Communications in
Analysis and Geometry
Volume 29, Number 8, 1847–1915, 2021

Proof of a null Penrose conjecture using a new quasi-local mass

Henri P. Roesch

We define an explicit quasi-local mass functional which is nondecreasing along *all* doubly convex foliations of null cones. Assuming the existence of a doubly convex foliation, we use this new functional to prove the Null Penrose Conjecture.

1 Introduction **1847**

2 Schwarzschild geometry **1859**

3 Propagation of ρ **1863**

4 Foliation comparison **1873**

5 Spherical symmetry **1894**

References **1912**

1. Introduction

A spacetime $(\mathcal{M}, g)$ is defined to be a four dimensional smooth manifold $\mathcal{M}$ equipped with a metric $g(\cdot,\cdot)$ (or $\langle\cdot,\cdot\rangle$) of Lorentzian signature $(-,+,+,+)$. We assume that the spacetime is both orientable and time orientable, i.e. admits a nowhere vanishing timelike vector field, defined to be future-pointing.

Throughout this paper, we will denote by Σ a spacelike embedding of a sphere in $\mathcal{M}$ with induced metric γ. It is well known that Σ has trivial normal bundle $T^\perp\Sigma$ with induced metric of signature $(-,+)$. From any choice of null section $\underline{L} \in \Gamma(T^\perp\Sigma)$, we have a unique null partner $L \in \Gamma(T^\perp\Sigma)$ satisfying $\langle \underline{L}, L\rangle = 2$, providing $T^\perp\Sigma$ with a null basis $\{L, \underline{L}\}$. We also notice that any 'boost' $\{\underline{L}, L\} \to \{\underline{L}_a, L_a\}$ given by:

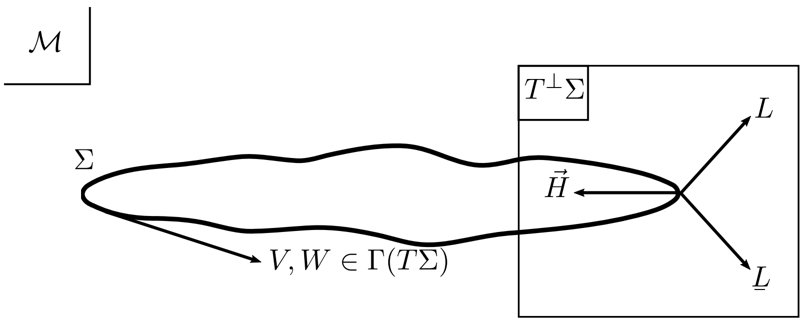
$$\underline{L}_a := a\underline{L}, \;\; L_a := \frac{1}{a}L$$

(for $a \in \mathcal{F}(\Sigma)$ a non-vanishing smooth function on Σ) insures $\langle \underline{L}_a, L_a\rangle = \langle \underline{L}, L\rangle = 2$.

Our convention for the second fundamental form II and mean curvature $\vec{H}$ of Σ are

$$\mathrm{II}(V,W) = D_V^{\perp}W, \quad \vec{H} = \mathrm{tr}_\Sigma\, \mathrm{II}$$

for $V, W \in \Gamma(T\Sigma)$ and D the Levi-Civita connection of the spacetime.



Definition 1.1. Given a choice of null basis $\{\underline{L}, L\}$, following the conventions of Sauter [32], we define the associated symmetric 2-tensors $\underline{\chi}, \chi$ and torsion (connection 1-form) ζ by

$$\begin{aligned} \underline{\chi}(V,W) &:= \langle D_V\underline{L}, W\rangle = -\langle \underline{L}, \mathrm{II}(V,W)\rangle \\ \chi(V,W) &:= \langle D_V L, W\rangle = -\langle L, \mathrm{II}(V,W)\rangle \\ \zeta(V) &:= \frac{1}{2}\langle D_V\underline{L}, L\rangle = -\frac{1}{2}\langle D_V L, \underline{L}\rangle \end{aligned}$$

where $V, W \in \Gamma(T\Sigma)$.

Denoting the exterior derivative on Σ by $\not{d}$, any boosted basis $\{\underline{L}_a, L_a\}$ produces the associated tensors of Definition 1.1:

$$\begin{aligned} \underline{\chi}_a(V,W) &:= \langle D_V(a\underline{L}), W\rangle = a\underline{\chi}(V,W) \\ \chi_a(V,W) &:= \langle D_V(\frac{1}{a}L), W\rangle = \frac{1}{a}\chi(V,W) \\ \zeta_a(V) &:= \frac{1}{2}\langle D_V(a\underline{L}), \frac{1}{a}L\rangle = \zeta(V) + V\log|a| = (\zeta + \not{d}\log|a|)(V). \end{aligned}$$

For a symmetric 2-tensor T on Σ its *trace-free* (or *trace-less*) part is given by

$$\hat{T} := T - \frac{1}{2}(\mathrm{tr}_\gamma T)\gamma$$

allowing us to decompose $\underline{\chi}$ into its *shear* and *expansion* components respectively:

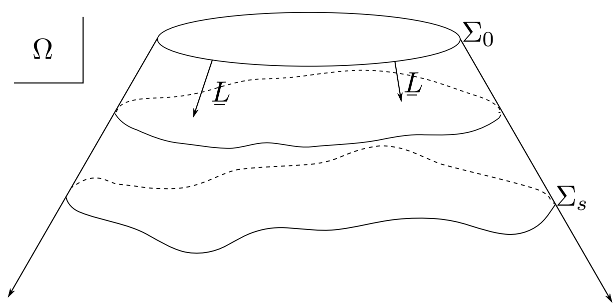
$$\underline{\chi} = \hat{\underline{\chi}} + \frac{1}{2}(\mathrm{tr}\,\underline{\chi})\gamma.$$

Definition 1.2. We say Σ *is expanding along* $\underline{L}$ for some null section $\underline{L} \in \Gamma(T^\perp\Sigma)$ provided that,

$$(\dagger) \qquad \langle -\vec{H}, \underline{L}\rangle = \mathrm{tr}\,\underline{\chi} > 0$$

on all of Σ.

Any infinitesimal flow of Σ along $\underline{L}$ gives, by first variation of area, $d\dot{A} = \langle -\vec{H}, \underline{L}\rangle dA = \mathrm{tr}\,\underline{\chi} dA$. So the flow is locally area expanding ($d\dot{A} > 0$) only if Σ "is expanding along $\underline{L}$":



Remark 1.3. In Section 4 we will show (Lemma 4.10), whenever Ω is past asymptotically flat inside a spacetime satisfying the dominant energy condition, a consequence of the famous Raychaudhuri equation (Section 3, (3.4)) is that any cross section $\Sigma \hookrightarrow \Omega$ is expanding along the past pointing null section $\underline{L} \in \Gamma(T^\perp\Sigma) \cap \Gamma(T\Omega)|_\Sigma$. So inequality ($\dagger$) holds for any foliation of Ω along $\underline{L}_a$ where $a > 0$ and we have an expanding null cone (as illustrated in the figure above).

For Σ expanding along some $\underline{L} \in \Gamma(T^\perp\Sigma)$ we are able to choose a canonical null basis $\{L^-, L^+\}$ by requiring that our flow along $L^- = a\underline{L}$ be uniformly area expanding ($d\dot{A} = dA$). From first variation of area, flowing along

$a\underline{L}$ gives

$$\dot{dA} = -\langle \vec{H}, a\underline{L}\rangle dA = a \operatorname{tr}\underline{\chi} dA.$$

So we achieve a uniformly area expanding null flow when $a = \frac{1}{\operatorname{tr}\underline{\chi}}$ giving:

Definition 1.4. For Σ expanding along some $\underline{L} \in \Gamma(T^{\perp}\Sigma)$ we call the associated canonical uniformly area expanding null basis $\{L^-, L^+\}$ given by

$$L^- := \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \;\; L^+ := \operatorname{tr}\underline{\chi} L$$

the *null inflation basis.*

We also define $\chi^{-(+)} := -\langle \mathrm{II}, L^{-(+)}\rangle$. From the comments immediately after Definition 1.1 we observe that

$$\begin{aligned} \operatorname{tr}\chi^- &= 1 \\ \operatorname{tr}\chi^+ &= \operatorname{tr}\underline{\chi}\operatorname{tr}\chi = \langle \vec{H}, \vec{H}\rangle \end{aligned}$$

and for $V \in \Gamma(T\Sigma)$ the torsion associated to this basis is given by

$$\tau(V) = \frac{1}{2}\langle D_V L^-, L^+\rangle = (\zeta - \not{d}\log\operatorname{tr}\underline{\chi})(V).$$

We will denote the induced covariant derivative on Σ by $\nabla\!\!\!\!/$.

Definition 1.5. Assuming Σ is expanding along $\underline{L}$, for some $\underline{L} \in \Gamma(T^{\perp}\Sigma)$, we define the geometric flux function

$$\rho = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \nabla\!\!\!\!/ \cdot \tau \tag{1.1}$$

where $\mathcal{K}$ represents the Gaussian curvature of Σ.
This allows us to define the associated quasi-local mass

$$m(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int_\Sigma \rho^{\frac{2}{3}} dA\Big)^{\frac{3}{2}}. \tag{1.2}$$

For the induced covariant derivative $\nabla\!\!\!\!/$ we denote the associated Laplacian on Σ by $\Delta\!\!\!\!/$.

Remark 1.6. Whenever $\operatorname{tr}\chi^+ = \langle \vec{H}, \vec{H}\rangle \neq 0$, Σ has two null inflation bases given by $\{L^-, L^+\}$ and $\{\frac{L^+}{\operatorname{tr}\chi^+}, \operatorname{tr}\chi^+ L^-\}$. As a result, we typically have two distinct flux functions

$$\rho_- = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \nabla\!\!\!/ \cdot \tau$$
$$\rho_+ = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \nabla\!\!\!/ \cdot \tau - \Delta\!\!\!/ \log|\langle \vec{H}, \vec{H}\rangle|$$

with associated mass functionals $m_\pm$. For the Bartnik datum α_H (see Section 1.1), we will see for a past pointing $\underline{L}$ that $\rho_- - \rho_+ = 2\nabla\!\!\!/ \cdot \alpha_H$ (Lemma 1.10). For $\langle \vec{H}, \vec{H}\rangle \neq 0$, whenever Σ is 'time-flat' (i.e. $\nabla\!\!\!/ \cdot \alpha_H = 0$) it follows that $\rho_- = \rho_+ \implies m_- = m_+$.

For a normal null flow off of some Σ with null flow vector $\underline{L}$, technically the flow speed is zero since $\langle \underline{L}, \underline{L}\rangle = 0$. In the case the Σ expands along $\underline{L}$ we define the *expansion speed*, σ, according to $\underline{L} = \sigma L^-$. We notice that $\sigma = \operatorname{tr}\underline{\chi}$. We are now ready to state our first result.

Theorem 1.7. *Let Ω be a null hypersurface foliated by spacelike spheres $\{\Sigma_s\}$ expanding along the null flow direction $\underline{L} = \sigma L^-$ such that $|\rho(s)| > 0$ for each s. Then the mass $m(s) := m(\Sigma_s)$ has rate of change*

$$\frac{8\pi}{(2m)^{\frac{1}{3}}}\frac{dm}{ds} = \int_{\Sigma_s} \frac{\sigma}{\rho^{\frac{1}{3}}}\bigg((|\hat{\chi}^-|^2 + G(L^-, L^-))(\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\Delta\!\!\!/ \log|\rho|) + \frac{1}{2}|\nu|^2 + G(L^-, N)\bigg)dA$$

where

$$G = Ric - \frac{1}{2}Rg, \qquad \nu := \frac{2}{3}\hat{\chi}^- \cdot d\!\!\!/ \log|\rho| - \tau,$$
$$\sigma = \operatorname{tr}\underline{\chi}, \qquad N := \frac{1}{9}|d\!\!\!/ \log|\rho||^2 L^- + \frac{1}{3}\nabla\!\!\!/ \log|\rho| - \frac{1}{4}L^+.$$

If we assume therefore that our spacetime $\mathcal{M}$ satisfies the dominant energy condition our mass functional $m(\Sigma_s)$ is non-decreasing for foliations $\{\Sigma_s\}$ of doubly convex 2-spheres:

Definition 1.8. A spacelike 2-sphere Σ is called *doubly convex* if it satisfies:

$$\rho > 0$$
$$\frac{1}{4}\langle \vec{H}, \vec{H}\rangle \geq \frac{1}{3}\Delta\!\!\!/ \log\rho.$$

In the case that the second inequality is strict, we say Σ is *strict doubly convex.*

So for a doubly convex foliation the dominant energy condition ensures the product of the first two terms of the integrand in Theorem 1.7 is non-negative. The second is non-negative since each Σ_s is spacelike and the last term is non-negative again from the dominant energy condition since $\langle N, N\rangle = 0$ and $\langle N, L^-\rangle = -\frac{1}{2} < 0$ (i.e. N is null and at every point $p \in \Sigma$ lies inside the same connected component of the null cone in $T_p\mathcal{M}$ as L^-).

We will assume in Sections 4 and 5 that $\underline{L}$ is past pointing. Adopting the same definitions as Mars and Soria [23] (see Section 4.1) we have our second main result:

Theorem 1.9. *Let Ω be a null hypersurface in a spacetime satisfying the dominant energy condition that extends to past null infinity. If $\{\Sigma_s\}$ is a doubly convex foliation we have*

$$m(0) \leq \lim_{s\to\infty} m(\Sigma_s) =: M$$

(for $M \leq \infty$). If, in addition, Ω is past asymptotically flat with strong flux decay and $\{\Sigma_s\}$ asymptotically geodesic (see Section 4) then

$$M \leq m_{TB}$$

where m_{TB} is the Trautman-Bondi mass of Ω. Moreover, in the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ we have the null Penrose inequality

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq m_{TB}.$$

Furthermore, if equality holds for some strict doubly convex foliation outside Σ_0, equality holds for all foliations of Ω and the data (γ, $\underline{\chi}$, $\operatorname{tr}\chi$ and ζ) is realized by some foliation of the standard null cone of the Schwarzschild spacetime.

1.1. Background

An interesting energy functional for a closed spacelike surface Σ introduced by Hawking [16] is defined by

$$E_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\Big(1 - \frac{1}{16\pi}\int_\Sigma \langle \vec{H}, \vec{H}\rangle dA\Big).$$

Named the *Hawking Energy* this functional provides a measure of the energy content within Σ. We also notice by the Gauss-Bonnet and Divergence

Theorems that

$$\int_{\Sigma} \rho dA = 8\pi \frac{E_H(\Sigma)}{\sqrt{\frac{|\Sigma|}{4\pi}}}$$

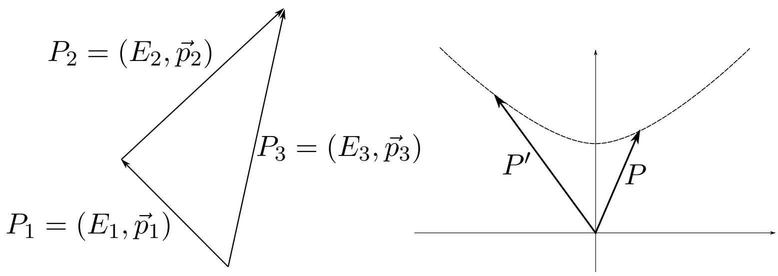
motivating in part why we call ρ a flux function.

Existence of a limit of the Hawking Energy along suitable 2-spheres foliating a null cone has been analyzed in work of Christodoulou-Klainermann [11], generalized by Bieri [4], also Klainerman-Nicolò [20], Chruściel-Paetz [12], and a setting we will adopt in Sections 3 and 4 due to Mars-Soria [23]. From this limit, one is able to define the notion of the total mass of a null cone Ω called the *Trautman-Bondi mass* (see Definition 4.18). Historically, the definition of mass at *null infinity* (see Section 3) can be traced back to Trautman [36], generalizing, and predating, a more explicit coordinate based construction by Bondi et al. [6, 31]. We refer the reader to [5] for more details, including how the Trautman-Bondi mass relates to the famous one of Arnowitt-Deser-Misner [2] at spacelike infinity.

In Minkowski spacetime, if Σ is chosen to be any cross-section of the null cone of a point, work of Sauter ([32], Section 4.5) shows that $E_H(\Sigma) = 0$, as expected of a flat vacuum spacetime. For Schwarzschild spacetime, the famous geometry modeling a static isolated black hole of fixed mass M, Sauter also shows when Σ is any cross-section of the so called 'standard null cone', $E_H(\Sigma) \geq M$ with equality if and only if Σ is 'time-symmetric' ([32], Lemma 4.4). This is reminiscent of the special relativistic understanding that an energy measurement $E = \sqrt{M^2 + |\vec{p}|^2}$ for a particle always over-estimates its mass M except when measured within its rest frame (i.e a frame where $\vec{p} = 0$).

Monotonicity properties of E_H were analyzed in detail by work of Bray-Hayward-Mars-Simon [7], and Bray-Jauregui-Mars [9]. In particular, these authors identify the existence of null flows for which E_H is non-decreasing. However, although the Hawking Energy enjoys monotonicity and convergence along certain flows, difficulty remains in assigning physical significance to the convergence of E_H due to the lack of control on the asymptotics of such flows [3, 21, 24, 32]. We expect these difficulties may very well be symptomatic of the fact that an energy functional is particularly susceptible to the plethora of ways boosts can develop along any given flow.

Given a fixed reference frame in special relativity, energy accumulates with 4-velocity (vector) addition $P_1 + P_2 = P_3 \implies E_3 = E_1 + E_2$. Analogously, we expect an expanding null flow off of Σ, 'within a fixed reference frame', to infinitesimally exhibit non-decreasing energy. However, with no a priori knowledge of the flow, we have no way to fix or even identify a



reference frame geometrically. Consequently, we have no way to account for any accumulation of 'phantom energy' coming from infinitesimal boosts along the flow. As shown above, this is analogous with a fluctuating reference frame in special relativity coming from boosts: $P \to P'$ (i.e energy increases) or $P' \to P$ (i.e. energy decreases). Geometrically, we expect this to manifest along the flow in a (local) 'tilting' of Σ. One may even expect a net decrease in energy as is evident in Schwarzschild spacetime (recall $E_H(\Sigma) \geq M$). This is not a problem, however, if we appeal instead to mass rather than energy. Leaning again on our special relativistic intuition, we see that the boost-separated 4-velocities $P' = (E', \vec{p}')$ and $P = (E, \vec{p})$ measure the same mass $M^2 = E^2 - |\vec{p}|^2 = (E')^2 - |\vec{p}'|^2 = (M')^2$. Moreover, by virtue of the Lorentzian triangle inequality (provided all vectors are timelike and future/past pointing), along any given flow the mass should always increase:

$$M_3 = |(E_1 + E_2, \vec{p}_1 + \vec{p}_2)| \geq |(E_1, \vec{p}_1)| + |(E_2, \vec{p}_2)| = M_1 + M_2.$$

Therefore, appealing instead to a quasi local mass functional we hope a larger class of flows will give rise to more generic monotonicity. We approach the problem of finding such a mass functional by first finding an optimal choice of flux function for E_H.

As early as 1962, in [6, 31] we find the use of a flux function called the *Bondi mass-aspect function* related to the Trautman-Bondi energy and mass (see [5] for a summary of the details). Another, used by Christodoulou-Klainermann in [11], is given by

$$\mu = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle - \nabla\!\!\!/ \cdot \zeta$$

dependent on the null basis $\{\underline{L}, L\} \subset \Gamma(T^{\perp}\Sigma)$. Using μ in his Ph.D thesis [32], Sauter showed the existence of flows on past null cones that render E_H non-decreasing, making explicit use of the fact that, under a boost, this mass aspect function changes via ζ according to

$$\zeta \to \zeta_a = \zeta + \not{d} \log |a| \implies \mu \to \mu_a = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle - \not{\nabla} \cdot \zeta - \not{\Delta} \log |a|.$$

From these observations, our divergence term given in (1.1) (up to a sign) is somewhat motivated by a desire of finding a flux function independent of boosts. In fact, in the case that Σ is *admissible*: $H^2 := \langle \vec{H}, \vec{H} \rangle > 0$, we're able to construct the orthonormal frame field

$$\{e_r = -\frac{\vec{H}}{H}, e_t\} : \ \langle e_r, e_r \rangle = 1, \ \langle e_t, e_t \rangle = -1, \ \langle e_r, e_t \rangle = 0$$

where e_t is future pointing. The associated connection 1-form is given by

$$\alpha_H(V) := \langle D_V e_r, e_t \rangle.$$

Thus, if $\underline{L}$ is past pointing, the following lemma allows us to give ρ in terms of the Bartnik data of Σ:

$$\rho = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle + \not{\nabla} \cdot \alpha_H - \not{\Delta} \log H.$$

Lemma 1.10. *For Σ admissible*

$$\tau = \pm \alpha_H - \not{d} \log H$$

from which we conclude that

$$\rho_{\mp} = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H} \rangle \pm \not{\nabla} \cdot \alpha_H - \not{\Delta} \log H$$

where $+/-$ indicates whether L^- is past/future pointing.

Proof. Since $-\vec{H} = \frac{1}{2}(\operatorname{tr}\chi \underline{L} + \operatorname{tr}\underline{\chi} L)$, we see $0 < H^2 = \operatorname{tr}\underline{\chi} \operatorname{tr}\chi$. We may therefore choose $\underline{L}$ so that Σ is expanding along $\underline{L}$. The inverse mean curvature vector is given by

$$\vec{I} := -\frac{\vec{H}}{H^2} = \frac{1}{2}\Big(\frac{\underline{L}}{\operatorname{tr}\underline{\chi}} + \frac{L}{\operatorname{tr}\chi}\Big).$$

As a result,

$$\begin{aligned}
\alpha_H(V) &= \langle D_V \frac{e_r}{H}, He_t\rangle = \langle D_V \frac{1}{2}\Big(\frac{\underline{L}}{\operatorname{tr}\underline{\chi}} + \frac{L}{\operatorname{tr}\chi}\Big), \mp\frac{1}{2}(\operatorname{tr}\chi\underline{L} - \operatorname{tr}\underline{\chi}L)\rangle \\
&= \pm\frac{1}{4}\Big(\langle D_V \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle - \langle D_V \frac{\operatorname{tr}\underline{\chi}L}{H^2}, H^2\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}\rangle\Big) \\
&= \pm\frac{1}{4}\Big(\langle D_V \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle + \langle D_V (H^2\frac{\underline{L}}{\operatorname{tr}\underline{\chi}}), \frac{\operatorname{tr}\underline{\chi}L}{H^2}\rangle\Big) \\
&= \pm\frac{1}{4}\Big(2\langle D_V \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}, \operatorname{tr}\underline{\chi}L\rangle + 2V\log H^2\Big)
\end{aligned}$$

we recall, $\tau(V) := \frac{1}{2}\langle D_V L^-, L^+\rangle$, with $L^- = \frac{\underline{L}}{\operatorname{tr}\underline{\chi}}$, and $L^+ = \operatorname{tr}\underline{\chi}L$. $\square$

We refer the reader to Section 2 for further motivation of (1.1) and (1.2) from an analysis of the Schwarzschild spacetime.

1.2. The Penrose conjecture

One important application for a quasi-local mass is to study the Penrose conjecture [28, 29]:

$$\sqrt{\frac{|\Sigma_i|}{16\pi}} \leq M$$

where $|\Sigma_i|$ is the area of an 'initial' black hole boundary, and M is the total mass of the system. In the appropriate setting, this provides not only a strengthened version of the famous Positive Mass Theorem of Schoen-Yau [33–35], and Witten [39], but also insight regarding the mathematical validity of the *weak cosmic censorship hypothesis* that Penrose employed in the formulation of his conjecture. We refer the reader to [22] for a more comprehensive introduction and survey. A rough summary of the heuristic argument is given in the following: according to the famous Hawking-Penrose singularity theorems ([15, 27]), a variety of 'physically reasonable' initial data for an isolated system (a cluster of stars, a galaxy, etc...) support solutions of the Einstein field equations that develop singularities. Under cosmic censorship, a spherical boundary forms prior to the singularity to 'wrap it up', hiding any chaotic physics likely to ensue. This boundary traces out the *event horizon*, acting as a semi-permeable barrier, trapping even light from escaping to the outside (a black hole). As matter continues to fall through the event horizon, the Hawking area theorem [16, 17] (see also [30]) describes that the area of the boundary expands to the future. Outside the horizon, the system approaches equilibrium via the dissipation of gravitational radiation. This is measured by the loss in Trautman-Bondi mass [14, 31] along null infinity. It is expected that the spacetime consequently settles towards a stationary vacuum solution of the field equations, namely a member of the Kerr family of rotating black holes. A Kerr black hole supports an event horizon where all cross-sections are isometric, called a *Killing Horizon*, each with ('final') area $|\Sigma_f|$. The area $|\Sigma_f|$ is also explicitly bounded above by the square of the irreducible Trautman-Bondi mass at null infinity:

$$|\Sigma_f| \leq 16\pi (m_{TB}^f)^2.$$

Therefore, returning to any initial configuration of total mass M, with a black hole boundary of area $|\Sigma_i|$, the Penrose conjecture is the resulting inequality:

$$\sqrt{\frac{|\Sigma_i|}{16\pi}} \leq \sqrt{\frac{|\Sigma_f|}{16\pi}} \leq m_{TB}^f \leq M.$$

The existence and exact location of the event horizon assumes knowledge of the complete future evolution of the spacetime and its structure. Consequently, cosmic censorship posits a fundamental mathematical structure to solutions of Einstein's field equations. The Penrose conjecture is therefore not only an interesting bound on a global geometric invariant (namely, the mass), it hints at the validity of cosmic censorship. In-fact, any physical system supporting data that contradicts the conjectured inequality would deal a crippling blow to the cosmic censorship hypothesis. In this paper we're concerned with a formulation of the conjecture for data supported on a null hypersurface. The original argument assumes an embedded asymptotically flat Riemannian hypersurface.

The first major breakthrough on the original formulation of the conjecture was for a class of geometrically standalone or 'time symmetric' initial data called the *Riemannian Penrose Inequality.* In 1997, Huisken-Ilmanen [19] were able to prove the Penrose conjecture for a single connected component of the black hole boundary. Their approach showed the existence of a suitably weak propagation of 2-spheres under *inverse mean curvature* for which E_H is non-decreasing. This flow interpolates between an outermost minimal surface (the black hole boundary component) where $E_H = \sqrt{\frac{|\Sigma_i|}{16\pi}}$, and spacelike infinity where the flow approximates round spheres, and E_H approaches the total ADM mass M_{ADM} ([2]). In 1999, Bray [8] extended the result to include multiple boundary components. By utilizing a completely different approach of conformally flowing the metric of the slice, Bray showed that the total mass of the space remains non-increasing while the total horizon area is non-decreasing. The horizon components eventually coalesce along the flow as the geometry approaches a time symmetric slice of the Schwarzschild space. The conjecture subsequently follows from the fact that the horizon area matches exactly the mass (by a factor of 16π) for this Schwarzschild slice.

For the null formulation, Sauter ([32], Theorem 4.10) proved the conjecture for a class of null hypersurfaces analogous to the time symmetric slices in the Riemannian case. In this paper, Theorem 1.9 extends Sauter's result to a class of null hypersurfaces admitting a doubly convex foliation as described in Definition 1.8.

1.3. Outline

This paper is organized as follows:

1) Section 1: Introduction
2) Section 2: Schwarzschild Geometry
   We motivate ρ and m from an analysis of the standard null cones of Schwarzschild geometry.
3) Section 3: Propogation of ρ
   We prove Theorem 1.7 by calculating the propagation of ρ along arbitrary null flows foliating a null cone Ω. We also study the restrictions placed on Ω in the case that a flow satisfies $\frac{dm}{ds} = 0$, the case of equality for Theorem 1.9.
4) Section 4: Foliation Comparison
   Given an arbitrary cross section Σ within a null hypersurface Ω, we find its flux ρ in terms of the data for a given background foliation. An analysis of this relationship yields Theorem 1.9 under the necessary decay.
5) Section 5: Spherical Symmetry
   For a class of perturbations of the black hole exterior in a spherically symmetric spacetime, we show the existence of asymptotically flat null cones of strong flux decay that allow strict doubly convex foliations.

2. Schwarzschild geometry

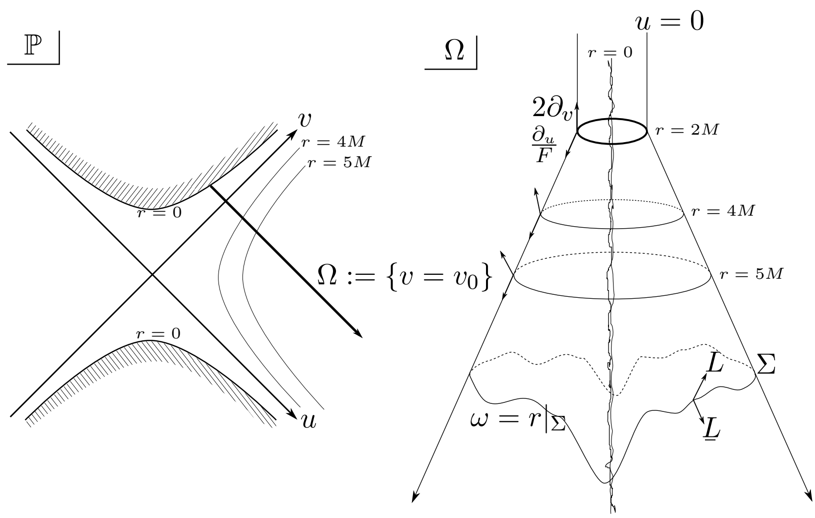
Schwarzschild spacetime models a static black hole of mass M given by the metric

$$g_S = -\mathfrak{h} dt \otimes dt + \mathfrak{h}^{-1} dr \otimes dr + r^2(d\vartheta \otimes d\vartheta + (\sin\vartheta)^2 d\varphi \otimes d\varphi)$$

where $\mathfrak{h} = 1 - \frac{2M}{r}$ for $2M > r > 0$, $r > 2M$. The maximal extension of this geometry is called the Kruskal spacetime $(\mathbb{P} \times_r \mathbb{S}^2, g_K)$ which is given by the warped product of the Kruskal Plane $\mathbb{P} := \{uv > -2Me^{-1}\}$ and the standard round $\mathbb{S}^2$ with warping function $r = g^{-1}(uv)$ for $g(r) = (r-2M)e^{\frac{r}{2M}-1}$, $r > 0$. The metric and its inverse are given by:

$$\begin{aligned} g_K &= F(r)(du \otimes dv + dv \otimes du) + r^2(d\vartheta \otimes d\vartheta + (\sin\vartheta)^2 d\varphi \otimes d\varphi) \\ g_K^{-1} &= \frac{1}{F}(\partial_v \otimes \partial_u + \partial_u \otimes \partial_v) + r^{-2}(\partial_\vartheta \otimes \partial_\vartheta + (\sin\vartheta)^{-2}\partial_\varphi \otimes \partial_\varphi) \end{aligned}$$

where $F(r) = \frac{8M^2}{r}e^{1-\frac{r}{2M}}$. We recover the Schwarzschild spacetime on $v > 0$, $u \neq 0$ with the coordinate change $t = 2M \log|\frac{v}{u}|$ ([25]).



Each round $\mathbb{S}^2$ has area $4\pi r^2$ so we interpret r as a 'radius' function and $F(r)$ gives rise to unbounded curvature at $r = 0$ and the 'black hole' singularity. A standard past null cone of Schwarzchild spacetime Ω is the hypersurface given by fixing the coordinate v, say $v = v_0$. Denoting the gradient of a function f by Df we recognize the null vector field $\frac{\partial_u}{F} = Dv$ restricts to Ω as both a tangent (since $\partial_u(v) = 0$) and normal (since $Dv \perp T\Omega$)

vector field. It follows that $Dv \in T^{\perp}\Omega \cap T\Omega$ and the induced metric on Ω is degenerate, so Ω is an example of a *null hypersurface*. From the identity $D_{Df}Df = \frac{1}{2}D|Df|^2$ we see $\frac{\partial_u}{F}$ is geodesic and Ω is realized as the past light cone of a spherically symmetric section of the event horizon ($r = 2M$) as shown above. Setting $\underline{L} = D(4M \log v) = \frac{4M}{v}\frac{\partial_u}{F}$ we see $\underline{L}(r) = \frac{4M}{v}\frac{r_u}{F} = \frac{4M}{v}\frac{v}{g'(r)F} = \frac{4M}{v}\frac{v}{4M} = 1$. We conclude that r restricts to an affine parameter along the geodesics generating Ω and therefore any cross section Σ can be given as a graph over $\mathbb{S}^2$ in Ω with graph function $\omega = r|_{\Sigma}$.

Lemma 2.1. *Given a cross section $\Sigma := \{r = \omega\}$ of the standard null cone $\Omega := \{v = v_0\}$ in Kruskal spacetime we have for the null vector field $\underline{L}$ satisfying $\underline{L}(r) = 1$ that:*

$$\gamma = \omega^2 \mathring{\gamma}, \qquad \underline{\chi} = \frac{1}{\omega}\gamma, \qquad \operatorname{tr}\underline{\chi} = \frac{2}{\omega}, \qquad \zeta = -\not{d}\log\omega,$$
$$\chi = \frac{1}{\omega}(1 - \frac{2M}{\omega} + |\nabla\!\!\!/\,\omega|^2)\gamma - 2H^{\omega}, \qquad \operatorname{tr}\chi = \frac{2}{\omega}\Big(1 - \frac{2M}{\omega} - \omega^2 \Delta\!\!\!/\,\log\omega\Big),$$
$$\rho = \frac{2M}{\omega^3}.$$

where γ is the metric, H^{ω} the Hessian of ω, and $\mathring{\gamma}$ the round metric on $\mathbb{S}^2$.

Proof. The above result is a special case of Lemma 5.1 setting $\beta = 0$ and $M(v, r) = M$ so we postpone the proof. □

2.1. Why $\nabla\!\!\!/ \cdot \tau$?

From Lemma 2.1 we see any cross-section $\Sigma := \{r = \omega\}$ of the standard null cone satisfies

$$\tau = \zeta - \not{d}\log\operatorname{tr}\underline{\chi} = -\not{d}\log\omega + \not{d}\log\omega = 0.$$

Restricting ourselves to Schwarzschild, the reader may be tempted into questioning the necessity of the divergence term in (1.1) given that it vanishes altogether. However, work of Wang, Wang and Zhang ([38] Theorem B') in n-dimensional Schwarzschild highlights an intimate relationship between τ on a codimension-2 spacelike surface Σ and its 'normal null geometry'. Namely, if Σ satisfies $\alpha_H = \not{d}\log H$ it must be constrained to a shear-free null hypersurface of spherical symmetry, or the standard null cone in dimension four. One could argue therefore that the vanishing of τ for any cross-section of the standard null cone in Schwarzschild simply serves to obscure its (expected) contribution to ρ. We will see in Section 3 that the ultimate contribution of $\nabla\!\!\!/ \cdot \tau$ is to the propagation of ρ (Theorem 3.5). It's inclusion removes various problematic terms along normal null flows off of our surface Σ culminating in the simple and physically relevant structure of Theorem 1.7.

2.2. Motivating $m(\Sigma)$

Motivating (1.2) is less subtle than (1.1) in Schwarzschild given the strikingly simple expression for ρ given in Lemma 2.1. For the cross-section $\Sigma := \{r = \omega\}$ with area form $dA = \omega^2 dS$ (dS the standard area form on a round $\mathbb{S}^2$), we notice that irrespective of ω we succeed in extracting precisely the black hole mass M as soon as we integrate ρ to the appropriate power

$$m(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int_\Sigma \rho^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} = \frac{1}{2}\Big(\frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{(2M)^{\frac{2}{3}}}{\omega^2}\omega^2 dS\Big)^{\frac{3}{2}} = M.$$

Lemma 2.2. *Suppose Σ is a compact Riemannian manifold, then for any $f \in \mathcal{F}(\Sigma)$*

$$\Big(\int f^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} = \inf_{\psi>0}\Big(\sqrt{\int \psi^2 dA}\int \frac{|f|}{\psi} dA\Big)$$

Proof. by choosing $\psi_\epsilon^3 = |f| + \epsilon$ for some $\epsilon > 0$ it's a simple verification that

$$\Big(\int (|f|+\epsilon)^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} \geq \sqrt{\int \psi_\epsilon^2 dA}\int \frac{|f|}{\psi_\epsilon} dA \geq \inf_{\psi>0}\sqrt{\int \psi^2 dA}\int \frac{|f|}{\psi} dA$$

so by the Dominated Convergence Theorem

$$\Big(\int f^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} = \lim_{\epsilon\to 0}\Big(\int (|f|+\epsilon)^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} \geq \inf_{\psi>0}\sqrt{\int \psi^2 dA}\int \frac{|f|}{\psi} dA.$$

We show the inequality holds in the opposite direction from Hölder's inequality

$$\int f^{\frac{2}{3}} dA = \int (\frac{f}{\psi})^{\frac{2}{3}} \psi^{\frac{2}{3}} dA \leq \Big(\sqrt{\int \psi^2 dA}\Big)^{\frac{2}{3}}\Big(\int \frac{|f|}{\psi} dA\Big)^{\frac{2}{3}}$$

where the result follows from raising both sides to the $\frac{3}{2}$ power and taking an infimum over all $\psi > 0$. $\square$

So given any 2-sphere in an arbitrary spacetime with non-negative flux $\rho \geq 0$, defining $E^{\psi}_H(\Sigma) := \frac{1}{8\pi}\sqrt{\frac{\int \psi^2 dA}{4\pi}} \int \frac{\rho}{\psi} dA$, we conclude that

$$m(\Sigma) = \inf_{\psi>0} E^{\psi}_H(\Sigma) \leq E_H(\Sigma)$$

as expected. Recalling our use of Hölder's inequality in the proof of Lemma 2.2, we see that $m(\Sigma) = E_H(\Sigma)$ if and only if ρ is constant on Σ. So for $\Sigma := \{r = \omega\} \subset \Omega$, where Ω is the standard null cone in Schwarzschild spacetime, we see that $m(\Sigma)$ underestimates the Hawking energy $E_H(\Sigma)$ with equality only if ρ hence ω is constant. Namely, the round spheres of intersection between time-symmetric slices given by $t = const > 0$ (hence $r = const > 0$ in Ω), as expected from Sauters work ([32], Lemma 4.4). One can show (see, for example, [23]) that a cross-section of the standard null cone is a round sphere with constant Gauss curvature $K = \frac{1}{r_0^2}$ if and only if ω solves the non-linear equation

$$1 - \left(\frac{\omega}{r_0}\right)^2 = \mathring{\Delta} \log\left(\frac{\omega}{r_0}\right)$$

for $\mathring{\Delta}$ the Laplace-Beltrami operator with respect to the metric $\mathring{\gamma}$. Solutions take the form $\omega(\vartheta, \varphi) = r_0 \frac{\sqrt{1-|\vec{v}|^2}}{1-\vec{v}\cdot\vec{n}(\vartheta,\varphi)}$, for some $\vec{v}$ inside the unit ball $\mathring{B}^3 \subset \mathbb{R}^3$, and $\vec{n}(\vartheta, \varphi)$ the unit position vector. We conclude that this round sphere $\Sigma^{r_0}_{\vec{v}} \hookrightarrow \Omega$ has energy $E_H(r_0, \vec{v}) = \frac{M}{\sqrt{1-|\vec{v}|^2}}$ which is precisely the observed (special relativistic) energy of a particle of mass M traveling at velocity $\vec{v}$ relative to its observer. It follows that the energy of an asymptotically round foliation $\{\Sigma_s := \{r = \omega_s\}\} \subset \Omega$ approaches the mass M only if $\vec{v} = 0$ 'at infinity'. Clearly this corresponds asymptotically to the $r = const$ foliation inside Ω i.e. the time symmetric spheres. Herein it seems the difficulty lies in finding foliations such that $E_H(\Sigma_s)$ *increases* to the Trautman-Bondi mass. Even in Schwarzschild spacetime, if insistent upon the use of E_H, our only choice of foliation increasing to the mass M is to foliate with time symmetric spheres. Not only is this flow highly specialized, it dictates strong restrictions on our initial choice of Σ. This is to be expected of a quasi-local energy as mentioned in Section 1.1 due to its inherent sensitivity to boosts in our abstract reference frame along the flow. In the next Section, with the proof of Theorem 1.7, we show that appealing to the mass $m(\Sigma)$ instead of energy we produce a non-decreasing quantity along *any* null flow off of a doubly convex 2-sphere.

3. Propagation of ρ

Our convention for constructing the Riemann curvature tensor is:

$$R_{XY}Z := D_{[X,Y]}Z - [D_X, D_Y]Z.$$

Given an orthonormal frame field $\{E_0, E_1, E_2, E_3\}$, whereby $\langle E_i, E_i\rangle = \epsilon_i$ for $\epsilon_0 = -\epsilon_{i>0} = -1$, the Ricci 2-tensor is given by

$$\mathrm{Ric}(X, Y) = \sum_{i=0}^{3} \epsilon_i \langle R_{XE_i}Y, E_i\rangle,$$

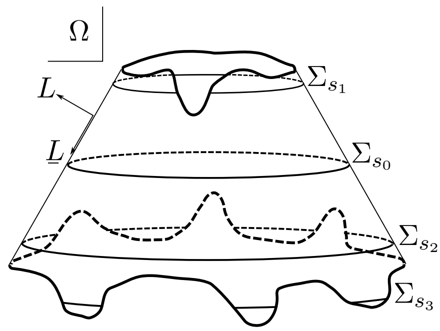
and the Ricci Scalar by $R = \sum_{i=0}^{3} \epsilon_i \mathrm{Ric}(E_i, E_i)$. In this section we will work towards proving Theorem 1.7 by finding the propagation of our flux function ρ along an arbitrary null flow.

3.1. Setup

We adopt the same setup as in [23] which we summarize here in order to introduce our notation:

Suppose Ω is a smooth connected, null hypersurface embedded in $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. Here we let $\underline{L}$ be a smooth, non-vanishing, null vector field of Ω, $\underline{L} \in \Gamma(T\Omega)$. It's a well known fact (see, for example, [10]) that the integral curves of $\underline{L}$ are pre-geodesic so we're able to find $\kappa \in \mathcal{F}(\Omega)$ such that $D_{\underline{L}}\underline{L} = \kappa\underline{L}$. We assume the existence of an embedded 2-sphere Σ in Ω such that any integral curve of $\underline{L}$ intersects Σ precisely once. As previously used, we will refer to such Σ as *cross sections* of Ω. This gives rise to a natural submersion $\pi : \Omega \to \Sigma$ sending $p \in \Omega$ to the intersection with Σ of the integral curve $\gamma_p^{\underline{L}}$ of $\underline{L}$ for which $\gamma_p^{\underline{L}}(0) = p$. Given $\underline{L}$ and a constant s_0 we may construct a function $s \in \mathcal{F}(\Omega)$ from $\underline{L}(s) = 1$ and $s|_\Sigma = s_0$. For $q \in \Sigma$, if $(s_-(q), s_+(q))$ represents the range of s along $\gamma_q^{\underline{L}}$ then letting $S_- = \sup_\Sigma s_-$ and $S_+ = \inf_\Sigma s_+$ we notice that the interval (S_-, S_+) is non-empty.

Given that $\underline{L}(s) = 1$ the Implicit Function Theorem gives for $t \in (S_-, S_+)$ that $\Sigma_t := \{p \in \Omega | s(p) = t\}$ is diffeomorphic to $\mathbb{S}^2$ through Σ. For $s < S_-$ or $s > S_+$, in the case that Σ_s is non-empty, although smooth it may no longer be connected. We have that the collection $\{\Sigma_s\}$ gives a foliation of Ω.We construct another null vector field L by assigning at every $p \in \Omega$ $L|p \in T_p\mathcal{M}$ be the unique null vector satisfying $\langle \underline{L}, L\rangle = 2$ and $\langle L, v\rangle = 0$ for any $v \in T_p\Sigma_{s(p)}$. As before each Σ_s is endowed with an induced metric γ_s, two null second fundamental forms $\underline{\chi}_s = -\langle \vec{\mathrm{II}}, \underline{L}\rangle$ and $\chi_s = -\langle \vec{\mathrm{II}}, L\rangle$ as well



$s_3 < s_2 < S_- < s_0 < S_+ < s_1$

as the connection 1-form (or torsion) $\zeta_s(V) = \frac{1}{2}\langle D_V \underline{L}, L\rangle$. We will need the following known result (see, for example [32]):

Lemma 3.1. *Given $V \in \Gamma(T\Sigma_s)$,*

$$\begin{aligned} D_V \underline{L} &= \vec{\underline{\chi}}_s(V) + \zeta_s(V)\underline{L} \\ D_V L &= \vec{\chi}_s(V) - \zeta_s(V) L \\ D_{\underline{L}} L &= -2\vec{\zeta}_s - \kappa L \end{aligned}$$

where, given $V, W \in \Gamma(T\Sigma_s)$, the vector fields $\vec{\zeta}_s$, $\vec{\underline{\chi}}_s(V)$, $\vec{\chi}_s(V) \in \Gamma(T\Sigma_s)$ are uniquely determined by $\langle \vec{\zeta}_s, V\rangle = \zeta_s(V)$, $\langle \vec{\underline{\chi}}_s(V), W\rangle = \underline{\chi}_s(V, W)$, and similarly for $\vec{\chi}_s$.

Proof. It suffices to check all identities agree by taking the metric inner product with vectors $\underline{L}, L$ and an extension W satisfying $W|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ keeping in mind that $[\underline{L}, W]|_{\Sigma_s} \in \Gamma(T\Sigma_s)$. We leave this verification to the reader. $\square$

For any cross section Σ of Ω and $v \in T_q(\Sigma)$ we may extend v along the generator $\gamma_q^{\underline{L}}$ according to

$$\begin{aligned} \dot{V}(s) &= D_{V(s)} \underline{L} \\ V(0) &= v. \end{aligned}$$

Since $x \in T_p\Omega \iff \langle \underline{L}|_p, x\rangle = 0$ we see from the fact that

$$\begin{aligned} (\langle V(\dot{s}), \underline{L}\rangle) = \langle D_{V(s)}\underline{L}, \underline{L}\rangle + \kappa\langle V(s), \underline{L}\rangle &= \frac{1}{2} V(s)\langle \underline{L}, \underline{L}\rangle + \kappa\langle V(s), \underline{L}\rangle \\ &= \kappa\langle V(s), \underline{L}\rangle \end{aligned}$$

and $\langle V(0), \underline{L}\rangle = 0$ we can solve to get $\langle V(s), \underline{L}\rangle = 0$ for all s. As a result any section $W \in \Gamma(T\Sigma)$ is extended to all of Ω satisfying $[\underline{L}, W] = 0$. We also notice along each generator $0 = [\underline{L}, W]s = \underline{L}(Ws) = \dot{W}s$ such that $Ws|_\Sigma = 0$ forces $Ws = 0$ on all of Ω. We conclude that $W|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ and denote by $E(\Sigma) \subset \Gamma(T\Omega)$ the set of such extensions off of Σ along $\underline{L}$. We also note that linear independence is preserved along generators by standard uniqueness theorems allowing us to extend basis fields $\{X_1, X_2\} \subset \Gamma(T\Sigma)$ off of Σ as well.

3.2. The structure equations

We will need to propagate the Christoffel symbols with the following known result (see, for example [32]):

Lemma 3.2. *Given $U, V, W \in E(\Sigma)$,*

$$\langle [\underline{L}, \nabla\!\!\!/_V W], U\rangle = (\nabla\!\!\!/_V \underline{\chi}_s)(W, U) + (\nabla\!\!\!/_W \underline{\chi}_s)(V, U) - (\nabla\!\!\!/_U \underline{\chi}_s)(V, W)$$

where $\nabla\!\!\!/$ the induced covariant derivative on each Σ_s.

We extend the tensors $\gamma_s, \underline{\chi}_s, \chi_s, \zeta_s$ to tensors on Ω in the natural way, specifically, for any $X, Y \in \Gamma(T\Omega)$:

$$\gamma(X, Y) := \langle X, Y\rangle,\ \underline{\chi}(X, Y) := \langle D_X \underline{L}, Y\rangle$$
$$\chi(X, Y) := \langle D_X L, Y\rangle,\ \zeta(X) := \frac{1}{2}\langle D_X \underline{L}, L\rangle.$$

Given any basis extension $\{X_1, X_2\} \subset E(\Sigma)$, we define $T_{ij} := T(X_i, X_j)$ for any 2-tensor T of Ω. Assuming Einstein's summation convention, we then uniquely define the components γ^{ij} with the specification $\gamma^{ik}\gamma_{kj} = \delta^i_j$ (δ^i_j the usual Kronecker delta components). Therefore, we may also extend the metric inverse of γ_s with $\gamma^{-1} := \gamma^{ij} X_i \otimes X_j$.

We wish to understand the propagation of the above tensors along $\{\Sigma_s\}$. To that end, for any tensor T on Ω we denote by T_s the restriction to the appropriate bundle over the leaf Σ_s, observing that $(\gamma)_s = \gamma_s$, $(\underline{\chi})_s = \underline{\chi}_s$, $(\chi)_s = \chi_s$, and $(\zeta)_s = \zeta_s$.

Proposition 3.3 (Structure Equations).

$$\underline{L}\mathcal{K} = -\operatorname{tr}\underline{\chi}_s\mathcal{K} - \frac{1}{2}\Delta\!\!\!/\operatorname{tr}\underline{\chi}_s + \nabla\!\!\!/\cdot(\nabla\!\!\!/\cdot\hat{\underline{\chi}}_s) \tag{3.1}$$

$$(\mathcal{L}_{\underline{L}}\gamma)_s = 2\underline{\chi}_s \tag{3.2}$$

$$(\mathcal{L}_{\underline{L}}\underline{\chi})_s = -\underline{\alpha}_s + \frac{1}{2}|\hat{\underline{\chi}}_s|^2\gamma_s + \operatorname{tr}\underline{\chi}_s\hat{\underline{\chi}}_s + \frac{1}{4}(\operatorname{tr}\underline{\chi}_s)^2\gamma_s + \kappa\underline{\chi}_s \tag{3.3}$$

$$\underline{L}\operatorname{tr}\underline{\chi} = -\frac{1}{2}(\operatorname{tr}\underline{\chi}_s)^2 - |\hat{\underline{\chi}}_s|^2 - G(\underline{L},\underline{L}) + \kappa\operatorname{tr}\underline{\chi}_s \tag{3.4}$$

$$\begin{aligned}(\mathcal{L}_{\underline{L}}\chi)_s = &\Big(\mathcal{K} + \hat{\underline{\chi}}_s\cdot\hat{\chi}_s + \frac{1}{2}G(\underline{L},L)\Big)\gamma_s + \frac{1}{2}\operatorname{tr}\underline{\chi}_s\hat{\chi}_s + \frac{1}{2}\operatorname{tr}\chi_s\hat{\underline{\chi}}_s \\ &- \hat{G} - 2S(\nabla\!\!\!/\zeta_s) - 2\zeta_s\otimes\zeta_s - \kappa\chi_s\end{aligned} \tag{3.5}$$

$$\underline{L}\operatorname{tr}\chi = G(\underline{L},L) + 2\mathcal{K} - 2\nabla\!\!\!/\cdot\zeta_s - 2|\zeta_s|^2 - \langle\vec{H},\vec{H}\rangle - \kappa\operatorname{tr}\chi_s \tag{3.6}$$

$$(\mathcal{L}_{\underline{L}}\zeta)_s = G_{\underline{L}} - \nabla\!\!\!/\cdot\hat{\underline{\chi}}_s - \operatorname{tr}\underline{\chi}_s\zeta_s + \frac{1}{2}d\!\!\!/\operatorname{tr}\underline{\chi}_s + d\!\!\!/\kappa \tag{3.7}$$

where $\underline{\alpha}_s(V,W) := \langle R_{\underline{L}V}\underline{L},W\rangle$, $S(T)$ *is the symmetric part of a 2-tensor* T, $G_{\underline{L}}(V) := G(\underline{L},V)$, *and* $\hat{G}(V,W) := G(V,W) - \frac{1}{2}(\operatorname{tr}\gamma_s G)\langle V,W\rangle$.

Proof. All the identities above are obtained by taking the $\underline{L}$ derivative of the appropriate tensor components under a basis extension $\{X_1, X_2\} \subset E(\Sigma)$. For (3.1)-(3.3) we refer the reader to [32], and for non-vacuum, (3.4)-(3.7) can be found in [13]. □

From Proposition 3.1 we have the propagation of the Gauss curvature, for the other terms in the definition of ρ, first we extend the torsion τ_s to $\tau = \zeta - d\log\operatorname{tr}\underline{\chi}$, and recall $\langle\vec{H},\vec{H}\rangle_s = \operatorname{tr}\underline{\chi}_s\operatorname{tr}\chi_s$. By a slight abuse of notation, we omit subscripts where the meaning is clear from context:

Corollary 3.4. *For* $\{\Sigma_s\}$ *expanding along* $\underline{L} = \sigma L^-$ *we have,*

$$\begin{aligned}(\mathcal{L}_{\underline{L}}\tau)_s =& -\sigma\tau - \nabla\!\!\!/\cdot(\sigma\hat{\chi}^-) + \sigma G_{L^-} + d\!\!\!/\Big(\sigma(|\hat{\chi}^-|^2 + G(L^-,L^-))\Big) \\ \underline{L}\nabla\!\!\!/\cdot\tau_s =& -2\sigma\nabla\!\!\!/\cdot\tau + \Delta\!\!\!/\Big(\sigma(|\hat{\chi}^-|^2 + G(L^-,L^-))\Big) - 2\nabla\!\!\!/\cdot(\sigma\hat{\chi}^-\cdot\tau) \\ &+ \nabla\!\!\!/\cdot(\sigma G_{L^-}) - \tau(\nabla\!\!\!/\sigma) - \nabla\!\!\!/\cdot\nabla\!\!\!/\cdot(\sigma\hat{\chi}^-) \\ \underline{L}\langle\vec{H},\vec{H}\rangle =& -\sigma\Big(\langle\vec{H},\vec{H}\rangle(|\hat{\chi}^-|^2 + G(L^-,L^-)) - G(L^-,L^+) + 2|\tau|^2\Big) \\ &- \frac{3}{2}\sigma\langle\vec{H},\vec{H}\rangle - 2\sigma\nabla\!\!\!/\cdot\tau + 2\sigma\mathcal{K} - 2\Delta\!\!\!/\sigma - 4\tau(\nabla\!\!\!/\sigma)\end{aligned}$$

Proof. By combining (3.4) and (3.7):

$$\begin{aligned}(\mathcal{L}_{\underline{L}}(\zeta - d\log\operatorname{tr}\underline{\chi}))_s &= (\mathcal{L}_{\underline{L}}\zeta)_s - \not{d}\underline{L}\log\operatorname{tr}\underline{\chi} \\ &= G_{\underline{L}} - \not{\nabla}\cdot\hat{\underline{\chi}} - \operatorname{tr}\underline{\chi}\zeta + \frac{1}{2}\not{d}\operatorname{tr}\underline{\chi} + \not{d}\kappa \\ &\quad - \not{d}\Big(-\frac{1}{2}\operatorname{tr}\underline{\chi} - \frac{|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}} + \kappa\Big) \\ &= -\operatorname{tr}\underline{\chi}(\zeta - \not{d}\log\operatorname{tr}\underline{\chi}) - \not{\nabla}\cdot\hat{\underline{\chi}} + G_{\underline{L}} + \not{d}\frac{|\hat{\underline{\chi}}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}\underline{\chi}}.\end{aligned}$$

The first result follows as soon as we switch to the inflation basis $\{L^-, L^+\}$. For the second, we note that for any 1-form η on Ω we have a 2-tensor $\not{\nabla}\eta_s$ on Σ_s. We extend this 2-tensor to all of Ω by extending $\not{\nabla}\eta_s$ trivially to all of $T\Omega|_{\Sigma_s}$ for each s. Assuming $V, W \in E(\Sigma)$ we therefore have,

$$\begin{aligned}\mathcal{L}_{\underline{L}}(\not{\nabla}\eta_s)(V,W) &= \underline{L}((\not{\nabla}_V\eta_s)(W)) = V\underline{L}\eta_s(W) - \underline{L}\eta_s(\not{\nabla}_V W) \\ &= V(\mathcal{L}_{\underline{L}}\eta)_s(W) - (\mathcal{L}_{\underline{L}}\eta)_s(\not{\nabla}_V W) - \eta([\underline{L}, \not{\nabla}_V W]) \\ &= \not{\nabla}_V(\mathcal{L}_{\underline{L}}\eta)_s(W) - \eta([\underline{L}, \not{\nabla}_V W]).\end{aligned}$$

For the inverse metric tensor γ^{-1}, we observe with the aid of a basis extension that $\underline{L}\gamma^{ij} = -\gamma^{ik}(\underline{L}\gamma_{kl})\gamma^{lj}$ so that 3.2 gives $\mathcal{L}_{\underline{L}}\gamma^{-1} = -2\underline{\chi}^{ij}X_i \otimes X_j$ (raising indices with γ^{ij}). Therefore, denoting by C^a_b contraction between the contravariant a-th and covariant b-th slots we find

$$\begin{aligned}\underline{L}(\not{\nabla}\cdot\eta_s) &= C^1_1 C^2_2(\mathcal{L}_{\underline{L}}\gamma^{-1}\otimes\not{\nabla}\eta_s + \gamma^{-1}\otimes\mathcal{L}_{\underline{L}}(\not{\nabla}\eta_s)) \\ &= -2\underline{\chi}\cdot\not{\nabla}\eta_s + \operatorname{tr}(\mathcal{L}_{\underline{L}}\not{\nabla}\eta_s) \\ &= -2(\hat{\underline{\chi}} + \frac{1}{2}\operatorname{tr}\underline{\chi}\gamma)\cdot\not{\nabla}\eta_s + \not{\nabla}\cdot(\mathcal{L}_{\underline{L}}\eta)_s - \eta(2\not{\nabla}\vec{\cdot}\,\hat{\underline{\chi}})\end{aligned}$$

where the last term comes from Lemma 3.2 after taking a trace over V, W. We conclude that

$$\underline{L}(\not{\nabla}\cdot\eta_s) = -\operatorname{tr}\underline{\chi}\not{\nabla}\cdot\eta_s - 2\not{\nabla}\cdot(\hat{\underline{\chi}}\cdot\eta_s) + \not{\nabla}\cdot(\mathcal{L}_{\underline{L}}\eta)_s.$$

The second equality of the corollary now straight forwardly follows from the first by setting $\eta = \tau$. For the final equality, since $\operatorname{tr}\chi = \frac{\langle\vec{H},\vec{H}\rangle}{\sigma}$ we have from

(3.6)

$$\underline{L}\frac{\langle \vec{H}, \vec{H}\rangle}{\sigma} = G(L^-, L^+) + 2\mathcal{K} - 2\nabla\!\!\!/ \cdot \tau - 2\Delta\!\!\!/ \log\sigma - 2|\tau|^2 - 2|d\!\!\!/ \log\sigma|^2$$
$$- 4\tau(\nabla\!\!\!/ \log\sigma) - \langle \vec{H}, \vec{H}\rangle - \kappa\frac{\langle \vec{H}, \vec{H}\rangle}{\sigma}$$

and also from (3.4)

$$\underline{L}\frac{\langle \vec{H}, \vec{H}\rangle}{\sigma} = \frac{1}{\sigma}\underline{L}\langle \vec{H}, \vec{H}\rangle - \frac{1}{\sigma^2}\langle \vec{H}, \vec{H}\rangle\Big(- \frac{1}{2}\sigma^2 - \sigma^2(|\hat{\chi}^-|^2 + G(L^-, L^-)) + \kappa\sigma\Big).$$

The result follows as soon as we combine these formulae and solve for $\underline{L}\langle \vec{H}, \vec{H}\rangle$ making use of the substitution

$$\frac{\Delta\!\!\!/ \sigma}{\sigma} = \Delta\!\!\!/ \log\sigma + |d\!\!\!/ \log\sigma|^2.$$

□

Theorem 3.5 (Propagation of ρ). *Assuming $\{\Sigma_s\}$ is expanding along the flow vector $\underline{L} = \sigma L^-$ we conclude that*

$$\underline{L}\rho + \frac{3}{2}\sigma\rho = \frac{\sigma}{2}\Big(\frac{1}{2}\langle \vec{H}, \vec{H}\rangle\Big(|\hat{\chi}^-|^2 + G(L^-, L^-)\Big) + |\tau|^2 - \frac{1}{2}G(L^-, L^+)\Big)$$
$$+ \Delta\!\!\!/ \Big(\sigma(|\hat{\chi}^-|^2 + G(L^-, L^-))\Big) - 2\nabla\!\!\!/ \cdot (\sigma\hat{\chi}^- \cdot \tau) + \nabla\!\!\!/ \cdot (\sigma G_{L^-})$$

Proof. From Proposition 3.3 and Corollary 3.4 the proof reduces to a simple exercise in algebraic manipulation

$$\begin{aligned}
\underline{L}\rho &= \underline{L}\mathcal{K} - \frac{1}{4}\underline{L}\langle \vec{H}, \vec{H}\rangle + \underline{L}\nabla\!\!\!/ \cdot \tau \\
&= -\sigma\mathcal{K} - \frac{1}{2}\Delta\!\!\!/ \sigma + \nabla\!\!\!/ \cdot \nabla\!\!\!/ \cdot (\sigma\hat{\chi}^-) \\
&\quad + \frac{\sigma}{2}\Big(\frac{1}{2}\langle \vec{H}, \vec{H}\rangle(|\hat{\chi}^-|^2 + G(L^-, L^-)) - \frac{1}{2}G(L^-, L^+) + |\tau|^2\Big) \\
&\quad + \frac{3}{2}\sigma\frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \frac{\sigma}{2}\nabla\!\!\!/ \cdot \tau - \frac{1}{2}\sigma\mathcal{K} + \frac{1}{2}\Delta\!\!\!/ \sigma + \tau(\nabla\!\!\!/ \sigma) \\
&\quad - 2\sigma\nabla\!\!\!/ \cdot \tau + \Delta\!\!\!/ \Big(\sigma(|\hat{\chi}^-|^2 + G(L^-, L^-))\Big) - 2\nabla\!\!\!/ \cdot (\sigma\hat{\chi}^- \cdot \tau) \\
&\quad + \nabla\!\!\!/ \cdot (\sigma G_{L^-}) - \tau(\nabla\!\!\!/ \sigma) - \nabla\!\!\!/ \cdot \nabla\!\!\!/ \cdot (\sigma\hat{\chi}^-)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma}{2}\Big(\frac{1}{2}\langle \vec{H}, \vec{H}\rangle \Big(|\hat{\chi}^-|^2 + G(L^-, L^-)\Big) + |\tau|^2 - \frac{1}{2}G(L^-, L^+)\Big) \\
&\quad + \not{\Delta}\Big(\sigma(|\hat{\chi}^-|^2 + G(L^-, L^-))\Big) - 2\not{\nabla}\cdot(\sigma\hat{\chi}^-\cdot\tau) + \not{\nabla}\cdot(\sigma G_{L^-}) \\
&\quad - \sigma\frac{3}{2}(\mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not{\nabla}\cdot\tau)
\end{aligned}
$$

□

Corollary 3.6. *For $\{\Sigma_s\}$ expanding along the flow vector $\underline{L} = \sigma L^-$ and any $u \in \mathcal{F}(\Sigma_s)$*

$$
\begin{aligned}
\int_{\Sigma_s} e^u\Big(\underline{L}\rho + \frac{3}{2}\sigma\rho\Big)dA &= \int_{\Sigma_s} \sigma e^u\Big(\Big(|\hat{\chi}^-|^2 + G(L^-, L^-)\Big)\Big(\frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not{\Delta}u\Big) \\
&\quad + \frac{1}{2}|2\hat{\chi}^-\cdot \not{d}u + \tau|^2 + G(L^-, |\not{\nabla}u|^2L^- - \not{\nabla}u - \frac{1}{4}L^+)\Big)dA
\end{aligned}
$$

Proof. We start by integrating by parts on the last three terms of Theorem 3.5

$$
\begin{aligned}
&\int e^u\Big(\not{\Delta}(\sigma(|\hat{\chi}^-|^2 + G(L^-, L^-))) - 2\not{\nabla}\cdot(\sigma\hat{\chi}^-\cdot\tau) + \not{\nabla}\cdot(\sigma G_{L^-})\Big)dA \\
&= \int \sigma e^u\Big(e^{-u}(\not{\Delta}e^u)(|\hat{\chi}^-|^2 + G(L^-, L^-)) + 2\hat{\chi}^-(\not{\nabla}u, \vec{\tau}) - G(L^-, \not{\nabla}u)\Big)dA \\
&= \int \sigma e^u\Big((\not{\Delta}u + |\not{\nabla}u|^2)(|\hat{\chi}^-|^2 + G(L^-, L^-)) + 2\hat{\chi}^-(\not{\nabla}u, \vec{\tau}) - G(L^-, \not{\nabla}u)\Big)dA \\
&= \int \sigma e^u\Big((|\hat{\chi}^-|^2 + G(L^-, L^-))\not{\Delta}u + |\hat{\chi}^-|^2|\not{\nabla}u|^2 + 2\underline{\hat{\chi}}(\not{\nabla}u, \vec{\tau}) \\
&\quad + G(L^-, |\not{\nabla}u|^2L^- - \not{\nabla}u)\Big)dA.
\end{aligned}
$$

As a result

$$
\begin{aligned}
\int e^u\Big(\underline{L}\rho + \frac{3}{2}\sigma\rho\Big)dA &= \int \sigma e^u\Big((|\hat{\chi}^-|^2 + G(L^-, L^-))\Big(\frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not{\Delta}u\Big) \\
&\quad + |\hat{\chi}^-|^2|\not{\nabla}u|^2 + 2\hat{\chi}^-(\not{\nabla}u, \vec{\tau}) + \frac{1}{2}|\tau|^2 + G(L^-, |\not{\nabla}u|^2L^- - \not{\nabla}u - \frac{1}{4}L^+)\Big)dA.
\end{aligned}
$$

Since $\hat{\chi}^-$ is symmetric and trace-free it follows that $|\hat{\chi}^-\cdot\not{d}u|^2 = \frac{1}{2}|\hat{\chi}^-|^2|\not{\nabla}u|^2$ from which the first three terms of the second line simplifies to give

$$
|\hat{\chi}^-|^2|\not{\nabla}u|^2 + 2\hat{\chi}^-(\not{\nabla}u, \vec{\tau}) + \frac{1}{2}|\tau|^2 = \frac{1}{2}|2\hat{\chi}^-\cdot\not{d}u + \tau|^2
$$

□

Remark 3.7. An interesting consequence of the above corollary in spacetimes satisfying the dominant energy condition is the fact that any $u \in \mathcal{F}(\Sigma)$ gives

$$\int e^u\Big(\underline{L}\rho + \frac{3}{2}\sigma\rho\Big)dA \geq \int \sigma e^u(|\hat{\underline{\chi}}^-|^2 + G(L^-, L^-))\Big(\frac{1}{4}\langle\vec{H}, \vec{H}\rangle + \Delta\!\!\!\!/\, u\Big)dA$$

The proof of Theorem 1.7 is a simple consequence of the following corollary:

Corollary 3.8. *Assuming $\{\Sigma_s\}$ is expanding along the flow vector $\underline{L} = \sigma L^-$ with each Σ_s of non-zero flux ($|\rho(s)| > 0$) then*

$$\begin{aligned}
\frac{d}{ds}\int_{\Sigma_s}\rho^{\frac{2}{3}}dA &= \int_{\Sigma_s}\underline{L}\rho^{\frac{2}{3}} + \sigma\rho^{\frac{2}{3}}dA \\
&= \frac{2}{3}\int_{\Sigma}\frac{\sigma}{\rho^{\frac{1}{3}}}\Big(\Big(|\hat{\underline{\chi}}^-|^2 + G(L^-, L^-)\Big)\Big(\frac{1}{4}\langle\vec{H}, \vec{H}\rangle - \frac{1}{3}\Delta\!\!\!\!/\,\log|\rho|\Big) \\
&+ \frac{1}{2}\Big|\frac{2}{3}\hat{\underline{\chi}}^- \cdot d\!\!\!/\,\log|\rho| - \tau\Big|^2 + G(L^-, \frac{1}{9}|\nabla\!\!\!\!/\,\log|\rho||^2 L^- \\
&+ \frac{1}{3}\nabla\!\!\!\!/\,\log|\rho| - \frac{1}{4}L^+)\Big)dA
\end{aligned}$$

Proof. From the first variation of Area formula

$$\underline{L}dA = -\langle\vec{H}, \underline{L}\rangle dA = -\sigma\langle\vec{H}, L^-\rangle dA = \sigma dA$$

we get the first equality. For the second we apply Corollary 3.6 with $e^u = \frac{2}{3}|\rho|^{-\frac{1}{3}}$, canceling the sign in the case that $\rho < 0$. $\square$

3.3. Case of equality

Theorem 3.9. *Let Ω be a null hypersurface in a spacetime satisfying the dominant energy condition with vector field $\underline{L}$ tangent to the null generators of Ω. Suppose $\{\Sigma_s\}$ is an expanding and strict doubly convex foliation defined as the level sets of a function $s : \Omega \to \mathbb{R}$ satisfying $\underline{L}(s) = 1$ and $\frac{dm}{ds} = 0$. Then all foliations achieve equality, moreover, we find an affine level set function $r \in \mathcal{F}(\Omega)$ with $r_0 := r|_{\Sigma_{s_0}} \circ \pi$ such that any surface $\Sigma := \{r = \omega \circ \pi\}$, for $\omega \in \mathcal{F}(\Sigma_{s_0})$, has data:*

$$\begin{aligned}
&\gamma = \omega^2\gamma_0, && \underline{\chi} = \omega\gamma_0, && \operatorname{tr}\underline{\chi} = \frac{2}{\omega}, \\
&\operatorname{tr}\chi = \frac{2}{\omega}(\mathcal{K}_0 - \frac{r_0}{\omega} - \omega^2\Delta\!\!\!\!/\,\log\omega), && \zeta = -d\!\!\!/\,\log\omega, && \rho = \frac{r_0}{\omega^3}.
\end{aligned}$$

where $r_0^2\gamma_0$ is the metric on Σ_{s_0} and $\mathcal{K}_0$ the Gaussian curvature associated to γ_0. In case $\operatorname{tr}\chi|_{\Sigma_{s_0}} = 0$ our data corresponds with the standard null cone in Schwarzschild spacetime of mass $M = \frac{r_0}{2}$.

Proof. Without loss of generality we assume $s_0 = 0$. Immediately from Corollary 3.8 we conclude for this particular foliation that

$$\begin{aligned} |\hat{\chi}^-|^2 + G(L^-, L^-) &= 0 \\ |\frac{2}{3}\hat{\chi}^- \cdot \not{d} \log\rho - \tau|^2 &= 0 \\ G(L^-, \frac{1}{9}|\nabla\!\!\!/ \log\rho|^2 L^- + \frac{1}{3}\nabla\!\!\!/ \log\rho - \frac{1}{4}L^+) &= 0. \end{aligned}$$

So from the first equality we have both $\hat{\chi}^- = 0$ and $G(L^-, L^-) = 0$. Combined with the second equality we conclude that $\tau = 0$ for this particular foliation and therefore Corollary 3.4 ensures that $G_{L^-} = 0$ as well. Finally, we may therefore utilize the final equality to conclude also that $G(L^+, L^-) = 0$ so that, for any $p \in \Omega$ and any $X \in T_pM$, we have

$$G(L^-, X) = 0.$$

From this and Corollary 3.4 we have for any foliation off of Σ_0 generated by some $\underline{L}_a$ ($a > 0$) that

$$\mathcal{L}_{\underline{L}_a}\tau^a + a\sigma\tau^a = 0.$$

Given that $\tau^a|_{\Sigma_0} = \tau|_{\Sigma_0} = 0$ this enforces $\tau^a = 0$ by standard uniqueness theorems.

We recognise this implies the case of equality for all foliations so without loss of generality we assume that $\underline{L}$ is geodesic. We are now in a position to show that the flux $\rho \in \mathcal{F}(\Omega)$ is independent of the foliation from which it is constructed. In particular, for any $a > 0$, foliating off of Σ_0 along the vector field $\underline{L}_a$ will construct a ρ_a which we would like to show agrees pointwise on Ω with ρ.

From Theorem 3.5 we have

$$\underline{L}\rho = -\frac{3}{2}\operatorname{tr}\underline{\chi}\rho = 3\rho\underline{L}\log\operatorname{tr}\underline{\chi}$$

so for any $p \in \Omega$, solving this ODE along the geodesic $\gamma^{\underline{L}}_{\pi(p)}(s)$ gives

$$\frac{\rho \circ s(p)}{\rho(0)} = \Big(\frac{\operatorname{tr}\underline{\chi}(p)}{\operatorname{tr}\underline{\chi}(0)}\Big)^3.$$

For the vector field $\underline{L}_a$, Theorem 3.5 gives

$$\begin{aligned}\underline{L}_a\rho_a = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\rho_a &= 3\rho_a(\underline{L}_a\log\operatorname{tr}\underline{\chi}_a - \kappa_a)\\ &= 3\rho_a\underline{L}_a(\log\operatorname{tr}\underline{\chi}_a - \log a)\\ &= 3\rho_a\underline{L}_a(\log\operatorname{tr}\underline{\chi})\end{aligned}$$

where the penultimate line comes from the fact that $\kappa_a\underline{L}_a = D_{\underline{L}_a}\underline{L}_a = a\underline{L}(a)\underline{L} = \underline{L}_a(\log a)\underline{L}_a$, and the final line from the fact that $\operatorname{tr}\underline{\chi}_a = a\operatorname{tr}\underline{\chi}$. Solving this ODE along the pregeodesic $\gamma^{\underline{L}_a}_{\pi(p)}(t)$, we have

$$\frac{\rho_a\circ t(p)}{\rho_a(0)} = \Big(\frac{\operatorname{tr}\underline{\chi}(p)}{\operatorname{tr}\underline{\chi}(0)}\Big)^3 = \frac{\rho\circ s(p)}{\rho(0)}.$$

Since we're foliating off of Σ_0 in both cases, and $\rho|_{\Sigma_0}$ is independent of our choice of null basis, we have $\rho(p) = \rho_a(p)$ as desired.
We therefore define the functions r_0 and r according to

$$\frac{1}{r_0^2} = \rho|_{\Sigma_{s_0}},\quad \frac{r_0\circ\pi}{r^3} = \rho$$

(i.e. $r|_{\Sigma_0} = r_0$) so that Theorem 3.5 gives $-3\frac{r_0\circ\pi}{r^4}\underline{L}_a(r) = \underline{L}_a(\rho) = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\rho = -\frac{3}{2}\operatorname{tr}\underline{\chi}_a\frac{r_0\circ\pi}{r^3}$ and therefore $\underline{L}_a(r) = \frac{1}{2}\operatorname{tr}\underline{\chi}_a r$. It follows that if we scale $\underline{L}$ such that $\operatorname{tr}\underline{\chi}|_{\Sigma_0} = \frac{2}{r_0}$ then $\underline{L}(\operatorname{tr}\underline{\chi}r) = -\frac{1}{2}(\operatorname{tr}\underline{\chi})^2r + \operatorname{tr}\underline{\chi}(\frac{1}{2}\operatorname{tr}\underline{\chi}r) = 0$ implies that $\operatorname{tr}\underline{\chi} = \frac{2}{r}$ and $\underline{L}(r) = 1$. So r is in fact our level set function. For $r_0^2\gamma_0$ the metric on Σ_0, by Lie dragging γ_0 along $\underline{L}$ to all of Ω we have

$$\mathcal{L}_{\underline{L}}(r^2\gamma_0) = 2r\gamma_0 = \frac{2}{r}(r^2\gamma_0) = \operatorname{tr}\underline{\chi}(r^2\gamma_0).$$

So from (3.2), $\mathcal{L}_{\underline{L}}(r^2\gamma_0 - \gamma) = \operatorname{tr}\underline{\chi}(r^2\gamma_0 - \gamma)$ and $r_0^2\gamma_0 - \gamma(r_0) = 0$ giving $\gamma(r) = r^2\gamma_0$ by uniqueness. We conclude that for any $0 \le \omega \in \mathcal{F}(\Sigma_0)$ the cross-section $\Sigma := \{r = \omega\circ\pi\}$ has metric $\gamma_\omega = \gamma(r)|_\Sigma = \omega^2\gamma_0$ with Gaussian curvature $\mathcal{K}_\omega = \frac{1}{\omega^2}\mathcal{K}_0 - \Delta\!\!\!/\log\omega$. Moreover,

$$\begin{aligned}\frac{r_0\circ\pi}{\omega^3} = \rho_\omega &= \mathcal{K}_\omega - \frac{1}{4}\langle\vec{H},\vec{H}\rangle\\ &= \frac{1}{\omega^2}\mathcal{K}_0 - \Delta\!\!\!/\log\omega - \frac{1}{2\omega}\operatorname{tr}\chi_\omega\end{aligned}$$

having used the fact that $\rho_\omega = \rho|_\Sigma$ (from independence of foliation) in the first line and $\operatorname{tr}\underline{\chi}_\omega = \operatorname{tr}\underline{\chi}|_\Sigma$ in the second. We conclude that,

$$\operatorname{tr}\chi_\omega = \frac{2}{\omega}(\mathcal{K}_0 - \frac{r_0 \circ \pi}{\omega} - \omega^2 \Delta\!\!\!/ \log\omega).$$

In the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ the strict doubly convex condition forces $\frac{1}{r_0^2} = \rho|_{\Sigma_0}$ to be constant by way of the maximum principle. From our expression for $\operatorname{tr}\chi_{r_0}$ we conclude that $\mathcal{K}_0 = 1$ and therefore γ_0 is a round metric on $\mathbb{S}^2$. $\square$

Remark 3.10. We bring to the attention of the reader that due the lack of information regarding the term $\hat{G}$ in (3.5) we are unable to conclude with any knowledge of the datum χ on Σ. In the case of vacuum, this no longer poses a problem, and one is able to correlate $\chi|_\Sigma$ with $\chi|_{\Sigma_{r_0}}$ as shown by Sauter ([32], Lemma 4.3).

4. Foliation comparison

In this section we show how the flux function ρ of an arbitrary cross section of Ω decomposes in terms of the flux of the background foliation. With the appropriate asymptotic decay on Ω this allows us to prove Theorem 1.9. We will need to adapt the Gauss and Codazzi equations (see, for example [25]) to the normal basis $\{\underline{L}, L\}$:

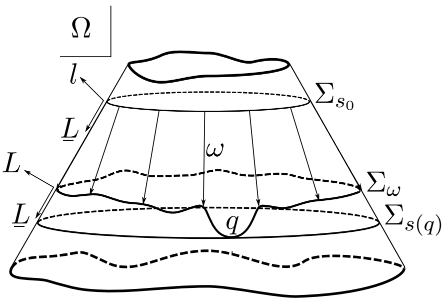
Proposition 4.1. *Suppose Σ is a co-dimension 2 semi-Riemannian submanifold of $\mathcal{M}^{n+1}$ that locally admits a normal null basis $\{\underline{L}, L\}$, $\langle\underline{L}, L\rangle = 2$. Then,*

(4.1)
$$(n-1)\mathcal{K} - \frac{n-2}{n-1}\langle\vec{H},\vec{H}\rangle + \hat{\underline{\chi}}\cdot\hat{\chi} = -R - 2G(\underline{L}, L) - \frac{1}{2}\langle R_{\underline{L}L}\underline{L}, L\rangle$$
(4.2)
$$\nabla\!\!\!/ \cdot \hat{\underline{\chi}}(V) - \hat{\underline{\chi}}(V,\vec{\zeta}) + \frac{n-2}{n-1}\operatorname{tr}\underline{\chi}\zeta(V) - \frac{n-2}{n-1}V tr\underline{\chi} = G(V,\underline{L}) - \frac{1}{2}\langle R_{\underline{L}V}L, \underline{L}\rangle$$

for $V \in \Gamma(T\Sigma)$ and $(n-1)\mathcal{K}$ the scalar curvature of Σ.

4.1. Additional setup

We follow once again the construction of [23] starting with a background foliation as constructed in Section 3 off of an initial cross-section Σ_{s_0}. As before, each Σ_s allows a null basis $\{\underline{L}, l\}$ such that $\langle \underline{L}, l\rangle = 2$. Also from section 3 we have the diffeomorphism $p \mapsto (\pi(p), s(p))$ of Ω onto its image. Therefore any cross-section with associated embedding $\Phi : \mathbb{S}^2 \to \Omega$ is equivalently realized with the map $\tilde{\Phi} = (\pi, s) \circ \Phi$. Expressing the component functions $\Psi := \pi \circ \Phi$ and $\omega := s \circ \Phi$ we recognize that $\Psi : \mathbb{S}^2 \to \Sigma_{s_0}$ is a diffeomorphism and therefore the embedding $\Phi : \mathbb{S}^2 \to \Omega$ is uniquely characterized as a graph over Σ_{s_0} with graph function $\omega \circ \Psi^{-1}$. Without confusion we will simply denote the graph function by ω and it's associated cross section by Σ_ω.



We wish to compare both the intrinsic and extrinsic geometry of Σ_ω at a point q with the geometry of the surface $\Sigma_{s(q)}$. We extend ω to all of Ω in the usual way by imposing it be constant along generators of $\underline{L}$, in other words, $\omega(p) := (\omega \circ \pi)(p)$. For the extrinsic geometry of Σ_ω we have the null-normal basis $\{\underline{L}, L\}$ whereby L is given by the conditions $\langle \underline{L}, L\rangle = 2$ and $\langle V, L\rangle = 0$ for any $V \in \Gamma(T\Sigma_\omega)$. As before, Σ_ω has second fundamental form decomposing into the null components $\underline{\chi}$ (associated to $\underline{L}$) and χ (associated to L) with torsion ζ. For each Σ_s we equivalently decompose the second fundamental form into the components K (associated to $\underline{L}$) and Q (associated to l) with torsion t. We will denote the induced covariant derivative on Σ_s by ∇ and on Σ_ω by $\nabla\!\!\!\!/$. The following lemma is known ([23],[32]):

Lemma 4.2. *Given $q \in \Sigma_\omega \cap \Sigma_{s(q)}$ the map given by*

$$T_\omega : T_q\Sigma_{s(q)} \to T_q\Sigma_\omega \quad \textit{where} \quad v \to \tilde{v} := v + v\omega\underline{L}$$

is a well defined isomorphism with natural extension $E(\Sigma_{s_0}) \to E(\Sigma_\omega)$. Moreover,

$$\begin{gathered}\gamma_\omega(\tilde{V}, \tilde{W}) = \gamma_s(V, W),\ \underline{\chi}(\tilde{V}, \tilde{W}) = K(V, W),\\ \zeta(\tilde{V}) = t(V) - K(V, \nabla\omega) + \kappa\langle V, \nabla\omega\rangle,\end{gathered}$$

$$\begin{aligned}\chi(\tilde{V},\tilde{W}) &= Q(V,W) - 2t(V)\langle W,\nabla\omega\rangle - 2t(W)\langle V,\nabla\omega\rangle - |\nabla\omega|^2 K(V,W)\\ &\quad - 2H^\omega(V,W) + 2K(V,\nabla\omega)\langle W,\nabla\omega\rangle + 2K(W,\nabla\omega)\langle V,\nabla\omega\rangle\\ &\quad - 2\kappa\langle V,\nabla\omega\rangle\langle W,\nabla\omega\rangle,\\ \operatorname{tr}\chi &= \operatorname{tr}Q - 4t(\nabla\omega) - 2(\Delta\omega - 2\hat{K}(\nabla\omega,\nabla\omega)) + \operatorname{tr}K|\nabla\omega|^2 - 2\kappa|\nabla\omega|^2.\end{aligned}$$

From Lemma 4.2, it follows that

$$L = l - |\nabla\omega|^2\underline{L} - 2\nabla\omega$$

since it is null, perpendicular to $T_\omega(v)$ for any $v \in T_q\Sigma_{s(q)}$, and satisfies $\langle \underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega\rangle = \langle \underline{L}, l\rangle = 2$. We are now ready to prove our first main result of this section. On Σ_ω we will denote the flux function (1.1) by $\not{\rho}$, and on Σ_s, by ρ. The following theorem provides comparison between the two:

Theorem 4.3 (Flux Comparison Theorem). *At any $q \in \Sigma_\omega \cap \Sigma_s$ we have*

$$\begin{aligned}\not{\rho} &= \rho + \not{\nabla}\cdot\Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}\not{\nabla}\omega\Big) + \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big)|\nabla\omega|^2\\ &\quad + \nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + G(\underline{L},\nabla\omega) - 2\hat{K}(\vec{t} - \nabla\log\operatorname{tr}K, \nabla\omega)\end{aligned}$$

Remark 4.4. Revisiting Theorem 3.9 and the case that $\hat{\underline{\chi}} = G(\underline{L},\cdot) = 0$, Theorem 4.3 provides an alternative proof that $\not{\rho}$ agrees with ρ point wise.

Proof. When used, we assume $V, W, U \in E(\Sigma_{s_0})(\implies \tilde{V}, \tilde{W}, \tilde{U} \in E(\Sigma_\omega))$. We will need to know how to relate the covariant derivatives between the two surfaces so first a lemma

Lemma 4.5. $T_\omega\Big(\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega\Big) = \not{\nabla}_{\tilde{V}}\tilde{W}$

Proof. Since $\not{\nabla}_{\tilde{V}}\tilde{W}|_q = (S + S\omega\underline{L})|_q = T_\omega(S|_q)$ for some $S \in \Gamma(T\Sigma_{s(q)})$ it follows that $\langle\not{\nabla}_{\tilde{V}}\tilde{W}, U\rangle = \langle S, U\rangle$ for any $U \in E(\Sigma_{s_0})$. We find

$$\begin{aligned}\langle\not{\nabla}_{\tilde{V}}\tilde{W}, U\rangle &= \langle D_{\tilde{V}}\tilde{W} + \frac{1}{2}\underline{\chi}(\tilde{V},\tilde{W})L + \frac{1}{2}\chi(\tilde{V},\tilde{W})\underline{L}, U\rangle\\ &= \langle D_{\tilde{V}}\tilde{W}, U\rangle + \frac{1}{2}K(V,W)\langle L, U\rangle\\ &= \tilde{V}\langle W, U\rangle - \langle\tilde{W}, D_{\tilde{V}}U\rangle + \frac{1}{2}K(V,W)\langle l - |\nabla\omega|^2\underline{L} - 2\nabla\omega, U\rangle\end{aligned}$$

$$
\begin{aligned}
&= (V + V\omega \underline{L})\langle W, U\rangle - \langle W + W\omega \underline{L}, D_{V+V\omega \underline{L}}U\rangle - K(V,W)U\omega \\
&= V\langle W, U\rangle + 2V\omega K(W,U) - \Big(\langle W, \nabla_V U\rangle + V\omega K(W,U) - W\omega K(V,U)\Big) \\
&\quad - K(V,W)U\omega \\
&= \Big(V\langle W, U\rangle - \langle W, \nabla_V U\rangle\Big) + K(W,U)V\omega + K(V,U)W\omega - K(V,W)U\omega \\
&= \langle \nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V,W)\nabla\omega, U\rangle
\end{aligned}
$$

so $S = \nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V,W)\nabla\omega$ since $E(\Sigma_{s_0})|_{\Sigma_{s(q)}} = \Gamma(T\Sigma_{s(q)})$. □

Now we proceed with the proof of Theorem 4.3 in 3 parts:

STEP 1: *Comparison between $\nabla\!\!\!/ \cdot \zeta$ and $\nabla \cdot t$:*

From Lemmas 4.2 and 4.5 we have

$$
\begin{aligned}
(\nabla\!\!\!/_{\tilde{V}}\zeta)(\tilde{W}) &= \tilde{V}(\zeta(\tilde{W})) - \zeta(\nabla\!\!\!/_{\tilde{V}}\tilde{W}) \\
&= (V + V\omega \underline{L})\Big(t(W) - K(W, \nabla\omega) + \kappa\langle W, \nabla\omega\rangle\Big) \\
&\quad - t\Big(\nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V,W)\nabla\omega\Big) \\
&\quad + K\Big(\nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V,W)\nabla\omega, \nabla\omega\Big) \\
&\quad - \kappa\langle \nabla_V W + V\omega \vec{K}(W) + W\omega \vec{K}(V) - K(V,W)\nabla\omega, \nabla\omega\rangle.
\end{aligned}
$$

Isolating the terms of the second line we get

$$
\begin{aligned}
&(V + V\omega \underline{L})(t(W) - K(W, \nabla\omega) + \kappa W\omega) \\
&\quad = Vt(W) + V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot \hat{K}(W) - \operatorname{tr} K t(W) + \frac{1}{2}W \operatorname{tr} K + W\kappa\Big) \\
&\qquad - VK(W, \nabla\omega) - V\omega(\mathcal{L}_{\underline{L}}K)(W, \nabla\omega) - V\omega K(W, [\underline{L}, \nabla\omega]) \\
&\qquad + V\kappa W\omega + \kappa V W\omega + V\omega \underline{L}\kappa W\omega
\end{aligned}
$$

where (3.7) was used to give the first line. To continue we'll need an expression for $[\underline{L}, \nabla\omega]$, so we use (3.2):

$$
\begin{aligned}
2K(\nabla\omega, V) = (\mathcal{L}_{\underline{L}}\gamma_s)(\nabla\omega, V) &= \underline{L}\langle \nabla\omega, V\rangle - \langle [\underline{L}, \nabla\omega], V\rangle \\
&= \underline{L}V\omega - \langle [\underline{L}, \nabla\omega], V\rangle = -\langle [\underline{L}, \nabla\omega], V\rangle
\end{aligned}
$$

since $[\underline{L}, \nabla\omega] \in \Gamma(T\Sigma_s)$, we conclude that $[\underline{L}, \nabla\omega] = -2\vec{K}(\nabla\omega)$. Substitution back into our calculation and using (3.3) (specifically $\mathcal{L}_{\underline{L}}K(V,W) =$

$-\underline{\alpha}(V,W) + \langle \vec{K}(V), \vec{K}(W)\rangle + \kappa K(V,W))$ gives

$$\begin{aligned}(V + V\omega \underline{L})(t(W) - K(W,\nabla\omega) + \kappa W\omega) &= Vt(W) - VK(W,\nabla\omega)\\ &+ V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot\hat{K}(W) - \operatorname{tr} K t(W) + \frac{1}{2}W\operatorname{tr}K + \underline{\alpha}(W,\nabla\omega)\\ &\qquad + \langle \vec{K}(W), \vec{K}(\nabla\omega)\rangle\Big)\\ &+ V\omega W\kappa - \kappa V\omega K(W,\nabla\omega) + V\kappa W\omega + \kappa VW\omega + \underline{L}\kappa V\omega W\omega.\end{aligned}$$

Collecting terms we get

$$\begin{aligned}(\not{\nabla}_{\tilde{V}}\zeta)(\tilde{W}) &= Vt(W) - t(\nabla_V W) + K(\nabla_V W,\nabla\omega) - VK(W,\nabla\omega)\\ &+ V\omega\Big(G_{\underline{L}}(W) - \nabla\cdot\hat{K}(W) - \operatorname{tr}Kt(W) + \frac{1}{2}W\operatorname{tr}K + \underline{\alpha}(W,\nabla\omega)\\ &\quad + \langle\vec{K}(W),\vec{K}(\nabla\omega)\rangle\Big)\\ &- V\omega K(W,\vec{t}) - W\omega K(V,\vec{t}) + K(V,W)t(\nabla\omega)\\ &+ V\omega\langle\vec{K}(W),\vec{K}(\nabla\omega)\rangle + W\omega\langle\vec{K}(V),\vec{K}(\nabla\omega)\rangle - K(V,W)K(\nabla\omega,\nabla\omega)\\ &+ V\omega W\kappa - \kappa V\omega K(W,\nabla\omega) + V\kappa W\omega + \kappa VW\omega + \underline{L}\kappa V\omega W\omega\\ &- \kappa\nabla_V W\omega - \kappa V\omega K(W,\nabla\omega) - \kappa W\omega K(V,\nabla\omega) + \kappa K(V,W)|\nabla\omega|^2.\end{aligned}$$

So taking a trace over V and W

$$\begin{aligned}\not{\nabla}\cdot\zeta &= \nabla\cdot t - \nabla\cdot(\vec{K}(\nabla\omega))\\ &+ \Big(G_{\underline{L}}(\nabla\omega) - (\nabla\cdot\hat{K})(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega)\\ &\qquad\qquad + \frac{1}{2}\nabla\omega\operatorname{tr}K + \underline{\alpha}(\nabla\omega,\nabla\omega) + |\vec{K}(\nabla\omega)|^2\Big)\\ &- 2K(\nabla\omega,\vec{t}) + \operatorname{tr}Kt(\nabla\omega) + 2|\vec{K}(\nabla\omega)|^2 - \operatorname{tr}KK(\nabla\omega,\nabla\omega)\\ &+ 2\nabla\omega\kappa - 3\kappa K(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 + \kappa\operatorname{tr}K|\nabla\omega|^2\\ &= \nabla\cdot t - \Big(\nabla\cdot(\vec{K}(\nabla\omega)) + (\nabla\cdot\hat{K})(\nabla\omega) - \frac{1}{2}\nabla\omega\operatorname{tr}K\Big)\\ &\qquad\qquad - 2\Big(K(\nabla\omega,\vec{t}) - \frac{1}{2}\operatorname{tr}Kt(\nabla\omega)\Big)\\ &+ 3|\vec{K}(\nabla\omega)|^2 - \operatorname{tr}KK(\nabla\omega,\nabla\omega) + G_{\underline{L}}(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega) + \underline{\alpha}(\nabla\omega,\nabla\omega)\\ &+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa\operatorname{tr}K|\nabla\omega|^2\\ &= \nabla\cdot t - \Big(2(\nabla\cdot\hat{K})(\nabla\omega) + H^\omega\cdot K\Big) - 2\hat{K}(\nabla\omega,\vec{t}) + 3|\vec{K}(\nabla\omega)|^2\\ &- \operatorname{tr}KK(\nabla\omega,\nabla\omega) + G_{\underline{L}}(\nabla\omega) - \operatorname{tr}Kt(\nabla\omega) + \underline{\alpha}(\nabla\omega,\nabla\omega)\end{aligned}$$

$$
\begin{aligned}
&+ 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2\\
&= \nabla\cdot t - 2(\nabla\cdot\hat{K})(\nabla\omega) - H^\omega\cdot\hat{K} - \frac{1}{2}\operatorname{tr} K\Delta\omega - 2\hat{K}(\nabla\omega,\vec{t}) + \frac{3}{2}|\hat{K}|^2|\nabla\omega|^2\\
&\quad + 2\operatorname{tr} K\hat{K}(\nabla\omega,\nabla\omega) + \frac{1}{4}(\operatorname{tr} K)^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) - \operatorname{tr} Kt(\nabla\omega)\\
&+ \underline{\alpha}(\nabla\omega,\nabla\omega) + 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega,\nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa\operatorname{tr} K|\nabla\omega|^2.
\end{aligned}
$$

STEP 2: *Comparison between* $\nabla\!\!\!/\cdot\zeta - \Delta\!\!\!/\log\operatorname{tr}\underline{\chi}$ *and* $\nabla\cdot t - \Delta\log\operatorname{tr} K$:

Since $\operatorname{tr}\underline{\chi} = \operatorname{tr} K|_{\Sigma_\omega}$ we start by comparing $\Delta\!\!\!/\log\operatorname{tr} K$ with $\Delta\log\operatorname{tr} K$

$$
H^{\log\operatorname{tr}\underline{\chi}}(\tilde{V},\tilde{W}) = \langle\nabla\!\!\!/_{\tilde{V}}\nabla\!\!\!/\log\operatorname{tr} K,\tilde{W}\rangle = \tilde{V}\tilde{W}\log\operatorname{tr} K - \nabla\!\!\!/_{\tilde{V}}\tilde{W}\log\operatorname{tr} K
$$

So isolating the first term we get

$$
\begin{aligned}
\tilde{V}\tilde{W}\log\operatorname{tr} K &= (V + V\omega\underline{L})(W + W\omega\underline{L})\log\operatorname{tr} K\\
&= VW\log\operatorname{tr} K + (VW\omega + V\omega W + W\omega V)\underline{L}\log\operatorname{tr} K\\
&\quad + V\omega W\omega\underline{L}\underline{L}\log\operatorname{tr} K
\end{aligned}
$$

and then the second

$$
\begin{aligned}
\nabla\!\!\!/_{\tilde{V}}\tilde{W}\log\operatorname{tr} K &= (\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega)\log\operatorname{tr} K\\
&\quad + (\nabla_V W + V\omega\vec{K}(W) + W\omega\vec{K}(V) - K(V,W)\nabla\omega)\omega\underline{L}\log\operatorname{tr} K
\end{aligned}
$$

having used Lemma 4.5. Collecting terms

$$
\begin{aligned}
&H^{\log\operatorname{tr} K}(\tilde{V},\tilde{W}) = VW\log\operatorname{tr} K - \nabla_V W\log\operatorname{tr} K\\
&\quad + (VW\omega - \nabla_V W\omega)\underline{L}\log\operatorname{tr} K + V\omega W\omega\underline{L}\underline{L}\log\operatorname{tr} K\\
&\quad - \Big(V\omega K(W,\nabla\log\operatorname{tr} K) + W\omega K(V,\nabla\log\operatorname{tr} K)\\
&\qquad\qquad - K(V,W)\langle\nabla\omega,\nabla\log\operatorname{tr} K\rangle\Big)\\
&\quad + \Big(K(V,W)|\nabla\omega|^2 - V\omega K(W,\nabla\omega) - W\omega K(V,\nabla\omega)\\
&\qquad\qquad + V\omega W + W\omega V\Big)\underline{L}\log\operatorname{tr} K.
\end{aligned}
$$

So that a trace over V and W yields

$$
\begin{aligned}
\Delta\!\!\!/\log\operatorname{tr} K &= \Delta\log\operatorname{tr} K + \Delta\omega\underline{L}\log\operatorname{tr} K + |\nabla\omega|^2\underline{L}\underline{L}\log\operatorname{tr} K\\
&\quad - 2\hat{K}(\nabla\omega,\nabla\log\operatorname{tr} K) - 2\hat{K}(\nabla\omega,\nabla\omega)\underline{L}\log\operatorname{tr} K + 2\nabla\omega\underline{L}\log\operatorname{tr} K.
\end{aligned}
$$

We take the opportunity at this point of the calculation to bring to the attention of the reader that we have not yet used any distinguishing characteristics of the function $\log \operatorname{tr} K$ in comparison to an arbitrary $f \in \mathcal{F}(\Omega)$. In particular, we notice if $f \in \mathcal{F}(\Omega)$ satisfies $\underline{L} f = 0$ switching with $\log \operatorname{tr} K$ above yields the fact $\not{\Delta} f = \Delta f - 2\hat{K}(\nabla\omega, \nabla f)$. As a result,

Lemma 4.6.

$$\not{\Delta} g = \Delta g + \not{\nabla} \cdot (\underline{L} g \not{\nabla}\omega) + \nabla\omega \underline{L} g - 2\hat{K}(\nabla\omega, \nabla g)$$

for any $g \in \mathcal{F}(\Omega)$.

Proof. We have

$$\begin{aligned}
\not{\Delta} g &= \Delta g + \Delta\omega \underline{L} g + |\nabla\omega|^2 \underline{L}\underline{L} g - 2\hat{K}(\nabla\omega, \nabla g) - 2\hat{K}(\nabla\omega, \nabla\omega)\underline{L} g + 2\nabla\omega \underline{L} g \\
&= \Delta g + (\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega))\underline{L} g + (\nabla\omega + |\nabla\omega|^2 \underline{L})\underline{L} g \\
&\quad + \nabla\omega \underline{L} g - 2\hat{K}(\nabla\omega, \nabla g) \\
&= \Delta g + \not{\Delta}\omega \underline{L} g + \not{\nabla}\omega \underline{L} g + \nabla\omega \underline{L} g - 2\hat{K}(\nabla\omega, \nabla g) \\
&= \Delta g + \not{\nabla} \cdot (\underline{L} g \not{\nabla}\omega) + \nabla\omega \underline{L} g - 2\hat{K}(\nabla\omega, \nabla g)
\end{aligned}$$

having used the fact that $\underline{L}\omega = 0$ and the comment immediately preceding the statement of Lemma 4.6 to get the third equality. □

Finishing up Step 2 we have

$$\begin{aligned}
&\not{\nabla} \cdot \zeta - \not{\Delta} \log \operatorname{tr} \underline{\chi} = \nabla \cdot t - \Delta \log \operatorname{tr} K \\
&\quad - 2(\nabla \cdot \hat{K})(\nabla\omega) - H^\omega \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K) - \operatorname{tr} K t(\nabla\omega) \\
&\qquad + \nabla\omega \operatorname{tr} K + \frac{3}{2}|\hat{K}|^2 |\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \underline{\alpha}(\nabla\omega, \nabla\omega) \\
&\quad - \Big(\frac{1}{2} \operatorname{tr} K \Delta\omega + \Delta\omega \underline{L} \log \operatorname{tr} K\Big) + \Big(\frac{1}{4}(\operatorname{tr} K)^2 - \underline{L}\underline{L} \log \operatorname{tr} K\Big)|\nabla\omega|^2 \\
&\qquad - \Big(\nabla\omega \operatorname{tr} K + 2\nabla\omega \underline{L} \log \operatorname{tr} K\Big) + 2\hat{K}(\nabla\omega, \nabla\omega)\Big(\operatorname{tr} K + \underline{L} \log \operatorname{tr} K\Big) \\
&\quad + 2\nabla\omega\kappa - 3\kappa\hat{K}(\nabla\omega, \nabla\omega) + \kappa\Delta\omega + \underline{L}\kappa|\nabla\omega|^2 - \frac{1}{2}\kappa \operatorname{tr} K |\nabla\omega|^2 \\
&= \nabla \cdot t - \Delta \log \operatorname{tr} K \\
&\quad - 2(\nabla \cdot \hat{K})(\nabla\omega) - H^\omega \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K) - \operatorname{tr} K t(\nabla\omega) \\
&\quad + \nabla\omega \operatorname{tr} K + |\hat{K}|^2 |\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \hat{\underline{\alpha}}(\nabla\omega, \nabla\omega) + \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L}, \underline{L})\Big)|\nabla\omega|^2 \\
&\quad + \Big(\Delta\omega - 2\hat{K}(\nabla\omega, \nabla\omega)\Big)\frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \Big(-\frac{1}{2}(|\hat{K}|^2 + G(\underline{L},\underline{L}) - \kappa \operatorname{tr} K) + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K}\Big)|\nabla\omega|^2 \\
&\qquad\quad + 2\nabla\omega \frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K} + \operatorname{tr} K \hat{K}(\nabla\omega,\nabla\omega) \\
&\quad - \kappa\hat{K}(\nabla\omega,\nabla\omega) - \frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2 \\
&= \nabla\cdot t - \Delta \log \operatorname{tr} K \\
&\quad - 2(\nabla\cdot\hat{K})(\nabla\omega) - H^\omega \cdot \hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla \log \operatorname{tr} K) - \operatorname{tr} K t(\nabla\omega) \\
&\qquad + \nabla\omega \operatorname{tr} K + |\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \underline{\hat{\alpha}}(\nabla\omega,\nabla\omega) \\
&\quad + \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K}|\nabla\omega|^2 \\
&\quad + 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr} K} + \operatorname{tr} K\hat{K}(\nabla\omega,\nabla\omega) - \kappa\hat{K}(\nabla\omega,\nabla\omega)
\end{aligned}
$$

having used (3.4) to get the last two lines in the second equality, Lemma 4.6 to get $\Delta\omega - 2\hat{K}(\nabla\omega,\nabla\omega) = \not{\Delta}\omega$ in the second equality followed by cancellation of the terms $\frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big)|\nabla\omega|^2$ and $\frac{1}{2}\kappa \operatorname{tr} K|\nabla\omega|^2$.

STEP 3: *Comparison between* $\not{\rho}$ *and* ρ:

If Σ_s has Gauss curvature $\mathcal{C}$ and mean curvature vector $\vec{h}$, we have from (4.1)

$$
\begin{aligned}
&\mathcal{K} - \frac{1}{4}\langle\vec{H},\vec{H}\rangle + \frac{1}{2}\hat{\underline{\chi}}\cdot\hat{\chi} = -\frac{1}{2}R - G(\underline{L},L) - \frac{1}{4}\langle R_{\underline{L}L}\underline{L},L\rangle \\
&\quad = -\frac{1}{2}R - G(\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega) \\
&\qquad\qquad\qquad\qquad - \frac{1}{4}\langle R_{\underline{L}\,l-|\nabla\omega|^2\underline{L}-2\nabla\omega}\underline{L}, l - |\nabla\omega|^2\underline{L} - 2\nabla\omega\rangle \\
&\quad = \mathcal{C} - \frac{1}{4}\langle\vec{h},\vec{h}\rangle + \frac{1}{2}\hat{K}\cdot\hat{Q} + |\nabla\omega|^2 G(\underline{L},\underline{L}) \\
&\qquad\qquad\qquad\qquad + 2G(\underline{L},\nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l,\underline{L}\rangle - \langle R_{\underline{L}\,\nabla\omega}\underline{L},\nabla\omega\rangle \\
&\quad = \mathcal{C} - \frac{1}{4}\langle\vec{h},\vec{h}\rangle + \frac{1}{2}\hat{K}\cdot\hat{Q} + \frac{1}{2}|\nabla\omega|^2 G(\underline{L},\underline{L}) \\
&\qquad\qquad\qquad\qquad + \Big(2G(\underline{L},\nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l,\underline{L}\rangle\Big) - \underline{\hat{\alpha}}(\nabla\omega,\nabla\omega)
\end{aligned}
$$

from this we conclude

$$
\Big(\mathcal{K} - \frac{1}{4}\langle\vec{H},\vec{H}\rangle + \not{\nabla}\cdot\zeta - \not{\Delta}\log \operatorname{tr}\underline{\chi}\Big) - \Big(\mathcal{C} - \frac{1}{4}\langle\vec{h},\vec{h}\rangle + \nabla\cdot t - \Delta\log\operatorname{tr} K\Big)
$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\hat{K}\cdot\hat{Q} - \hat{\underline{\chi}}\cdot\hat{\chi}\Big) + \frac{1}{2}|\nabla\omega|^2 G(\underline{L},\underline{L}) + \Big(2G(\underline{L},\nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle\Big) \\
&\quad - \hat{\underline{\alpha}}(\nabla\omega,\nabla\omega) - 2(\nabla\cdot\hat{K})(\nabla\omega) - H^\omega\cdot\hat{K} - 2\hat{K}(\nabla\omega, \vec{t} - \nabla\log\operatorname{tr}K) \\
&\qquad - \operatorname{tr}K t(\nabla\omega) + \nabla\omega\operatorname{tr}K + |\hat{K}|^2|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) + \hat{\underline{\alpha}}(\nabla\omega,\nabla\omega) \\
&\quad + \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}|\nabla\omega|^2 \\
&\quad + 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \operatorname{tr}K\hat{K}(\nabla\omega,\nabla\omega) - \kappa\hat{K}(\nabla\omega,\nabla\omega)
\end{aligned}
$$

From Lemma 4.2 we have

$$
\begin{aligned}
\hat{K}\cdot\hat{Q} - \hat{\underline{\chi}}\cdot\hat{\chi} &= \hat{K}\cdot\hat{Q} - \Big(\hat{K}\cdot\hat{Q} - |\nabla\omega|^2|\hat{K}|^2 - 4\hat{K}(\nabla\omega,\vec{t}) + 2|\hat{K}|^2|\nabla\omega|^2 \\
&\qquad + 2\operatorname{tr}K\hat{K}(\nabla\omega,\nabla\omega) - 2\hat{K}\cdot H^\omega - 2\kappa\hat{K}(\nabla\omega,\nabla\omega)\Big) \\
&= -|\hat{K}|^2|\nabla\omega|^2 - 2\operatorname{tr}K\hat{K}(\nabla\omega,\nabla\omega) + 2\hat{K}\cdot H^\omega + 4\hat{K}(\nabla\omega,\vec{t}) \\
&\quad + 2\kappa\hat{K}(\nabla\omega,\nabla\omega)
\end{aligned}
$$

so that we finally have

$$
\begin{aligned}
\not{\rho} - \rho &= \frac{1}{2}|\nabla\omega|^2\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big) + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\nabla\omega,\vec{t} - \nabla\log\operatorname{tr}K) \\
&\quad + \Big(2G_{\underline{L}}(\nabla\omega) - \langle R_{\underline{L}\,\nabla\omega} l, \underline{L}\rangle - 2(\nabla\cdot\hat{K})(\nabla\omega) \\
&\qquad\qquad + 2\hat{K}(\nabla\omega,\vec{t}) - \operatorname{tr}K t(\nabla\omega) + \nabla\omega\operatorname{tr}K\Big) \\
&\quad + \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}|\nabla\omega|^2 + 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}.
\end{aligned}
$$

Amazingly, the second line cancels due to (4.2) giving,

$$
\begin{aligned}
\not{\rho} - \rho &= \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big)|\nabla\omega|^2 + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\nabla\omega,\vec{t} - \nabla\log\operatorname{tr}K) \\
&\quad + \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}|\nabla\omega|^2 + 2\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}
\end{aligned}
$$

and the result then follows from the fact that $\not{\nabla}\omega = \nabla\omega + |\nabla\omega|^2\underline{L}$ as well as

$$
\not{\nabla}\cdot\Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}\not{\nabla}\omega\Big) = \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \not{\nabla}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}.
$$

$\square$

4.2. Asymptotic flatness

In this section we wish to study the limiting behaviour of our mass functional in the setting of asymptotic flatness constructed by Mars and Soria [23]. Beyond the assumption that we have a cross section Σ_{s_0} of Ω we also assume for some (hence any) choice of past-directed null geodesic field $\underline{L}$ that $S_+ = \infty$. So all geodesics $\gamma_q^{\underline{L}}$ are 'past complete' with domain $(s_-(q), \infty)$. We now take $s_0 = 0$ ignoring all points p satisfying $s(p) \leq S_-$ and conclude that $\Omega \cong \mathbb{S}^2 \times (S_-, \infty)$. Although the value of S_- will depend on our choice of geodesic field $\underline{L}$ our interest lies only on the past of Σ_0 (i.e. $\mathbb{S}^2 \times (0, \infty)$) so we ignore this subtlety. A null hypersurface Ω with all the above properties is called *extending to past null infinity.*

In order to impose decay conditions of various *transversal tensors* (i.e. tensors satisfying $T(\underline{L}, \cdots) = \cdots = T(\cdots, \underline{L}) = 0$) we choose a local basis on Σ_0 and extend it to a basis field $\{X_i\} \subset E(\Sigma_0)$. Given a transversal k-tensor $T(s)$ we say,

- $T = O(1)$ iff $T_{i_1 \dots i_k} := T(X_{i_1}, ..., X_{i_k})$ is uniformly bounded and $T = O_n(s^{-m})$ iff
$$s^{m+j}(\mathcal{L}_{\underline{L}})^j T(s) = O(1) \quad (0 \leq j \leq n)$$
- $T = o(s^{-m})$ iff $\lim_{s \to \infty} s^m T(s)_{i_1 \dots i_k} = 0$ and $T = o_n(s^{-m})$ iff
$$s^{m+j}(\mathcal{L}_{\underline{L}})^j T(s) = o(1) \quad (0 \leq j \leq n)$$
- $T = o_n^X(s^{-m})$ iff
$$s^m \mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_j}} T(s) = o(1) \quad (0 \leq j \leq n).$$

Now we're ready to define asymptotic flatness for Ω as given by the authors of [23]:

Definition 4.7. We say Ω is *past asymptotically flat* if it extends to past null infinity and there exists a choice of cross section Σ_0 and null geodesic field $\underline{L}$ with corresponding level set function s satisfying the following:

1) There exists two symmetric 2-covariant transversal and $\underline{L}$ Lie constant tensor fields $\mathring{\gamma}$ and γ_1 such that
$$\tilde{\gamma} := \gamma - s^2 \mathring{\gamma} - s\gamma_1 = o_1(s) \cap o_2^X(s)$$

2) There exists a transversal and $\underline{L}$ Lie constant one-form t_1 such that

$$\tilde{t} := t - \frac{t_1}{s} = o_1(s^{-1})$$

3) There exist $\underline{L}$ Lie constant functions θ_0 and θ such that

$$\tilde{\theta} := \operatorname{tr} Q - \frac{\theta_0}{s} - \frac{\theta}{s^2} = o(s^{-2})$$

4) The scalar $\langle R_{X_{i_1} X_{i_2}} X_{i_3}, X_{i_4} \rangle$ is such that $\lim_{s \to \infty} \frac{1}{s^2} \langle R_{X_{i_1} X_{i_2}} X_{i_3}, X_{i_4} \rangle$ exists while its double trace satisfies $-\frac{1}{2} R - G(\underline{L}, l) - \frac{1}{4} \langle R_{\underline{L} l} \underline{L}, l \rangle = o(s^{-2})$.

We will have the need to supplement the notion of asymptotic flatness of Ω with a stronger version of the *energy flux decay condition* (as given in [23]) with the following:

Definition 4.8. Suppose Ω is past asymptotically flat. We say Ω has *strong flux decay* if

$$G(\underline{L}, X_i) = o(s^{-2}), \ \tilde{t} = o_1^X(s^{-1}) \text{ and } \mathcal{L}_{\underline{L}}^j \tilde{\gamma} = o_{3-j}^X(s^{1-j}) \text{ for } 1 \le j \le 3$$

and *strong decay* if the condition on $G_{\underline{L}}$ is dropped.

We will also need some results from [23] (Proposition 3, Lemma 2, Section 4) resulting directly from the asymptotically flat restriction on Ω. One particularly valuable consequence is the ability to choose our geodesic field $\underline{L}$ to give any conformal change on the 'metric at null infinity', which turns out to be given by the 2-tensor, $\mathring{\gamma}$. By the Uniformization Theorem we conclude that this covers all possible metrics on a Riemannian 2-sphere. We will denote the covariant derivative coming from $\mathring{\gamma}$ by $\mathring{\nabla}$.

Proposition 4.9. *Suppose Ω is past asymptotically flat with a choice of null geodesic field $\underline{L}$ and corresponding level set function s. Letting $\gamma(s)^{ij}$ denote the inverse of $\gamma(s)_{ij}$,*

$$\gamma(s)^{ij} = \frac{1}{s^2} \mathring{\gamma}^{ij} - \frac{1}{s^3} \mathring{\gamma}_1{}^{ij} + o(s^{-3}) \tag{4.3}$$

$$K_{ij} = s \mathring{\gamma}_{ij} + \frac{1}{2} \gamma_{1ij} + o(1) \tag{4.4}$$

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + o(s^{-2}) \tag{4.5}$$

$$\operatorname{tr} Q = \frac{2\mathring{\mathcal{K}}}{s} + \frac{\theta}{s^2} + o(s^{-2}) \tag{4.6}$$

$$\operatorname{tr} K = \frac{2}{s} + \frac{\underline{\theta}}{s^2} + o(s^{-2}) \tag{4.7}$$

where $\mathring{\gamma}^{ij}$ is the inverse of $\mathring{\gamma}_{ij}$, tensors with a ring indicate that indices have been raised with $\mathring{\gamma}$, and $\underline{\theta} = -\frac{1}{2}\mathring{\operatorname{tr}}\gamma_1$.
It follows in case $\mathcal{L}_{\underline{L}}\tilde{\gamma} = o_1^X(1)$ that

$$t_1 = \frac{1}{2}\mathring{\nabla} \cdot \gamma_1 + \not{d}\underline{\theta} \iff G(\underline{L}, X_i) = o(s^{-2})$$

Proof. We refer the reader to [23] (Proposition 3) for proof. □

As promised in Remark 1.3 we are now able to prove the following well known result:

Lemma 4.10. *Suppose Ω extends to past null infinity with null geodesic field $\underline{L}$. Then any cross section $\Sigma \hookrightarrow \Omega$ satisfies $\operatorname{tr} K \geq 0$. If Ω is past asymptotically flat then Σ is expanding along $\underline{L}$.*

Proof. For $\omega \in \mathcal{F}(\Omega)$ constructed by Lie dragging $s|_\Sigma$ along $\underline{L}$ we have $\Sigma = \Sigma_1$ for the geodesic foliation $\{\Sigma_\lambda\}$ given by $s = \omega\lambda$. So it suffices to prove the result along an arbitrary geodesic foliation for Ω. From (3.4), if $\operatorname{tr} K(s_i) < 0$ for some initial s_i, then in a neighborhood we have

$$\underline{L}\Big(\frac{1}{\operatorname{tr} K}\Big) = \frac{1}{2} + \frac{|\hat{K}|^2 + G(\underline{L}, \underline{L})}{\operatorname{tr} K^2} \geq \frac{1}{2}$$

as long as $\operatorname{tr} K(s)$ remains negative. Therefore,

$$\frac{1}{\operatorname{tr} K}(s) \geq \frac{1}{\operatorname{tr} K}(s_i) + \frac{s - s_i}{2}$$

for any such $s \geq s_i$. So we can find an $s_f > s_i$ such that $\operatorname{tr} K(s) \xrightarrow{s \to s_f^-} -\infty$. Since this contradicts smoothness we must have that $\operatorname{tr} K \geq 0$ on all of Ω. If Ω is past asymptotically flat it follows from Proposition 4.9 that $\operatorname{tr} K(s) > 0$ for sufficiently large s. Since (3.4) gives

$$\underline{L}(\operatorname{tr} K) = -\frac{1}{2}(\operatorname{tr} K)^2 - |\hat{K}|^2 - G(\underline{L}, \underline{L}) \leq 0$$

we conclude that $\operatorname{tr} K(s_i) \geq \operatorname{tr} K(s_f)$ for all $s_i \leq s_f$. It follows that $\operatorname{tr} K > 0$ on all of Ω. □

Lemma 4.11. *On each Σ_s the difference tensor*

$$\mathcal{D}(V,W) := \nabla_V W - \mathring{\nabla}_V W$$

admits the decomposition

$$\mathcal{D}^k_{ij} = \frac{1}{2}(\mathring{\nabla}_i \mathring{\gamma}_1{}^k{}_j + \mathring{\nabla}_j \mathring{\gamma}_1{}^k{}_i - \mathring{\nabla}^k \mathring{\gamma}_{1ij})\frac{1}{s} + O(s^{-2}).$$

Moreover, if $f \in \mathcal{F}(\Omega)$ is Lie constant along $\underline{L}$ then

$$\Delta f = \frac{1}{s^2}\mathring{\Delta} f + (-\mathring{\gamma}_1^{ij}\mathring{\nabla}_i\mathring{\nabla}_j f - (\mathring{\nabla}_i \mathring{\gamma}_1^{ij}) f,_j + (\mathring{\nabla}^i \underline{\theta}) f,_i)\frac{1}{s^3} + o(s^{-3}).$$

Proof. The result follows from the well known fact (see, for example [37]) that we will need later on:

$$\begin{aligned}\langle \mathcal{D}(V,W), U\rangle &= \frac{1}{2}(\mathring{\nabla}_V \gamma(W,U) + \mathring{\nabla}_W \gamma(V,U) - \mathring{\nabla}_U \gamma(V,W)) \\ &= \frac{s}{2}(\mathring{\nabla}_V \gamma_1(W,U) + \mathring{\nabla}_W \gamma_1(V,U) - \mathring{\nabla}_U \gamma_1(V,W)) \\ &\quad + \frac{1}{2}(\mathring{\nabla}_V \tilde{\gamma}(W,U) + \mathring{\nabla}_W \tilde{\gamma}(V,U) - \mathring{\nabla}_U \tilde{\gamma}(V,W)).\end{aligned}$$

The second is a simple consequence of the first, we refer the reader to [23] (Lemma 2) for proof. □

In the next Proposition we show that the decomposition of the metric given in Definition 4.7 part 1 allows us to find $\mathcal{K}_{\gamma(s)}$ up to $O(s^{-4})$:

Proposition 4.12. *For a decomposition of the metric $\gamma(s) = s^2\mathring{\gamma} + s\gamma_1 + \tilde{\gamma}$ for some fixed s we have:*

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + \frac{1}{s^3}(\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}) + O(s^{-4}) \tag{4.8}$$

Proof. First we take the opportunity to show that $V, W \in E(\Sigma_0)$ gives $\mathring{\nabla}_V W \in E(\Sigma_0)$. Starting with the Koszul formula

$$\begin{aligned}2\mathring{\gamma}(\mathring{\nabla}_V W, U) &= V\mathring{\gamma}(W,U) + W\mathring{\gamma}(U,V) - U\mathring{\gamma}(V,W) - \mathring{\gamma}(V,[W,U]) \\ &\quad + \mathring{\gamma}(W,[U,V]) + \mathring{\gamma}(U,[V,W])\end{aligned}$$

and the fact that $\mathring{\gamma}$ is Lie constant along $\underline{L}$ we conclude that $\underline{L}\mathring{\gamma}(\mathring{\nabla}_V W, U) = \mathring{\gamma}([\underline{L}, \mathring{\nabla}_V W], U)$ on the left, applying $\underline{L}$ on the right we find everything vanishes since $V, W \in E(\Sigma_0) \implies [V,W] \in E(\Sigma_0)$. Thus, $\mathring{\gamma}([\underline{L}, \mathring{\nabla}_V W], U) = 0$.

Since $[\underline{L}, \mathring{\nabla}_V W] \in \Gamma(T\Sigma_s)$ and $\mathring{\gamma}$ is non-degenerate, $[\underline{L}, \mathring{\nabla}_V W] = 0$ and $\mathring{\nabla}_V W \in E(\Sigma_0)$. To show the decomposition of $\mathcal{K}_{\gamma(s)}$ we start by finding the decomposition of the Riemann curvature tensor on Σ_s:

$$\begin{aligned}
&\langle R^s_{X_iX_j} X_k, X_m\rangle = \langle \nabla_{[X_i,X_j]} X_k, X_m\rangle - X_i\langle \nabla_{X_j} X_k, X_m\rangle \\
&\quad + \langle \nabla_{X_j} X_k, \nabla_{X_i} X_m\rangle + X_j\langle \nabla_{X_i} X_k, X_m\rangle - \langle \nabla_{X_i} X_k, \nabla_{X_j} X_m\rangle \\
&= \langle \mathring{\nabla}_{[X_i,X_j]} X_k, X_m\rangle - X_i\langle \mathring{\nabla}_{X_j} X_k, X_m\rangle \\
&\quad + \langle \mathring{\nabla}_{X_j} X_k, \mathring{\nabla}_{X_i} X_m\rangle + X_j\langle \mathring{\nabla}_{X_i} X_k, X_m\rangle - \langle \mathring{\nabla}_{X_i} X_k, \mathring{\nabla}_{X_j} X_m\rangle \\
&\qquad + \langle \mathcal{D}([X_i,X_j],X_k),X_m\rangle - X_i\langle \mathcal{D}(X_j,X_k),X_m\rangle + \langle \mathcal{D}(X_j,X_k),\mathring{\nabla}_{X_i} X_m\rangle \\
&\qquad + \langle \mathring{\nabla}_{X_j} X_k, \mathcal{D}(X_i,X_m)\rangle + X_j\langle \mathcal{D}(X_i,X_k),X_m\rangle - \langle \mathcal{D}(X_i,X_k),\mathring{\nabla}_{X_j} X_m\rangle \\
&\qquad\quad - \langle \mathring{\nabla}_{X_i} X_k, \mathcal{D}(X_j,X_m)\rangle \\
&\qquad + \langle \mathcal{D}(X_j,X_k),\mathcal{D}(X_i,X_m)\rangle - \langle \mathcal{D}(X_i,X_k),\mathcal{D}(X_j,X_m)\rangle.
\end{aligned}$$

Using the decomposition $\gamma_s = s^2\mathring{\gamma} + O(s)$ we recognize the leading order term, combining lines 3 and 4, is $s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m)$. In order to find the next to leading order term the fact that $\langle R^s_{X_iX_j}X_k, X_m\rangle - s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m)$ defines a 4-tensor on each Σ_s allows us to search independently of our choice of basis $\{X_1, X_2\}$. In particular, we may assume that $\mathring{\nabla}_{X_i}X_j = 0$ at $q \in \Sigma_s$ (hence on all of $\gamma^{\underline{L}}_q$, since $\mathring{\nabla}_{X_i}X_j \in E(\Sigma_0)$). So assuming restriction to the generator through q and using Lemma 4.11 we have

$$\begin{aligned}
&\langle R^s_{X_iX_j} X_k, X_m\rangle - s^2\mathring{\gamma}(\mathring{R}_{X_iX_j}X_k, X_m) \\
&= -sX_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + sX_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) \\
&\quad - \frac{s}{2}X_i(\mathring{\nabla}_{X_j}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_k}\gamma_1(X_j,X_m) - \mathring{\nabla}_{X_m}\gamma_1(X_j,X_k)) \\
&\quad + \frac{s}{2}X_j(\mathring{\nabla}_{X_i}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_k}\gamma_1(X_i,X_m) - \mathring{\nabla}_{X_m}\gamma_1(X_i,X_k)) \\
&\quad + O(1) \\
&= -sX_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + sX_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) \\
&\quad \frac{s}{2}\Big(\mathring{\nabla}_{X_j}\mathring{\nabla}_{X_i}\gamma_1(X_k,X_m) + \mathring{\nabla}_{X_j}\mathring{\nabla}_{X_k}\gamma_1(X_i,X_m) - \mathring{\nabla}_{X_j}\mathring{\nabla}_{X_m}\gamma_1(X_i,X_k) \\
&\quad - \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}\gamma_1(X_k,X_m) - \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_k}\gamma_1(X_j,X_m) + \mathring{\nabla}_{X_i}\mathring{\nabla}_{X_m}\gamma_1(X_j,X_k)\Big) \\
&\quad + O(1).
\end{aligned}$$

It remains to simplify the two terms of the first line in the second equality. Since

$$X_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) = \mathring{\nabla}_{X_i}\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + \gamma_1(\mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}X_k, X_m)$$

we conclude that

$$-X_i\gamma_1(\mathring{\nabla}_{X_j}X_k, X_m) + X_j\gamma_1(\mathring{\nabla}_{X_i}X_k, X_m) = \gamma_1(\mathring{R}_{X_iX_j}X_k, X_m).$$

Moreover, it is easily shown using our choice of basis extension that

$$\begin{aligned}\frac{1}{2}\mathring{\nabla}_{X_j}\mathring{\nabla}_{X_i}\gamma_1(X_k, X_m) - \frac{1}{2}\mathring{\nabla}_{X_i}\mathring{\nabla}_{X_j}\gamma_1(X_k, X_m) + \gamma_1(\mathring{R}_{X_iX_j}X_k, X_m)&\\ = \frac{1}{2}(\gamma_1(\mathring{R}_{X_iX_j}X_k, X_m) - \gamma_1(\mathring{R}_{X_iX_j}X_m, X_k)).&\end{aligned}$$

So we finally have from the fact that Σ_s is of dimension 2 that

$$\begin{aligned}&\langle R^s_{X_iX_j}X_k, X_m\rangle\\ &= s^2\mathring{\mathcal{K}}(\mathring{\gamma}_{ik}\mathring{\gamma}_{jm} - \mathring{\gamma}_{im}\mathring{\gamma}_{jk}) + \frac{s}{2}\mathring{\mathcal{K}}(\mathring{\gamma}_{ik}\gamma_{1jm} - \mathring{\gamma}_{im}\gamma_{1jk} + \mathring{\gamma}_{jm}\gamma_{1ik} - \mathring{\gamma}_{jk}\gamma_{1im})\\ &\quad + \frac{s}{2}(\mathring{\nabla}_j\mathring{\nabla}_k\gamma_{1im} - \mathring{\nabla}_j\mathring{\nabla}_m\gamma_{1ik} - \mathring{\nabla}_i\mathring{\nabla}_k\gamma_{1jm} + \mathring{\nabla}_i\mathring{\nabla}_m\gamma_{1jk}) + O(1).\end{aligned}$$

Using (4.3) to take a trace over i, k:

$$\begin{aligned}(Ric^s)_{jm} &= \mathring{\mathcal{K}}\mathring{\gamma}_{jm} - \frac{1}{s}\mathring{\mathcal{K}}\underline{\theta}\mathring{\gamma}_{jm} + \frac{1}{2s}(\mathring{\nabla}_j(\mathring{\nabla}\cdot\gamma_1)_m + 2\mathring{\nabla}_j\mathring{\nabla}_m\underline{\theta} - (\mathring{\nabla}^2\gamma_1)_{jm}\\ &\quad + (\mathring{\nabla}\cdot(\mathring{\nabla}\gamma_1))_{mj}) + \frac{\mathring{\mathcal{K}}}{s}(2\underline{\theta}\mathring{\gamma}_{jm} + \gamma_{1jm}) + O(s^{-2})\\ &= \mathring{\mathcal{K}}\mathring{\gamma}_{jm} + \frac{1}{s}\Big(\mathring{\mathcal{K}}\underline{\theta}\mathring{\gamma}_{jm} + \mathring{\mathcal{K}}\gamma_{1jm} + \frac{1}{2}\mathring{\nabla}_j(\mathring{\nabla}\cdot\gamma_1)_m + \frac{1}{2}(\mathring{\nabla}\cdot(\mathring{\nabla}\gamma_1))_{mj} + \mathring{\nabla}_j\mathring{\nabla}_m\underline{\theta}\\ &\quad - \frac{1}{2}(\mathring{\nabla}^2\gamma_1)_{jm}\Big) + O(s^{-4})\end{aligned}$$

and then over j, m:

$$2\mathcal{K}_{\gamma(s)} = \frac{2}{s^2}\mathring{\mathcal{K}} + \frac{1}{s^3}\Big(2\mathring{\mathcal{K}}\underline{\theta} - 2\mathring{\mathcal{K}}\underline{\theta} + \mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + 2\mathring{\Delta}\underline{\theta}\Big) + \frac{2}{s^3}\mathring{\mathcal{K}}\underline{\theta} + O(s^{-4})$$

giving the result. □

Remark 4.13. Interestingly, in the case that Ω is asymptotically flat satisfying the energy flux decay condition we conclude that

$$\mathcal{K}_{\gamma(s)} = \frac{\mathring{\mathcal{K}}}{s^2} + \frac{1}{s^3}(\mathring{\mathcal{K}}\underline{\theta} + \mathring{\nabla}\cdot t_1) + O(s^{-4})$$

according to Proposition 4.12.

Definition 4.14. For Ω past asymptotically flat with background geodesic foliation $\{\Sigma_s\}$ we say a foliation $\{\Sigma_{s_\star}\}$ is *asymptotically geodesic* provided

$$s = \phi s_\star + \xi$$

with *scale factor* $\phi > 0$ a Lie constant function along $\underline{L}$ and $\underline{L}^i \xi = o^X_{2-i}(s^{1-i})$ for $0 \leq i \leq 2$. In addition (similarly to [23]), we will say $\{\Sigma_{s_\star}\}$ *approaches large spheres* provided the class of geodesic foliations measuring $\phi = 1$ also induce $\mathring{\gamma}$ to be the round metric on $\mathbb{S}^2$.

Remark 4.15. Given a basis extension $\{X_i\} \subset E(\Sigma_0)$ (on $\{\Sigma_s\}$) and a foliation $\{\Sigma_{s_\star}\}$ as in Definition 4.14, Lie dragging $s|_{\Sigma_{s_\star}}$ along $\underline{L}$ to give $\omega \in \mathcal{F}(\Omega)$ we see at $q \in \Sigma_{s_\star}$:

$$\begin{aligned}
\omega_i &= \phi_i s_\star + \xi_s \omega_i + \xi_i \\
\omega_{ij} &= \phi_{ij} s_\star + \xi_{ss}\omega_i\omega_j + \xi_{sj}\omega_i + \xi_s\omega_{ij} + \xi_{ij}
\end{aligned}$$

where $\omega_i := X_i\omega$, $\omega_{ij} := X_j X_i \omega$, $\xi_s := \underline{L}\xi$, $\xi_{ss} := \underline{L}\underline{L}\xi$, $\xi_i := X_i(\xi|_{\Sigma_s(q)})$, $\xi_{si} = X_i(\xi_s|_{\Sigma_s})$ and $\xi_{ij} = X_j X_i(\xi|_{\Sigma_s})$. The decay on ξ therefore gives us that:

$$\begin{aligned}
\omega_i &= \frac{\phi_i s_\star + \xi_i}{1-\xi_s} = \phi_i s_\star + o(s_\star) \\
\omega_{ij} &= \frac{1}{1-\xi_s}\Big(\phi_{ij}s_\star + \xi_{ss}\Big(\frac{\xi_i + \phi_i s_\star}{1-\xi_s}\Big)\Big(\frac{\xi_j + \phi_j s_\star}{1-\xi_s}\Big) + \xi_{sj}\Big(\frac{\xi_i + \phi_i s_\star}{1-\xi_s}\Big) + \xi_{ij}\Big) \\
&= \phi_{ij}s_\star + o(s_\star).
\end{aligned}$$

From (4.3) and Lemma 4.11 we conclude that

$$\begin{aligned}
d\omega|_{\Sigma_{s_\star}} &= (s_\star\phi)^2\big(\frac{-1}{s_\star} d\phi^{-1}|_{\Sigma_{s_\star}} + o(s_\star^{-1})\big) \\
\Delta\omega|_{\Sigma_{s_\star}} &= \frac{1}{\phi^2 s_\star}\mathring{\Delta}\phi + o(s_\star^{-1}).
\end{aligned}$$

We will need the following result found by the authors of [23] (Theorem 1):

Proposition 4.16. *Suppose Ω is past asymptotically flat and $\{\Sigma_{s_\star}\}$ is an asymptotically geodesic foliation with scale factor $\phi > 0$. Then*

$$\lim_{s_\star\to\infty} E_H(\Sigma_{s_\star}) = \frac{1}{16\pi}\sqrt{\frac{\int \phi^2 d\mathring{A}}{4\pi}} \int \frac{1}{\phi}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big) d\mathring{A}$$

with $\mathring{\gamma}$, $\mathring{\mathcal{K}}$, $\underline{\theta}$, θ and t_1 associated with the background geodesic foliation.

Remark 4.17. Using the terminology of [23], suppose Ω is a past asymptotically flat null hypersurface with a background geodesic foliation $\{\Sigma_s\}$ approaching large spheres (i.e $\mathring{\gamma}$ is the round metric at infinity). Then, for any other geodesic foliation of scale factor ψ, it follows that the metric at infinity is $\psi^2\mathring{\gamma}$ (see [23], Section 4), and $\{\Sigma_s\}$ approaches large spheres if and only if ψ solves the equation (recall, Section 2.2)

$$1 - \psi^2 = \mathring{\Delta} \log \psi. \tag{4.9}$$

Proposition 4.16 shows all asymptotically geodesic foliations $\{\Sigma_{s_\star}\}$ of the same scale factor ϕ share the limit

$$E(\phi) = \lim_{s_\star \to \infty} E_H(\Sigma_{s_\star})$$

which measures a Trautman-Bondi energy $E_{TB}(\psi)$ if ψ solves (4.9).

Definition 4.18. The Trautman-Bondi mass is therefore given by

$$m_{TB} = \inf_{\psi}\{E_{TB}(\psi) | 1 - \psi^2 = \mathring{\Delta} \log \psi\}.$$

We observe that the conditions defining asymptotic flatness imposes a sense of convergence upon the data of Ω towards the standard null cone in Schwarzschild spacetime. As such, Definition 4.18 clearly identifies the Schwarzschild mass of Ω according to the discussion of Section 2.2. Including the references in Section 1.1 on the history of the Trautman-Bondi mass and how it relates to other definitions of mass, we also refer the reader to [32] (Section 4.1) for further motivation and context for Definition 4.18.

Theorem 4.19. *Suppose Ω is a past asymptotically flat null hypersurface inside a spacetime satisfying the dominant energy condition. Then given the existence of an asymptotically geodesic doubly convex foliation $\{\Sigma_{s_\star}\}$ approaching large spheres we have*

$$m(0) \leq E_{TB}$$

for E_{TB} the Trautman-Bondi energy of Ω associated to $\{\Sigma_{s_\star}\}$. If equality is achieved along a strict doubly convex foliation then $E_{TB} = m_{TB}$ the Trautman-Bondi mass of Ω. In the case that $\operatorname{tr}\chi|_{\Sigma_0} = 0$ we conclude instead *with the weak Null Penrose inequality*

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq E_{TB}$$

where equality along a strict doubly convex foliation enforces that any foliation of Ω *shares its data* (γ, $\underline{\chi}$, $\operatorname{tr}\chi$ *and* ζ) *with some foliation of the standard null cone of Schwarzschild spacetime.*

Proof. Since any asymptotically geodesic doubly convex foliation has non-decreasing mass from Theorem 1.7 and $m(\Sigma_{s_\star}) \leq E_H(\Sigma_{s_\star})$ from Lemma 2.2, it follows from Proposition 4.16 that $m(\Sigma_{s_\star})$ converges since $E_H(\Sigma_{s_\star})$ does. Moreover, $\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) \leq \lim_{s_\star\to\infty} E_H(\Sigma_{s_\star})$ and from [23] (Corollary 3) it follows that $\lim_{s_\star\to\infty} E_H(\Sigma_{s_\star})$ is the Trautman-Bondi energy associated to the abstract reference frame coupled to the foliation $\{\Sigma_{s_\star}\}$. Given the case of equality, Theorem 1.7 enforces that $m(0) = m(\Sigma_{s_\star})$ for all $s_\star$. So Theorem 3.9 applies and we conclude that $m(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int r_0^{\frac{2}{3}} dA_0\Big)^{\frac{3}{2}}$ (for some positive function r_0 on Σ_0 of area form $r_0^2 dA_0$) irrespective of the cross-section $\Sigma \subset \Omega$. This gives, according to Remark 4.17 and Lemma 2.2,

$$\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) = \frac{1}{2}\Big(\frac{1}{4\pi}\int r_0^{\frac{2}{3}} dA_0\Big)^{\frac{3}{2}} = E_{TB} \leq \inf_{\phi>0} E(\phi) \leq m_{TB}.$$

Since $E_{TB} \leq \inf E(\phi) \leq m_{TB} \leq E_{TB}$ all must be equal.

If $\operatorname{tr}\chi|_{\Sigma_0} = 0$ the doubly convex conditions gives

$$0 \geq \not{\Delta} \log \not{\phi}|_{\Sigma_0}$$

and the maximum principle implies $\not{\phi}|_{\Sigma_0} = \mathcal{K} + \not{\nabla}\cdot\tau$ is constant. From the Gauss-Bonnet and Divergence Theorems we conclude that $\not{\phi}|_{\Sigma_0} = \frac{4\pi}{|\Sigma_0|}$ and therefore $m(0) = \sqrt{\frac{|\Sigma_0|}{16\pi}}$. Under this restriction Theorem 3.9 enforces that any foliation of Ω corresponds with a foliation of the standard null cone in Schwarzschild with respect to the data γ, $\underline{\chi}$, $\operatorname{tr}\chi$ and ζ. $\square$

From Proposition 4.16 and Lemma 2.2

$$\inf_{\phi>0} E(\phi) = \frac{1}{4}\Big(\frac{1}{4\pi}\int (\mathcal{K}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)^{\frac{2}{3}} d\mathring{A}\Big)^{\frac{3}{2}}$$

provided $\mathcal{K}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla} \cdot t_1 \geq 0$. We show, given that Ω satisfies the strong flux decay condition, this quantity is in fact $\lim_{s_\star \to \infty} m(\Sigma_{s_\star})$. We will need the following proposition to do so:

Proposition 4.20. *Suppose Ω is past asymptotically flat with strong decay. Given a choice of null geodesic field $\underline{L}$ and corresponding level set function s we have*

$$\nabla \cdot \nabla \cdot K = -\frac{1}{2s^4}\mathring{\nabla} \cdot \mathring{\nabla} \cdot \gamma_1 + o(s^{-4}) \tag{4.10}$$

$$\Delta \operatorname{tr} K = \frac{\mathring{\Delta}\underline{\theta}}{s^4} + o(s^{-4}) \tag{4.11}$$

$$\nabla \cdot t = \frac{1}{s^3}\mathring{\nabla} \cdot t_1 + o(s^{-3}) \tag{4.12}$$

Proof. From Lemma 4.11 and (4.4)

$$\begin{aligned}
\nabla_i K_{jm} &= \mathring{\nabla}_i(s\mathring{\gamma}_{jm} + \frac{1}{2}\gamma_{1jm}) - \mathcal{D}^k_{ij}K_{km} - \mathcal{D}^k_{im}K_{jk} + o^X_1(1)\\
&= \frac{1}{2}\mathring{\nabla}_i\gamma_{1jm} - \mathring{\nabla}_i\gamma_{1jm} + o^X_1(1)\\
&= -\frac{1}{2}\mathring{\nabla}_i\gamma_{1jm} + o^X_1(1).
\end{aligned}$$

where the first term of the second line comes from the fact that $\mathring{\nabla}\mathring{\gamma} = 0$. Next we compute

$$\begin{aligned}
\nabla_i\nabla_j K_{mn} &= \mathring{\nabla}_i\nabla_j K_{mn} - \mathcal{D}^k_{ij}\nabla_k K_{mn} - \mathcal{D}^k_{im}\nabla_j K_{kn} - \mathcal{D}^k_{in}\nabla_j K_{mk}\\
&= -\frac{1}{2}\mathring{\nabla}_i\mathring{\nabla}_j\gamma_{1mn} + o(1)
\end{aligned}$$

So contracting with (4.3) over j, m followed by i, n we get (4.9) and contracting instead over m, n and then i, j (4.10) follows. For (4.11)

$$\begin{aligned}
\nabla_i t_j &= \mathring{\nabla}_i t_j - \mathcal{D}^k_{ij}t_k\\
&= \frac{1}{s}\mathring{\nabla}_i t_{1j} + o(s^{-1})
\end{aligned}$$

and the result follows as soon as we contract with (4.3) over i, j. □

Theorem 4.21. *Suppose Ω is past asymptotically flat with strong flux decay and $\{\Sigma_s\}$ is some background geodesic foliation. Then for any asymptotically* *geodesic foliation* $\{\Sigma_{s_\star}\}$ *with scale factor* $\phi > 0$ *we have*

$$s_\star^3 \not{\rho}(s_\star) = \frac{1}{2\phi^3}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big) + o(s_\star^0)$$

Proof. First let us remind ourselves of Theorem 4.3

$$\begin{aligned}\not{\rho} = \rho + \not{\nabla}\cdot\Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}\not{\nabla}\omega\Big) + \frac{1}{2}\Big(|\hat{K}|^2 + G(\underline{L},\underline{L})\Big)|\nabla\omega|^2 \\ + \nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + G_{\underline{L}}(\nabla\omega) - 2\hat{K}(\vec{t} - \nabla\log\operatorname{tr}K, \nabla\omega).\end{aligned}$$

Denoting the exterior derivative on Σ_s by d_s, since $\operatorname{tr}K = \frac{2}{s} + \frac{\underline{\theta}}{s^2} + o(s^{-2})$, we conclude that $d_s\log\operatorname{tr}K = \frac{1}{2s}d\underline{\theta}|_{\Sigma_s} + o(s^{-1})$ giving

$$\hat{K}(\vec{t} - \nabla\log\operatorname{tr}K, \nabla\omega)|_{\Sigma_{s_\star}} = o(s_\star^{-3}).$$

Since $\mathcal{L}^2_{\underline{L}}\tilde{\gamma} = o_1^X(s^{-1}) \cap o_1(s^{-1})$ we also see that

$$\begin{aligned}|\hat{K}|^2 + G(\underline{L},\underline{L}) &= -\underline{L}\operatorname{tr}K - \frac{1}{2}(\operatorname{tr}K)^2 \\ &= -(-\frac{2}{s^2} - \frac{2}{s^3}\underline{\theta}) - \frac{1}{2}(\frac{2}{s} + \frac{\underline{\theta}}{s^2})^2 + o_1^X(s^{-3}) \cap o_1(s^{-3}) \\ &= o_1^X(s^{-3}) \cap o_1(s^{-3})\end{aligned}$$

and therefore, from Remark 4.15:

$$\begin{aligned}\not{\nabla}\cdot\Big(\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}\not{\nabla}\omega\Big) &= \not{\nabla}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + \not{\Delta}\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} \\ &= \Big(\nabla\omega\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} + |\nabla\omega|^2\underline{L}\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K} \\ &\quad + (\Delta\omega - 2\hat{K}(\nabla\omega,\nabla\omega))\frac{|\hat{K}|^2 + G(\underline{L},\underline{L})}{\operatorname{tr}K}\Big)\Big|_{\Sigma_{s_\star}} \\ &= o(s_\star^{-3}).\end{aligned}$$

From the strong flux decay condition we have $G_{\underline{L}}(\nabla\omega)|_{\Sigma_{s_\star}} = o(s_\star^{-3})$ also. From (4.10) we have

$$\Delta\log\operatorname{tr}K = \frac{\Delta\operatorname{tr}K}{\operatorname{tr}K} - \frac{|\nabla\operatorname{tr}K|^2}{(\operatorname{tr}K)^2} = \frac{\mathring{\Delta}\underline{\theta}}{2s^3} + o(s^{-3})$$

and combining this with Propositions 4.12 and 4.20:

$$\begin{aligned}\not{\rho} = \rho|_{\Sigma_{s_\star}} + o(s_\star^{-3}) &= \frac{1}{\omega^3}\Big(\frac{1}{2}\mathring{\mathcal{K}}\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} - \frac{1}{2}\theta\Big) + \frac{1}{\omega^3}\mathring{\nabla}\cdot t_1 \\ &\quad - \frac{1}{2\omega^3}\mathring{\Delta}\underline{\theta} + o(s_\star^{-3}) \\ &= \frac{1}{2\omega^3}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big) + o(s_\star^{-3})\end{aligned}$$

having used Proposition 4.9, specifically $\frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta} = \mathring{\nabla}\cdot t_1$, to obtain the final line. □

Remark 4.22. We refer the reader to [23] (Proposition 3) to observe the derivation of 4.6 from Definition 4.7, part 4, for an arbitrary geodesic background foliation.

The only time we use 4.6 in Theorem 4.21 is in the penultimate equality of the proof. Subsequently, any *other* geodesic foliation inherits 4.6 as a result of Theorem 4.21, Propositions 4.12, 4.20, and identity 4.7. In Section 5 we will show 4.6 directly from the metric structure for perturbations from spherical symmetry. Consequently, we circumvent the use of Definition 4.7, part 4 altogether.

Corollary 4.23. *With the same hypotheses as in Theorem 4.21 we have*

$$\lim_{s_\star\to\infty} m(\Sigma_{s_\star}) = \frac{1}{4}\Big(\frac{1}{4\pi}\int(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1)^{\frac{2}{3}}d\mathring{A}\Big)^{\frac{3}{2}}$$

Proof. From Theorem 4.21 we directly conclude

$$\begin{aligned}4\pi(4m(\Sigma_{s_\star}))^{\frac{2}{3}} &= \int(2\not{\rho})^{\frac{2}{3}}dA_\omega \\ &= \int\frac{1}{\omega^2}\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1 + o(1)\Big)^{\frac{2}{3}}f\omega^2 d\mathring{A}\end{aligned}$$

where $f = \sqrt{\frac{\det(\gamma_\omega)_{ij}}{\omega^2\det(\mathring{\gamma})_{ij}}} = 1 + o(1)$, giving

$$4\pi(4\lim_{s_\star\to\infty} m(\Sigma_{s_\star}))^{\frac{2}{3}} = \int\Big(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla}\cdot t_1\Big)^{\frac{2}{3}}d\mathring{A}$$

by the Dominated Convergence Theorem. □

Finally we're ready to prove Theorem 1.9:

Proof. (Theorem 1.9) The first claim of Theorem 1.9 is a simple consequence of Theorem 1.7. The doubly convex condition and Theorem 4.21 enforces that

$$0 \leq \lim_{s_\star \to \infty} s_\star^3 \not{\phi} = \frac{1}{2\phi^3}(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla} \cdot t_1)$$

and therefore Theorem 1.7, Corollary 4.23, Lemma 2.2 and Proposition 4.16 gives

$$\begin{aligned} m(\Sigma_0) \leq \lim_{s_\star \to \infty} m(\Sigma_{s_\star}) &= \frac{1}{4}\Big(\frac{1}{4\pi}\int(\mathring{\mathcal{K}}\underline{\theta} - \theta - \mathring{\Delta}\underline{\theta} + 4\mathring{\nabla} \cdot t_1)^{\frac{2}{3}} d\mathring{A}\Big)^{\frac{3}{2}} \\ &= \inf_{\phi > 0} E(\phi) \leq m_{TB}. \end{aligned}$$

The rest of the proof is settled identically as in Theorem 4.19. □

5. Spherical symmetry

The Null Penrose conjecture in spherical symmetry is known (see [18]). We start in this section by providing a proof in order to utilize our results from prior sections. This will also serve as a template from which to study perturbations of the spherically symmetric metric. We will uncover smooth null cones Ω on which the asymptotic flatness and strong flux decay conditions are maintained. We also show the existence of an asymptotically geodesic and strict doubly convex foliation for a certain class of perturbations of the black hole exterior.

5.1. The metric

In polar areal coordinates [26] the metric takes the form

$$g = -a(t,r)^2 dt \otimes dt + b(t,r)^2 dr \otimes dr + r^2 \mathring{\gamma}$$

for $\mathring{\gamma}$ the standard round metric on $\mathbb{S}^2$. From which the change in coordinates $(t,r) \to (v(t,r), r)$ given by

$$bv_t = av_r$$

produces the metric and metric inverse given by

$$\begin{aligned} g &= -\mathfrak{h} e^{2\beta} dv \otimes dv + e^{\beta}(dv \otimes dr + dr \otimes dv) + r^2 \mathring{\gamma} \\ g^{-1} &= e^{-\beta}(\partial_v \otimes \partial_r + \partial_r \otimes \partial_v) + \mathfrak{h}\partial_r \otimes \partial_r + \frac{1}{r^2}\mathring{\gamma}^{-1} \end{aligned}$$

where $\mathfrak{h} := (1 - \frac{2M(t,r)}{r}) = b^{-2}$ and $e^{\beta} = abv_t^{-1}$.

It's a well known fact when $M(t,r) = m_0 > 0$ and $\beta(t,r) = 0$ for m_0 a constant the above metric covers the upper-half region in Kruskal space-time (given by $\{v > 0\}$, where the coordinate v is different from above) namely, the Schwarzschild geometry in 'ingoing-Eddington-Finkelstein' coordinates. In spherical symmetry, we will therefore refer to the null hypersurfaces $\Omega := \{v = v_0\}$ as the *standard null cones* as they agree with the similarly named hypersurfaces in the Schwarzschild case.

5.2. Calculating ρ

We approach the calculation similarly to the case of Schwarzschild. Denoting the gradient of v by Dv we use the identity $D_{Dv}Dv = \frac{1}{2}D|Dv|^2$ to see $\underline{L} := Dv = e^{-\beta}\partial_r$ satisfies $D_{\underline{L}}\underline{L} = 0$ providing us our choice of geodesic field for Ω and level set function s (as in Section 3). For convenience we will choose our background foliation $\{\Sigma_r\}$ of Ω to be the level sets of the coordinate r. Since β only depends on r, the foliation $\{\Sigma_r\}$ is simply a re-parametrization of the geodesic foliation $\{\Sigma_s\}$ coming from $\underline{L}$. Moreover, the vector field extensions $0 = [\underline{L}, V] = e^{\beta}[\partial_r, V]$ remain unchanged. An arbitrary cross section Σ of Ω is therefore given as a graph over Σ_{r_0} (for some r_0) which we Lie drag along ∂_r to the rest of Ω giving some $\omega \in \mathcal{F}(\Omega)$. On Σ we therefore have the linearly independent normal vector fields

$$\begin{aligned}\underline{L} &= e^{-\beta}\partial_r\\ D(r-\omega) &= e^{-\beta}\partial_v + \mathfrak{h}\partial_r - \nabla\omega\end{aligned}$$

where in this subsection (5.2) ∇ will temporarily denote the induced covariant derivative on Σ_r. We wish to find the null section $L \in \Gamma(T^{\perp}\Sigma)$ satisfying $\langle L, \underline{L}\rangle = 2$. Since $L = c_1\underline{L} + c_2 D(r-\omega)$ we have

$$\begin{aligned}2 &= \langle L, \underline{L}\rangle = c_2 e^{-\beta}\partial_r(r-\omega) = c_2 e^{-\beta}\\ 0 &= \langle L, L\rangle = 2c_1c_2\langle \underline{L}, D(r-\omega)\rangle + c_2^2\langle D(r-\omega), D(r-\omega)\rangle\\ &= 2c_1c_2e^{-\beta} + c_2^2(e^{-2\beta}\langle\partial_v,\partial_v\rangle + |\nabla\omega|^2 + 2e^{-\beta}\mathfrak{h}\langle\partial_v,\partial_r\rangle)\\ &= 2c_1c_2e^{-\beta} + c_2^2(\mathfrak{h} + |\nabla\omega|^2)\end{aligned}$$

giving $c_2 = 2e^{\beta}$ and $c_1 = -e^{2\beta}(\mathfrak{h} + |\nabla\omega|^2)$ so that

$$\begin{aligned}L &= -e^{\beta}(\mathfrak{h} + |\nabla\omega|^2)\partial_r + 2\partial_v + 2\mathfrak{h}e^{\beta}\partial_r - 2e^{\beta}\nabla\omega\\ &= 2\partial_v + e^{\beta}(\mathfrak{h} - |\nabla\omega|^2)\partial_r - 2e^{\beta}\nabla\omega\end{aligned}$$

$$
\begin{aligned}
&= 2\partial_v + e^{\beta}(\mathfrak{h} - |\nabla\!\!\!/ \omega|^2)\partial_r - 2e^{\beta}(\nabla\!\!\!/ \omega - |\nabla\!\!\!/ \omega|^2\partial_r) \\
&= 2\partial_v + e^{\beta}(\mathfrak{h} + |\nabla\!\!\!/ \omega|^2)\partial_r - 2e^{\beta}\nabla\!\!\!/ \omega
\end{aligned}
$$

having used the fact that $\nabla\!\!\!/ \omega = \nabla\omega + |\nabla\omega|^2\partial_r$ to get the third equality. We note from the warped product structure (as for Kruskal spacetime) that $E(\Sigma_{r_0}) = \mathcal{L}(\mathbb{S}^2)|_{\Omega}$ where $\mathcal{L}(\mathbb{S}^2)$ is the set of lifted vector fields from the $\mathbb{S}^2$ factor of the spacetime product manifold. As a result we may globally extend $V \in E(\Sigma_{r_0})$ to satisfy $[\partial_v, V] = 0$. The following facts are a direct application of the Koszul formula, we refer the reader to [25] (pg 206) for the details:

$$
D_{\partial_r}\partial_v = -\frac{1}{2}\partial_r(\mathfrak{h}e^{2\beta})e^{-\beta}\partial_r \tag{5.1}
$$

$$
D_V\partial_v = 0 \tag{5.2}
$$

$$
D_{\partial_r}\partial_r = \partial_r\beta\partial_r \tag{5.3}
$$

$$
D_V\partial_r = \frac{1}{r}V. \tag{5.4}
$$

Lemma 5.1. *Suppose $\Omega = \{v = v_0\}$ is the standard null cone in a spherically symmetric spacetime of metric*

$$
g = -\mathfrak{h}e^{2\beta(v,r)}dv \otimes dv + e^{\beta}(v,r)(dv \otimes dr + dr \otimes dv) + r^2\mathring{\gamma}
$$

where $\mathfrak{h} = (1 - \frac{2M(v,r)}{r})$ and $\mathring{\gamma}$ is the round metric on $\mathbb{S}^2$. Then for some cross section $\Sigma_{r_0} \subset \Omega$ and $\omega \in \mathcal{F}(\Sigma_{r_0})$, $\Sigma := \{r = \omega \circ \pi\}$ produces the data (writing $\omega \circ \pi$ as ω):

$$
\gamma = \omega^2\mathring{\gamma}, \qquad \underline{\chi} = \frac{e^{-\beta(v_0,\omega)}}{\omega}\gamma, \qquad \operatorname{tr}\underline{\chi} = \frac{2e^{-\beta(v_0,\omega)}}{\omega}, \qquad \zeta = -d\!\!\!/ \log\omega,
$$

$$
\begin{aligned}
\chi &= e^{\beta(v_0,\omega)}\Big((\mathfrak{h} + |\nabla\!\!\!/ \omega|^2)\frac{\gamma}{\omega} - 2\tilde{H}^{\omega} - 2\beta_r d\!\!\!/ \omega \otimes d\!\!\!/ \omega\Big), \\
\operatorname{tr}\chi &= \frac{2e^{\beta(v_0,\omega)}}{\omega}(\mathfrak{h} - \omega^2\Delta\!\!\!\!/ \log\omega - \omega\beta_r|\nabla\!\!\!/ \omega|^2), \\
\rho &= \frac{2M(v_0,\omega)}{\omega^3} + \Delta\!\!\!\!/ \beta + \frac{\beta_r}{\omega}|\nabla\!\!\!/ \omega|^2.
\end{aligned}
$$

Proof. For any $V \in E(\Sigma_{r_0})$ Lemma 4.2 gives $\tilde{V} := V + V\omega\partial_r|_{\Sigma} \in \Gamma(T\Sigma)$ so that the first identity follows directly from the metric restriction. From (5.4):

$$
D_{\tilde{V}}\underline{L} = e^{-\beta}D_V(\partial_r) + e^{\beta}V\omega D_{\underline{L}}\underline{L} = \frac{e^{-\beta}}{r}V
$$

so the second identity is given by

$$\underline{\chi}(\tilde{V},\tilde{W}) = \langle D_{\tilde{V}}\underline{L},\tilde{W}\rangle = \frac{e^{-\beta}}{r}\langle V,W\rangle$$

and a trace over V,W gives the third so that $\not{\Delta}\log\operatorname{tr}\underline{\chi} = -\not{\Delta}\beta - \not{\Delta}\log\omega$. For the forth identity:

$$\begin{aligned}\chi(\tilde{V},\tilde{W}) &= 2\langle D_{\tilde{V}}\partial_v,\tilde{W}\rangle + e^{\beta}(\mathfrak{h}+|\not{\nabla}\omega|^2)\langle D_{\tilde{V}}\partial_r,\tilde{W}\rangle - 2e^{\beta}\langle D_{\tilde{V}}\not{\nabla}\omega,\tilde{W}\rangle \\ &\quad - 2\beta_r e^{\beta}\tilde{V}\omega\tilde{W}\omega \\ &= e^{\beta}(\mathfrak{h}+|\not{\nabla}\omega|^2)\frac{1}{\omega}\langle\tilde{V},\tilde{W}\rangle - 2e^{\beta}\tilde{H}^{\omega}(\tilde{V},\tilde{W}) - 2\beta_r e^{\beta}(\not{d}\omega\otimes\not{d}\omega)(\tilde{V},\tilde{W})\end{aligned}$$

where $\langle D_{\tilde{V}}\partial_v,\tilde{W}\rangle = 0$ from (5.1) and (5.2) to give the second equality. Taking a trace over $\tilde{V},\tilde{W}$ we conclude with the fifth identity:

$$\begin{aligned}\operatorname{tr}\chi|_{\Sigma} &= \frac{2e^{\beta(v_0,\omega)}}{\omega}(\mathfrak{h}+|\not{\nabla}\omega|^2-\omega\not{\Delta}\omega) - 2\beta_r e^{\beta(v_0,\omega)}|\not{\nabla}\omega|^2 \\ &= \frac{2e^{\beta}}{\omega}(\mathfrak{h}-\omega^2(\frac{\not{\Delta}\omega}{\omega}-\frac{|\not{\nabla}\omega|^2}{\omega^2})) - 2\beta_r e^{\beta}|\not{\nabla}\omega|^2 \\ &= \frac{2e^{\beta}}{\omega}(\mathfrak{h}-\omega^2\not{\Delta}\log\omega - \omega\beta_r|\not{\nabla}\omega|^2).\end{aligned}$$

As a result we have that

$$\langle\vec{H},\vec{H}\rangle = \operatorname{tr}\underline{\chi}\operatorname{tr}\chi = \frac{4}{\omega^2}(\mathfrak{h}-\omega^2\not{\Delta}\log\omega-\omega\beta_r|\not{\nabla}\omega|^2).$$

Since the metric on Σ is given by $\omega^2\mathring{\gamma}$ we conclude that it has Gaussian curvature

$$\mathcal{K} = \frac{1}{\omega^2}(1-\mathring{\Delta}\log\omega) = \frac{1}{\omega^2} - \not{\Delta}\log\omega$$

and therefore

$$\mathcal{K} - \frac{1}{4}\langle\vec{H},\vec{H}\rangle = \frac{2M(v_0,\omega)}{\omega^3} + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2.$$

Moreover, the torsion is given by

$$\zeta(\tilde{V}) = \frac{1}{2}\langle D_{\tilde{V}}\underline{L},L\rangle = \frac{e^{-\beta}}{2r}\langle V,L\rangle = -\frac{1}{r}V\omega = -\frac{1}{r}\tilde{V}\omega$$

from which we conclude $\zeta(\tilde{V})|_{\Sigma} = -\tilde{V}\log\omega$ and $\not{\nabla}\cdot\zeta = -\not{\Delta}\log\omega$, giving

$$\rho = \frac{2M(v_0,\omega)}{\omega^3} + \not{\Delta}\beta + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2. \qquad \square$$

Remark 5.2. We recover the data of Lemma 2.1 as soon as we set $m_0 = M$, $\beta = 0$ and $r_0 = 2m_0$ as expected.

So in comparison to Schwarzschild spacetime we have the additional terms $\not{\Delta}\beta + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2$ in the flux function ρ. It turns out that a non-trivial $G(\underline{L}, \underline{L})$ is responsible. Since $\not{\Delta}\beta = \beta_{rr}|\not{\nabla}\omega|^2 + \beta_r \not{\Delta}\omega$ and

$$\begin{aligned} G(\underline{L}, \underline{L}) &= -\underline{L}\,\mathrm{tr}\,K - \frac{1}{2}(\mathrm{tr}\,K)^2 - |\hat{K}|^2 \\ &= -e^{-\beta}\partial_r(\frac{2e^{-\beta}}{r}) - \frac{1}{2}\frac{4e^{-2\beta}}{r^2} = \frac{2\beta_r}{r}e^{-2\beta} \end{aligned}$$

it follows, for arbitrary ω, that $\not{\Delta}\beta(\omega) + \frac{\beta_r}{\omega}|\not{\nabla}\omega|^2 = 0$ if and only if β is independent of the r-coordinate and therefore $G(\underline{L}, \underline{L}) = 0$. For the function $M(v_0, r)$ we look to $G(\underline{L}, L)$ along the foliation $\{\Sigma_r\}$ since:

$$\begin{aligned} G(\underline{L}, L) &= \underline{L}\,\mathrm{tr}\,\chi - 2\mathcal{K}_s + 2\nabla \cdot t + 2|\vec{t}|^2 + \langle \vec{H}, \vec{H} \rangle_s \\ &= e^{-\beta}\partial_r(\frac{2e^{\beta}}{r}(1 - \frac{2M}{r})) - \frac{2}{r^2} + \frac{4}{r^2}(1 - \frac{2M}{r}) \\ &= \frac{2\beta_r}{r}(1 - \frac{2M}{r}) - \frac{4M_r}{r^2}. \end{aligned}$$

It follows from Corollary 3.4, on Σ_r, that

$$G_{\underline{L}} = 0.$$

Since these components are all that contribute to the monotonicity of (1.2) for the foliation $\{\Sigma_r\}$ we see that our need of the dominant energy condition reduces to

$$0 \leq \mathfrak{h}\beta_r \leq \frac{2M_r}{r}$$

on $\{\mathfrak{h} \geq 0\} \cap \Omega$. Next we show that $\{\Sigma_r\}$ is a re-parametrization of a geodesic and strict doubly convex foliation:

5.3. Asymptotic flatness

We now wish to choose the necessary decay on β and M in order to employ Theorem 1.9. For $\underline{L} = e^{-\beta}\partial_r$ the geodesic foliation $\{\Sigma_s\}$ has level set function given by

$$s(r) = \int_{r_0}^{r} e^{\beta(t)} dt$$

for which $\omega = const. \iff s = const.$ and therefore

$$\rho(s) = \frac{2M(r(s))}{r(s)^3}.$$

It follows from Lemma 5.1 that $\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\Delta\!\!\!/ \log \rho = \frac{\mathfrak{h}}{r(s)^2} > 0$ is equivalent to $r(s) > r_0 = 2M(v_0, r_0)$ as in Schwarzschild.

Lemma 5.3. *Choosing $|\beta(v_0, r)| = o_2(r^{-1})$ integrable, $M(v_0, r) = m_0 + o(r^0)$ for some constant m_0, Ω is asymptotically flat with strong flux decay.*

Proof. We've already verified that $G_{\underline{L}} = 0$. Since $\frac{ds}{dr} = e^{\beta(r)} = (1 + \beta\frac{e^{\beta}-1}{\beta})$, $|\beta|$ is integrable and $\frac{e^{\beta}-1}{\beta}$ is bounded it follows that $\frac{ds}{dr} = 1 + f$ where $|f| = o_2(r^{-1})$ is integrable. As a result

$$s = r - r_0 + \int_{r_0}^{\infty} f(t)dt - \int_{r}^{\infty} f(t)dt = r - c_0 + o_3(r^0)$$

where $\beta_0 = \int_{r_0}^{\infty} f(t)dt$ and $c_0 = r_0 - \beta_0$. We conclude that $r(s) = s + c_0 + o_3(s^0)$ since our assumptions on β imply that $\int_{r(s)}^{\infty} f(t)dt = o_3(s^0)$. From the fact that

$$\begin{aligned}\gamma_s &= r^2\mathring{\gamma}|_{\Sigma_s} = (s + c_0 + o_3(1))^2\mathring{\gamma} = s^2(1 + \frac{c_0}{s} + o_3(s^{-1}))^2\mathring{\gamma} \\ &= s^2\mathring{\gamma} + 2c_0 s\mathring{\gamma} + o_3(s)\mathring{\gamma}\end{aligned}$$

we see $\tilde{\gamma} = o_3(s)\mathring{\gamma}$ ensuring condition 1 of Definition 4.7 holds up to strong decay given that all dependence on tangential derivatives falls on the $\underline{L}$ Lie constant tensor $\mathring{\gamma}$. Since $\vec{t} = 0$ for this foliation condition 2 follows trivially up to strong decay. If we assume that $M(v_0, r) = m_0 + o(1)$ for some constant m_0 we see directly from Lemma 5.1

$$\begin{aligned}\operatorname{tr} Q = \operatorname{tr}\chi|_{\Sigma_s} &= \frac{2}{r}(1 - \frac{2M}{r})|_{\Sigma_s} + o(s^{-2}) = \frac{2}{s}(1 - \frac{c_0}{s})(1 - \frac{2m_0}{s}) + o(s^{-2}) \\ &= \frac{2}{s} - 2\frac{c_0 + 2m_0}{s^2} + o(s^{-2})\end{aligned}$$

giving us the third condition of Definition 4.7.

As mentioned in Remark 4.22, strong flux decay bypasses our need of the forth condition of Definition 4.7, since $\operatorname{tr} Q = \frac{2\mathring{\mathcal{K}}}{s} + o(s^{-1})$ is verified above. $\square$

From Lemma 5.3, Theorem 1.9, Theorem 3.9 and the comments immediately proceeding Remark 5.2 we have the following proof of the known (see [18]) null Penrose conjecture in spherical symmetry:

Theorem 5.4. *Suppose $\Omega := \{v = v_0\}$ is a standard null cone of a spherically symmetric spacetime of metric*

$$ds^2 = -\Big(1 - \frac{2M(v,r)}{r}\Big)e^{2\beta(v,r)}dv^2 + 2e^{\beta(v,r)}dvdr + r^2\Big(d\vartheta^2 + \sin\vartheta^2 d\varphi^2\Big)$$

where

1) $|\beta(v_0, r)| = o_2(r^{-1})$ *is integrable*
2) $M(v_0, r) = m_0 + o(r^0)$ *for some constant* $m_0 > 0$
3) $0 \le \mathfrak{h}\beta_r \le \frac{2M_r}{r}$

Then,

$$\sqrt{\frac{|\Sigma|}{16\pi}} \le m_0$$

for m_0 the Trautman-Bondi mass of Ω and $\Sigma := \{r_0 = 2M(v_0, r_0)\}$. In the case of equality we have $\beta = 0$ and $M = m_0$ so that Ω is a standard null cone of Schwarzschild spacetime.

5.4. Perturbing spherical symmetry

We wish to study perturbations off of the spherically symmetric metric given in Theorem 5.4 for the coordinate chart $(v, r, \vartheta, \varphi)$. We start by choosing a 1-form η such that $\eta(\partial_r(\partial_v)) = \mathcal{L}_{\partial_v}\eta = 0$ and a 2-tensor γ satisfying $\gamma(\partial_r(\partial_v), \cdot) = \mathcal{L}_{\partial_v}\gamma = 0$ with restriction $\gamma|_{(v,r)\times\mathbb{S}^2}$ positive definite. Finally, we choose smooth functions M, β and α. Defining $\vec{\eta}$ to be the unique vector field satisfying $r^2\gamma(\vec{\eta}, X) = \eta(X)$ for arbitrary $X \in \Gamma(TM)$ and $|\vec{\eta}|^2 := r^2\gamma(\vec{\eta}, \vec{\eta})$ the spacetime metric and its inverse are given by

$$\begin{aligned} g &= -(\mathfrak{h} + \alpha)e^{2\beta}dv \otimes dv + e^{\beta}(dv \otimes (dr + \eta) + (dr + \eta) \otimes dv) + r^2\gamma \\ g^{-1} &= e^{-\beta}(\partial_v \otimes \partial_r + \partial_r \otimes \partial_v) + (\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r \otimes \partial_r \\ &\qquad - (\vec{\eta} \otimes \partial_r + \partial_r \otimes \vec{\eta}) + \frac{1}{r^2}\gamma^{-1}. \end{aligned}$$

We see that $\Omega := \{v = v_0\}$ remains a null hypersurface with $\underline{L}(= Dv) = e^{-\beta}\partial_r \in \Gamma(T\Omega) \cap \Gamma(T^\perp\Omega)$. Our metric resembles the perturbed metric used by Alexakis [1]. We'll need the following to specify our decay conditions:

Definition 5.5. Suppose Ω extends to past null infinity with level set function s for some null field $\underline{L}$. For a transversal k-tensor T

- We say $T(s,\delta) = \delta o_n^X(s^{-m})$ if $T = o_n^X(s^{-m})$ and

$$\limsup_{\delta\to 0}\sup_{\Omega}\frac{1}{\delta}|s^m(\mathcal{L}_{X_{i_1}}...\mathcal{L}_{X_{i_j}}T)(s,\delta)| < \infty \text{ for } 0 \le j \le n$$

- We define

$$|T|^2_{\mathring{H}^m} = |T|^2_{\mathring{\gamma}} + |\mathring{\nabla}T|^2_{\mathring{\gamma}} + \cdots + |\mathring{\nabla}^m T|^2_{\mathring{\gamma}}.$$

Decay Conditions on Ω:

1) $r^2\gamma = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ where:
   a) $\mathring{\gamma}$ is the ∂_r-Lie constant, transversal standard round metric on $\mathbb{S}^2$ independent of δ
   b) γ_1 is a ∂_r-Lie constant, transversal 2-tensor independent of δ
   c) $\tilde{\gamma}$ is a transversal 2-tensor satisfying $(\mathcal{L}_{\partial_r})^i\tilde{\gamma} = \delta o^X_{5-i}(r^{1-i})$ for $0 \le i \le 3$

2) $\alpha = \delta\frac{\alpha_0}{r} + \tilde{\alpha}$ where α_0 is a ∂_r-Lie constant function independent of δ and $|\tilde{\alpha}|_{\mathring{H}^2} \le \delta h_1(r)$ for $h_1 = o(r^{-1})$

3) β satisfies:
   a) $|\beta| = o_2(r^{-1})$ is r-integrable
   b) $|\mathring{\nabla}\beta|_{\mathring{H}^3} \le \delta h_2(r)$ for some integrable $h_2 = o(r^{-1})$
   c) $|\mathring{\nabla}\beta_r|_{\mathring{H}^2} = O(r^{-1})$

4) $M = m_0 + \tilde{m}$ where $m_0 > 0$ is a constant independent of δ and $|\tilde{m}|_{\mathring{H}^2} \le \delta h_3(r)$ for $h_3 = o(1)$

5) η is a transversal 1-form satisfying:
   a) $\eta = o_2(1)$
   b) $|\eta|_{\mathring{H}^3} + r|\mathcal{L}_{\partial_r}\eta|_{\mathring{H}^3} \le \delta h_4(r)$ for $h_4 = o(1)$.

5.4.1. The geodesic foliation. As in the spherically symmetric case we identify the null geodesic field $Dv = e^{-\beta}\partial_r$. We will again for convenience take the background foliation to be level sets of the coordinate r. We wish therefore to relate the given decay in r to the geodesic foliation given by

the field $\underline{L} := Dv$ in order to show Ω is asymptotically flat with strong flux decay.
Once again $\frac{ds}{dr} = e^{\beta} = 1 + f$ where $f = \beta\frac{e^{\beta}-1}{\beta}$ is r-integrable due to decay condition 3. Taking local coordinates (ϑ, φ) on Σ_{r_0} (for some r_0) we have

$$s = r - c_0(\vartheta, \varphi) - \beta_1(r, \vartheta, \varphi) \tag{5.5}$$

for $\beta_0(\vartheta, \varphi) := \int_{r_0}^{\infty} f(t, \vartheta, \varphi)dt$, $c_0 = r_0 - \beta_0$ and $\beta_1(r, \vartheta, \varphi) = \int_{r}^{\infty} f(t, \vartheta, \varphi)dt$. Since each Σ_r is compact, an m-th order partial derivative of f is bounded by $C|\mathring{\nabla} f|_{\mathring{H}^{m-1}}$ for some constant C independent of r (from decay condition 3). From decay condition 3, provided $m \leq 4$, derivatives in ϑ, φ of β_0 and β_1 pass into the integral (for fixed r) onto f and are bounded. On any Σ_s (i.e fixed s) it follows from (5.5) that

$$\partial_{\vartheta(\varphi)} r = -\frac{\int_{r_0}^{r} \partial_{\vartheta(\varphi)} f(t, \vartheta, \varphi)dt}{1 + f} = -e^{-\beta} \int_{r_0}^{r} \beta_{\vartheta(\varphi)} e^{\beta} dt$$

with bounded derivatives up to third order. It's a simple verification in local coordinates, from

$$r(s, \vartheta, \varphi) = s + c_0(\vartheta, \varphi) + \beta_1(r(s, \vartheta, \varphi), \vartheta, \varphi),$$

that $\partial_s^i \beta_1 = o_{3-i}^X(s^{-i})$ for $0 \leq i \leq 3$. Coupled with the fact that $\mathcal{L}_{\underline{L}} = e^{-\beta}\mathcal{L}_{\partial_r}$ on transversal tensors we conclude that $(\mathcal{L}_{\underline{L}})^i \tilde{\gamma} = o_{3-i}^X(s^{1-i})$ for $0 \leq i \leq 3$ and therefore

$$\gamma_s = r^2 \gamma|_{\Sigma_s} = s^2 \mathring{\gamma} + s\Gamma_1 + \tilde{\Gamma} \tag{5.6}$$

where

$$\begin{aligned} \Gamma_1 &= 2c_0 \mathring{\gamma} + \delta\gamma_1 \\ \tilde{\Gamma} &= \Big(2s\beta_1 + (\beta_1 + c_0)^2\Big)\mathring{\gamma} + \delta(\beta_1 + c_0)\gamma_1 + \tilde{\gamma} \end{aligned}$$

satisfies the requirements towards strong decay.

5.4.2. Calculating $\boldsymbol{\rho}$. Since we will compare computations for the foliation $\{\Sigma_r\}$ with the geodesic foliation of 5.4.1 we will revert back to denoting the covariant derivative on Σ_s by ∇ and the covariant derivative on Σ_r by $\nabla\!\!\!/$. Beyond Definition 5.5, we will also need to explicitly refer to vector field extensions $[\partial_r, V] = 0$ off of some Σ_{r_0} which, given $\beta = \beta(r, \vartheta, \varphi)$, distinguishes from extensions along $\underline{L}$ (denoted by $E(\Sigma_{r_0})$). We therefore contrast by using the subscript $V \in E_{\partial_r}(\Sigma_{r_0})$.

For the foliation $\{\Sigma_r\}$ we have the linearly independent normal vector fields

$$\begin{aligned}\underline{L} &= e^{-\beta}\partial_r\\ Dr &= e^{-\beta}\partial_v + (\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r - \vec{\eta}\end{aligned}$$

from which similar calculations as in spherical symmetry yield the unique null normal satisfying $\langle \underline{L}, L\rangle = 2$ to be given by

$$L = 2\partial_v + e^{\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\partial_r - 2e^{\beta}\vec{\eta}.$$

Lemma 5.6. *We have*

$$\begin{aligned}\underline{\chi} &= e^{-\beta}(r\mathring{\gamma} + \frac{\delta}{2}\gamma_1 + \frac{1}{2}(\mathcal{L}_{\partial_r}\tilde{\gamma}))\\ \operatorname{tr}\underline{\chi} &= e^{-\beta}(\frac{2}{r} + \frac{\delta\underline{\theta}}{r^2}) + \delta o_4^X(r^{-2})\end{aligned}$$

for $\underline{\theta} := -\frac{1}{2}\mathring{\operatorname{tr}}\gamma_1$*. Moreover,*

$$\not\nabla^m\underline{\chi} = -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}^m\gamma_1 + \delta o_{4-m}^X(1),\ \ 0 \le m \le 4.$$

Proof. First we extend $V, W \in E_{\partial_r}(\Sigma_{r_0})$ off of Ω such that $[\partial_v, V(W)] = 0$. Then for $\underline{\chi}$:

$$\begin{aligned}\underline{\chi}(V,W) &= \langle D_V(e^{-\beta}\partial_r), W\rangle = e^{-\beta}\langle D_V\partial_r, W\rangle = e^{-\beta}\frac{1}{2}\partial_r\langle V, W\rangle\\ &= e^{-\beta}(r\mathring{\gamma}(V,W) + \frac{\delta}{2}\gamma_1(V,W) + \frac{1}{2}\mathcal{L}_{\partial_r}\tilde{\gamma}(V,W))\end{aligned}$$

having used the Koszul formula to get the third equality. So using a basis extension $\{X_1, X_2\} \subset E_{\partial_r}(\Sigma_{r_0})$ Proposition 4.9 provides the inverse metric $\gamma(r)^{ij} = \frac{1}{r^2}\mathring{\gamma}^{ij} - \frac{\delta}{r^3}\mathring{\gamma}_1{}^{ij} + \delta o_5^X(r^{-3})$ and $\operatorname{tr}\underline{\chi}$ follows by contracting $\gamma(r)^{-1}$ with $\underline{\chi}$. For the final identity we note from Lemma 4.11 we have for the decomposition $\gamma(r) = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ the difference tensor

$$\begin{aligned}\langle \mathcal{D}(V,W), U\rangle &= \langle \not\nabla_V W - \mathring{\nabla}_V W, U\rangle\\ &= \frac{r\delta}{2}\Big(\mathring{\nabla}_V\gamma_1(W,U) + \mathring{\nabla}_W\gamma_1(V,U) - \mathring{\nabla}_U\gamma_1(V,W)\Big)\\ &\quad + \frac{1}{2}\Big(\mathring{\nabla}_V\tilde{\gamma}(W,U) + \mathring{\nabla}_W\tilde{\gamma}(V,U) - \mathring{\nabla}_U\tilde{\gamma}(V,W)\Big)\end{aligned}$$

for $V, W, U \in E_{\partial_r}(\Sigma_{r_0})$. So proceeding as in Proposition 4.20

$$\begin{aligned}
\underline{\nabla}_i \underline{\chi}_{jk} &= \mathring{\nabla}_i \underline{\chi}_{jk} - \mathcal{D}^m_{ij}\underline{\chi}_{mk} - \mathcal{D}^m_{ik}\underline{\chi}_{jm} \\
&= \mathring{\nabla}_i (re^{-\beta}\mathring{\gamma}_{jk} + e^{-\beta}\frac{\delta}{2}\gamma_{1jk} + e^{-\beta}\frac{1}{2}(\mathcal{L}_{\partial_r}\tilde{\gamma})_{jk}) - e^{-\beta}\delta\mathring{\nabla}_i(\gamma_{1jk}) + \delta o_4^X(1) \\
&= r\mathring{\gamma}_{jk}\mathring{\nabla}_i(e^{-\beta}) - e^{-\beta}\frac{\delta}{2}\mathring{\nabla}_i\gamma_{1jk} + \frac{\delta}{2}\gamma_{1jk}\mathring{\nabla}_i(e^{-\beta}) + \delta o_3^X(1) \\
&= -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}_i\gamma_{1jk} + \delta o_3^X(1).
\end{aligned}$$

Iteration provides our result

$$\underline{\nabla}^m \underline{\chi} = -e^{-\beta}\frac{\delta}{2}\mathring{\nabla}^m\gamma_1 + \delta o_{4-m}^X(1), \;\; 1 \le m \le 4$$

from decay condition 3. □

For χ we have

$$\begin{aligned}
\chi(V,W) &= 2\langle D_V\partial_v, W\rangle + e^{\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\langle D_V\partial_r, W\rangle - 2\langle D_V e^{\beta}\vec{\eta}, W\rangle \\
&= 2\langle D_V\partial_v, W\rangle + e^{2\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\underline{\chi}(V,W) - 2\underline{\nabla}_V(e^{\beta}\eta)(W)
\end{aligned}$$

and using the Koszul formula on the first term we see

$$\begin{aligned}
2\langle D_V\partial_v, W\rangle &= V(e^{\beta}\eta(W)) + \partial_v\langle V, W\rangle - W(e^{\beta}\eta(V)) - \langle V, [\partial_v, W]\rangle \\
&\quad + \langle \partial_v, [W,V]\rangle + \langle W, [V,\partial_v]\rangle \\
&= \underline{\nabla}_V(e^{\beta}\eta)(W) - \underline{\nabla}_W(e^{\beta}\eta)(V) \\
&= \mathrm{curl}(e^{\beta}\eta)(V,W)
\end{aligned}$$

so that a trace over V, W yields $\mathrm{tr}\,\chi = e^{2\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\,\mathrm{tr}\,\underline{\chi} - 2\underline{\nabla}\cdot(e^{\beta}\eta)$ and therefore

$$\begin{aligned}
\langle \vec{H}, \vec{H}\rangle &= e^{2\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)(\mathrm{tr}\,\underline{\chi})^2 - 2\underline{\nabla}\cdot(e^{\beta}\eta)\,\mathrm{tr}\,\underline{\chi} \\
&= \Big(1 - \frac{2M}{r} + \delta\frac{\alpha_0}{r}\Big)\Big(\frac{2}{r} + \frac{\delta\underline{\theta}}{r^2}\Big)^2 + \delta o_2^X(r^{-3}) \\
&= \Big(1 - \frac{2M}{r} + \delta\frac{\alpha_0}{r}\Big)\Big(\frac{4}{r^2} + \frac{4\delta\underline{\theta}}{r^3}\Big) + \delta o_2^X(r^{-3}) \\
&= \frac{4}{r^2}\Big(1 - \frac{2m_0}{r} + \delta\frac{\underline{\theta}}{r} + \delta\frac{\alpha_0}{r}\Big) + \delta o_2^X(r^{-3})
\end{aligned}$$

from decay conditions 2-5. For ζ we have

$$\zeta(V) = \langle D_V(e^{-\beta}\partial_r), \partial_v\rangle - e^{\beta}\langle D_V(e^{-\beta}\partial_r), \vec{\eta}\rangle$$

$$= -V\beta + e^{-\beta}\langle D_V\partial_r, \partial_v\rangle - \langle D_V\partial_r, \vec{\eta}\rangle$$
$$= -V\beta + e^{-\beta}\langle D_V\partial_r, \partial_v\rangle - e^{\beta}\underline{\chi}(V, \vec{\eta}).$$

From the Koszul formula

$$\begin{aligned}2\langle D_V\partial_r, \partial_v\rangle &= V\langle\partial_r, \partial_v\rangle + \partial_r\langle V, \partial_v\rangle - \partial_v\langle V, \partial_r\rangle - \langle V, [\partial_r, \partial_v]\rangle + \langle\partial_r, [\partial_v, V]\rangle \\ &\quad + \langle\partial_v, [V, \partial_r]\rangle \\ &= e^{\beta}V\beta + \partial_r(e^{\beta}\eta(V)) = e^{\beta}V\beta + \mathcal{L}_{\partial_r}(e^{\beta}\eta)(V)\end{aligned}$$

from which we conclude that $\zeta(V) = -\frac{1}{2}V(\beta) + \frac{e^{-\beta}}{2}\mathcal{L}_{\partial_r}(e^{\beta}\eta)(V) - e^{\beta}\underline{\chi}(V, \vec{\eta})$ and

$$\begin{aligned}\nabla\!\!\!/ \cdot \zeta &= -\frac{1}{2}\Delta\!\!\!/\beta + \frac{1}{2}\nabla\!\!\!/ \cdot (e^{-\beta}\mathcal{L}_{\partial_r}(e^{\beta}\eta)) - \nabla\!\!\!/ \cdot (e^{\beta}\underline{\chi}(\vec{\eta})) \\ &= -\frac{1}{2}\Delta\!\!\!/\beta + \frac{1}{2}\nabla\!\!\!/ \cdot (\beta_r\eta) + \frac{1}{2}\nabla\!\!\!/ \cdot (\mathcal{L}_{\partial_r}\eta) - e^{\beta}\underline{\chi}(\nabla\!\!\!/\beta, \vec{\eta}) - e^{\beta}\nabla\!\!\!/ \cdot (\underline{\chi}(\vec{\eta})) \\ &= \delta o_2^X(r^{-3})\end{aligned}$$

having used decay conditions 3, 5, and Lemma 5.6 for the final line.

Lemma 5.7. *Ω satisfies conditions 1, 2 and 3 of Definition 4.7. Ω additionally satisfies the strong flux decay condition if and only if*

$$\frac{1}{2}\mathring{\nabla} \cdot \gamma_1 + d\underline{\theta} = 0$$

for $\underline{\theta} = -\frac{1}{2}\mathring{\mathrm{tr}}\gamma_1$ and is subsequently past asymptotically flat.

Proof. Having already verified condition 1 up to strong decay for γ_s of our geodesic foliation $\{\Sigma_s\}$ we continue to show conditions 2 and 3.
Given $V \in E_{\partial_r}(\Sigma_{r_0})$ Lemma 4.2 ensures $V - Vs\underline{L}|_{\Sigma_s} \in \Gamma(T\Sigma_s)$ and we see that

$$\begin{aligned}[V - Vs\underline{L}, \underline{L}] &= [V, \underline{L}] + \underline{L}Vs\underline{L} = e^{\beta}V(e^{-\beta})\underline{L} + e^{-\beta}V(\partial_r s)\underline{L} \\ &= (e^{\beta}V(e^{-\beta}) + e^{-\beta}V(e^{\beta}))\underline{L} = 0.\end{aligned}$$

So $V - Vs\underline{L} \in E(\Sigma_0)$ and Lemma 4.2 gives

$$\begin{aligned}t(V - Vs\underline{L}) &= t(V) = \zeta(V) + \underline{\chi}(V, \nabla\!\!\!/ s) \\ &= -\frac{1}{2}V(\beta) + \frac{1}{2}\beta_r\eta(V) + \frac{1}{2}(\mathcal{L}_{\partial_r}\eta)(V) - e^{\beta}\underline{\chi}(V, \vec{\eta}) + \underline{\chi}(V, \nabla\!\!\!/ s)\end{aligned}$$

$$\begin{aligned}
&= (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V) + \underline{\chi}(V, \nabla\!\!\!/ s)\\
&= (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V) + re^{-\beta}\mathring{\gamma}(V, \frac{1}{r^2}\mathring{\nabla} s)\\
&= \frac{e^{-\beta}}{r} V\beta_0 + (\delta o_3^X(r^{-1}) \cap o_1(r^{-1}))(V)
\end{aligned}$$

having used decay conditions 3 and 5 to get the second line, Lemma 5.6 for the third and (5.5) for the last. Moreover,

$$\begin{aligned}
(\mathcal{L}_{V-Vs\underline{L}}t)(W - Ws\underline{L}) &= (V - Vs\underline{L})(t(W - Ws\underline{L})) - t([V, W])\\
&= (\mathcal{L}_V t)(W) - Vs\underline{L}(t(W))\\
&= (\mathcal{L}_V - e^{-\beta}Vs\mathcal{L}_{\partial_r})(\frac{d\beta_0}{r})(W) + o(r^{-1})\\
&= \frac{1}{r}\mathcal{L}_V(d\beta_0)(W) + o(r^{-1})\\
&= \frac{1}{r}(\mathcal{L}_{V-Vs\underline{L}}d\beta_0)(W - Ws\underline{L}) + o(r^{-1})
\end{aligned}$$

where the last line follows since β_0 is $\underline{L}$-Lie constant. With a basis extension $\{X_i\} \subset E(\Sigma_0)$ we therefore conclude that $\mathcal{L}_{X_i}t = \frac{1}{s}\mathcal{L}_{X_i}d\beta_0 + o(s^{-1})$ so that condition 2 for asymptotic flatness is satisfied up to strong decay with $t_1 = d\beta_0$. From Proposition 4.9 and (5.6):

$$\begin{aligned}
\operatorname{tr} K &= \frac{2}{s} - \frac{1}{2s^2}\mathring{\operatorname{tr}}\Gamma_1 + o(s^{-2}) = \frac{2}{s} - \frac{1}{2s^2}\mathring{\operatorname{tr}}(2c_0\mathring{\gamma} + \delta\gamma_1) + o(s^{-2})\\
&= \frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2} + o(s^{-2})
\end{aligned}$$

and

$$\begin{aligned}
\hat{K} &= K - \frac{1}{2}\operatorname{tr} K\gamma_s = s\mathring{\gamma} + \frac{1}{2}\Gamma_1 - \frac{1}{2}\Big(\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2} + o(s^{-2})\Big)\gamma_s + o(1)\\
&= s\mathring{\gamma} + \frac{1}{2}(2c_0\mathring{\gamma} + \delta\gamma_1) - \frac{1}{2}(\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2})(s^2\mathring{\gamma} + s(2c_0\mathring{\gamma} + \delta\gamma_1)) + o(1)\\
&= -\frac{\delta}{2}(\gamma_1 + \underline{\theta}\mathring{\gamma}) + o(1).
\end{aligned}$$

For condition 3 we take $r|_{\Sigma_s} \in \mathcal{F}(\Sigma_s)$ and Lie drag it to the the rest of Ω along ∂_r (hence $\underline{L}$) to give $r_s \in \mathcal{F}(\Omega)$. Using Lemma 4.2 from the vantage point of the cross section Σ_s amongst the background foliation $\{\Sigma_r\}$:

$$e^{-\beta}\operatorname{tr} Q = e^{-\beta}\operatorname{tr}\chi - 4(\zeta + d\!\!\!/ \log e^{\beta})(\nabla\!\!\!/ r_s) - 2\Delta r_s + |\nabla\!\!\!/ r_s|^2 e^{\beta}\operatorname{tr}\underline{\chi} - 2\beta_r|\nabla\!\!\!/ r_s|^2$$

From the expression of $r(s)$ in 5.4.1, recalling Remark 4.15, we see $dr_s = -d\beta_0 + o(1)$ from which Lemma 4.11 implies that $\Delta r_s = -\frac{1}{s^2}\mathring{\Delta}\beta_0 + o(s^{-2})$. From decay conditions 3, 5, and Lemma 5.6 we have

$$\begin{aligned}
\operatorname{tr} Q &= \operatorname{tr}\chi|_{\Sigma_s} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \Big(e^{2\beta}(\mathfrak{h} + \alpha + |\vec{\eta}|^2)\operatorname{tr}\underline{\chi} - 2\nabla\!\!\!\!/ \cdot (e^{\beta}\eta)\Big)|_{\Sigma_s} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= (\frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0}{s^2})(1 - \frac{2M}{s} + \delta\frac{\alpha_0}{s}) + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \frac{2}{s} + \frac{\delta\underline{\theta} - 2c_0 - 4M + 2\delta\alpha_0}{s^2} + 2\frac{\mathring{\Delta}\beta_0}{s^2} + o(s^{-2}) \\
&= \frac{2}{s} - 2\frac{c_0 + 2M}{s^2} + \frac{1}{s^2}(\delta\underline{\theta} + 2\mathring{\Delta}\beta_0 + 2\delta\alpha_0) + o(s^{-2})
\end{aligned}$$

and condition 3 follows as soon as we set $M = m_0 + \delta o_2^X(1)$. As in the spherically symmetric case the highest order term for $\operatorname{tr} Q$ agrees with $\frac{2\mathring{\mathcal{K}}}{s}$ where $\mathring{\mathcal{K}} = 1$ is the Gaussian curvature of $\mathring{\gamma}$. We recall that our need for condition 4 depends on whether Ω has strong flux decay (Remark 4.22). From Proposition 4.9 and 5.6 we will have strong flux decay if and only if

$$\begin{aligned}
d\beta_0 = t_1 &= \frac{1}{2}\mathring{\nabla}\cdot\Gamma_1 - \frac{1}{2}d\!\!\!/\,\mathring{\operatorname{tr}}\Gamma_1 \\
&= \frac{1}{2}\mathring{\nabla}\cdot(2c_0\mathring{\gamma} + \delta\gamma_1) + d\!\!\!/(\delta\underline{\theta} - 2c_0) \\
&= \frac{1}{2}d(-2\beta_0) + \frac{\delta}{2}\mathring{\nabla}\cdot\gamma_1 + \delta d\underline{\theta} + 2d\beta_0 \\
&= d\beta_0 + \delta(\frac{1}{2}\mathring{\nabla}\cdot\gamma_1 + d\underline{\theta})
\end{aligned}$$

which in turn holds if and only if $\frac{1}{2}\mathring{\nabla}\cdot\gamma_1 + d\underline{\theta} = 0$. □

Henceforth we will assume the conditions of Lemma 5.7 for Ω. From Proposition 4.12

$$\begin{aligned}
\mathcal{K}_{r^2\gamma} &= \frac{1}{r^2} + \frac{\delta}{r^3}\Big(\underline{\theta} + \frac{1}{2}\mathring{\nabla}\cdot\mathring{\nabla}\cdot\gamma_1 + \mathring{\Delta}\underline{\theta}\Big) + \delta o_4^X(r^{-3}) \\
&= \frac{1}{r^2} + \frac{\delta}{r^3}\underline{\theta} + \delta o_4^X(r^{-3}).
\end{aligned}$$

From Lemma 5.6 we have

$$\nabla\!\!\!\!/_i\nabla\!\!\!\!/_j\underline{\chi}_{mn} = -\frac{\delta}{2}\mathring{\nabla}_i\mathring{\nabla}_j\gamma_{1mn} + \delta o_2^X(1)$$

so that contraction with $\gamma(r)^{-1}$ first in mn then ij gives

$$\not\Delta \operatorname{tr} \underline{\chi} = \frac{\delta}{r^4} \mathring{\Delta} \underline{\theta} + \delta o_2^X(r^{-4})$$

which we use in $\not\Delta \log \operatorname{tr} \underline{\chi} = \frac{\not\Delta \operatorname{tr} \underline{\chi}}{\operatorname{tr} \underline{\chi}} - \frac{|\not\nabla \operatorname{tr} \underline{\chi}|^2}{(\operatorname{tr} \underline{\chi})^2}$ to conclude

$$\not\Delta \log \operatorname{tr} \underline{\chi} = \frac{\delta}{2r^3} \mathring{\Delta} \underline{\theta} + \delta o_2^X(r^{-3}).$$

Finally we have ρ

$$\begin{aligned}
\rho &= \mathcal{K}_{r^2\gamma} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \not\nabla \cdot \zeta - \not\Delta \log \operatorname{tr} \underline{\chi} \\
&= \frac{1}{r^2} + \frac{\delta}{r^3}\underline{\theta} - \frac{1}{r^2} + \frac{2m_0}{r^3} - \delta \frac{\underline{\theta}}{r^3} - \delta \frac{\alpha_0}{r^3} - \frac{\delta}{2r^3} \mathring{\Delta} \underline{\theta} + \delta o_2^X(r^{-3}) \\
&= \frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3}) \\
&= \frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\not\Delta \log \rho = {}& \frac{1}{r^2}\Big(1 - \frac{2m_0}{r} + \delta\frac{\underline{\theta}}{r} + \delta\frac{\alpha_0}{r}\Big) \\
& - \frac{1}{3}\not\Delta \log\Big(\frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})\Big) + \delta o_2^X(r^{-3}).
\end{aligned}$$

We may now use Lemma 4.11 to decompose the last term

$$\begin{aligned}
&\not\Delta \log\Big(\frac{2m_0}{r^3} - \frac{\delta}{r^3}(\frac{5}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3})\Big) \\
&\qquad = \frac{1}{r^2}\mathring{\Delta} \log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(1)\Big) + \delta o(r^{-2}) \\
&\qquad = \frac{1}{r^2}\mathring{\Delta} \log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big) + \delta o(r^{-2})
\end{aligned}$$

giving

$$\begin{aligned}
\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\not\Delta \log \rho = {}& \frac{1}{r^2}\Big(1 - \frac{2m_0}{r} - \frac{1}{3}\mathring{\Delta} \log\Big(1 - \frac{\delta}{2m_0}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big)\Big) \\
& + \delta o(r^{-2}).
\end{aligned}$$

Since $m_0 > 0$ we notice for sufficiently small δ our perturbation ensures $\rho > 0$ for all $r > 0$. However, from our construction so far it's not yet possible to conclude that some $\delta > 0$ will enforce $\frac{1}{4}\langle \vec{H}, \vec{H}\rangle \geq \frac{1}{3}\Delta\!\!\!/ \log\rho$ along the foliation. Moreover, the existence of a horizon ($\operatorname{tr}\chi = 0$) is equally questionable.

5.4.3. Smoothing to spherical symmetry. We will solve this difficulty by 'smoothing' away all perturbations in a neighborhood of the (desired) horizon in order to obtain spherical symmetry on $r < r_1$ for some $r_1 > 0$ yet to be chosen. The resulting spherical symmetry will uncover the horizon at $r = r_0 < r_1$ and will also provide a choice of $\delta > 0$ so that $\frac{1}{4}\langle \vec{H}, \vec{H}\rangle > \frac{1}{3}\Delta\!\!\!/ \log\rho$ away from it, causing the foliation $\{\Sigma_r\}$ to be strict doubly convex.

We will use a smooth step function $0 \leq S_\delta(r) \leq 1$ such that $S_\delta(r) = 0$ for $r < r_1$ and $S_\delta(r) = 1$ for $r > r_2$ for some finite $r_2(\delta)$ chosen to ensure $|S'_\delta(r)| \leq \delta$. We start off by choosing *parameter functions*, $\tilde{\beta}(v, r)$, $\tilde{M}(v, r)$, according to the conditions of Theorem 5.4 for the desired spherically symmetric region. Therefore, $r_0 = 2\tilde{M}(v_0, r_0)$, $2\tilde{M}(v_0, r) \leq r$ for $r > r_0$, $\tilde{M}_r \geq 0$, and $\tilde{M} = m_0 + o(1)$. We induce spherical symmetry on $r < r_1$ with the following substitutions:

$$\begin{aligned}
\tilde{\gamma} &\to \delta r(S_\delta(r) - 1)\gamma_1 + S_\delta(r)\tilde{\gamma}\\
\beta(r, \vartheta, \varphi) &\to S_\delta(r)\beta(r, \vartheta, \varphi) + (1 - S_\delta(r))\tilde{\beta}(v_0, r)\\
M(r, \vartheta, \varphi) &\to S_\delta(r)M(r, \vartheta, \varphi) + (1 - S_\delta(r))\tilde{M}(v_0, r)\\
\tilde{\alpha} &\to S_\delta(r)\tilde{\alpha} - (1 - S_\delta(r))\frac{\delta\alpha_0}{r}\\
\eta &\to S_\delta(r)\eta.
\end{aligned}$$

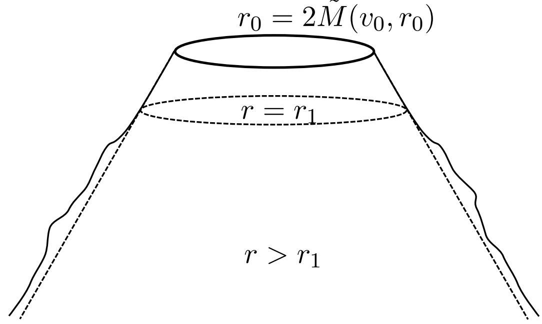
We leave the reader the simple verification that these changes to our perturbation tensors $\tilde{\gamma}$, β, M, $\tilde{\alpha}$ and η maintain the decay conditions 1-5. Clearly for $r > r_2$ our substitutions leave the metric unchanged while inducing spherical symmetry on $r < r_1$ with the spherical parameter functions $\tilde{\beta}\,, \tilde{M}$:

An example $S_\delta(r)$ is given by the function

$$S_\delta(r) = \begin{cases} 0 & r \leq r_1\\ \dfrac{e^{\frac{k}{r_1 - r}}}{e^{\frac{k}{r_1 - r}} + e^{\frac{k}{r - r_2}}} & r_1 < r < r_2\\ 1 & r_2 \leq r \end{cases}$$

where $k = \frac{4}{\delta}$ and $r_2(\delta) = r_1 + k$. Since $S_\delta(r) = P(\frac{1}{r_1 - r} + \frac{1}{r_2 - r})$, $P(r) = \frac{e^{kr}}{1 + e^{kr}}$ satisfies the logistic equation

$$P'(r) = kP(1 - P)$$



we have

$$\begin{aligned}
S'_\delta(r) &= kS_\delta(r)(1 - S_\delta(r))(\frac{1}{(r-r_1)^2} + \frac{1}{(r-r_2)^2}) \\
&= k\frac{S_\delta(r)}{(r-r_1)^2}(1 - S_\delta(r)) + kS_\delta(r)\frac{1 - S_\delta(r)}{(r-r_2)^2} \\
&\leq k\Big(\frac{S_\delta(r)}{(r-r_1)^2} + \frac{1 - S_\delta(r)}{(r-r_2)^2}\Big).
\end{aligned}$$

Elementary analysis reveals on the interval $r_1 < r < r_2$ that

$$\begin{aligned}
0 \leq \frac{e^{\frac{k}{r_1 - r}}}{(r-r_1)^2} \leq \frac{4}{k^2 e^2} \\
0 \leq \frac{1}{e^{\frac{k}{r_1 - r}} + e^{\frac{k}{r - r_2}}} \leq \frac{1}{2}e^2
\end{aligned}$$

yielding from simple symmetry arguments that both $\frac{S_\delta(r)}{(r-r_1)^2}, \frac{1-S_\delta(r)}{(r-r_2)^2} \leq \frac{2}{k^2}$ and therefore

$$0 \leq S'_\delta(r) \leq k\frac{4}{k^2} = \delta$$

as desired. Denoting $m(r, \delta) := S_\delta(r)m_0 + (1 - S_\delta(r))\tilde{M}(r)$ the new metric gives

$$\rho = \begin{cases} \frac{2\tilde{M}(v_0,r)}{r^3}, & r < r_1 \\ \frac{2m(r,\delta)}{r^3} - \frac{\delta}{r^3}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0) + \delta o_2^X(r^{-3}), & r_1 \leq r \end{cases}$$

and

$$\frac{1}{4}\langle \vec{H}, \vec{H}\rangle - \frac{1}{3}\not{\Delta}\log\rho =$$
$$\begin{cases} \frac{1}{r^2}(1 - \frac{2\tilde{M}(v_0,r)}{r}), & r < r_1 \\ \frac{1}{r^2}\Big(1 - \frac{2m(r,\delta)}{r} - \frac{1}{3}\mathring{\Delta}\log\Big(1 - \frac{\delta}{2m(r,\delta)}(\frac{1}{2}\mathring{\Delta}\underline{\theta} + \alpha_0)\Big)\Big) + \delta o(r^{-2}), & r_1 \leq r. \end{cases}$$

Since $m_0 \geq m(r,\delta) \geq \frac{r_0}{2}$, we see for any choice of $r_1 > 2m_0$, sufficiently small δ induces strict double convexity on the foliation $\{\Sigma_r\}$. If we therefore restrict to perturbations satisfying the dominant energy condition on Ω, then Theorem 1.9 implies the following:

Theorem 5.8. *Let g_δ be a metric perturbation off of spherical symmetry given by*

$$g_\delta = -(\mathfrak{h}+\alpha)e^{2\beta}dv\otimes dv + e^{\beta}(dv\otimes(dr+\eta)+(dr+\eta)\otimes dv)+r^2\gamma$$

where

1) $r^2\gamma = r^2\mathring{\gamma} + r\delta\gamma_1 + \tilde{\gamma}$ *is trasversal with* $\mathring{\gamma}$ *the transversal* ∂_r*-Lie constant round metric on* $\mathbb{S}^2$ *independent of* δ*,* γ_1 *a transversal* ∂_r*-Lie constant 2-tensor independent of* δ *satisfying* $\mathring{\nabla}\cdot\gamma_1 = d(\mathring{\mathrm{tr}}\gamma_1)$ *and* $(\mathcal{L}_{\partial_r})^i\tilde{\gamma} = \delta o^X_{5-i}(r^{1-i})$ *for* $0\leq i\leq 3$*.*
2) $\alpha = \delta\frac{\alpha_0}{r}+\tilde{\alpha}$ *where* α_0 *is a* ∂_r*-constant function independent of* δ *and* $|\tilde{\alpha}|_{\mathring{H}^2}\leq \delta h_1(r)$ *for* $h_1 = o(r^{-1})$
3) β *satisfies:*
   a) $|\beta| = o_2(r^{-1})$ *is r-integrable*
   b) $|\mathring{\nabla}\beta|_{\mathring{H}^3}\leq \delta h_2(r)$ *for some integrable* $h_2 = o(r^{-1})$
   c) $|\mathring{\nabla}\beta_r|_{\mathring{H}^2} = O(r^{-1})$
4) $M = m_0+\tilde{m}$ *where* $m_0>0$ *is a constant, independent of* δ *and* $|\tilde{m}|_{\mathring{H}^2}\leq\delta h_3(r)$ *for* $h_3 = o(1)$
5) η *is a transversal 1-form satisfying:*
   a) $\eta = o_2(1)$
   b) $|\eta|_{\mathring{H}^3} + r|\mathcal{L}_{\partial_r}\eta|_{\mathring{H}^3}\leq\delta h_4(r)$ *for* $h_4 = o(1)$*.*

Then, for sufficiently small δ*,* $\Omega := \{v = v_0\}$ *is past asymptotically flat with strong flux decay. In addition, for any choice of spherical parameters* $\tilde{\beta}$*,* $\tilde{M}$ *as in Theorem 5.4, smoothing to spherical symmetry with a step function* $S_\delta(r)$*, such that* $S'_\delta(r)\leq\delta$*, whereby:*

$$\begin{aligned}
\tilde{\gamma} &\to \delta r(S_\delta(r)-1)\gamma_1 + S_\delta(r)\tilde{\gamma}\\
\beta(r,\vartheta,\varphi) &\to S_\delta(r)\beta(r,\vartheta,\varphi) + (1-S_\delta(r))\tilde{\beta}(r)\\
M(r,\vartheta,\varphi) &\to S_\delta(r)M(r,\vartheta,\varphi) + (1-S_\delta(r))\tilde{M}(r)\\
\tilde{\alpha} &\to S_\delta(r)\tilde{\alpha} - (1-S_\delta(r))\frac{\delta\alpha_0}{r}\\
\eta &\to S_\delta(r)\eta
\end{aligned}$$

we have that $\Sigma := \{r_0 = 2\tilde{M}(v_0, r_0)\}$ *is marginally outer trapped and the coordinate spheres* $\{\Sigma_r\}_{r\geq r_0}$ *are strict doubly convex. Moreover, if* g_δ *respects the dominant energy condition on* Ω *we have the Penrose inequality:*

$$\sqrt{\frac{|\Sigma|}{16\pi}} \leq m_{TB}$$

where m_{TB} *is the Trautman-Bondi mass of* Ω*.*

Acknowledgments

The author would like to deeply thank Hubert L. Bray for his supervision and support during the development of these ideas as well as Marc Mars for insightful conversation and his invaluable comments upon a close reading of this paper. The author is also very appreciative for the financial support for his last year of graduate school and for travel to conferences provided by NSF grant DMS-1406396.

References

[1] S. Alexakis, *The Penrose inequality on perturbations of the Schwarzschild exterior*, arXiv preprint `arXiv:1506.06400` (2015).

[2] R. Arnowitt, S. Deser, and C. W. Misner, *Republication of: The dynamics of general relativity*, General Relativity and Gravitation **40** (2008), no. 9, 1997–2027.

[3] G. Bergqvist, *On the Penrose inequality and the role of auxiliary spinor fields*, Classical and Quantum Gravity **14** (1997), no. 9, 2577.

[4] L. Bieri, *An extension of the stability theorem of the Minkowski space in general relativity*, J. Differential Geom. **86** (2010), no. 1, 17–70.

[5] L. Bieri and P. T. Chruściel, *Future-complete null hypersurfaces, interior gluings, and the Trautman-Bondi mass*, arXiv preprint `arXiv:1612.04359` (2016).

[6] H. Bondi, M. Van der Burg, and A. Metzner, *Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1962) 21–52.

[7] H. Bray, S. Hayward, M. Mars, and W. Simon, *Generalized inverse mean curvature flows in spacetime*, Communications in mathematical physics **272** (2007), no. 1, 119–138.

[8] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, Journal of Differential Geometry **59** (2001), no. 2, 177–267.

[9] H. L. Bray, J. L. Jauregui, and M. Mars, *Time flat surfaces and the monotonicity of the spacetime Hawking mass II*, Ann. Henri Poincare **17** (2016), no. 6, 1457–1475.

[10] D. Christodoulou, *Mathematical problems of general relativity theory I and II*, Lecture Notes, to be published.

[11] D. Christodoulou and S. Klainerman, The Global Nonlinear Stability of the Minkowski Space (PMS-41), Princeton University Press (2014).

[12] P. T. Chruściel and T.-T. Paetz, *The mass of light-cones*, Class. Quantum Grav. **31** (2014) 102001.

[13] E. Gourgoulhon and J. L. Jaramillo, *A 3+ 1 perspective on null hypersurfaces and isolated horizons*, Physics Reports **423** (2006), no. 4, 159–294.

[14] A. M. H. Bondi, M.G.J. van der Berg, *Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems*, in Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 269, 21–52, The Royal Society (1962).

[15] S. W. Hawking, *The occurrence of singularities in cosmology* **294** (1966), no. 1439,.

[16] S. W. Hawking, *Gravitational radiation in an expanding universe*, Journal of Mathematical Physics **9** (1968), no. 4, 598–604.

[17] S. W. Hawking, *Black holes in general relativity*, Communications in Mathematical Physics **25** (1972) 152–166.

[18] S. A. Hayward, *Gravitational energy in spherical symmetry*, Physical Review D **53** (1996), no. 4, 1938.

[19] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, Journal of Differential Geometry **59** (2001), no. 3, 353–437.

[20] S. Klainerman and F. Nicolò, The Evolution Problem in General Relativity, Progress in Mathematical Physics, Birkhäuser (2003), ISBN 9780817642549.

[21] M. Ludvigsen and J. Vickers, *An inequality relating total mass and the area of a trapped surface in general relativity*, Journal of Physics A: Mathematical and General **16** (1983), no. 14, 3349.

[22] M. Mars, *Present status of the Penrose inequality*, Classical and Quantum Gravity **26** (2009), no. 19, 193001.

[23] M. Mars and A. Soria, *The asymptotic behaviour of the Hawking energy along null asymptotically flat hypersurfaces*, Classical and Quantum Gravity **32** (2015), no. 18, 185020.

[24] M. Mars and A. Soria, *On the Penrose inequality along null hypersurfaces*, Classical and Quantum Gravity **33** (2016), no. 11, 115019.

[25] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Vol. 103, Academic press (1983).

[26] A. R. Parry, *A survey of spherically symmetric spacetimes*, Analysis and Mathematical Physics **4** (2014), no. 4, 333–375.

[27] R. Penrose, *Gravitational Collapse and Space-Time Singularities*, Phys. Rev. Lett. **14** (1965) 57–59.

[28] R. Penrose, *Gravitational collapse: The role of general relativity*, Technical report, Birkbeck Coll., London (1969).

[29] R. Penrose, *Naked singularities*, Annals of the New York Academy of Sciences **224** (1973), no. 1, 125–134.

[30] G. G. R. H. P.T Chruściel, E. Delay, *Regularity of horizons and the area theorem*, Annales Henri Poincaré **42** (2001) 109–178.

[31] R. K. Sachs, *Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time*, in Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 270, 103–126, The Royal Society (1962).

[32] J. Sauter, Foliations of null hypersurfaces and the Penrose inequality, Ph.D. thesis, ETH Zurich (2008).

[33] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Communications in Mathematical Physics **65** (1979), no. 1, 45–76.

[34] R. Schoen and S.-T. Yau, *Proof of the positive mass theorem. II*, Communications in Mathematical Physics **79** (1981), no. 2, 231–260.

[35] R. Schoen and S. T. Yau, *Proof that the Bondi mass is positive*, Physical Review Letters **48** (1982), no. 6, 369.

[36] A. Trautman, *Radiation and boundary conditions in the theory of gravitation*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **6** (1958) 407.

[37] R. M. Wald, General relativity, University of Chicago press (2010).

[38] M.-T. Wang, Y.-K. Wang, and X. Zhang, *Minkowski formulae and Alexandrov theorems in spacetime*, Journal of Differential Geometry **105** (2017), no. 2, 249–290.

[39] E. Witten, *A new proof of the positive energy theorem*, Communications in Mathematical Physics **80** (1981), no. 3, 381–402.

Department of Mathematics, Columbia University
2990 Broadway, New York, NY, 10027, USA
E-mail address: roesch@math.columbia.edu

Received November 26, 2018
Accepted May 23, 2019