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Convergence result and blow-up examples for the Guan–Li mean curvature flow on warped product spaces

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We examine the question of convergence of solutions to a geometric flow which was introduced by Guan and Li [6] for starshaped hypersurfaces in space forms and generalized by Guan, Li, and Wang [7] to the case of warped product spaces. We obtain a convergence result under a condition on the optimal modulus of continuity of the initial data. Moreover we show by examples that this condition is optimal at least in the one-dimensional case.

1. Introduction and main results

Let $n \geq 1$, $(\mathbb{S}^n, g_{\mathbb{S}^n})$ be the standard n–sphere, $I \subset \mathbb{R}$ be a closed interval, and $(N, \overline{g})$ be the warped product of $\mathbb{S}^n$ and I equipped with

$$\overline{g} = \phi(\rho)^2 g_{\mathbb{S}^n} + d\rho^2,$$

where $\phi : I \to (0, \infty)$ is a smooth warping function. We consider the following flow which was introduced by Guan and Li [6] in the case of space forms and generalized by Guan, Li, and Wang [7] to the case of warped product spaces (see also Cant [5] in case $n = 1$):

$$\partial_t F = \big(n\phi' - Hu\big)\,\nu,$$

where $T \in (0, \infty]$, $(F(\cdot, t))_{t \in [0,T)}$ is a smooth family of embeddings into N which defines smooth hypersurfaces $(M_t)_{t \in [0,T)}$ and H, u, ν are the mean curvature, support function, and outward unit normal vector field, respectively, of the hypersurfaces $(M_t)_{t \in [0,T)}$. A crucial property of this flow is that it preserves the volume enclosed by the initial hypersurface while monotonically decreasing the area (see [6, Proposition 3.5]).

Throughout this paper, we assume that the hypersurfaces $(M_t)_{t \in [0,T)}$ are starshaped, i.e. for every $t \in [0, T)$, M_t is the graph of a function $\rho(\cdot, t)$:

$\mathbb{S}^n \to \mathbb{R}$. We then obtain (see the formulas in [7, Section 3]) that ρ solves the initial value problem

$$(1.1) \quad \begin{cases} \partial_t \rho = \operatorname{div}\left(\dfrac{\nabla \rho}{\sqrt{\phi(\rho)^2 + |\nabla\rho|^2}}\right) + n\dfrac{\phi'(\rho)}{\phi(\rho)}\dfrac{|\nabla\rho|^2}{\sqrt{\phi(\rho)^2+|\nabla\rho|^2}} & \text{in } D_T \\ \rho(\cdot,0) = \rho_0 & \text{on } \mathbb{S}^n, \end{cases}$$

where $\operatorname{div} := \operatorname{div}_{g_{\mathbb{S}^n}}$, $|\cdot| = |\cdot|_{g_{\mathbb{S}^n}}$, $D_T := \mathbb{S}^n \times (0,T) \to \mathbb{R}$, and ρ_0 is the radial function of M_0. It follows from classical theory of parabolic equations that for every $\rho_0 \in C^\infty(\mathbb{S}^n, I)$, there exists a unique solution $\rho \in C^\infty\left(\overline{D_T}\right)$ of (1.1) for small $T > 0$. Moreover, a straightforward application of the maximum principle gives $\rho(D_T) \subseteq I$.

The following result has been obtained by Guan and Li [6] in the case of space forms and generalized by Guan, Li, and Wang [7] to the case of warped product spaces:

Theorem 1.1. *(Guan and Li [6], Guan, Li, and Wang [7]) Let $I \subset \mathbb{R}$ be a closed interval, $\phi \in C^\infty(I,(0,\infty))$, and $n \geq 1$. Assume that*

$$(1.2) \qquad \phi'^2 - \phi\phi'' \geq 0 \quad \text{in } I.$$

Then for any $\rho_0 \in C^\infty(\mathbb{S}^n, I)$, the solution of (1.1) exists for all time and converges exponentially to a constant i.e. $T = \infty$ and there exist $\rho_\infty \in I$, $C, \eta > 0$ such that $|\rho(x,t) - \rho_\infty| \leq Ce^{-\eta t}$ for all $(x,t) \in D_\infty$.

This result has been successfully used in [5–7] to solve isoperimetric problems in warped product spaces. As is explained in [7, Proposition 6.1], the condition (1.2) is strongly related to the notion of photon sphere in general relativity.

In this paper, we investigate the case where the condition (1.2) is not satisfied. In this case, we obtain a convergence result under a barrier condition on the optimal modulus of continuity of ρ_0, namely

$$\omega_{\rho_0}(\theta) := \sup\left\{|\rho_0(y) - \rho_0(x)| \,:\, x, y \in \mathbb{S}^n \text{ and } \operatorname{dist}_{\mathbb{S}^n}(x,y) = \theta\right\}$$

for all $\theta \in [0,\pi]$. Here $\operatorname{dist}_{\mathbb{S}^n}$ denotes the distance on $\mathbb{S}^n$ with respect to the standard metric. We obtain the following result:

Theorem 1.2. *Let $I \subset \mathbb{R}$ be a closed interval, $\phi \in C^\infty(I,(0,\infty))$, and $n \geq 1$. Then there exists $\lambda_0 > 0$ such that for any $\rho_0 \in C^\infty(\mathbb{S}^n, I)$, if*

$$\omega_{\rho_0}(\theta) \leq \lambda_0 \theta^{1/2} \qquad \forall \theta \in [0,\pi], \tag{1.3}$$

then the solution of (1.1) *exists for all time and converges exponentially to a constant.*

We prove Theorem 1.2 in Section 2 by using an approach based on Kruzhkov's doubling variable technique [8] and inspired by the works of Andrews and Clutterbuck [1–4]. As in the papers of Cant [5], Guan and Li [6], and Guan, Li, and Wang [7], Theorem 1.2 can be applied to solve isoperimetric problems in the warped product space $(N,\overline{g})$ provided $\phi'^2 - \phi\phi'' \leq 1$ in I, which is a necessary condition for the isoperimetric inequality (see Li and Wang [9]).

The following result, obtained in case $n = 1$, shows the optimality of the exponent $1/2$ in (1.3):

Theorem 1.3. *Assume that $n = 1$, $0 \in I$, ϕ is even, and $\phi''(0) > 0$. Then for any $\sigma \in (0,1/2)$ $\lambda > 0$, there exist $\rho_0 \in C^\infty(\mathbb{S}^n, I)$ such that*

$$\omega_{\rho_0}(\theta) \leq \lambda \theta^{\sigma} \qquad \forall \theta \in [0,\pi] \tag{1.4}$$

and the solution of (1.1) *is such that $\partial_x \rho$ blows up in finite time i.e. $\sup_{D_t} |\partial_x \rho| \to \infty$ as $t \to T$ for some $T \in (0,\infty)$.*

We prove Theorem 1.3 in Section 3. As far as the author knows, this is the first existence result of blowing-up solutions for (1.1). The high non-linearity of the flow makes it difficult to construct examples of blowing-up solutions. Here, the solutions that we construct are periodic, with a large number of oscillations. Our existence result relies on the construction of a suitable family of barrier functions on a small arc of $\mathbb{S}^1$ with zero boundary condition. We then exploit the symmetry of the warping function to extend our solutions to the whole $\mathbb{S}^1$.

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2. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. As in the paper of Guan and Li [6], it will be convenient to use the change of functions

$$\gamma = \Gamma(\rho) := \int_{\overline{\rho}}^{\rho} \frac{ds}{\phi(s)} \qquad \text{and} \qquad \psi := \phi \circ \Gamma^{-1}, \tag{2.1}$$

where $\overline{\rho} \in I$ is fixed. By differentiating, we obtain $\nabla \rho = \psi(\gamma) \nabla \gamma$ and $\phi'(\rho) = \psi'(\gamma)/\psi(\gamma)$. Hence the problem (1.1) becomes

$$\begin{cases} \partial_t \gamma = \dfrac{1}{\psi(\gamma)} \operatorname{div}\left(\dfrac{\nabla \gamma}{\sqrt{1+|\nabla\gamma|^2}}\right) + n \dfrac{\psi'(\gamma)}{\psi(\gamma)^2} \dfrac{|\nabla\gamma|^2}{\sqrt{1+|\nabla\gamma|^2}} & \text{in } D_T \\ \gamma(\cdot, 0) = \gamma_0 & \text{on } \mathbb{S}^n, \end{cases} \tag{2.2}$$

where

$$\gamma_0 := \int_{\overline{\rho}}^{\rho_0} \frac{ds}{\phi(s)}.$$

A straightforward application of the maximum principle gives $\gamma(D_T) \subseteq \gamma_0(\mathbb{S}^n)$ i.e.

$$\min_{\mathbb{S}^n} \gamma_0 \leq \gamma(x,t) \leq \max_{\mathbb{S}^n} \gamma_0 \qquad \forall (x,t) \in D_T.$$

We assume that $\rho_0(\mathbb{S}^n) \subseteq I$ and we let $\lambda > 0$ be such that

$$\omega_{\rho_0}(\theta) \leq \lambda \sqrt{\theta} \qquad \forall \theta \in [0, \pi]. \tag{2.3}$$

Since $\rho_0 \in C^\infty(\mathbb{S}^n, I)$, we obtain that there exists $\Lambda > 0$ such that

$$\omega_{\rho_0}(\theta) \leq \Lambda \theta \qquad \forall \theta \in [0, \pi]. \tag{2.4}$$

For every $\delta \in (0, \lambda^2/\Lambda^2)$, an easy study of functions gives that

$$\Lambda \theta \leq 2\lambda\big(\sqrt{\delta+\theta} - \sqrt{\delta}\big) \qquad \forall \theta \in \left(0, \frac{4\lambda}{\Lambda^2}\big(\lambda - \Lambda\sqrt{\delta}\big)\right) \tag{2.5}$$

and

$$\sqrt{\theta} \leq 2\big(\sqrt{\delta+\theta} - \sqrt{\delta}\big) \qquad \forall \theta > \frac{16}{9}\delta. \tag{2.6}$$

It follows from (2.3)–(2.6) that we can choose $\delta \in (0,1)$ small enough such that

$$\omega_{\rho_0}(\theta) \le 2\lambda\big(\sqrt{\delta+\theta} - \sqrt{\delta}\big) \qquad \forall \theta \in [0,\pi]\,. \tag{2.7}$$

By using the mean value theorem, it follows from (2.7) that

$$\omega_{\gamma_0}(\theta) \le 2\overline{\lambda}\big(\sqrt{\delta+\theta} - \sqrt{\delta}\big) \quad \forall \theta \in [0,\pi]\,, \tag{2.8}$$

where

$$\overline{\lambda} := \lambda \sup_I \frac{1}{\phi}.$$

We will show that if λ is smaller than a constant λ_0 depending only on I, ϕ, and n, then $|\nabla\gamma|$ is bounded above by an exponentially decaying function. We will use an approach based on Kruzhkov's doubling variable technique [8]. This approach was successfully used in the works of Andrews and Clutterbuck [1–4] to obtain sharp estimates on the gradient and modulus of continuity of solutions to quasilinear parabolic equations. We fix $\eta > 0$ and we define

$$\kappa(\theta,t) := 2\overline{\lambda}\big(\sqrt{\delta+\theta} - \sqrt{\delta}\big)e^{-\eta t} \qquad \forall (\theta,t) \in [0,\pi]\times[0,T]\,,$$

where δ and $\overline{\lambda}$ are as above. We then define

$$Z(x,y,t) := \gamma(y,t) - \gamma(x,t) - \kappa(d(x,y),t) \qquad \forall (x,y,t) \in U_T,$$

where $U_T := (\mathbb{S}^n)^2 \times [0,T]$. It follows from (2.8) that $Z(x,y,0) \le 0$ for all $x,y \in \mathbb{S}^n$. In what follows, we will show that if η and λ are small enough, then $Z(x,y,t) \le 0$ for all $(x,y,t) \in U_T$. We assume by contradiction that Z is not everywhere nonpositive in U_T. Then we obtain that for small $\varepsilon > 0$, there exists $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in U_T$ such that

$$Z(x_\varepsilon,y_\varepsilon,t_\varepsilon) = \max\big(\{Z(x,y,t) :\ x,y \in \mathbb{S}^n \text{ and } t \le t_\varepsilon\}\big) = \varepsilon. \tag{2.9}$$

We define $\theta_\varepsilon := d(x_\varepsilon, y_\varepsilon)$.

As a first step, we obtain the following result:

Step 2.1. *$\varepsilon = o(\theta_\varepsilon)$ as $\varepsilon \to 0$.*

Proof of Step 2.1. Assume by contradiction that there exists a sequence $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}}$ such that $\varepsilon_\alpha > 0$, $\theta_{\varepsilon_\alpha} = \mathrm{O}(\varepsilon_\alpha)$, and $\varepsilon_\alpha \to 0$ as $\alpha \to \infty$. Since

$\kappa(0,t_{\varepsilon_\alpha})=0$, by applying the mean value theorem, we obtain that there exist $\zeta_\alpha,\xi_\alpha\in(0,\theta_{\varepsilon_\alpha})$ such that

$$\kappa(\theta_{\varepsilon_\alpha},t_{\varepsilon_\alpha})=\partial_\theta\kappa(\zeta_\alpha,t_{\varepsilon_\alpha})\,\theta_{\varepsilon_\alpha} \tag{2.10}$$

and

$$\gamma(y_{\varepsilon_\alpha},t_{\varepsilon_\alpha})-\gamma(x_{\varepsilon_\alpha},t_{\varepsilon_\alpha})=\big\langle\nabla_x\gamma(\tau_{\varepsilon_\alpha}(\xi_\alpha),t_{\varepsilon_\alpha}),\tau'_{\varepsilon_\alpha}(\xi_\alpha)\big\rangle\,\theta_{\varepsilon_\alpha}, \tag{2.11}$$

where $\tau_{\varepsilon_\alpha}:[0,\theta_{\varepsilon_\alpha}]\to\mathbb{S}^n$ is a minimizing geodesic from x_{ε_α} to y_{ε_α}. It follows from (2.9), (2.10), (2.11), and Cauchy–Schwartz inequality that $\theta_{\varepsilon_\alpha}\neq0$ and

$$\frac{\varepsilon_\alpha}{\theta_{\varepsilon_\alpha}}\le|\nabla_x\gamma(\tau_{\varepsilon_\alpha}(\xi_\alpha),t_{\varepsilon_\alpha})|-\partial_\theta\kappa(\zeta_\alpha,t_{\varepsilon_\alpha}). \tag{2.12}$$

Since $(t_{\varepsilon_\alpha})_{\alpha\in\mathbb{N}}$ is decreasing, we obtain $t_{\varepsilon_\alpha}\to t_0$ for some $t_0\ge0$. Since $\theta_{\varepsilon_\alpha}\to0$, we obtain $\zeta_\alpha,\xi_\alpha\to0$. Moreover up to a subsequence $x_{\varepsilon_\alpha},y_{\varepsilon_\alpha}\to x_0\in\mathbb{S}^n$. By passing to the limit into (2.12), we then obtain

$$\limsup_{\alpha\to\infty}\frac{\varepsilon_\alpha}{\theta_{\varepsilon_\alpha}}\le|\nabla_x\gamma(x_0,t_0)|-\partial_\theta\kappa(0,t_0). \tag{2.13}$$

On the other hand, by passing to the limit into (2.9), first as $\varepsilon\to0$ and then as $x,y\to x_0$, we obtain

$$|\nabla_x\gamma(x_0,t_0)|\le\partial_\theta\kappa(0,t_0). \tag{2.14}$$

By putting together (2.13) and (2.14), we obtain a contradiction with $\theta_{\varepsilon_\alpha}=\mathrm{O}(\varepsilon_\alpha)$. This ends the proof of Step 2.1. □

We then prove the following result:

Step 2.2. $\theta_\varepsilon<\pi$.

Proof of Step 2.2. Assume by contradiction that $\theta_\varepsilon=\pi$. Then it follows from (2.9) that

$$\frac{d}{d\theta}\left[Z\left(x_\varepsilon,\exp_{x_\varepsilon}(\theta v),t_\varepsilon\right)\right]\big|_{\theta=\pi}=\langle\nabla_x\gamma(y_\varepsilon,t_\varepsilon),\nu_\varepsilon(v)\rangle-\partial_\theta\kappa(\pi,t_\varepsilon)=0$$

for all $v\in T_{x_\varepsilon}\mathbb{S}^n$ such that $|v|=1$, where $\nu_\varepsilon(v)=\frac{d}{d\theta}\exp_{x_\varepsilon}(\theta v)\big|_{\theta=\pi}$. By observing that $\nu_\varepsilon(-v)=-\nu_\varepsilon(v)$, we then obtain a contradiction with $\partial_\theta\kappa(\pi,t_\varepsilon)>0$. This ends the proof of Step 2.2. □

Remark that it follows from Steps 2.1 and 2.2 that for small ε, the function Z is differentiable in a neighborhood of the point $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$.

Our next result is as follows:

Step 2.3. *There exists a constant* $\Lambda_0 = \Lambda_0 (I, \phi, n) > 0$ *such that*

$$\partial_t \kappa (\theta_\varepsilon, t_\varepsilon) \leq \frac{\Lambda_0^{-1} \partial_\theta^2 \kappa (\theta_\varepsilon, t_\varepsilon)}{\left(1 + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)^2\right)^{3/2}} + \frac{\Lambda_0 \left(\theta_\varepsilon^{-1} \kappa (\theta_\varepsilon, t_\varepsilon) + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)\right) \kappa (\theta_\varepsilon, t_\varepsilon) \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)}{\left(1 + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)^2\right)^{1/2}} \tag{2.15}$$

for small $\varepsilon > 0$.

Proof of Step 2.3. We let $\tau_\varepsilon : [0, \theta_\varepsilon] \to \mathbb{S}^n$ be a minimizing geodesic from x_ε to y_ε. It follows from (2.9) that

$$\nabla_x Z (x_\varepsilon, y_\varepsilon, t_\varepsilon) = 0, \quad \nabla_y Z (x_\varepsilon, y_\varepsilon, t_\varepsilon) = 0, \quad \text{and} \quad \partial_t Z (x_\varepsilon, y_\varepsilon, t_\varepsilon) \geq 0$$

which give

$$\begin{cases} \nabla_x \gamma (x_\varepsilon, t_\varepsilon) = \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon) \tau_\varepsilon' (0) \\ \nabla_x \gamma (y_\varepsilon, t_\varepsilon) = \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon) \tau_\varepsilon' (\theta_\varepsilon) \\ \partial_t \gamma (y_\varepsilon, t_\varepsilon) - \partial_t \gamma (x_\varepsilon, t_\varepsilon) \geq \partial_t \kappa (\theta_\varepsilon, t_\varepsilon) . \end{cases} \tag{2.16}$$

By using (2.2) and (2.16), we obtain

$$\partial_t \kappa (\theta_\varepsilon, t_\varepsilon) \leq \frac{A_\varepsilon}{\left(1 + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)^2\right)^{3/2}} + \frac{B_\varepsilon + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)^2 C_\varepsilon}{\left(1 + \partial_\theta \kappa (\theta_\varepsilon, t_\varepsilon)^2\right)^{1/2}}, \tag{2.17}$$

where

$$A_\varepsilon := \frac{\left[\nabla_x^2 \gamma (y_\varepsilon, t_\varepsilon)\right] (\tau_\varepsilon' (\theta_\varepsilon), \tau_\varepsilon' (\theta_\varepsilon))}{\psi (\gamma (y_\varepsilon, t_\varepsilon))} - \frac{\left[\nabla_x^2 \gamma (x_\varepsilon, t_\varepsilon)\right] (\tau_\varepsilon' (0), \tau_\varepsilon' (0))}{\psi (\gamma (x_\varepsilon, t_\varepsilon))},$$

$$B_\varepsilon := \frac{\Delta_x \gamma (y_\varepsilon, t_\varepsilon) - \left[\nabla_x^2 \gamma (y_\varepsilon, t_\varepsilon)\right] (\tau_\varepsilon' (\theta_\varepsilon), \tau_\varepsilon' (\theta_\varepsilon))}{\psi (\gamma (y_\varepsilon, t_\varepsilon))} - \frac{\Delta_x \gamma (x_\varepsilon, t_\varepsilon) - \left[\nabla_x^2 \gamma (x_\varepsilon, t_\varepsilon)\right] (\tau_\varepsilon' (0), \tau_\varepsilon' (0))}{\psi (\gamma (x_\varepsilon, t_\varepsilon))},$$

$$C_\varepsilon := n\left(\frac{\psi'(\gamma(y_\varepsilon,t_\varepsilon))}{\psi(\gamma(y_\varepsilon,t_\varepsilon))^2}-\frac{\psi'(\gamma(x_\varepsilon,t_\varepsilon))}{\psi(\gamma(x_\varepsilon,t_\varepsilon))^2}\right).$$

Since $\kappa(\theta_\varepsilon,t_\varepsilon), \partial_\theta\kappa(\theta_\varepsilon,t_\varepsilon)>0$ and $\partial_\theta^2\kappa(\theta_\varepsilon,t_\varepsilon)<0$, in order to obtain (2.15), it remains to prove that there exist constants $c_1,c_2,c_3>0$ depending only on I, ϕ, and n such that

$$\begin{cases} A_\varepsilon \le c_1\partial_\theta^2\kappa(\theta_\varepsilon,t_\varepsilon)\\ B_\varepsilon \le c_2\theta_\varepsilon^{-1}\kappa(\theta_\varepsilon,t_\varepsilon)^2\,\partial_\theta\kappa(\theta_\varepsilon,t_\varepsilon)\\ C_\varepsilon\le c_3\kappa(\theta_\varepsilon,t_\varepsilon)\end{cases}\tag{2.18}$$

for small ε. We begin with proving the last estimate in (2.18). Remark that by using Step 2.1, we obtain

$$\frac{\kappa(\theta_\varepsilon,t_\varepsilon)}{\varepsilon}=\frac{1}{\varepsilon}\int_0^{\theta_\varepsilon}\partial_\theta\kappa(s,t_\varepsilon)\,ds=\frac{1}{\varepsilon}\int_0^{\theta_\varepsilon}\frac{\overline{\lambda}e^{-\eta t_\varepsilon}}{\sqrt{\delta+s}}ds\ge\frac{\overline{\lambda}e^{-\eta t_\varepsilon}\theta_\varepsilon}{\varepsilon\sqrt{\delta+\theta_\varepsilon}}\to\infty$$

as $\varepsilon\to 0$, which, together with (2.9), implies that

$$\gamma(y_\varepsilon,t_\varepsilon)-\gamma(x_\varepsilon,t_\varepsilon)\le 2\kappa(\theta_\varepsilon,t_\varepsilon)\tag{2.19}$$

for small ε. Since $\gamma(D_T)\subseteq\gamma_0(\mathbb{S}^n)$ and $\rho_0(\mathbb{S}^n)\subseteq I$, by applying the mean value theorem together with (2.19), we obtain

$$C_\varepsilon\le 2n\sup_{\gamma_0(\mathbb{S}^n)}\left(\frac{\psi'}{\psi^2}\right)'\kappa(\theta_\varepsilon,t_\varepsilon)\le 2n\sup_I\left(\frac{\phi'}{\phi}\right)'\kappa(\theta_\varepsilon,t_\varepsilon)\tag{2.20}$$

for small ε which gives the last estimate in (2.18). Now we prove the first two estimates in (2.18). We let $(v_{\varepsilon,1}(0),\dots,v_{\varepsilon,n}(0))$ be an orthonormal basis of $T_{x_\varepsilon}\mathbb{S}^n$ such that $v_{\varepsilon,n}(0)=\tau_\varepsilon'(0)$. For any $i\in\{1,\dots,n\}$, we let $\varphi_{\varepsilon,i}$ be a smooth function on $[0,\theta_\varepsilon]$ such that

$$\varphi_{\varepsilon,i}(0)=\frac{1}{\sqrt{\psi(\gamma(x_\varepsilon,t_\varepsilon))}}\quad\text{and}\quad\varphi_{\varepsilon,i}(\theta_\varepsilon)=\frac{\delta_i}{\sqrt{\psi(\gamma(y_\varepsilon,t_\varepsilon))}}\tag{2.21}$$

with $\delta_i:=1$ in case $i\ne n$ and $\delta_i:=-1$ in case $i=n$. For any $r\ge 0$ and $\theta\in[0,\theta_\varepsilon]$, we define

$$\tau_{\varepsilon,i}(r,\theta):=\exp_{\tau_\varepsilon(\theta)}(r\varphi_{\varepsilon,i}(\theta)\,v_{\varepsilon,i}(\theta)),$$

where $v_{\varepsilon,i} : [0, \theta_\varepsilon] \to T\mathbb{S}^n$ is the parallel transport of $v_{\varepsilon,i}(0)$ along τ_ε. By using (2.21), we obtain

$$A_\varepsilon = \frac{d^2}{dr^2} \left[\gamma\left(\tau_{\varepsilon,n}\left(r, \theta_\varepsilon\right), t_\varepsilon\right) - \gamma\left(\tau_{\varepsilon,n}\left(r, 0\right), t_\varepsilon\right)\right]\Big|_{r=0} \tag{2.22}$$

and

$$B_\varepsilon = \sum_{i=1}^{n-1} \frac{d^2}{dr^2} \left[\gamma\left(\tau_{\varepsilon,i}\left(r, \theta_\varepsilon\right), t_\varepsilon\right) - \gamma\left(\tau_{\varepsilon,i}\left(r, 0\right), t_\varepsilon\right)\right]\Big|_{r=0}. \tag{2.23}$$

On the other hand, for any $i \in \{1, \dots, n\}$, since $\partial_\theta \kappa\left(\cdot, t_\varepsilon\right) > 0$ on $(0, \pi)$, it follows from (2.9) that

$$\begin{aligned} \gamma\left(\tau_{\varepsilon,i}\left(r, \theta_\varepsilon\right), t_\varepsilon\right) - \gamma\left(\tau_{\varepsilon,i}\left(r, 0\right), t_\varepsilon\right) - \varepsilon &\leq \kappa\left(d\left(\tau_{\varepsilon,i}\left(r, 0\right), \tau_{\varepsilon,i}\left(r, \theta_\varepsilon\right)\right), t_\varepsilon\right) \\ &\leq \kappa\left(\int_0^{\theta_\varepsilon} \left|\partial_\theta \tau_{\varepsilon,i}\left(r, \theta\right)\right| d\theta, t_\varepsilon\right) \end{aligned} \tag{2.24}$$

for all $i \in \{1, \dots, n\}$ for small $r \geq 0$ with equality in case $r = 0$. Moreover, by using (2.16), we obtain

$$\begin{aligned} &\frac{d}{dr} \left[\gamma\left(\tau_{\varepsilon,i}\left(r, \theta_\varepsilon\right), t_\varepsilon\right) - \gamma\left(\tau_{\varepsilon,i}\left(r, 0\right), t_\varepsilon\right)\right]\Big|_{r=0} \\ &\quad = \partial_\theta \kappa\left(\theta_\varepsilon, t_\varepsilon\right)\left(\varphi_{\varepsilon,i}\left(\theta_\varepsilon\right)\left\langle v_{\varepsilon,n}\left(\theta_\varepsilon\right), v_{\varepsilon,i}\left(\theta_\varepsilon\right)\right\rangle - \varphi_{\varepsilon,i}(0)\left\langle v_{\varepsilon,n}(0), v_{\varepsilon,i}(0)\right\rangle\right) \\ &\quad = \partial_\theta \kappa\left(\theta_\varepsilon, t_\varepsilon\right) \int_0^{\theta_\varepsilon} \left\langle v_{\varepsilon,n}(\theta), \varphi_{\varepsilon,i}'(\theta)\, v_{\varepsilon,i}(\theta)\right\rangle d\theta \\ &\quad = \partial_\theta \kappa\left(\theta_\varepsilon, t_\varepsilon\right) \int_0^{\theta_\varepsilon} \left\langle \partial_\theta \tau_{\varepsilon,i}(0, \theta), \partial_r \partial_\theta \tau_{\varepsilon,i}(0, \theta)\right\rangle d\theta \\ &\quad = \frac{d}{dr}\left[\kappa\left(\int_0^{\theta_\varepsilon} \left|\partial_\theta \tau_{\varepsilon,i}(r, \theta)\right| d\theta, t_\varepsilon\right)\right]\Bigg|_{r=0}. \end{aligned} \tag{2.25}$$

It follows from (2.24) and (2.25) that

$$\begin{aligned} &\frac{d^2}{dr^2} \left[\gamma\left(\tau_{\varepsilon,i}\left(r, \theta_\varepsilon\right), t_\varepsilon\right) - \gamma\left(\tau_{\varepsilon,i}\left(r, 0\right), t_\varepsilon\right)\right]\Big|_{r=0} \\ &\quad \leq \frac{d^2}{dr^2}\left[\kappa\left(\int_0^{\theta_\varepsilon} \left|\partial_\theta \tau_{\varepsilon,i}(r, \theta)\right| d\theta, t_\varepsilon\right)\right]\Bigg|_{r=0} \end{aligned}$$

$$
\begin{aligned}
&= \partial_\theta^2 \kappa\left(\theta_\varepsilon, t_\varepsilon\right)\left(\frac{d}{dr}\left[\int_0^{\theta_\varepsilon}\left|\partial_\theta \tau_{\varepsilon,i}\left(r,\theta\right)\right| d\theta\right]\bigg|_{r=0}\right)^2 \\
&\quad + \partial_\theta \kappa\left(\theta_\varepsilon, t_\varepsilon\right)\frac{d^2}{dr^2}\left[\int_0^{\theta_\varepsilon}\left|\partial_\theta \tau_{\varepsilon,i}\left(r,\theta\right)\right| d\theta\right]\bigg|_{r=0}.
\end{aligned}
\tag{2.26}
$$

By proceeding as in (2.25) and using (2.21), we obtain

$$
\begin{aligned}
&\frac{d}{dr}\left[\int_0^{\theta_\varepsilon}\left|\partial_\theta \tau_{\varepsilon,i}\left(r,\theta\right)\right| d\theta\right]\bigg|_{r=0} \\
&\quad = \varphi_{\varepsilon,i}\left(\theta_\varepsilon\right)\left\langle v_{\varepsilon,n}\left(\theta_\varepsilon\right), v_{\varepsilon,i}\left(\theta_\varepsilon\right)\right\rangle - \varphi_{\varepsilon,i}\left(0\right)\left\langle v_{\varepsilon,n}\left(0\right), v_{\varepsilon,i}\left(0\right)\right\rangle \\
&\quad = \begin{cases} 0 & \text{if } i \neq n \\ \dfrac{1}{\sqrt{\psi\left(\gamma\left(x_\varepsilon, t_\varepsilon\right)\right)}} + \dfrac{1}{\sqrt{\psi\left(\gamma\left(y_\varepsilon, t_\varepsilon\right)\right)}} & \text{if } i = n. \end{cases}
\end{aligned}
\tag{2.27}
$$

Moreover, since $\gamma\left(D_T\right) \subseteq \gamma_0\left(\mathbb{S}^n\right)$ and $\rho_0\left(\mathbb{S}^n\right) \subseteq I$, we obtain

$$
\frac{1}{\sqrt{\psi\left(\gamma\left(x_\varepsilon, t_\varepsilon\right)\right)}} + \frac{1}{\sqrt{\psi\left(\gamma\left(y_\varepsilon, t_\varepsilon\right)\right)}} \geq 2 \inf_{\gamma_0\left(\mathbb{S}^n\right)} \frac{1}{\sqrt{\psi}} \geq 2 \inf_I \frac{1}{\sqrt{\phi}}.
\tag{2.28}
$$

By differentiating twice, we obtain

(2.29)

$$
\begin{aligned}
\frac{d^2}{dr^2}\left[\int_0^{\theta_\varepsilon}\left|\partial_\theta \tau_{\varepsilon,i}\left(r,\theta\right)\right| d\theta\right]\bigg|_{r=0} &= \int_0^{\theta_\varepsilon}\Big(\left|\partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right)\right|^{-1}\Big(\left|\partial_r\partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right)\right|^2 \\
&\quad + \left\langle \partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right), \partial_r^2\partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right)\right\rangle\Big) \\
&\quad - \left|\partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right)\right|^{-3}\left\langle \partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right), \partial_r\partial_\theta \tau_{\varepsilon,i}\left(0,\theta\right)\right\rangle^2\Big) d\theta \\
&= \int_0^{\theta_\varepsilon}\Big(\left|v_{\varepsilon,n}\left(\theta\right)\right|^{-1}\Big(\varphi'_{\varepsilon,i}\left(\theta\right)^2\left|v_{\varepsilon,i}\left(\theta\right)\right|^2 \\
&\quad - \left\langle v_{\varepsilon,n}\left(\theta\right), R\left(\varphi_{\varepsilon,i}\left(\theta\right)v_{\varepsilon,i}\left(\theta\right), v_{\varepsilon,n}\left(\theta\right)\right)\varphi_{\varepsilon,i}\left(\theta\right)v_{\varepsilon,i}\left(\theta\right)\right\rangle\Big) \\
&\quad - \left|v_{\varepsilon,n}\left(\theta\right)\right|^{-3}\left\langle v_{\varepsilon,n}\left(\theta\right), \varphi'_{\varepsilon,i}\left(\theta\right)v_{\varepsilon,i}\left(\theta\right)\right\rangle^2\Big) d\theta \\
&= \begin{cases} \displaystyle\int_0^{\theta_\varepsilon}\left(\varphi'_{\varepsilon,i}\left(\theta\right)^2 - \varphi_{\varepsilon,i}\left(\theta\right)^2\right) d\theta & \text{if } i \neq n \\ 0 & \text{if } i = n, \end{cases}
\end{aligned}
$$

where R is the curvature tensor of $\left(\mathbb{S}^n, g_{\mathbb{S}^n}\right)$. Since $\partial_\theta^2 \kappa\left(\theta_\varepsilon, t_\varepsilon\right) \leq 0$, the first estimate in (2.18) follows from (2.22) and (2.26)–(2.29). Now, we prove the second estimate in (2.18). In case $i \neq n$, by integrating by parts, we obtain

$$\begin{aligned}(2.30)\quad \int_0^{\theta_\varepsilon} \left(\varphi'_{\varepsilon,i}(\theta)^2 - \varphi_{\varepsilon,i}(\theta)^2\right) d\theta &= \varphi_{\varepsilon,i}(\theta_\varepsilon)\,\varphi'_{\varepsilon,i}(\theta_\varepsilon) - \varphi_{\varepsilon,i}(0)\,\varphi'_{\varepsilon,i}(0) \\ &\quad - \int_0^{\theta_\varepsilon} \left(\varphi''_{\varepsilon,i}(\theta) + \varphi_{\varepsilon,i}(\theta)\right) \varphi_{\varepsilon,i}(\theta)\, d\theta.\end{aligned}$$

By using (2.30) with the function $\varphi_{\varepsilon,i}$ defined as

$$\varphi_{\varepsilon,i}(\theta) := \frac{1}{\sin(\theta_\varepsilon)} \left(\frac{\sin(\theta_\varepsilon - \theta)}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))}} + \frac{\sin(\theta)}{\sqrt{\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \right) \quad \forall \theta \in [0, \theta_\varepsilon],$$

we obtain

$$\begin{aligned}\int_0^{\theta_\varepsilon} &\left(\varphi'_{\varepsilon,i}(\theta)^2 - \varphi_{\varepsilon,i}(\theta)^2\right) d\theta = \varphi_{\varepsilon,i}(\theta_\varepsilon)\,\varphi'_{\varepsilon,i}(\theta_\varepsilon) - \varphi_{\varepsilon,i}(0)\,\varphi'_{\varepsilon,i}(0) \\ &= \frac{1}{\sin(\theta_\varepsilon)} \Bigg[\left(\frac{1}{\psi(\gamma(x_\varepsilon, t_\varepsilon))} + \frac{1}{\psi(\gamma(y_\varepsilon, t_\varepsilon))} \right) \cos(\theta_\varepsilon) \\ &\quad - \frac{2}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))\,\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \Bigg] \\ &= \frac{\cos(\theta_\varepsilon)}{\sin(\theta_\varepsilon)} \left(\frac{1}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))}} - \frac{1}{\sqrt{\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \right)^2 \\ &\quad - \frac{2\tan(\theta_\varepsilon/2)}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))\,\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \\ (2.31)\quad &\leq \frac{1}{\theta_\varepsilon} \left(\frac{1}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))}} - \frac{1}{\sqrt{\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \right)^2.\end{aligned}$$

By proceeding as in (2.20), we obtain

$$(2.32)\quad \left| \frac{1}{\sqrt{\psi(\gamma(x_\varepsilon, t_\varepsilon))}} - \frac{1}{\sqrt{\psi(\gamma(y_\varepsilon, t_\varepsilon))}} \right| \leq \sup_I \left(\frac{\phi'}{\sqrt{\phi}} \right) \kappa(\theta_\varepsilon, t_\varepsilon)$$

for small ε. By using (2.29), (2.31), and (2.32), we obtain

$$(2.33)\quad \sum_{i=1}^{n-1} \frac{d^2}{dr^2} \left[\int_0^{\theta_\varepsilon} |\partial_\theta \tau_{\varepsilon,i}(r, \theta)|\, d\theta \right] \Bigg|_{r=0} \leq \sup_I \left(\frac{\phi'}{\sqrt{\phi}} \right)^2 \frac{\kappa(\theta_\varepsilon, t_\varepsilon)^2}{\theta_\varepsilon}$$

for small ε. The second estimate in (2.18) then follows from (2.23), (2.26), (2.27), and (2.33). This ends the proof of Step 2.3. □

We can now end the proof of Theorem 1.2.

End of proof of Theorem 1.2. By applying Step 2.3 and observing that $\kappa(\theta_\varepsilon, t_\varepsilon) \leq 2\theta_\varepsilon \partial_\theta \kappa(\theta_\varepsilon, t_\varepsilon)$ and $\kappa(\theta_\varepsilon, t_\varepsilon) \partial_\theta \kappa(\theta_\varepsilon, t_\varepsilon) \leq 2\overline{\lambda}^2 e^{-2\eta t_\varepsilon}$, we obtain

(2.34)
$$-2\eta\left(\sqrt{\delta+\theta_\varepsilon}-\sqrt{\delta}\right) \leq \frac{-\Lambda_0^{-1}}{2\left(\delta+\theta_\varepsilon+\overline{\lambda}^2 e^{-2\eta t_\varepsilon}\right)^{3/2}} + \frac{6\Lambda_0\overline{\lambda}^2 e^{-2\eta t_\varepsilon}}{\left(\delta+\theta_\varepsilon+\overline{\lambda}^2 e^{-2\eta t_\varepsilon}\right)^{1/2}}$$

Since $\delta < 1$, $\theta_\varepsilon < \pi$, and $e^{-2\eta t_\varepsilon} \leq 1$, it follows from (2.34) that

$$1 \leq 4\Lambda_0\left(1+\pi+\overline{\lambda}^2\right)\left(\eta\sqrt{(1+\pi)\left(1+\pi+\overline{\lambda}^2\right)}+3\Lambda_0\overline{\lambda}^2\right).$$

which gives a contradiction when $\overline{\lambda}$ and η are smaller than some constants depending only on I, ϕ, and n. This proves that for such values of $\overline{\lambda}$ and η, we have $Z \leq 0$ in U_T and so

$$\sup_{\mathbb{S}^n} |\nabla\gamma(\cdot,t)| \leq \overline{\lambda}\delta^{-1/2}e^{-\eta t} \qquad \forall t \in [0,T]. \tag{2.35}$$

Since $|\nabla\rho| = \phi(\rho)|\nabla\gamma|$, it follows from (2.35) that

$$\sup_{\mathbb{S}^n} |\nabla\rho(\cdot,t)| \leq \sup_I (\phi)\, \overline{\lambda}\delta^{-1/2}e^{-\eta t} \qquad \forall t \in [0,T]. \tag{2.36}$$

It then follows from classical theory of parabolic equations that $\rho(\cdot,t)$ exists for all $t \geq 0$. Moreover, it follows from (2.36) that $\rho(\cdot,t)$ converges exponentially to a constant. This ends the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We first prove the following result:

Lemma 3.1. *Assume that* $n = 1$, $0 \in I$, ϕ *is even, and* $\phi''(0) > 0$. *Let* ψ *and* Γ *be as in* (2.1) *with* $\overline{\rho} = 0$, $J := \Gamma(I)$, *and for any* $\tau > 0$ *and* $k \in \mathbb{N}\setminus\{0\}$, $D_{1,k,\tau} := [0,\pi/(2k)) \cup (\pi/(2k),\pi/k] \times [0,\tau)$ *and* $D_{2,k,\tau} := [0,\pi/k] \times [0,\tau)$. *Then for any* $\sigma \in (0,1/2)$ *and* $\mu > 0$, *there exists* $\varepsilon_0 > 0$ *such that for any* $\tau > 0$ *and* $k \in \mathbb{N}$ *such that* $k^2\tau < \varepsilon_0$ *and* $k > 1/\varepsilon_0$, *there exist* $\zeta_1 \in$

$C^\infty(D_{1,k,\tau},J)\cap C^0(D_{2,k,\tau},J)$ *and* $\zeta_2\in C^\infty(D_{2,k,\tau},J)$ *such that*

$$\partial_t\zeta_i\le\frac{1}{\psi(\zeta_i)}\frac{\partial_\theta^2\zeta_i}{\left(1+(\partial_\theta\zeta_i)^2\right)^{3/2}}+\frac{\psi'(\zeta_i)}{\psi(\zeta_i)^2}\frac{(\partial_\theta\zeta_i)^2}{\sqrt{1+(\partial_\theta\zeta_i)^2}}\quad \text{in } D_{i,k,\tau} \tag{3.1}$$

for $i\in\{1,2\}$ *and the function* $\zeta:=\max(\zeta_1,\zeta_2)$ *is such that*

(A1) $\zeta_1(\pi/(2k),t)<\zeta_2(\pi/(2k),t)\,(=\zeta(\pi/(2k),t))\quad\forall t\in[0,\tau)$,

(A2) $\left|\zeta(\theta,0)-\zeta(\theta',0)\right|<\mu\left|\theta-\theta'\right|^\sigma\quad\forall\theta,\theta'\in[0,\pi/k]$,

(A3) $\zeta(0,t)=\zeta(\pi/k,t)=0\quad\forall t\in[0,\tau)$,

(A4) $\partial_\theta\zeta(0,t)\to\infty$ *and* $\partial_\theta\zeta(\pi/k,t)\to-\infty$ *as* $t\to\tau$.

Proof of Lemma 3.1. We fix $p\in(2/(1-\sigma),4)$. We let $\tau>0$ and $k\in\mathbb{N}\backslash\{0\}$ to be chosen later on so that k is large and $k^2\tau$ is small. For any $(\theta,t)\in D_{2,k,\tau}$, we define

$$\zeta_1(\theta,t):=\min\left(\frac{c_1\theta}{((\tau-t)^p+\theta^2)^{1/p}},\frac{c_1(\pi/k-\theta)}{\left[(\tau-t)^p+(\pi/k-\theta)^2\right]^{1/p}}\right)$$

and

$$\zeta_2(\theta,t):=c_1A_k\left(\sin(k\theta)-c_2k^2t\right),$$

where $A_k:=2^{2/p}(\pi/k)^{1-2/p}$ and c_1 and c_2 are positive constants independent of θ, t, k, and τ to be fixed later on. Note that $1-2/p>\sigma$. It is easy to check that $\zeta_1\in C^\infty(D_{1,k,\tau},J)\cap C^0(D_{2,k,\tau},J)$, $\zeta_2\in C^\infty(D_{2,k,\tau},J)$, and (A2)–(A4) hold true for small τ and large k. If moreover $k^2\tau$ is small, then we obtain that (A1) holds true. It remains to prove that (3.1) holds true. Since ϕ is even and $\phi''(0)>0$, we obtain that ψ is also even and $\psi''(0)>0$. By applying the mean value theorem and since $\psi'(0)=0$ and $\zeta_1(\theta,t)\ge0$, we obtain

$$\frac{\psi'(\zeta_1(\theta,t))}{\psi(\zeta_1(\theta,t))^2}\ge\inf_{\zeta_1(D_{2,k,\tau})}\left(\frac{\psi'}{\psi^2}\right)'\zeta_1(\theta,t) \tag{3.2}$$

for all $(\theta,t)\in D_{2,k,\tau}$. Moreover, direct calculations give

$$\partial_t\zeta_1(\theta,t)=\frac{c_1\theta(\tau-t)^{p-1}}{((\tau-t)^p+\theta^2)^{1+1/p}}\le\frac{c_1\theta}{((\tau-t)^p+\theta^2)^{2/p}}, \tag{3.3}$$

$$\frac{\partial_\theta^2\zeta_1(\theta,t)}{\left(1+(\partial_\theta\zeta_1(\theta,t))^2\right)^{3/2}}$$
$$= -\frac{2c_1\theta\left(3(\tau-t)^p+(1-2/p)\theta^2\right)\left((\tau-t)^p+\theta^2\right)^{1+2/p}}{p\left(\left((\tau-t)^p+\theta^2\right)^{2+2/p}+c_1^2\left((\tau-t)^p+(1-2/p)\theta^2\right)^2\right)^{3/2}}$$
$$\geq -\frac{6\theta}{c_1^2p(1-2/p)^3\left((\tau-t)^p+\theta^2\right)^{1-2/p}}, \tag{3.4}$$

and

$$\frac{\zeta_1(\theta,t)(\partial_\theta\zeta_1(\theta,t))^2}{\sqrt{1+(\partial_\theta\zeta_1(\theta,t))^2}}$$
$$= \frac{c_1^3\theta\left((\tau-t)^p+(1-2/p)\theta^2\right)^2\left((\tau-t)^p+\theta^2\right)^{-1-2/p}}{\left(\left((\tau-t)^p+\theta^2\right)^{2+2/p}+c_1^2\left((\tau-t)^p+(1-2/p)\theta^2\right)^2\right)^{1/2}}$$
$$\geq \frac{c_1^3(1-2/p)^2\theta}{\sqrt{\left(\tau^p+(\pi/(2k))^2\right)^{2/p}+c_1^2}\left((\tau-t)^p+\theta^2\right)^{2/p}} \tag{3.5}$$

for all $(\theta,t)\in[0,\pi/(2k))\times[0,\tau)$. Since $p<4$, it follows from (3.2)–(3.5) that

$$\frac{1}{\psi(\zeta_1(\theta,t))}\frac{\partial_\theta^2\zeta_1(\theta,t)}{\left(1+(\partial_\theta\zeta_1(\theta,t))^2\right)^{3/2}}+\frac{\psi'(\zeta_1(\theta,t))}{\psi(\zeta_1(\theta,t))^2}\frac{(\partial_\theta\zeta_1(\theta,t))^2}{\sqrt{1+(\partial_\theta\zeta_1(\theta,t))^2}}$$
$$-\partial_t\zeta_1(\theta,t)\geq\frac{\theta}{\left((\tau-t)^p+\theta^2\right)^{2/p}}\Bigg(-\frac{6\left(\tau^2+(\pi/(2k))^2\right)^{-1+4/p}}{c_1^2p(1-2/p)^3}\sup_{\zeta_1(D_{2,k,\tau})}\frac{1}{\psi}$$
$$+\frac{c_1^3(1-2/p)^2}{\sqrt{\left(\tau^p+(\pi/(2k))^2\right)^{2/p}+c_1^2}}\inf_{\zeta_1(D_{2,k,\tau})}\left(\frac{\psi'}{\psi^2}\right)'-c_1\Bigg) \tag{3.6}$$

for all $(\theta,t)\in[0,\pi/(2k))\times[0,\tau)$ provided

$$\inf_{\zeta_1(D_{2,k,\tau})}\left(\frac{\psi'}{\psi^2}\right)'\geq 0.$$

Moreover, direct calculations give

$$\zeta_1(D_{2,k,\tau})=\left[0,c_1(\pi/k)^{1-2/p}\right). \tag{3.7}$$

Since $2 < p < 4$, we obtain

$$1 - \frac{2}{p} > 0 \qquad \text{and} \qquad -1 + \frac{4}{p} > 0. \tag{3.8}$$

Since $\psi'(0) = 0$, and $\psi''(0) > 0$, it follows from (3.6)–(3.8) that (3.1) holds true for $i = 1$ for small τ and large k provided the constant c_1 is chosen large enough so that $c_1 > (1 - 2/p)^{-2} \psi(0)^2 / \psi''(0)$. With regard to the function ζ_2, we obtain

$$\partial_t \zeta_2(\theta, t) = -c_1 c_2 k^2 A_k, \tag{3.9}$$

$$\frac{\partial_\theta^2 \zeta_2(\theta, t)}{\left(1 + (\partial_\theta \zeta_2(\theta, t))^2\right)^{3/2}} = -\frac{c_1 k^2 A_k \sin(k\theta)}{\left(1 + c_1^2 k^2 A_k^2 \cos(k\theta)^2\right)^{3/2}}$$
$$\geq -c_1 k^2 A_k\,, \tag{3.10}$$

and

$$\frac{(\partial_\theta \zeta_2(\theta, t))^2}{\sqrt{1 + (\partial_\theta \zeta_2(\theta, t))^2}} = \frac{c_1^2 k^2 A_k^2 \cos(k\theta)^2}{\sqrt{1 + c_1^2 k^2 A_k^2 \cos(k\theta)^2}} \in \left[0, c_1^2 k^2 A_k^2\right] \tag{3.11}$$

for all $(\theta, t) \in D_{2,k,\tau}$. It follows from (3.9)–(3.11) that

(3.12)
$$\frac{1}{\psi(\zeta_2(\theta, t))} \frac{\partial_\theta^2 \zeta_2(\theta, t)}{\left(1 + (\partial_\theta \zeta_2(\theta, t))^2\right)^{3/2}} + \frac{\psi'(\zeta_2(\theta, t))}{\psi(\zeta_2(\theta, t))^2} \frac{(\partial_\theta \zeta_2(\theta, t))^2}{\sqrt{1 + (\partial_\theta \zeta_2(\theta, t))^2}}$$
$$-\partial_t \zeta_2(\theta, t) \geq c_1 k^2 A_k \left(-\sup_{\zeta_2(D_{2,k,\tau})} \frac{1}{\psi} + A_k c_1 \min\left(\inf_{\zeta_2(D_{2,k,\tau})} \frac{\psi'}{\psi^2}, 0 \right) + c_2 \right)$$

for all $(\theta, t) \in D_{2,k,\tau}$. Moreover, direct calculations give

$$\zeta_2(D_{2,k,\tau}) = \left(-c_1 c_2 A_k k^2 \tau, c_1 A_k \right]. \tag{3.13}$$

It follows from (3.8) and (3.13) that for every $\varepsilon > 0$, if $k^2 \tau < \varepsilon$ and $k > 1/\varepsilon$, then

$$\zeta_2(D_{2,k,\tau}) \subset \left(-2^{2/p} \pi^{1-2/p} \varepsilon^{2-2/p} c_1 c_2, 2^{2/p} \pi^{1-2/p} \varepsilon^{1-2/p} c_1 \right). \tag{3.14}$$

By continuity of $1/\psi$ and ψ'/ψ^2 and since $\psi(0) > 0$, $A_k \to 0$ as $k \to \infty$ and $\varepsilon^{1-2/p} \to 0$ as $\varepsilon \to 0$, it follows from (3.12) and (3.14) that if the constant

c_2 is chosen so that $c_2 > 1/\psi(0)$, then there exists $\varepsilon_0 > 0$ such that

$$-\sup_{\zeta_2(D_{2,k,\tau})} \frac{1}{\psi} + A_k c_1 \min\left(\inf_{\zeta_2(D_{2,k,\tau})} \frac{\psi'}{\psi^2}, 0\right) + c_2 > 0 \tag{3.15}$$

for all $\tau > 0$ and $k \in \mathbb{N}$ such that $k^2\tau < \varepsilon_0$ and $k > 1/\varepsilon_0$. By putting together (3.12) and (3.15), we obtain that (3.1) holds true with $i = 2$. This ends the proof of Lemma 3.1. □

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. We fix $\sigma < 1/2$, $\lambda > 0$, and we define

$$\mu := \lambda \inf_I \frac{1}{\phi}. \tag{3.16}$$

We let ψ and Γ be as in (2.1) and J, τ, k, $D_{1,k,\tau}$, $D_{2,k,\tau}$, ζ_1, ζ_2, and ζ be as in Lemma 3.1. By using (A1)–(A3) and since $\zeta_1 \in C^\infty(D_{1,k,\tau}, J) \cap C^0(D_{2,k,\tau}, J)$ and $\zeta_2 \in C^\infty(D_{2,k,\tau}, J)$, we obtain that there exists $\varepsilon_0, a_0, b_0 \in \mathbb{R}$ such that

$$b_0 < \min\left(\frac{a_0\pi}{2k}, (\mu - \varepsilon_0)\left(\frac{\pi}{2k}\right)^\sigma, \sup J\right) \tag{3.17}$$

and

$$\zeta(\theta, 0) < \widetilde{\gamma}_0(\theta) \qquad \forall \theta \in (0, \pi/k), \tag{3.18}$$

where

$$\widetilde{\gamma}_0(\theta) := \begin{cases} \min\left(a_0\theta, (\mu - \varepsilon_0)\theta^\sigma, b_0\right) & \text{if } 0 \le \theta \le \dfrac{\pi}{2k} \\ \widetilde{\gamma}_0\left(\dfrac{\pi}{k} - \theta\right) & \text{if } \dfrac{\pi}{2k} < \theta \le \dfrac{\pi}{k}. \end{cases}$$

For any $\varepsilon > 0$ and $\theta \in [0, \pi/k]$, we then define

$$\widetilde{\gamma}_0^{(\varepsilon)}(\theta) := \begin{cases} f_\varepsilon\left(a_0\theta, f_\varepsilon\left((\mu - \varepsilon_0)\theta^\sigma, b_0\right)\right) & \text{if } 0 \le \theta \le \dfrac{\pi}{2k} \\ \widetilde{\gamma}_0^{(\varepsilon)}\left(\dfrac{\pi}{k} - \theta\right) & \text{if } \dfrac{\pi}{2k} < \theta \le \dfrac{\pi}{k}, \end{cases}$$

where

$$f_\varepsilon(\xi_1, \xi_2) := \frac{1}{2}\left[\xi_1 + \xi_2 - \varepsilon\,\eta\left(\frac{\xi_2 - \xi_1}{\varepsilon}\right)\right] \qquad \forall \xi_1, \xi_2 \in \mathbb{R}$$

and $\eta : \mathbb{R} \to (0, \infty)$ is a smooth, even cutoff function such that $\eta(\theta) = \theta$ for all $\theta \in [1, \infty)$ and $\eta'(\theta) > 0$ for all $\theta \in (0, 1)$. By using (3.18), it is easy to see that for small ε, $\widetilde{\gamma}_0^{(\varepsilon)} \in C^\infty([0,\pi/k], J)$ and $\widetilde{\gamma}_0^{(\varepsilon)} \to \widetilde{\gamma}_0$ in $C^{0,1}([0,\pi/k])$. Hence, by using (3.17) and remarking that $\widetilde{\gamma}_0^{(\varepsilon)}(\theta) = \widetilde{\gamma}_0(\theta) = a_0\theta$ for small θ, we obtain that for small ε, $\widetilde{\gamma}_0^{(\varepsilon)}$ is such that

(B1) $\zeta(\theta,0) \le \widetilde{\gamma}_0^{(\varepsilon)}(\theta) \quad \forall \theta \in [0,\pi/k]$,

(B2) $\left|\widetilde{\gamma}_0^{(\varepsilon)}(\theta) - \widetilde{\gamma}_0^{(\varepsilon)}(\theta')\right| < \mu\left|\theta-\theta'\right|^\sigma \quad \forall \theta,\theta' \in [0,\pi/k]$,

(B3) $\widetilde{\gamma}_0^{(\varepsilon)}(0) = \widetilde{\gamma}_0^{(\varepsilon)}(\pi/k) = \widetilde{\gamma}_0^{(\varepsilon)\prime\prime}(0) = \widetilde{\gamma}_0^{(\varepsilon)\prime\prime}(\pi/k) = 0$.

In what follows, we fix ε small enough so that (B1)–(B3) hold true. Since $\widetilde{\gamma}_0^{(\varepsilon)} \in C^\infty([0,\pi/k], J)$, the classical theory of parabolic equations (see for instance Lieberman [10, Theorem 8.2]) gives the existence of a solution $\widetilde{\gamma} \in C^\infty([0,\pi/k]\times[0,T))$ of the problem

(3.19)
$$\begin{cases} \partial_t\widetilde{\gamma} = \dfrac{1}{\psi(\widetilde{\gamma})}\dfrac{\partial_\theta^2\widetilde{\gamma}}{\left(1+(\partial_\theta\widetilde{\gamma})^2\right)^{3/2}} + \dfrac{\psi'(\widetilde{\gamma})}{\psi(\widetilde{\gamma})^2}\dfrac{(\partial_\theta\widetilde{\gamma})^2}{\sqrt{1+(\partial_\theta\widetilde{\gamma})^2}} & \text{in } [0,\pi/k]\times[0,T) \\ \widetilde{\gamma}(\cdot,0) = \widetilde{\gamma}_0^{(\varepsilon)} & \text{on } [0,\pi/k] \\ \widetilde{\gamma}(0,\cdot) = \widetilde{\gamma}(\pi/k,\cdot) = 0 & \text{on } [0,T), \end{cases}$$

where $T \in (0,\infty]$ is the maximal existence time for $\widetilde{\gamma}$. Moreover, since $\widetilde{\gamma}_0^{(\varepsilon)}([0,\pi/k]) \subseteq J$, it follows from the maximum principle that

$$\widetilde{\gamma}([0,\pi/k]\times[0,T)) \subseteq J.$$

By using (A1) and (3.1) and integrating by parts, we obtain that ζ is a weak subsolution of the equation in (3.19), i.e.

$$\int_0^{\tau'}\int_0^{\pi/k}\left(\eta\partial_t\zeta + \frac{1}{\psi(\zeta)}\frac{\partial_\theta\zeta\partial_\theta\eta}{\sqrt{1+(\partial_\theta\zeta)^2}} - 2\frac{\psi'(\zeta)}{\psi(\zeta)^2}\frac{(\partial_\theta\zeta)^2\eta}{\sqrt{1+(\partial_\theta\zeta)^2}}\right)d\theta dt \le 0$$

for all $\tau' \in (0,\tau)$ and $\eta \in C^1(D_{2,k,\tau})$ such that $\eta \ge 0$ in $D_{2,k,\tau}$ and $\eta(0,\cdot) = \eta(\pi/k,\cdot) = 0$ on $[0,\tau)$. We define $\omega := \zeta - \widetilde{\gamma}$. It follows from (A3), (B1), and (3.19) that $\omega \le 0$ on $\{0,\pi/k\}\times[0,\min(T,\tau))$ and $[0,\pi/k]\times\{0\}$. By applying the mean value theorem, we obtain that for any $\tau' \in (0,\min(T,\tau))$, there exist $a_1,a_2,b_1,b_2 \in L^\infty(D_{2,k,\tau'})$ such that $\inf\{a_1(\theta,t) : (\theta,t) \in D_{2,k,\tau'}\} > 0$

and

$$\int_0^{\tau'}\int_0^{\pi/k}\left(\eta\partial_t\omega+\left(a_1\partial_\theta\omega+a_2\omega\right)\partial_\theta\eta+\left(b_1\partial_\theta\omega+b_2\omega\right)\eta\right)d\theta dt\leq 0 \tag{3.20}$$

for all $\eta\in C^1\left(D_{2,k,\tau'}\right)$ such that $\eta\geq 0$ in $D_{2,k,\tau'}$ and $\eta\left(0,\cdot\right)=\eta\left(\pi/k,\cdot\right)=0$ on $[0,\tau')$. By applying a weak comparison principle (see for instance Lieberman [10, Corollary 6.16]), it follows from (3.20) that

$$\zeta\left(\theta,t\right)\leq\widetilde{\gamma}\left(\theta,t\right) \tag{3.21}$$

for all $(\theta,t)\in D_{2,k,\min(T,\tau)}$. It follows from (A3), (A4), and (3.21) that $T\leq\tau$. Note that by using similar arguments as in (3.20)–(3.21), we obtain that $\widetilde{\gamma}$ is the unique solution of (3.19). It then follows from classical theory of parabolic equations that

$$\lim_{t\to T}\sup_{D_{2,k,t}}\left|\partial_\theta\widetilde{\gamma}\right|=\infty. \tag{3.22}$$

Indeed, if (3.22) is not true, then $T=\infty$ (see for instance Lieberman [10, Theorems 8.3 and 12.1]), which is in contradiction with $T\leq\tau$. We let $\gamma:\mathbb{S}^1\times[0,T)\to\mathbb{R}$ be the function defined as

$$\gamma\left(\left(\cos\theta,\sin\theta\right),t\right):=\begin{cases}\widetilde{\gamma}\left(\theta-j\pi/k,t\right) & \text{if } j \text{ is even}\\ -\widetilde{\gamma}\left(\left(j+1\right)\pi/k-\theta,t\right) & \text{if } j \text{ is odd}\end{cases}$$

for all $(\theta,t)\in[j\pi/k,(j+1)\pi/k)\times[0,\min(T,\tau))$, $j\in\{0,\dots,2k-1\}$. Since $\psi'(0)=0$, it follows from (3.19) and (B3) that $\partial_\theta^2\widetilde{\gamma}\left(j\pi/k,t\right)=0$ for all $t\in[0,T)$ and $j\in\{0,\dots,2k-1\}$ which implies that γ is a smooth solution of (2.2). By using (B2), (3.16), (3.22), and the change of functions (2.1), we then obtain the existence of $\rho_0\in C^\infty\left(\mathbb{S}^n,I\right)$ such that (1.4) holds true, the solution of (1.1) exists and $\partial_x\rho\left(\cdot,t\right)$ blows up as $t\to T$. This ends the proof of Theorem 1.3. □

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