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A generalization of the Escobar-Riemann mapping-type problem to smooth metric measure spaces

JHOVANNY MUÑOZ POSSO

In this article, we introduce a problem analogous to the Yamabe-type problem considered by Case in [4], which generalizes the Escobar-Riemann mapping problem for smooth metric measure spaces with boundary. For this purpose, we consider the generalization of the Sobolev trace inequality deduced by Bolley et al. in [3]. This trace inequality allows us to introduce an Escobar quotient and its infimum. We call this infimum the weighted Escobar constant. The Escobar-Riemann mapping-type problem for smooth metric measure spaces in manifolds with boundary consists of finding a function that attains the weighted Escobar constant. Furthermore, we resolve this problem when the weighted Escobar constant is negative. Finally, we obtain an Aubin-type inequality, connecting the weighted Escobar constant for compact smooth metric measure spaces and the optimal constant for the trace inequality in [3].

1. Introduction

When (M^n, g) is a Riemannian manifold with boundary, we denote by ∂M the boundary of M and by H_g the trace of the second fundamental form of ∂M. The Escobar-Riemann mapping problem for manifolds with boundary is concerned with finding a metric g that has scalar curvature $R_g \equiv 0$ in M and constant H_g on ∂M in the conformal class of the initial metric g (see [8]). Since this problem in the Euclidean half-space reduces to finding the minimizers in the sharp Sobolev trace inequality, we consider a particular case of the trace Gagliardo-Nirenberg-Sobolev inequality in [3].

To present the trace Gagliardo-Nirenberg-Sobolev inequality, let $\mathbb{R}^n_+ = \{(x,t) : x \in \mathbb{R}^{n-1}, t \geq 0\}$ denote the half-space and $\partial\mathbb{R}^n_+ = \{(x,0) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}\}$ denote its boundary. We identify $\partial\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ whenever necessary.

Theorem 1. *[3] Fix $m \geq 0$. For all $w \in W^{1,2}(\mathbb{R}^n_+) \cap L^{\frac{2(m+n-1)}{m+n-2}}(\mathbb{R}^n_+)$, it holds that*

(1)
$$\Lambda_{m,n} \left(\int_{\partial\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{2m+n-2}{m+n-1}} \leq \left(\int_{\mathbb{R}^n_+} |\nabla w|^2 \right) \left(\int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{m}{m+n-1}}$$

where the constant $\Lambda_{n,m}$ is given by

$$\Lambda_{m,n} = (m+n-2)^2 \left(\frac{Vol(S^{2m+n-1})^{\frac{1}{2m+n-1}}}{2(2m+n-2)} \right)^{\frac{2m+n-1}{m+n-1}} \times \left(\frac{\Gamma(2m+n-1)}{\pi^m \Gamma(m+n-1)} \right)^{\frac{1}{m+n-1}} \tag{2}$$

and $Vol(S^{2m+n-1})$ is the volume of the $2m+n-1$-dimensional unit sphere. Moreover, equality holds if and only if w is a constant multiple of the function w_{ϵ,x_0} defined on $\mathbb{R}^n_+$ by

$$w_{\epsilon,x_0}(x,t) := \left(\frac{2\epsilon}{(\epsilon+t)^2 + |x-x_0|^2} \right)^{\frac{m+n-2}{2}} \tag{3}$$

where $\epsilon > 0$ and $x_0 \in \mathbb{R}^{n-1}$.

Note that the special case $m = 0$ in Theorem 1 recovers the well-known sharp Sobolev trace inequality of Beckner and Escobar (see [1] and [7]).

On the other hand, Del Pino and Dolbeaut studied the sharp Gagliardo-Nirenberg-Sobolev inequalities. Based on Del Pino and Dolbeaut's result, Case, in [4], considered a Yamabe-type problem for smooth metric measure spaces in manifolds without boundary, which generalizes the Yamabe problem when $m = 0$. Then, using Theorem 1 instead of Gagliardo-Nirenberg-Sobolev inequalities and following ideas similar to those in [4], we will introduce an Escobar-Riemann mapping-type problem for smooth metric measure spaces in manifolds with boundary. Thus, it is necessary to consider the notion of a *smooth metric measure space with boundary* defined by a five-tuple $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$, where dV_g and $d\sigma_g$ are the volume form induced by the metric g in M and on the boundary ∂M, respectively; ϕ is a function such that $\phi \in C^\infty(M)$; and m is a parameter such that $m \in [0, \infty)$. In addition, if $m = 0$, we require $\phi = 0$.

Let us denote the scalar curvature, the Laplacian and the gradient associated with the metric g by R_g, Δ_g, and ∇_g, respectively. The *weighted scalar curvature* R^m_ϕ of a smooth metric measure space for $m = 0$ is $R^m_\phi = R_g$, and that for $m \neq 0$ is the function $R^m_\phi := R_g + 2\Delta_g\phi - \frac{m+1}{m}|\nabla_g\phi|^2$. The *weighted Escobar quotient* for this smooth metric measure is defined by

(4)
$$\mathcal{Q}(w) = \frac{\int_M (|\nabla w|^2 + \frac{m+n-2}{4(m+n-1)}R^m_\phi w^2)e^{-\phi}dV_g + \int_{\partial M} \frac{m+n-2}{2(m+n-1)}H^m_\phi w^2 e^{-\phi}d\sigma_g}{(\int_{\partial M}|w|^{\frac{2(m+n-1)}{m+n-2}}e^{-\phi}d\sigma_g)^{\frac{2m+n-2}{m+n-1}}(\int_M |w|^{\frac{2(m+n-1)}{m+n-2}}e^{-\frac{(m-1)\phi}{m}}dV_g)^{-\frac{m}{m+n-1}}},$$

where we denote by $H^m_\phi = H_g - \dfrac{\partial\phi}{\partial\eta}$ the Gromov mean curvature and $\dfrac{\partial}{\partial\eta}$ is the outer normal derivative.

The *weighted Escobar constant* $\Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] \in \mathbb{R} \cup \{-\infty\}$ is defined by

$$(5)\quad \Lambda := \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] = \inf\{\mathcal{Q}(w) : w \in H^1(M, e^{-\phi}dV_g)\}.$$

If $m = 0$, the quotient (4) coincides with the Sobolev quotient considered by Escobar in the Escobar-Riemann mapping problem. We prove the existence of a minimizer of the weighted Escobar constant when this constant is finite and negative. The exact statement is as follows:

Theorem A. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary such that $m \geq 0$ and the weighted Escobar constant satisfies $-\infty < \Lambda < 0$. Then, there exists a positive function $w \in C^\infty(M)$ such that*

$$\mathcal{Q}(w) = \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m].$$

Using Theorem 1, we prove that the weighted Escobar constant for a compact smooth measure space with boundary is always less than or equal to the weighted Escobar constant of the model case $(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)$.

Theorem B. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary such that $m \geq 0$. Then*

$$\Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}. \tag{6}$$

We recall that in the Escobar-Riemann mapping problem ($m = 0$), if the inequality (6) is strict, it follows the existence of the minimizer. The same result is expected for the Escobar-Riemann-type problem. For that reason, we conjecture the following:

Conjecture. *Let $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ be a compact smooth metric measure space with boundary such that $m \geq 0$ and*

$$-\infty < \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m] < \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}. \tag{7}$$

Then, there exists a positive function $w \in C^\infty(M)$ such that

$$\mathcal{Q}(w) = \Lambda[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m].$$

The conjecture is one way to show that the weighted Escobar constant is minimized when Λ is nonnegative (see Remark 6 below for another way). Additionally, Case, in [4], proved a similar result for the weighted Yamabe constant on smooth metric measure spaces without boundary. He used this result to obtain the existence of minimizers of the weighted Yamabe constant for $m \in \mathbb{N}$. Additionally, Muñoz in [11] used this to obtain the existence of minimizers of the weighted Yamabe constant for the nonlocally conformally flat manifolds, with small parameter m and dimension $n \geq 6$. To follow Case's ideas for proving the conjecture, it is necessary to obtain a Liouville-type result for the inequality (1), which, to the best of our knowledge, has not yet been proven.

This paper is organized as follows. In section 2, we give a different proof for Theorem 1 than that given in [3] for the particular case of $2m \in \mathbb{N} \cup \{0\}$. In sections 3 and 4, we consider our notion of smooth metric measure spaces with boundary and other concepts to introduce our Escobar-Riemann-type problem. In sections 5 and 6, we prove Theorem A and B, respectively.

2. General trace inequality

In this section, we give a proof for Theorem 1 for the case of $2m \in \mathbb{N} \cup \{0\}$, which is different from the proof in [3]. As we mentioned in the introduction, the trace Gagliardo-Nirenberg-Sobolev inequality facilitates an introduction of our Escobar-Riemann-type problem. The proof that we present depends on the Sobolev trace inequality in $\mathbb{R}^{n+2m}$ and its minimizers. This idea is based on Bakry et al. (see [2]).

Remark 1. *To facilitate the reading of the proof of Theorem 1, we recall that the case $m = 0$ in inequality (1) recovers the sharp Sobolev trace inequality (see [1] and [7])*

$$\Lambda_{0,n}\left(\int_{\partial\mathbb{R}^n_+} |w|^{\frac{2(n-1)}{n-2}}\right)^{\frac{n-2}{n-1}} \leq \left(\int_{\mathbb{R}^n_+} |\nabla w|^2\right), \tag{8}$$

where $\Lambda_{0,n} = \frac{n-2}{2}(vol(S^{n-1}))^{\frac{1}{n-1}}$. Equality in (8) holds if and only if w is a positive constant multiple of the functions of the form

$$w = \left(\frac{\epsilon}{(\epsilon + t)^2 + |x - x_0|^2}\right)^{\frac{n-2}{2}}. \tag{9}$$

Lemma 1. *Let p, q, B, and C be positive numbers and define $h(\tau) = B\tau^p + C\tau^{-q}$ for $\tau > 0$. Then, h attains the infimum in $\tau_0 = (\frac{qB}{pA})^{\frac{1}{p+q}}$ and*

$$\inf_{\tau>0} h(\tau) = h(\tau_0) = B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} \left(\frac{q}{p}\right)^{\frac{p}{p+q}} \left(\frac{q+p}{p}\right).$$

Proof. Since h is a positive continuous function for $\tau > 0$ and

$$\lim_{\tau\to 0^+} h(\tau) = \lim_{\tau\to\infty} h(\tau) = \infty,$$

it follows that h attains the infimum for some $\tau_0 > 0$. A direct computation shows that $h'(\tau) = \tau^{p-1}(pB - qC\tau^{-p-q})$. Therefore, $\tau_0 = (\frac{qC}{pB})^{\frac{1}{p+q}}$ and

$$
(10) \qquad \begin{aligned}
h((\tfrac{qC}{pB})^{\frac{1}{p+q}}) &= B(\tfrac{qC}{pB})^{\frac{p}{p+q}} + C(\tfrac{qC}{pB})^{\frac{-q}{p+q}} \\
&= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} + B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{-q}{p+q}} \\
&= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} (1+\tfrac{p}{q}) \\
&= B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} (\tfrac{q}{p})^{\frac{p}{p+q}} (\tfrac{p+q}{q}).
\end{aligned}
$$

□

Remark 2. *If $m \to \infty$, inequality* (1) *takes the form*

$$
(11) \qquad \Lambda_{\infty,n}\left(\int_{\partial\mathbb{R}^n_+}|w|^2\right)^2 \le \left(\int_{\mathbb{R}^n_+}|\nabla w|^2\right)\left(\int_{\mathbb{R}^n_+}|w|^2\right)
$$

where $\lim_{m\to\infty}\Lambda_{m,n} = \Lambda_{\infty,n}$.

Inequality (11) *is equivalent to the trace inequality* $H^1(M) \to L^2(\partial M)$

$$
(12) \qquad 2(\Lambda_{\infty,n})^{\frac{1}{2}}\left(\int_{\partial\mathbb{R}^n_+}|w|^2dx\right) \le \int_{\mathbb{R}^n_+}|\nabla w|^2dxdt + \int_{\mathbb{R}^n_+}|w|^2dxdt.
$$

In fact, suppose inequality (12) *holds. For $\tau > 0$, define the function $w_\tau(x,t) = w(\frac{1}{\tau}(x,t))$. The change in variable $(y,s) = \frac{1}{\tau}(x,t)$ implies*

$$
\int_{\partial\mathbb{R}^n_+}|w_\tau|^2(x,0)dx = \tau^{n-1}\int_{\partial\mathbb{R}^n_+}|w|^2(y,0)dy,
$$

$$
\int_{\mathbb{R}^n_+}|\nabla w_\tau|^2(x,t)dxdt = \tau^{n-2}\int_{\mathbb{R}^n_+}|\nabla w|^2(y,s)dyds
$$

and

$$
\int_{\mathbb{R}^n_+}|w_\tau|^2(x,t)dxdt = \tau^{n}\int_{\mathbb{R}^n_+}|w|^2(y,s)dyds.
$$

Then, using w_τ and the equalities above in inequality (12), *we obtain*

$$
(13) \qquad 2(\Lambda_{\infty,n})^{\frac{1}{2}}\left(\int_{\partial\mathbb{R}^n_+}|w|^2(y,0)dy\right) \le \tau B + \tau^{-1}C,
$$

where $B = \int_{\mathbb{R}^n_+} |w|^2(y,s)dyds$ *and* $C = \int_{\mathbb{R}^n_+} |\nabla w|^2(y,s)dyds$*. Lemma 1 yields that for* $\tau_0 = (\frac{C}{B})^{\frac{1}{2}}$*, the following holds*

$$(14) \qquad \tau_0 B + \tau_0^{-1} C = 2B^{\frac{1}{2}}C^{\frac{1}{2}} = 2\left(\int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} |w|^2 dxdt\right)^{\frac{1}{2}}.$$

Since inequality (13) *is true for every* $\tau > 0$*, in particular, it is true for* $\tau_0 = (\frac{C}{B})^{\frac{1}{2}}$*, and by* (14)*, we have*

$$(15) \qquad 2(\Lambda_{\infty,n})^{\frac{1}{2}} \left(\int_{\partial\mathbb{R}^n_+} |w|^2\right) \leq 2\left(\int_{\mathbb{R}^n_+} |\nabla w|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n_+} |w|^2\right)^{\frac{1}{2}},$$

which is equivalent to (11)*.*

Now, suppose that inequality (11) *holds; then, inequality* (15) *holds. In addition, inequality* (12) *is a consequence of inequality* $2ab \leq a^2 + b^2$*.*

In our proof for Theorem 1, we use the following Lemma, which was taken from [4].

Lemma 2. *Fix* $k,\ l \geq 0$*,* $2m \in \mathbb{N}$*, and constants* $a,\ \tau > 0$*. Then*

$$\int_{\mathbb{R}^{2m}} \frac{|y|^{2l} dy}{(a + \frac{|y|^2}{\tau})^{2m+k}} = \frac{\pi^m \Gamma(m+l)\Gamma(m+k-l)\tau^{m+l}}{\Gamma(m)\Gamma(2m+k)a^{m+k-l}}.$$

Proof of Theorem 1. We are able to prove inequality (1) only for $2m \in \mathbb{N}$. For this purpose, consider sharp Sobolev trace inequality (8) for $\mathbb{R}^{n+2m}_+$. The proof consists of using this inequality for the special function

$$(16) \qquad f(y,x,t) := \left(w^{\frac{-2}{m+n-2}}(x,t) + \frac{|y|^2}{\tau}\right)^{-\frac{2m+n-2}{2}} \in C^\infty(\mathbb{R}^{n+2m}_+),$$

where $(x,t) \in \mathbb{R}^n_+$, $y \in \mathbb{R}^{2m}$ and $\tau > 0$.

Suppose f is of the form (16). First, we analyze the on the left-hand side of inequality (8). Fixing (x,t), we note that $f^{\frac{2(2m+n-1)}{2m+n-2}}$ takes the form of the function considered in Lemma 2 with $a = w^{\frac{-2}{m+n-2}}(x,t)$. Fubini's Theorem, Lemma 2 with $k = n-1$ and $l = 0$, and some calculation yield

$$\int_{\partial\mathbb{R}_+^{2m+n}} f^{\frac{2(2m+n-1)}{2m+n-2}} dxdy = \frac{\pi^m \Gamma(m+n-1)\tau^m}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}_+^n} w^{\frac{2(m+n-1)}{m+n-2}} dx. \tag{17}$$

To analyze the term on the right-hand side of inequality (8), we compute

$$|\nabla f|^2 = \frac{\left(\frac{2m+n-2}{2}\right)^2 \left(\left(\frac{2}{m+n-2}\right)^2 w^{-\frac{2(m+n)}{m+n-2}} |\nabla w|^2 + 4\frac{|y|^2}{\tau^2}\right)}{\left(w^{-\frac{2}{m+n-2}} + \frac{|y|^2}{\tau}\right)^{2m+n}}.$$

Lemma 2 leads to

$$\begin{aligned} \int_{\mathbb{R}_+^{2m+n}} |\nabla f|^2 dydxdt = & \left(\frac{2m+n-2}{m+n-2}\right)^2 \left(\frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)}\right) \\ & \times \int_{\mathbb{R}_+^n} |\nabla w|^2 dxdt \\ & + \left(\frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)}\right) \\ & \times \int_{\mathbb{R}_+^n} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt. \end{aligned} \tag{18}$$

Using equalities (17) and (18) in inequality (8), we determine that

$$\begin{aligned} & \Lambda_{0,2m+n} \left(\frac{\pi^m \Gamma(m+n-1)\tau^m}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}_+^n} w^{\frac{2(m+n-1)}{m+n-2}} dx\right)^{\frac{2m+n-2}{2m+n-1}} \\ & \leq \left(\frac{2m+n-2}{m+n-2}\right)^2 \left(\frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)}\right) \int_{\mathbb{R}_+^n} |\nabla w|^2 dxdt \\ & \quad + \left(\frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)}\right) \int_{\mathbb{R}_+^n} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt. \end{aligned} \tag{19}$$

Rewriting (19), we obtain

$$\Lambda_{0,2m+n} \left(\frac{\pi^m \Gamma(m+n-1)}{\Gamma(2m+n-1)} \int_{\partial\mathbb{R}_+^n} w^{\frac{2(m+n-1)}{m+n-2}} dx\right)^{\frac{2m+n-2}{2m+n-1}} A \leq h(\tau), \tag{20}$$

where

$$A = \frac{\Gamma(2m+n)}{(2m+n-2)^2\pi^m\Gamma(m+n)},$$

$$h(\tau) = B\tau^{\frac{m}{2m+n-1}} + C\tau^{-\frac{m+n-1}{2m+n-1}},$$

$$B = \frac{1}{(m+n-2)^2}\int_{\mathbb{R}^n_+} |\nabla w|^2 dxdt,$$

and

$$C = \frac{m}{m+n-1}\int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt.$$

Lemma 1 implies that the function h minimizes for $\tau_0 = (\frac{(m+n-1)C}{mB})^{\frac{m+n-2}{2m+n-1}}$ and

$$\Lambda_{2m+n,0}\left(\frac{\pi^m\Gamma(m+n-1)}{\Gamma(2m+n-1)}\int_{\partial\mathbb{R}^n_+} w^{\frac{2(m+n-1)}{m+n-2}} dx\right)^{\frac{2m+n-2}{2m+n-1}} A \leq h(\tau_0). \tag{21}$$

Inequality (21) proves inequality (1) with $\Lambda_{m,n}$ as in (2). Next, we characterize the functions that achieve equality in (1). Note that for $\mathbb{R}^{n+2m}_+$ and f defined in (16), the equality in (8) holds if and only if

$$f(y,x,t) = \left(\frac{(t+\epsilon)^2 + |x-x_0|^2 + |y|^2}{\tau}\right)^{-\frac{2m+n-2}{2}}, \quad \text{for} \quad \tau > 0,$$

i.e

$$w^{\frac{-2}{m+n-2}}(x,t) = \tau((t+\epsilon)^2 + |x-x_0|^2)$$

(see Escobar [7] and Beckner [1]). Then, the family of functions $\{w_{\epsilon,x_0}\}$ in (3) is the only one that satisfies the equality in (1). □

3. Smooth metric measure spaces with boundary and the weighted conformal Laplacian

Our approach is based on [4] and [5]. The first step is to introduce the definition of a smooth metric measure space with boundary.

Definition 1. *Let (M^n, g) be a Riemannian manifold, and let us denote by dV_g and $d\sigma_g$ the volume form induced by g in M and ∂M, respectively. Let $\phi \in C^\infty(M)$ be a smooth function, and let $m \in [0,\infty)$ be a dimensional parameter. In the case of $m = 0$, we require that $\phi = 0$. A smooth metric measure space with boundary is the five-tuple $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$.*

As in [4], sometimes we denote by the four-tuple $(M^n, g, v^m dV_g, v^m d\sigma_g)$ a smooth metric measure space where v and ϕ are related by $v^m = e^{-\phi}$. We denote by R_g the scalar curvature of (M, g) and Ric and the Ricci tensor of (M, g), with η as the outer unit normal on ∂M and $\dfrac{\partial}{\partial \eta}$ as the outer normal derivative. Additionally, we denote the second fundamental form, the trace of the second fundamental form, and the mean curvature on the boundary ∂M by h_{ij}, $H_g := g^{ij} h_{ij}$, and $h_g = \frac{H_g}{n-1}$, respectively.

Definition 2. *Given a smooth metric measure space* $(M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)$. *The weighted scalar curvature* R^m_ϕ*, the Bakry-Émery Ricci curvature* Ric^m_ϕ*, and the Gromov mean curvature are the tensors*

$$R^m_\phi := R_g + 2\Delta\phi - \frac{m+1}{m}|\nabla\phi|^2, \tag{22}$$

$$Ric^m_\phi := Ric + \nabla^2\phi - \frac{1}{m} d\phi \otimes d\phi, \tag{23}$$

and

$$H^m_\phi = H_g - \frac{\partial \phi}{\partial \eta}, \tag{24}$$

respectively.

Definition 3. *Let* $(M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)$ *and* $(M^n, \hat{g}, e^{-\hat{\phi}} dV_{\hat{g}}, e^{-\hat{\phi}} d\sigma_{\hat{g}}, m)$ *be smooth metric measure spaces with boundary. We say they are pointwise conformally equivalent if there is a function* $\sigma \in C^\infty(M)$ *such that*

$$\hat{g} = e^{\frac{2\sigma}{m+n-2}} g \quad \text{and} \quad \hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi. \tag{25}$$

$(M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)$ *and* $(\hat{M}^n, \hat{g}, e^{-\hat{\phi}} dV_{\hat{g}}, e^{-\hat{\phi}} d\sigma_{\hat{g}}, m)$ *are conformally equivalent if there is a diffeomorphism* $F : \hat{M} \to M$ *such that the new smooth metric measure space with boundary* $(F^{-1}(M), F^* g, F^*(e^{-\phi} dV_g), F^*(e^{-\phi} d\sigma_g), m)$ *is pointwise conformally equivalent to* $(\hat{M}^n, \hat{g}, e^{-\hat{\phi}} dV_{\hat{g}}, e^{-\hat{\phi}} d\sigma_{\hat{g}}, m)$.

Remark 3. *The equalities in* (25) *imply*

$$e^{-\hat{\phi}} dV_{\hat{g}} = e^{\frac{m+n}{m+n-2}\sigma} e^{-\phi} dV_g \quad \text{and} \quad e^{-\hat{\phi}} d\sigma_{\hat{g}} = e^{\frac{m+n-1}{m+n-2}\sigma} e^{-\phi} d\sigma_g.$$

Definition 4. *Given a smooth metric measure space* $(M^n, g, e^{-\phi}dV_g, m)$, *the weighted Laplacian* $\Delta_\phi : C^\infty(M) \to C^\infty(M)$ *is the operator defined by*

$$\Delta_\phi u = \Delta u - \nabla u \cdot \nabla \phi$$

where $u \in C^\infty(M)$, Δ *is the usual Laplacian associated with the metric* g *and* ∇ *is the gradient calculated in the metric* g.

Definition 5. *Given a smooth metric measure space* $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$, *the weighted conformal Laplacian is given by*

$$L^m_\phi = -\Delta_\phi + \frac{m+n-2}{4(m+n-1)} R^m_\phi \qquad in \quad M, \tag{26}$$

and the weighted conformal Robin operator, B^m_ϕ *is given by*

$$B^m_\phi = \frac{\partial}{\partial \eta} + \frac{m+n-2}{2(m+n-1)} H^m_\phi \qquad on \quad \partial M, \tag{27}$$

respectively.

Proposition 1. *Let* $(M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)$ *and* $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$ *be two pointwise conformally equivalent smooth metric measure space such that* $\hat{g} = e^{\frac{2\sigma}{m+n-2}} g$ *and* $\hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi$. *Let us denote by* L^m_ϕ *and* $\hat{L}^m_{\hat{\phi}}$ *their respective weighted conformal Laplacians. Similarly, we denote with a hat all quantities computed with respect to the smooth metric measure space* $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$. *Then, we have* $\hat{v} = e^{\frac{\sigma}{m+n-2}} v$ *and the following transformation rules*

$$\hat{L}^m_{\hat{\phi}}(w) = e^{-\frac{m+n+2}{2(m+n-2)}\sigma} L^m_\phi(e^{\frac{\sigma}{2}} w), \quad \hat{B}^m_{\hat{\phi}}(w) = e^{-\frac{m+n}{2(m+n-2)}\sigma} B^m_\phi(e^{\frac{\sigma}{2}} w). \tag{28}$$

Proof. Let us denote with a hat the terms associated with the smooth metric measure space $(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, m)$. The identity $\hat{v} = e^{\frac{\sigma}{m+n-2}} v$ follows from the relations $\hat{v}^m = e^{-\hat{\phi}}$, $v^m = e^{-\phi}$, and $\hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi$. The equality on the left-hand side of (28) appears in [4].

To prove the identity on the right-hand side of (28), we recall that

$$\hat{B}^m_{\hat{\phi}}(w) = \frac{\partial w}{\partial \hat{\eta}} + \frac{m+n-2}{2(m+n-1)} (H_{\hat{g}} w - \frac{\partial \phi}{\partial \hat{\eta}} w) \tag{29}$$

where $\dfrac{\partial}{\partial \hat{\eta}}$ is the outer normal derivative associated with the metric $\hat{g}$.

The equalities

$$\hat{\eta} = e^{\frac{-\sigma}{m+n-2}}\eta \quad \text{and} \quad \nabla_{\hat{g}} = e^{-\frac{2\sigma}{m+n-2}}\nabla_g$$

imply

$$\frac{\partial w}{\partial \hat{\eta}} = \hat{g}(\nabla_{\hat{g}} w, \hat{\eta}) = e^{-\frac{\sigma}{(m+n-2)}}\frac{\partial w}{\partial \eta}. \tag{30}$$

On the other hand, since $H_g = (n-1)h_g$ and

$$h_{\hat{g}} = e^{-\frac{\sigma}{(m+n-2)}}\left(h_g + \frac{1}{m+n-2}\frac{\partial \sigma}{\partial \eta}\right)$$

(see equation (1.4) in [8]), then

$$H_{\hat{g}} = e^{-\frac{\sigma}{(m+n-2)}}\left(H_g + \frac{n-1}{m+n-2}\frac{\partial \sigma}{\partial \eta}\right). \tag{31}$$

Finally, using the equalities (29) and (31) and the relation (30) for w and $\hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi$, we obtain

(32)

$$\begin{aligned} \hat{B}^m_{\hat{\phi}}(w) &= e^{-\frac{\sigma}{(m+n-2)}}\left(\frac{\partial w}{\partial \eta} + \frac{w}{2}\frac{\partial \sigma}{\partial \eta} + \frac{m+n-2}{2(m+n-1)}(H_g w - \frac{\partial \phi}{\partial \eta}w)\right) \\ &= e^{-\frac{(m+n)\sigma}{(m+n-2)}}\hat{B}^m_{\phi}(e^{-\frac{\sigma}{2}}w). \end{aligned}$$

□

We denote by $(w,\varphi)_M = \int_M w.\varphi\, v^m dV_g$ the inner product in $L^2(M, v^m dV_g)$. Additionally, we denote by $||.||_{2,M}$ the norm in the space $L^2(M, v^m dV_g)$, and in some cases, we use the notation $||.||$ for this norm. $H^1(M, v^m dV_g)$ denotes the closure of $C^\infty(M)$ with respect to the norm

$$\int_M |\nabla w|^2 + |w|^2.$$

Here and subsequently, the integrals are computed using the measure $v^m dV_g$.

4. Preliminaries for the Escobar-Riemann-type problem

In this section, we define the weighted Escobar quotient, which generalizes the quotient considered by Escobar in [8], and we consider a suitable $\mathcal{W}$-functional. In general, the weighted Escobar quotient is not necessarily finite.

Similar to [4], we define the energies of these functionals and give some of their properties.

4.1. The weighted Escobar quotient

We start with the definition of the weighted Escobar quotient.

Definition 6. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. The weighted Escobar quotient $\mathcal{Q} : C^\infty(M) \to \mathbb{R}$ is defined by*

$$(33)\qquad \mathcal{Q}(w) = \frac{((L_\phi^m w, w)_M + (B_\phi^m w, w)_{\partial M})(\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} v^{-1})^{\frac{m}{m+n-1}}}{(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}})^{\frac{2m+n-2}{m+n-1}}}.$$

The weighted Escobar constant $\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R}$ of the smooth metric measure space $(M^n, g, v^m dV_g, v^m d\sigma_g, m)$ is

$$(34)\quad \Lambda[M^n, g, v^m dV_g, v^m d\sigma_g, m] = \inf\{\mathcal{Q}(w) : w \in H^1(M, v^m dV_g, v^m d\sigma_g)\}.$$

Remark 4. *In some cases, when the context is clear, we will not write the dependence of the smooth metric measure space with boundary; for example, we write $\mathcal{Q}$ and Λ instead of $\mathcal{Q}[M^n, g, v^m dV_g, v^m d\sigma_g]$ and $\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g]$, respectively. We note that since $C^\infty(M)$ is dense in $H^1(M, v^m dV_g)$ and $\mathcal{Q}(|w|) = \mathcal{Q}(w)$, it is sufficient to consider the weighted Escobar constant by minimizing over the space of nonnegative smooth functions on M; subsequently, we make this assumption without further comment.*

The weighted Escobar quotient satisfies similar properties to the weighted Yamabe quotient introduced by Case in [4]; for example, we observe that the weighted Escobar quotient is continuous in $m \in [0, \infty)$, and it is conformal invariant in the sense of Definition 3.

Proposition 2. *Let (M^n, g) be a compact Riemannian manifold with boundary. Fix $\phi \in C^\infty(M)$ and $m \in [0, \infty)$. Given any $w \in C^\infty(M)$, it holds that*

$$(35)\qquad \begin{aligned} &\lim_{k\to m} \mathcal{Q}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, k](w) \\ &\qquad = \mathcal{Q}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m](w). \end{aligned}$$

Proof. The proof follows by direct computation. □

Proposition 3. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. For any* σ, $w \in C^\infty(M)$, *it holds that*

$$\begin{aligned}&\mathcal{Q}[M^n, e^{\frac{2}{m+n-2}\sigma} g, e^{\frac{m+n}{m+n-2}\sigma} v^m dV_g, e^{\frac{m+n-1}{m+n-2}\sigma} v^m d\sigma_g](w)\\ &= \mathcal{Q}[M^n, g, v^m dV_g, v^m d\sigma_g](e^{\frac{\sigma}{2}} w).\end{aligned} \tag{36}$$

Proof. A straightforward computation shows that the integrals

$$\int_M |w|^{\frac{2(m+n-1)}{m+n-2}} v^{m-1} dV_g \qquad \text{and} \qquad \int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g \tag{37}$$

are invariant under the conformal transformation

$$(g, v^m dV_g, v^m d\sigma_g, w) \to (e^{\frac{2}{m+n-2}\sigma} g, e^{\frac{m+n}{m+n-2}\sigma} v^m dV_g, e^{\frac{m+n-1}{m+n-2}\sigma} v^m d\sigma_g, e^{-\frac{\sigma}{2}} w). \tag{38}$$

By Proposition 1, the term $(L_\phi^m w, w) + (B_\phi^m w, w)$ is invariant under (38). □

Similar to the smooth metric measure spaces, we have some behavior for the boundary volume. Note that in the boundary, the integral

$$\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g$$

measures the boundary volume $\int_{\partial M} \hat{v}^m d\sigma_{\hat{g}}$ of

$$\begin{aligned}&(M^n, \hat{g}, \hat{v}^m dV_{\hat{g}}, \hat{v}^m d\sigma_{\hat{g}}, m)\\ &\quad = (M^n, w^{\frac{4}{m+n-2}} g, w^{\frac{2(m+n)}{m+n-2}} v^m dV_g, w^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g, m).\end{aligned} \tag{39}$$

Additionally, with the same purpose, to simplify the calculus and to avoid the trivial noncompactness of the weighted Escobar-Riemann-type problem, we give the next definition of a volume-normalized function on the boundary.

Definition 7. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary. We say that a positive function* $w \in C^\infty(M)$ *is volume-normalized on the boundary if*

$$\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g = 1.$$

4.2. $\mathcal{W}$-functional

We introduce a $\mathcal{W}$-functional with similar properties as the $\mathcal{W}$-functional considered by Case in [4] and Perelman in [12].

Definition 8. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. The $\mathcal{W}$-functional $\mathcal{W} : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$ is defined by*

$$\begin{aligned}\mathcal{W}(w,\tau) &= \mathcal{W}[M^n, g, v^m dV_g, v^m d\sigma_g](w,\tau) \\ &= \tau^{\frac{m}{2(m+n-1)}} \left((L_\phi^m w, w) + (B_\phi^m w, w) \right) \\ &\quad + \int_M \tau^{-\frac{1}{2}} w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} - \int_{\partial M} w^{\frac{2(m+n-1)}{m+n-2}}\end{aligned} \tag{40}$$

when $m \in [0, \infty)$.

As in the weighted Escobar quotient and the $\mathcal{W}$-functional considered by Case in [4], the $\mathcal{W}$-functional defined above is continuous in m, conformally invariant, and scale invariant in τ.

Proposition 4. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. Then,*

$$\begin{aligned}&\lim_{k \to m} \mathcal{W}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, k](w,\tau) \\ &\qquad = \mathcal{W}[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m](w,\tau).\end{aligned}$$

Proof. The proof follows by straightforward computation. □

Proposition 5. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. The $\mathcal{W}$-functional is conformally invariant in its first component:*

$$\begin{aligned}&\mathcal{W}[M^n, e^{2\sigma} g, e^{(m+n)\sigma} v^m dV_g, e^{(m+n-1)\sigma} v^m d\sigma_g](w,\tau) \\ &\qquad = \mathcal{W}[M^n, g, v^m dV_g, v^m d\sigma_g](e^{\frac{(m+n-2)}{2}\sigma} w, \tau)\end{aligned} \tag{41}$$

for all σ, $w \in C^\infty(M)$ and $\tau > 0$. It is scale invariant in its second component:

$$\begin{aligned}&\mathcal{W}[M^n, cg, v^m dV_{cg}, v^m d\sigma_{cg}](w,\tau) \\ &\qquad = \mathcal{W}[M^n, g, v^m dV_g, v^m d\sigma_g](c^{\frac{(n-1)(m+n-2)}{4(m+n-1)}} w, c^{-1}\tau).\end{aligned} \tag{42}$$

Proof. Equality (41) follows as in Proposition 3, and equality (42) follows by direct computation. □

Since we are interested in minimizing the weighted Escobar quotient, it is natural to define the following energies as infima using the $\mathcal{W}$-functional and relating one of these energies to the weighted Escobar constant.

Definition 9. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. Given $\tau > 0$, the τ-energy $\nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau)$ is the number defined by*

(43)
$$\begin{aligned}\nu(\tau) &= \nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau)\\ &= \inf\left\{\mathcal{W}(w,\tau) : w \in H^1(M, v^m dV_g, v^m d\sigma_g), \int_{\partial M} w^{\frac{2(m+n-1)}{m+n-2}} = 1\right\}.\end{aligned}$$

The energy $\nu[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R} \cup \{-\infty\}$ is defined by

$$\nu = \nu[M^n, g, v^m dV_g, v^m d\sigma_g] = \inf_{\tau>0} \nu[g, v^m dV_g, v^m d\sigma_g](\tau).$$

The conformal invariance in the $\mathcal{W}$-functional is transferred to the energies.

Proposition 6. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. Then,*

$$\begin{aligned}&\nu[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}, e^{(m+n-1)\sigma}v^m d\sigma_{cg}](c\tau)\\ &\quad= \nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau),\end{aligned}$$

$$\nu[M^n, ce^{2\sigma}g, e^{(m+n)\sigma}v^m dV_{cg}, e^{(m+n-1)\sigma}v^m d\sigma_{cg}] = \nu[M^n, g, v^m dV_g, v^m d\sigma_g]$$

for all $\sigma \in C^\infty(M)$ and $c > 0$.

The following proposition shows that it is equivalent to consider the energy instead of the weighted Escobar constant when the latter is positive.

Proposition 7. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary, and denote by Λ and ν the weighted Escobar constant and the energy, respectively.*

- $\Lambda \in [-\infty, 0)$ *if and only if* $\nu = -\infty$;
- $\Lambda = 0$ *if and only if* $\nu = -1$; *and*

- $\Lambda > 0$ *if and only if* $\nu > -1$. *Moreover, in this case, we have*

$$\nu = \frac{2m+n-1}{m}\left[\frac{m\Lambda}{m+n-1}\right]^{\frac{m+n-1}{2m+n-1}} - 1 \tag{44}$$

and w *is a volume-normalized minimizer of* Λ *if and only if* (w,τ) *is a volume-normalized minimizer of* ν *for*

$$\tau = \left[\frac{m\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}}{(m+n-1)((L^m_\phi w, w) + (B^m_\phi w, w))}\right]^{\frac{m+n-1}{2(2m+n-1)}}. \tag{45}$$

Proof. If $\Lambda \in [-\infty, 0)$, then there is a volume-normalized function $w \in C^\infty(M)$ such that $(L^m_\phi w, w)+(B^m_\phi w, w)<0$. Then, it is clear that $\mathcal{W}(w,\tau)\to -\infty$ as $\tau\to\infty$, and it follows that $\nu = -\infty$. Reciprocally, if $\nu = -\infty$, there exists a volume-normalized function w and $\tau > 0$ such that $\mathcal{W}(w,\tau) < -1$, and it follows that $(L^m_\phi w, w) + (B^m_\phi w, w) < 0$ and $\Lambda \in [-\infty, 0)$.

Suppose $\Lambda \geq 0$. Lemma 1 shows that if A, $B > 0$, then

$$\inf_{x>0}\{Ax^{\frac{m}{m+n-1}} + Bx^{-1}\} = \frac{2m+n-1}{m}\left[\frac{m}{m+n-1}AB^{\frac{m}{m+n-1}}\right]^{\frac{m+n-1}{2m+n-1}} \tag{46}$$

for all $x > 0$, with equality if and only if

$$x = \left[\frac{mB}{(m+n-1)A}\right]^{\frac{m+n-1}{2m+n-1}}. \tag{47}$$

Note that equality (46) is achieved in the case of $A = 0$. Then, from equality (46), the definitions of Λ and ν and taking minimizing sequences of these infima, we obtain the remaining equivalences. When $\Lambda > 0$, using (46) and (47), we obtain that (44) and (45) hold. □

4.3. Variational formulae for the weighted energy functionals

The next proposition contains the computation of the Euler-Lagrange equations for minimizing the weighted Escobar quotient. We will use it in the proof of Theorem A on the regularity part.

Proposition 8. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary, and suppose that* $0 \leq w \in H^1(M)$ *is a volume-normalized minimizer of the weighted Escobar constant* Λ. *Then,* w *is a weak* *solution of*

$$
\text{(48)}\qquad \begin{aligned} L^m_\phi w + c_1 w^{\frac{m+n}{m+n-2}} v^{-1} &= 0, \quad \text{in} \quad M,\\ B^m_\phi w = c_2 w^{\frac{m+n}{m+n-2}}, & \qquad\quad \text{on} \quad \partial M \end{aligned}
$$

where

$$
c_1 = \frac{m\Lambda}{m+n-2}\left(\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}\right)^{-\frac{2m+n-1}{m+n-1}}
$$

and

$$
c_2 = \frac{(2m+n-2)\Lambda}{m+n-2}\left(\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}\right)^{-\frac{m}{m+n-1}}.
$$

Proof. This proposition follows immediately from the fact that the Dirichlet form associated to (L^m_ϕ, B^m_ϕ) is symmetric and from the definition of the weighted Escobar constant. □

Remark 5. *If $\Lambda = 0$, then we have in the proposition above that $c_1 = 0$ and $c_2 = 0$. In this case, it follows that the equations in* (48) *coincide with the equations for finding a new conformal smooth metric measure space such that $\hat{R}^m_\phi \equiv 0$ and $\hat{H}^m_\phi \equiv 0$. Moreover, the problem to find a conformal smooth metric measure space with $\hat{R}^m_\phi \equiv 0$ and $\hat{H}^m_\phi \equiv C$ is solved by a direct compact argument on the functional*

$$
\check{Q}(w) = \frac{(L^m_\phi w, w)_M + (B^m_\phi w, w)_{\partial M}}{\left(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{m+n-2}{m+n-1}}}
$$

due to $\frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}$ for $m > 0$.

Next, we consider the Euler Lagrange equation on the $\mathcal{W}$-functional, and we will use it in the proof of Theorem B.

Proposition 9. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary, fix $\tau > 0$, and suppose that $w \in H^1(M)$ is a nonnegative critical point of the map $\xi \to \mathcal{W}(\xi, \tau)$ acting on the space of* *volume-normalized elements of* $H^1(M)$. *Then,* w *is a weak solution of*

$$(49)\qquad \begin{aligned} \tau^{\frac{m}{2(m+n-1)}} L_\phi^m w + \tfrac{m+n-1}{m+n-2}\tau^{-\frac{1}{2}} w^{\frac{m+n}{m+n-2}} v^{-1} &= 0 && \text{in} \quad M, \\ \tau^{\frac{m}{2(m+n-1)}} B_\phi^m w &= c_3 w^{\frac{m+n}{m+n-2}} && \text{on} \quad \partial M, \end{aligned}$$

where

$$c_3 = (\nu(\tau)+1) + \frac{\tau^{-\frac{1}{2}}}{m+n-2}\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}.$$

If, additionally, (w,τ) *is a minimizer of the energy, then*

$$(50)\qquad c_3 = \frac{(m+n-1)(2m+n-2)}{(m+n-2)(2m+n-1)}(\nu+1).$$

Proof. The equality (49) follows immediately from the definition of $\mathcal{W}$. If (w,τ) is a critical point of the map $(w,\tau)\to\mathcal{W}(w,\tau)$, then

$$(51)\qquad \frac{m}{m+n-1}\tau^{\frac{m}{2(m+n-1)}}((L_\phi^m w, w) + (B_\phi^m w, w)) = \tau^{-\frac{1}{2}}\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}.$$

Using this identity, we can express ν and c_3 in terms of $(L_\phi^m w, w) + (B_\phi^m w, w)$, and these expressions yield (50). □

4.4. Euclidean half-space as the model space of the weighted Escobar problem

Theorem 1 gives a complete classification of the minimizers for the weighted Escobar quotient in the model space $(\mathbb{R}_+^n, dt^2+dx^2, dV, d\sigma, m)$ for m non-negative integers. In this subsection, we take a new (τ, x_0)-parametric family of functions as in (3) with $\tau>0$ and $x_0\in\mathbb{R}^{n-1}$.

To define the (τ,x_0)-parametric family of functions, fix $n\geq 3$ and $m>0$. Given any $x_0\in\mathbb{R}^{n-1}$ and $\tau>0$, define the function $w_{x_0,\tau}\in C^\infty(\mathbb{R}_+^n)$ by
(52)

$$w_{x_0,\tau}(t,x) = \tau^{-\frac{(n-1)(m+n-2)}{4(m+n-1)}}\left[\left(1+\left(\frac{c(m,n)}{\tau}\right)^{\frac{1}{2}} t\right)^2 + \frac{c(m,n)|x-x_0|^2}{\tau}\right]^{-\frac{m+n-2}{2}}$$

where $c(m,n) = \frac{m+n-1}{m(m+n-2)^2}$. By a change of variables we obtain

$$(53)\qquad V = \int_{\partial\mathbb{R}_+^n} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} 1^m d\sigma = \int_{\partial\mathbb{R}_+^n} w_{0,1}^{\frac{2(m+n-1)}{m+n-2}} 1^m d\sigma.$$

A straightforward computation shows that

(54)
$$\begin{aligned} -\tau^{\frac{m}{2(m+n-1)}}\Delta w_{x_0,\tau} + \tfrac{m+n-1}{m+n-2}\tau^{-\frac{1}{2}} w_{x_0,\tau}^{\frac{m+n}{m+n-2}} &= 0 && \text{in } \mathbb{R}^n_+,\\ \tau^{\frac{m}{2(m+n-1)}}\frac{\partial w_{x_0,\tau}}{\partial \eta} &= \left(\tfrac{m+n-1}{m}\right)^{\frac{1}{2}} w_{x_0,\tau}^{\frac{m+n}{m+n-2}} && \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

(55)
$$\sup_{(x,t)\in\mathbb{R}^n_+} w_{x_0,\tau}(x,t) = w_{x_0,\tau}(x_0,0) = \tau^{-\frac{(n-1)(m+n-2)}{4(m+n-1)}},$$

and for any $x \neq x_0$,

(56)
$$\lim_{\tau\to 0^+} w_{x_0,\tau}(x,t) = 0.$$

Define $\tilde{w}_{x_0,\tau} = V^{-\frac{m+n-2}{2(m+n-1)}} w_{x_0,\tau}$ with V as in (53). Since $\tilde{w}_{x_0,\tau}$ achieves the weighted Escobar quotient, by Proposition 7, there exits $\tilde{\tau} > 0$ such that

(57)
$$\begin{aligned} &\nu(\mathbb{R}^n_+, dt^2+dx^2, dV, d\sigma, m) + 1\\ &\quad = \mathcal{W}(\mathbb{R}^n_+, dt^2+dx^2, dV, d\sigma, m)(\tilde{w}_{x_0,\tau}, \tilde{\tau}) + 1\\ &\quad = \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V^{\frac{m+n-2}{m+n-1}}}\int_{\mathbb{R}^n_+} |\nabla w_{x_0,\tau}|^2 dV + \tilde{\tau}^{-\frac{1}{2}} V^{-1}\int_{\mathbb{R}^n_+} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dV. \end{aligned}$$

Then, Proposition 9 yields $\tilde{\tau} = \tau V^{-\frac{2}{2m+n-1}}$.

5. The Escobar-type problem for the negative weighted Escobar constant

In this section, we prove Theorem A by a direct compactness argument. For this purpose, we develop some estimations for the below bound for the Laplacian term in the Escobar quotient and some properties of Dirichlet eigenvalues and eigenfunctions.

In this section, C is a real constant that depends only on the smooth metric measure space $(M^n, g, v^m dV_g, v^m d\sigma_g)$ and possibly changes from line to line.

5.1. A below bound for the weighted conformal Laplacian term

All functions in the family $\{w_{\epsilon,0}\}$ as in (3) are minimizers of the weighted Escobar problem. Note that these functions are not uniformly bounded in

$H^1(M)$ as $\epsilon \to 0$. That shows that, in general, there is no reason to find a minimizing function by direct arguments in the weighted Escobar quotient. It is possible that if the weighted Escobar quotient is finite and we try to minimize it with a sequence of volume-normalized functions, then the terms involved in the numerator of the weighted Escobar quotient evaluated in these functions are not bounded uniformly. The next lemma addresses the control of one of those terms from below.

Lemma 3. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary, and suppose that Λ is finite; then, there exists a real constant C such that any volume-normalized function $\varphi \in H^1(M)$ satisfies*

$$(L_\phi^m \varphi, \varphi) + (B_\phi^m \varphi, \varphi) > C. \tag{58}$$

Proof. Suppose that there exists a sequence of functions $\{\varphi_i\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} (L_\phi^m \varphi_i, \varphi_i) + (B_\phi^m \varphi_i, \varphi_i) = -\infty \quad \text{and} \quad \int_{\partial M} \varphi_i^{\frac{2(m+n-1)}{m+n-2}} = 1. \tag{59}$$

Since Λ is finite, there exists a real constant C such that every volume-normalized function φ satisfies

$$C \le \Lambda(\varphi) = \left((L_\phi^m \varphi, \varphi) + (B_\phi^m \varphi, \varphi)\right) \left(\int_M \varphi^{\frac{2(m+n-1)}{m+n-2}}\right)^{\frac{m}{m+n-1}}.$$

From the last inequality, it follows that $\lim_{i\to\infty} \int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} = 0$, and by the Hölder inequality, it follows that $\int_M \varphi_i^2 < C$ for any i. Similarly, using the volume-normalized function φ_i and the Hölder inequality, we obtain $\int_{\partial M} \varphi_i^2 < C$. Using these L^2 estimates and integration by parts, we obtain

$$(L_\phi^m \varphi_i, \varphi_i) + (B_\phi^m \varphi_i, \varphi_i) > C,$$

contradicting the assumption (59). □

5.2. Dirichlet eigenvalues for the weighted conformal Laplacian

To state the following lemma, we say that a real number ρ is a *Dirichlet-type eigenvalue* if ρ satisfies for some $\varphi \in H_0^1(M) = \{\varphi | \, \varphi \in H^1(M), \, \varphi \equiv$

0 on $\partial M\}$

$$L^m_\phi \varphi = \rho\varphi \quad \text{in} \quad M. \tag{60}$$

We also call φ a *Dirichlet-type eigenfunction* if it satisfies (60). Let us denote by ρ_1 the first Dirichlet-type eigenvalue; then, ρ_1 admits a variational characterization as

$$\rho_1 = \inf_{\varphi \in H^1_0(M)} \frac{\int_M |\nabla\varphi|^2 + \frac{m+n-2}{4(m+n-1)} R^m_\phi \varphi^2}{\int_M \varphi^2}. \tag{61}$$

We know ρ_1 is finite, and we can choose a Dirichlet-type eigenfunction φ associated with this eigenvalue such that $\varphi \geq 0$. Moreover, using the maximum principle, we can take $\varphi > 0$ in $M \setminus \partial M$.

Lemma 4. *Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and $m > 0$. Then, $\Lambda = -\infty$ if and only if $\rho_1 \leq 0$.*

Proof. First, let us assume $\rho_1 \leq 0$. Let φ be a first eigenfunction of the problem (60) such that $\varphi > 0$ in $M \setminus \partial M$. Let us define

$$\psi_t = \frac{t\varphi + 1}{\sqrt{D}} \quad \text{where} \quad D = \left(\int_{\partial M} 1\right)^{\frac{m+n-2}{(m+n-1)}}$$

and observe that for some constant $C > 0$, we have

$$\int_{\partial M} \psi_t^{\frac{2(m+n-1)}{m+n-2}} = 1 \qquad \text{and} \qquad \int_M \psi_t^{\frac{2(m+n-1)}{m+n-2}} \geq C > 0. \tag{62}$$

Claim 1. $(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) \to -\infty$ *when* $t \to \infty$.

To prove this claim, we argue as Garcia and Muñoz in [9, Proposition 1]. First, we consider the case of $\rho_1 < 0$; using $\varphi \equiv 0$ on ∂M, we obtain

$$\begin{aligned}(L^m_\phi \psi_t, \psi_t) &+ (B^m_\phi \psi_t, \psi_t) \\ &= \frac{1}{D}\left[t^2\left(\rho_1 \int_M \varphi^2\right) + t\left(\frac{m+n-2}{2(m+n-1)}\int_M \varphi R^m_\phi\right) + E\right]\end{aligned} \tag{63}$$

where

$$E = \frac{m+n-2}{4(m+n-1)}\int_M R^m_\phi + \frac{m+n-2}{2(m+n-1)}\int_M H^m_\phi.$$

Since $\rho_1 < 0$, the quadratic term for t on the right-hand side of (63) is negative. Letting $t \to \infty$, we obtain the conclusion in this case.

Now, we suppose that $\rho_1 = 0$; then,

$$(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) = \frac{1}{D}\left[t\left(\frac{m+n-2}{2(m+n-1)} \int_M \varphi R^m_\phi \right) + E \right] \tag{64}$$

where E is defined as in the previous case. Since $\varphi \equiv 0$ on ∂M, by Hopf's Lemma, $\dfrac{\partial \varphi}{\partial \eta} < 0$. Then, using $\rho_1 = 0$ and integrating by parts, we obtain

$$\frac{m+n-2}{4(m+n-1)} \int_M \varphi R^m_\phi = \int_M \Delta_\phi \varphi = \int_{\partial M} \frac{\partial \varphi}{\partial \eta} < 0.$$

Then, the linear term for t on the right-hand side of (64) is negative. Taking $t \to \infty$, we obtain the conclusion in this case, and we finish the claim's proof.

Finally, from the estimate (62) and Claim 1, we obtain that $\mathcal{Q}(\psi_t) \to -\infty$ as $t \to \infty$; therefore, we conclude $\Lambda = -\infty$.

Next, we assume that $\Lambda = -\infty$, and we prove that $\rho_1 \leq 0$.

Claim 2. *R^m_ϕ is not identically zero.*

Arguing by contradiction, we suppose that $R^m_\phi \equiv 0$, and since $\Lambda = -\infty$, there exists a minimizing sequence of functions $\{\xi_i\}_{i=1}^\infty$ of Λ such that

$$\int_{\partial M} \xi_i^{\frac{2(m+n-1)}{m+n-2}} = 1 \tag{65}$$

and

$$(L^m_\phi \xi_i, \xi_i) + (B^m_\phi \xi_i, \xi_i) = \int_M |\nabla \xi_i|^2 + \int_{\partial M} \frac{m+n-2}{2(m+n-1)} H^m_\phi \xi_i^2 \leq 0.$$

By the equality (65) and the Hölder inequality, we obtain that $\int_{\partial M} \xi_i^2 < C$; then,

$$\begin{aligned} \int_M |\nabla \xi_i|^2 &= (L^m_\phi \xi_i, \xi_i) + (B^m_\phi \xi_i, \xi_i) - \int_{\partial M} \frac{m+n-2}{2(m+n-1)} H^m_\phi \xi_i^2 \\ &< \frac{m+n-2}{2(m+n-1)} \sup |H^m_\phi| \int_{\partial M} \xi_i^2 \\ &< C. \end{aligned} \tag{66}$$

By the above L^2 estimates and the Sobolev trace inequality

$$\int_M \xi_i^{\frac{2(m+n-1)}{m+n-2}} \leq C\left(\int_M |\nabla \xi_i|^2 + \int_{\partial M} \xi_i^2 \right), \tag{67}$$

we obtain

$$\int_M \xi_i^{\frac{2(m+n-1)}{m+n-2}} < C. \tag{68}$$

On the other hand,

$$(L_\phi^m \xi_i, \xi_i) + (B_\phi^m \xi_i, \xi_i) > \frac{-\sup|H_\phi^m|(m+n-2)}{2(m+n-1)} \int_{\partial M} \xi_i^2 > C. \tag{69}$$

Equality (65) and estimates (68) and (69) imply that $\mathcal{Q}(\xi_i) > C$ for every i, which is a contradiction with $\lim_{i\to\infty} \mathcal{Q}(\xi_i) = -\infty$, and we obtain that R_ϕ^m is not identically zero.

Continuing with the proof of $\rho_1 \leq 0$, let us take a minimizing sequence of functions $\{\varphi_i\}_{i=1}^\infty$ of Λ such that

$$\int_{\partial M} \varphi_i^{\frac{2(m+n-1)}{m+n-2}} = 1 \quad \text{and} \quad (L_\phi^m \varphi_i, \varphi_i) + (B_\phi^m \varphi_i, \varphi_i) \leq 0.$$

Claim 3. $\displaystyle\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \to \infty$ *when* $i \to \infty$.

Arguing by contradiction, we assume that there exists a constant $C > 0$ such that $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} < C$; then, by the Hölder inequality, we obtain that $\int_M \varphi_i^2 < C$ for every i. On the other hand, we have that $(L_\phi^m \varphi_i, \varphi_i) + (B_\phi^m \varphi_i, \varphi_i) \to -\infty$ when $i \to \infty$ since $\lim_{i\to\infty} \mathcal{Q}(\varphi_i) = -\infty$. Using this limit, Claim 2 and that φ_i is volume-normalized, we obtain $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$, which is a contradiction with the initial assumption. Hence,

$$\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \to \infty.$$

Claim 4. $\displaystyle\int_M \varphi_i^2 \to \infty$ *when* $i \to \infty$.

Arguing by contradiction, suppose that there exists a constant $C > 0$ such that $\int_M \varphi_i^2 < C$. Then

$$\int_M |\nabla \varphi_i|^2 \leq (L_\phi^m \varphi_i, \varphi_i) + (B_\phi^m \varphi_i, \varphi_i) + C(||\varphi_i||_{2,M}^2 + ||\varphi_i||_{2,\partial M}^2) < C. \tag{70}$$

On the other hand, by the Sobolev inequality, we obtain that there exists a constant C such that

$$\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \leq C\left(\int_M |\nabla\varphi_i|^2 + \int_M \varphi_i^2\right). \tag{71}$$

Then, inequalities (70) and (71) yield $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \leq C$. This is a contradiction of Claim 3, and we conclude that $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$.

Now, we are able to conclude the proof of the lemma. For this purpose, we argue as in the last part of Proposition 1 in Garcia and Muñoz [9]. Let us define the functions $\psi_i = \dfrac{\varphi_i}{||\varphi_i||_{2,M}}$. Then,

$$||\psi_i||_{2,M} = 1, \quad \lim_{i\to\infty} ||\psi_i||_{2,\partial M} = 0, \quad \text{and} \quad \mathcal{Q}(\psi_i) < 0. \tag{72}$$

A similar argument to that presented in (70) yields $\int_M |\nabla\varphi_i|^2 \leq C$.

Since $\{\psi_i\}_{i=1}^{\infty}$ is uniformly bounded in H^1, there exists a subsequence that we call again $\{\psi_i\}_{i=1}^{\infty}$, which converges weakly to a function ψ in $H^1(M)$ and converges strongly in $L^2(M)$ and $L^2(\partial M)$. Using the two first properties in (72), we conclude that $||\psi||_{2,M} = 1$ and $\psi \in H_0^1$. Finally, we conclude the proof of the lemma since ψ satisfies

$$\rho_1 \leq \int_M |\nabla\psi|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m \psi^2 \leq \liminf_{i\to\infty} (L_\phi^m \psi_i, \psi_i) + (B_\phi^m \psi_i, \psi_i) \leq 0.$$

□

5.3. Proof of Theorem A

In this subsection, we prove Theorem A using the previous lemmas presented in this section.

Proof of Theorem A. Let $\{w_i\}_{i=1}^{\infty}$ be a sequence of positive functions such that $\int_{\partial M} w_i^{\frac{2(m+n-1)}{m+n-2}} = 1$, $\mathcal{Q}(w_i) \leq 0$ and $\mathcal{Q}(w_i) \to \Lambda$ when $i \to \infty$. Then,

$$0 \geq (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i) \geq ||\nabla w_i||_{2,M}^2 - C(||w_i||_{2,M}^2 + ||w_i||_{2,\partial M}^2). \tag{73}$$

First, we consider the case of $||w_i^2||_{2,M} < C$; then, the last inequality yields that $\{w_i\}_{i=1}^{\infty}$ are uniformly bounded in $H^1(M)$. Recall that if $m > 0$, then $1 < \frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}$; i.e., $\frac{2(m+n-1)}{m+n-2}$ is less than the critical trace's inequality exponent. By the Sobolev and trace embedding theorems, there exists a function w and a subsequence $\{w_i\}_{i=1}^{\infty}$ that converges to w in $L^2(M)$,

$L^{\frac{2(m+n-1)}{m+n-2}}(M)$ and $L^{\frac{2(m+n-1)}{m+n-2}}(\partial M)$ and $\{w_i\}_{i=1}^{\infty}$ converges weakly to w in $H^1(M)$. It follows that there exists a constant C such that

$$\int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \geq C \quad \text{and} \quad ||w||_{\frac{2(m+n-1)}{m+n-2},\partial M} = 1.$$

Then, by construction, w minimizes the weighted Escobar quotient, and by Proposition 8, w is a nonnegative weak solution of

$$\begin{aligned} &L_\phi^m w + c_1 w^{\frac{m+n}{m+n-2}} v^{-1} = 0 \qquad &&\text{in} \quad M, \\ &B_\phi^m w = c_2 w^{\frac{m+n}{m+n-2}} &&\text{on} \quad \partial M. \end{aligned} \tag{74}$$

Since $1 < \frac{m+n-1}{m+n-2} < \frac{n-1}{n-2}$, the usual elliptic regularity argument for subcritical equations allows us to conclude that w is in fact smooth and positive, as we desired.

Below, we prove that we do not have the case when $||w_i||_{2,M} \to \infty$ is unbounded. Arguing by contradiction, we assume that $||w_i||_{2,M} \to \infty$ when $i \to \infty$. Consider the L^2 renormalized sequence of functions $\tilde{w}_i = \frac{w_i}{||w_i||_{2,M}}$. It follows that $||\tilde{w}_i||_{\frac{2(m+n-1)}{m+n-2},\partial M} \to 0$ when $i \to \infty$. Since $\tilde{w}_i$ satisfy the inequality (73) for every i, we know that $\{\tilde{w}_i\}_{i=1}^{\infty}$ is uniformly bounded in $H^1(M)$.

By the Sobolev and trace embedding theorems, there exists a function w and a subsequence $\{\tilde{w}_i\}_{i=1}^{\infty}$ that converges to w in $L^2(M)$, $L^{\frac{2(m+n-1)}{m+n-2}}(M)$ and $L^{\frac{2(m+n-1)}{m+n-2}}(\partial M)$ and weakly in $H^1(M)$. In consequence, $||w||_{2,M} = 1$ and using again that $||\tilde{w}_i||_{\frac{2(m+n-1)}{m+n-2},\partial M} \to 0$ when $i \to \infty$, we obtain that $w \equiv 0$ in ∂M.

On the other hand, Lemma 3 yields

$$0 > (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i) > -C. \tag{75}$$

Therefore, $(L_\phi^m \tilde{w}_i, \tilde{w}_i) + (B_\phi^m \tilde{w}_i, \tilde{w}_i) \to 0$ when $i \to \infty$. Using w as a test function in (61), we conclude that

$$\rho_1 \leq \int_M |\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m w \leq \liminf_{i\to\infty} (L_\phi^m \tilde{w}_i, \tilde{w}_i) + (B_\phi^m \tilde{w}_i, \tilde{w}_i) = 0.$$

However, $\rho_1 \leq 0$ contradicts Lemma 4 because Λ is finite by hypothesis.

□

Remark 6. *The argument presented in the proof of Theorem A works if there exists a minimizer sequence of volume-normalized* $\{w_i\}_{i=1}^{\infty}$ *and a constant* C *such that* $C > (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i)$ *for every* i. *Note that the previous statement does not apply for all smooth measure metric spaces with* $\Lambda \geq 0$ *since minimizing sequences of volume-normalized functions* $\{w_i\}_{i=1}^{\infty}$ *of* Λ *can satisfy*

$$\lim_{i\to\infty} (L_\phi^m w_i, w_i) + (B_\phi^m w_i, w_i) = \infty \quad \textit{and} \quad \lim_{i\to\infty} \int_M w_i^{\frac{2(m+n-1)}{m+n-2}} = 0. \tag{76}$$

6. Aubin-type inequality for weighted Escobar constants

In this section, we find an upper bound for the τ-energy as τ approaches zero; Theorem B is a consequence of this estimate. To prove this estimate, we use Theorem 1 and the family $\{w_{0,\tau}\}$ in (52) as test functions in the $\mathcal{W}$-functional. Actually, Theorem 1 is the reason that the weighted Escobar constant for the Euclidean half-space appears on the right-hand side of the inequality (6). Similar ideas to prove Theorem B appeared in [11]. As in the previous section, C is a real constant that depends only on the smooth metric measure space $(M^n, g, v^m dV_g, v^m d\sigma_g)$ and possibly changes from line to line or within the same line.

Lemma 5. *Let* $(M^n, g, v^m dV_g, v^m d\sigma_g)$ *be a compact smooth metric measure space with boundary and* $m \geq 0$; *then,*

$$\limsup_{\tau\to 0} \nu(\tau) \leq \nu[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m].$$

Proof. First, define $\tilde{w}_{x_0,\tau} = V^{-\frac{m+n-2}{2(m+n-1)}} w_{x_0,\tau}$ with V as in (53). By Theorem 1, we know that $\tilde{w}_{x_0,\tau}$ achieves the weighted Escobar quotient; hence, by Proposition 7, there exists $\tilde{\tau} > 0$ such that

$$\begin{aligned} &\nu(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m) + 1 \\ &\quad = \mathcal{W}(\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)(\tilde{w}_{x_0,\tau}, \tilde{\tau}) + 1 \\ &\quad = \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V^{\frac{m+n-2}{m+n-1}}} \int_{\mathbb{R}^n_+} |\nabla w_{x_0,\tau}|^2 dV + \tilde{\tau}^{-\frac{1}{2}} V^{-1} \int_{\mathbb{R}^n_+} w_{x_0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dV. \end{aligned} \tag{77}$$

Then, Proposition 9 yields $\tilde{\tau} = \tau V^{-\frac{2}{2m+n-1}}$.

On the other hand, fix a point $p \in \partial M$ and let (x_i, t) be the Fermi coordinates in some fixed neighborhood U of $p = (0, ..., 0)$. Let $1 > \epsilon > 0$ be such that $B(p, 2\epsilon) \subset U$. Let $\eta : M \to [0, 1]$ be a cutoff function such that

$\eta \equiv 1$ on B_ϵ^+, $supp(\eta) \subset B_{2\epsilon}^+$ and $|\nabla \eta|^2 < C\epsilon^{-1}$ in $A_\epsilon^+ = B_{2\epsilon}^+ \setminus B_\epsilon^+$. For each $0 < \tau < 1$, define $f_\tau : M \to \mathbb{R}$ by $f_\tau(x_1, ..., x_{n-1}, t) = \eta w_{0,\tau}(x_1, ..., x_{n-1}, t)$, and set $\tilde{f}_\tau = V_\tau^{-\frac{m+n-2}{2(m+n-1)}} f_\tau$ for

$$V_\tau = \int_{\partial M} f_\tau^{\frac{2(m+n-1)}{m+n-2}}.$$

Proposition 5 implies that if w is a volume-normalized function with the metric $v^{-2}g$, then

$$\begin{aligned}&\mathcal{W}[M^n, v^{-2}g, dV_{v^{-2}g}, d\sigma_{v^{-2}g}, m](w, \tau)\\ &\quad = \mathcal{W}[M^n, g, v^m dV_g, v^m d\sigma_g](v^{-\frac{m+n-2}{2}} w, \tau).\end{aligned}$$

This equality allows us to consider without loss of generality that $v \equiv 1$. Computing as in [10, Lemma 3.4] and using $dV_g = (1 + O(r))dxdt$ and $d\sigma_g = (1 + O(r))dx$, we obtain

(78)

$$\begin{aligned}&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1\\ &= \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}} \left(\int_{B_{2\epsilon}^+} |\nabla f_\tau|_g^2 + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dV_g \right.\\ &\quad \left. + \int_{B_{2\epsilon}^+ \cap \partial M} \frac{m+n-1}{2(m+n-2)} H_g f_\tau^2 d\sigma_g \right) + \tilde{\tau}^{-\frac{1}{2}} V_\tau^{-1} \int_{B_{2\epsilon}^+} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dV_g\\ &\leq (1 + C\epsilon) \left\{ \frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}} \left(\int_{B_{2\epsilon}^+} |\nabla f_\tau|_g^2 + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dxdt \right.\right.\\ &\quad \left.\left. + \int_{B_{2\epsilon}^+ \cap \partial M} \frac{m+n-1}{2(m+n-2)} H_g f_\tau^2 dx \right) + \tilde{\tau}^{-\frac{1}{2}} V_\tau^{-1} \int_{B_{2\epsilon}^+} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dxdt \right\}.\end{aligned}$$

Let us recall that $c(m, n) = \frac{m+n-1}{m(m+n-2)^2}$. By fixing $\epsilon < 1$ and taking $\sqrt{\tau} \leq \sqrt{c(m, n)} 2\epsilon$ we obtain

$$\begin{aligned}&\int_{B_{2\epsilon}^+} R_g f_\tau^2 dxdt \leq C \int_{B_{2\epsilon}^+} w_{0,\tau}^2 dxdt\\ &\qquad = C\tau^{-\frac{(n-1)(m+n-2)}{2(m+n-1)}} \int_{B_{2\epsilon}^+} \frac{dxdt}{((1 + (\frac{c(m,n)}{\tau})^{\frac{1}{2}} t)^2 + \frac{c(m,n)}{\tau} |x|^2)^{m+n-2}} \qquad (79)\\ &\qquad = C\tau^{\frac{n-1}{2(m+n-1)} + \frac{1}{2}} \int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \frac{dydt}{((1 + s)^2 + |y|^2)^{m+n-2}}.\end{aligned}$$

Similar to [10, Lemma 3.5], we obtain

$$\int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \frac{dydt}{((1+s)^2+|y|^2)^{m+n-2}} \tag{80}$$
$$= \begin{cases} C & \text{if} \quad 4-n-2m<0, \\ O(\tau^{m-\frac{1}{2}}) & \text{if} \quad n=3,\ \frac{1}{2}-m>0 \text{ and} \\ O(\log(\tau)) & \text{if} \quad n=3,\ \frac{1}{2}-m=0. \end{cases}$$

Then,

$$\int_{B^+_{2\epsilon}} R_g f_\tau^2 dxdt = E_1 = \begin{cases} O(\tau^{\frac{n-1}{2(m+n-1)}+\frac{1}{2}}) & \text{if}\, 4-n-2m<0, \\ O(\tau^{\frac{n-1}{2(m+n-1)}+m}) & \text{if}\, n=3,\ m<\frac{1}{2} \text{ and} \\ O(\tau^{\tau^{\frac{n-1}{2(m+n-1)}+\frac{1}{2}}}\log(\tau)) & \text{if}\, n=3,\ \frac{1}{2}-m=0. \end{cases} \tag{81}$$

Now, we estimate the integrals on the right-hand side in the second inequality of (78)

$$\begin{aligned} \int_{B^{n-1}_{2\epsilon}} H_g f_\tau^2 dx &\leq C \int_{B^{n-1}_{2\epsilon}} w_{0,\tau}^2 dx \\ &= C\tau^{\frac{n-1}{2(m+n-1)}} \int_{B^{n-1}_{2\epsilon}} (1+|y|^2)^{-(m+n-2)} dx \\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}}, \end{aligned} \tag{82}$$

$$\int_{B^+_{2\epsilon}} f_\tau^{\frac{2(m+n-1)}{m+n-2}} dxdt \leq \int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt. \tag{83}$$

Let us estimate the gradient integral in $A^+_\epsilon = B^+_{2\epsilon} \setminus B^+_\epsilon$. Observe that

$$|\nabla f_\tau|^2_{\tilde{g}} \leq C|\nabla f_\tau|^2 \leq C(\eta^2|\nabla w_{0,\tau}|^2 + |\nabla\eta|^2 w_{0,\tau}^2). \tag{84}$$

Now, we obtain

$$\begin{aligned} \int_{A^+_\epsilon} |\nabla\eta|^2 w_{0,\tau}^2 dxdt &\leq C\epsilon^{-2} \int_{A^+_\epsilon} w_{0,\tau}^2 dxdt \\ &\leq C\epsilon^{-2}\tau^{\frac{-(n-1)(m+n-2)}{2(m+n-1)}+\frac{n}{2}} \int_{A^+_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \left(\frac{1}{s^2+|y|^2}\right)^{m+n-2} dxdt \\ &\leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}} \end{aligned} \tag{85}$$

and

$$\begin{aligned}(86)\qquad &\int_{A_\epsilon^+} \eta^2 |\nabla w_{0,\tau}|^2 dxdt \\ &\leq C\tau^{\frac{-(n-1)(m+n-2)}{2(m+n-1)}+\frac{n}{2}-1} \int_{A^+_{\frac{\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \left(\frac{1}{s^2+|y|^2}\right)^{m+n-1} dxdt \\ &\leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}}.\end{aligned}$$

Then,

$$(87)\qquad \int_{A_\epsilon^+} |\nabla f_\tau|_g^2 dxdt \quad \leq C\epsilon^{2-n-2m}\tau^{\frac{n-1}{2(m+n-1)}+m+\frac{n-3}{2}}.$$

Since for the Fermi coordinates around p we obtain $g^{tt}=1$, $g^{ti}=0$ and $g^{ij}=\delta_{ij}+O(|x,t|)$, where $1\leq i,j\leq n-1$, it follows

$$\begin{aligned}(88)\qquad \int_{B_\epsilon} |\nabla f_\tau|_g^2 dxdt \quad &\leq \int_{B_\epsilon} |\nabla w_{0,\tau}|^2 dxdt + C\int_{B_\epsilon} |x,t|(w_{0,\tau})_i(w_{0,\tau})_j \\ &\leq \int_{B_\epsilon} |\nabla w_{0,\tau}|^2 dxdt + C\tau^{\frac{n-1}{2(m+n-1)}}.\end{aligned}$$

We already have the second inequality of (88) because

$$\begin{aligned}(89)\qquad &\int_{B_\epsilon} |x,t|(w_{0,\tau})_i(w_{0,\tau})_j \\ &\leq C\tau^{-\frac{(n-1)(m+n-2)}{2(m+n-1)}-2} \int_{B_\epsilon} \frac{|x,t|x_ix_j dxdt}{((1+(\frac{c(m,n)}{\tau})^{\frac{1}{2}}t)^2+\frac{c(m,n)}{\tau}|x|^2)^{m+n}} \\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}} \int_{B^+_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}} \frac{|y,s|^3 dydt}{((1+s)^2+|y|^2)^{m+n}} \\ &\leq C\tau^{\frac{n-1}{2(m+n-1)}}.\end{aligned}$$

Using inequalities (81), (82), (87) and (88) in inequality (78), we obtain that

(90)

$$\begin{aligned}&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1\\ &\leq (1 + C\epsilon)\left\{\frac{\tilde{\tau}^{\frac{m}{2(m+n-1)}}}{V_\tau^{\frac{m+n-2}{m+n-1}}}\left(\int_{\mathbb{R}^n_+} |\nabla w_{0,\tau}|^2 dxdt + C\tau^{\frac{n-1}{2(m+n-1)}}\right.\right.\\ &\qquad\left.\left. + C\tau^{\frac{(n-1)(2m+n-1)}{2(m+n-1)}+m+\frac{n-3}{2}}\epsilon^{2-n-2m} + E_1\right) + \tilde{\tau}^{-\frac{1}{2}} V_\tau^{-1}\int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt\right\}.\end{aligned}$$

Now, using inequality (77) we conclude

$$\begin{aligned}&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1\\ &\leq (1 + C\epsilon)\nu[\mathbb{R}^n_+, dt^2 + dx^2, dV_g, d\sigma_g, m]\\ &\quad + (1 + C\epsilon)\left\{\tilde{\tau}^{\frac{m}{2(m+n-1)}} V_\tau^{-\frac{m+n-2}{m+n-1}}\right.\\ &\quad \times\left(C\tau^{\frac{n-1}{2(m+n-1)}} + C\tau^{\frac{(n-1)(2m+n-1)}{2(m+n-1)}+m+\frac{n-3}{2}}\epsilon^{2-n-2m} + E_1\right)\\ &\quad + \tilde{\tau}^{\frac{m}{2(m+n-1)}}(V_\tau^{-\frac{m+n-2}{m+n-1}} - V^{-\frac{m+n-2}{m+n-1}})\int_{\mathbb{R}^n_+} |\nabla w_{0,\tau}|^2 dxdt\\ &\quad \left. + \tilde{\tau}^{-\frac{1}{2}}(V_\tau^{-1} - V^{-1})\int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt\right\}.\end{aligned} \tag{91}$$

On the other hand, we obtain

$$\begin{aligned}V - V_\tau &\leq \int_{\mathbb{R}^{n-1}\setminus B^{n-1}_\epsilon} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dx\\ &= \tau^{-\frac{n-1}{2}}\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_\epsilon}(1 + \frac{c(m,n)}{\tau}|x|^2)^{-(m+n-1)}dx\\ &= C\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_{\frac{2\epsilon\sqrt{c(m,n)}}{\sqrt{\tau}}}}(1 + |y|^2)^{-(m+n-1)}dy\\ &\leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}}.\end{aligned} \tag{92}$$

In particular, we obtain that the constants V_τ are uniformly bounded away from zero. Using estimate (92) and the Taylor expansion for the functions $x^{-\frac{m+n-2}{m+n-1}}$ and x^{-1}, we obtain

$$V_\tau^{-\frac{m+n-2}{m+n-1}} - V^{-\frac{m+n-2}{m+n-1}} \leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}} \tag{93}$$

and

$$V_\tau^{-1} - V^{-1} \leq C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}}. \tag{94}$$

Additionally, equality (77) implies the following estimates

$$\tilde{\tau}^{\frac{m}{2(m+n-1)}} \int_{\mathbb{R}^n_+} |\nabla w_{0,\tau}|^2 dxdt \leq C \quad \text{and} \quad \tilde{\tau}^{-\frac{1}{2}} \int_{\mathbb{R}^n_+} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} dxdt \leq C. \tag{95}$$

The substitution $\tilde{\tau} = \tilde{\tau} V^{\frac{1}{2m+n-1}}$, and the inequalities (93), (94), (95) and (91) yield

$$\begin{aligned}
&\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](\tilde{f}_\tau, \tilde{\tau}) + 1 \\
&\quad \leq (1 + C\epsilon)\nu[\mathbb{R}^n_+, dt^2 + dx^2, 1^m dV_g, 1^m d\sigma_g] \\
&\qquad + (1 + C\epsilon)\Big\{ V^{-\frac{m+n-2}{m+n-1} - \frac{m}{2(2m+n-1)(m+n-1)}} \\
&\qquad \times \Big(C\tau^{\frac{1}{2}} + C\tau^{\frac{1}{2}+m+\frac{n-3}{2}}\epsilon^{2-n-2m} \\
&\qquad + \tau^{\frac{m}{2(m+n-1)}} E_1 \Big) + C\epsilon^{1-n-2m}\tau^{m+\frac{n}{2}-\frac{1}{2}} \Big\}.
\end{aligned} \tag{96}$$

Finally, taking $\tau \to 0$ and after taking $\epsilon \to 0$ in (96), the conclusion follows. □

Proof of Theorem B. By the definition of ν and Lemma 5, we obtain that

$$\nu[M^n, g, v^m dV_g, v^m d\sigma_g] \leq \nu[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m]. \tag{97}$$

By Proposition 7, we conclude

$$\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m]. \quad \blacksquare \tag{98}$$

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Universidad del Valle, Departamento de Matemáticas
Universidad del Valle, Calle 13 No. 100 - 00, Cali, Colombia
E-mail address: jhovanny.munoz@correounivalle.edu.co

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