongs to two 5-cycles P_1 and P_2 such that $P_1 \cap P_2 = xy$.*



Since G contains no cycle of length 3 or 4, any C_5 containing the edge xy is uniquely determined by a 3-path passing through xy. Since $d_x = d_y = 3$, there are four 3-paths of form x_ixyy_j for $i, j = 1, 2$. Here x_1, x_2 are two neighbors of x other than y and y_1, y_2 are two neighbors of y other than x. We say two C_5's are *opposite* to each other at xy if one C_5 passes through x_ixyy_j and the other one passes through $x_{3-i}xyy_{3-j}$. The above lemma says that there is a pair of opposite C_5's sharing the edge xy. We say an edge xy is *irregular* if there are three or four C_5 passing through it.

From this lemma, we have the following corollary.

Corollary 1. *If G is a 3-regular Ricci-flat graph and contains no irregular edge, then G can be embedded into a surface so that all faces are pentagons.*

Proof. View G as 1-dimension skeleton and glue pentagons to G recursively. Starting with any C_5 and glue a pentagon to it as a face, call the two-dimensional region M.

Let xy be a boundary edge of M, that is, an edge belonging to only one pentagon f in M. This pentagon f determines an opposite C_5 of G with respect to the edge xy. We glue a pentagon face to the opposite C_5 at xy

to enlarge M. Since G contains no irregular edge, every edge must be in exactly two pentagons. Therefore, the process will continue until M has no boundary edge. When this process ends, we get an embedding of G into some surface so that every face is a C_5. □

We are ready to fix the proof of Theorem 1 in [1].

Proof of Theorem 1: In the original proof of Theorem 1 in [1], we have taken care of all the cases except that G is 3-regular and contains an irregular edge xy. The edge xy is either in three C_5's or four C_5's. We will show that the first case leads to the Triplex graph while the second case leads to the Petersen graph.

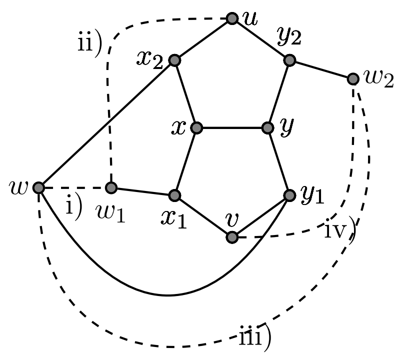
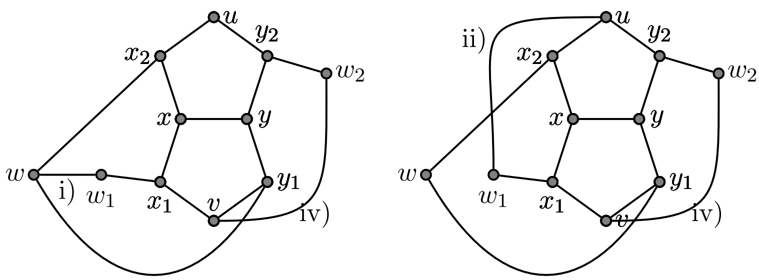


Figure 2. Starting configuration and possible extensions

First assume the edge xy is contained in three C_5's: ux_2xyy_2u, vx_1xyy_1v, and wx_2xyy_1w. The path x_1xyy_2 is not in any C_5. Let w_1 be the third neighbor of x_1, and w_2 be the third neighbor of y_2. Then w_1, w_2 are two distinct vertices, and they cannot be coincident with any vertex on the three C_5's. This is our starting configuration (See Figure 2 with solid lines).

Now consider the edge xx_1. Observe that the path w_1x_1xy is not on any C_5. Thus, the path $w_1x_1xx_2$ must be extended to a C_5. Either w_1u is an edge or w_1w is an edge. Similarly, by considering the edge yy_2, either w_2w or w_2v is an edge. These four possible edges are shown as dashed lines i), ii), iii), and iv) in Figure 2. There are four combinations: i)+iii), i)+iv), ii)+iii), ii)+iv). The combination i)+iii) is impossible since $d_w = 3$. The two cases i)+iv) and ii)+iii) are symmetric. Essentially we have two cases to consider:

Case: i)+iv): Now consider the edge w_1x_1. By Lemma 1, there are a pair of opposite C_5 sharing the edge w_1x_1. Such a pair of opposite pentagons can be obtained only by adding a new vertex w_3 as the third neighbor of w_1 and connecting w_3 to w_2, since connecting w_1 to u would cause a C_4. Now $x_1w_1w_3w_2vx_1$ and $x_1w_1wx_2xx_1$ are the two opposite pentagons at x_1w_1. But, in order to have two opposite pentagons also at the new edge w_1w_3 we must have w_3u as an edge, which then creates a C_4: $w_3uy_2w_2w_3$. Contradiction!



(a) Case 1 (b) Case 2

Figure 3. Two non-isomorphic ways to continue

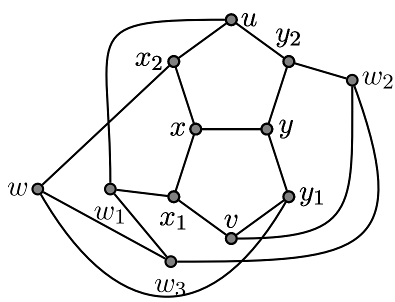


Figure 4. Unique way to complete into the Triplex graph.

Case: ii)+iv). Let w_3 be the third neighbor of w. (w_3 is distinct from w_1 and w_2 since the girth of G is at least 5.) Applying Lemma 1 on the edge wx_2, we must have a pair of opposite C_5's passing through wx_2. This will force w_3w_1 to be an edge. Similarly, by considering wy_1, we conclude that w_3w_2 must be an edge. This completes a 3-regular graph. It is easy to check this is the Triplex graph.

Now we assume xy is in four C_5's. For $i = 1, 2$ and $j = 1, 2$, write $k = 2(i-1) + j$ and let u_k be the vertex in the C_5 extending the path x_ixyy_j. Observe that connecting any pair u_1u_2, u_2u_4, u_4u_3, or u_1u_3 results a triangle. So only u_2u_3 and u_1u_4 can be connected (See Figure 5).

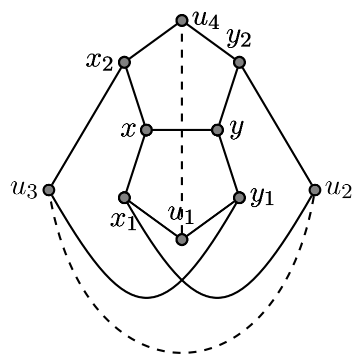


Figure 5. Starting configuration and possible extension when xy is in four C_5's.

Note that yy_1 are in two non-opposite C_5's: $xyy_1u_1x_1x$ and $xyy_1u_3x_2x$. So either u_2u_3 or u_1u_4 must be an edge.

If both u_1u_4 and u_2u_3 are edges, then the graph is completed and it is the Petersen graph.

If only one of them is an edge, by symmetry, we can assume u_1u_4 is an edge but u_2u_3 are not. Then u_2 must have an new neighbor, called z. Now the edge zu_2 can not be in any C_5. Otherwise, say zu_2XYZz is the C_5. We must have $X \in \{x_1, y_2\}$, $Y \in \{x, u_1, y, u_4\}$, and $Z \in \{x_2, y_1\}$. But now Zz is not an edge. Contradiction! $\square$

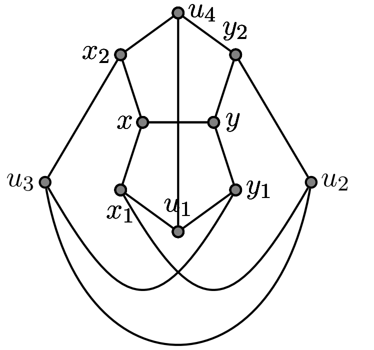


Figure 6. The Petersen Graph.

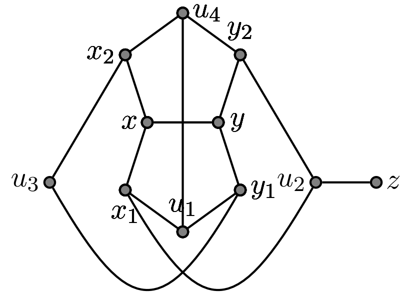


Figure 7. The edge zu_2 is not in any C_5. Contradiction!

References

[1] Y. Lin, L. Lu, and S.-T. Yau, *Ricci-flat graphs with girth at least five*, Comm. Anal. Geom. **22** (2014), no 4, 671–687.

[2] D. Cushing, R. Kangaslampi, Y. Lin, S. Liu, L. Lu, and S.-T. Yau, *Ricci-flat cubic graphs with girth five*, Comm. Anal. Geom. **29** (2021), no. 7, 1559–1570.

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