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# A high order Hopf lemma for mappings into classical domains and applications

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We establish a high order Hopf lemma type result for holomorphic mappings into classical domains, and find its applications in studying the boundary behavior of holomorphic isometric and proper maps.



## 1. Introduction

The classical Hopf lemma for subharmonic functions has been applied extensively in complex analysis to study holomorphic maps between domains or CR maps between real hypersurfaces. It is often used to obtain transversality of the map at a boundary point when the target domain or hypersurface satisfies certain types of "convexity" conditions. See Proposition 2.1 for a typical application of the Hopf lemma. Much effort has been made to generalize the classical Hopf lemma to more applicable settings to study the transversality and the unique continuation problem for holomorphic maps. Here we mention the papers [A2], [ABR], [BR1-2], [HKMP], [HK], [L], [BH] and references therein.

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In this paper we establish a Hopf lemma type result of different flavor from the aforementioned literature — a high order Hopf lemma for holomorphic mappings into classical domains. The mapping problem between bounded symmetric domains is an important subject in complex analysis and geometry. Due to the special geometric structure of bounded symmetric domains, many striking phenomena such as rigidity property have been discovered for proper and isometric maps since the classical result of Poincaré [Po]. For many results along this line, see [A1], [Fr], [Hu], [HJ], [DX1-2], [TH], [Ts], [Ng3], [M1-5], [YZ], [KZ1-2], [Km] and references therein. See also recent survey articles [NTY], [CXY] for more detailed account and references on this subject. Our Hopf lemma may shed new light on the study of the boundary behavior of mappings into bounded symmetric domains. As a first exploration, we will restrict ourselves to the case of classical domains in this article.

To explain our results, we first recall the notion of bounded symmetric domains. A complex manifold $X$ with a Hermitian metric $h$ is said to be a Hermitian symmetric space if, for every point $p \in X$, there exists an involutive holomorphic isometry $\sigma_p$ of $X$ such that $p$ is an isolated fixed point. An irreducible Hermitian symmetric space of noncompact type can be, by the Harish-Chandra embedding (See [Wo]), realized as a bounded domain in some complex Euclidean space. Such domains are convex, circular and sometimes called bounded symmetric domains. Irreducible bounded symmetric domains can be classified into Cartan's four types of classical domains and two exceptional domains (See [M1]). See more details on classical domains in Section 3.

The rank $r$ of a bounded symmetric domain $D$, can be defined as the dimension of the maximal polydisc that can be totally geodesically embedded into $D$. Write $K_D(Z,\overline{Z})$ for the Bergman kernel of an irreducible bounded symmetric domain $D$. Then there is a Hermitian polynomial $Q_D(Z,\overline{Z})$ such that $K_D(Z,\overline{Z}) = \frac{1}{Q_D(Z,\overline{Z})}$. Moreover, $Q_D(Z,\overline{Z}) = A_D\rho(Z,\overline{Z})^n$, where $A_D$ is a positive constant, $n$ is a positive integer both depending on $D$, and $\rho(Z,W)$ is an irreducible holomorphic polynomial satisfying $\rho(Z,\overline{Z}) > 0$ in $D$ and $\rho(Z,\overline{Z}) = 0$ on the boundary $\partial D$. Furthermore, let $P$ be the maximal polydisc $\Delta^r \times \{\mathbf{0}\}$ in $\Omega$ in Harish-Chandra coordinates, then for $\hat{Z} = (z_1, \cdots, z_r; \mathbf{0}) \in P$, we have

$$\rho(\hat{Z}, \overline{\hat{Z}}) = \prod_{i=1}^{r}(1 - |z_i|^2).$$

See [M1], [H], [FK], [Lo] for more details on $K_D$ and $\rho$. The function $\rho$ is sometimes called the generic norm associated to the domain $D$. We include the explicit formulas of $\rho$ for classical domains in Section 3. Write $K \subset \mathrm{Aut}_0(D)$ for the isotropy subgroup at $0 \in D$. An important property of $\rho$ is the invariance under $K : \rho(Z, \overline{Z}) = \rho(\gamma(Z), \overline{\gamma(Z)})$ for every $\gamma \in K$.

We next recall the boundary fine structure of an irreducible bounded symmetric domain $D$ (cf. [Wo]). By Borel embedding (See [M1], [Wo]), $D$ can be canonically embedded into its dual Hermitian symmetric manifolds $X$ of compact type. Under the embedding, every automorphism $g \in \mathrm{Aut}(D)$ extends to an automorphism of $X$ and $D$ becomes an open orbit under the action of $\mathrm{Aut}(D)$ on $X$. Moreover, denoting the rank of $D$ by $r$, the topological boundary $\partial D$ of $D$ decomposes into exactly $r$ orbits under the action of the identity component $\mathrm{Aut}_0(D)$ of $\mathrm{Aut}(D) : \partial D = \cup_{i=1}^r E_i$, where $E_k$ lies in the closure of $E_l$ if $k > l$ (The explicit formulas to define $E_k'$s can be found in Section 3 for classical domains). Moreover, $E_k$ is the smooth part of the semi-analytic variety $\cup_{j=k}^r E_j$ (See the proof of Lemma 2.2.3 in [MN]). In particular $E_1$ is the unique open orbit, which is indeed the smooth part of $\partial D$, and $E_r$ is the Shilov boundary. We recall in general for a bounded domain $D$, its Shilov boundary $S$ is defined to be the (unique) smallest closed subset of $\partial D$ such that for every holomorphic function $f$ defined in a neighborhood of $\overline{D}$, it holds that

$$\sup_{z \in \overline{D}} |f(z)| = \sup_{z \in S} |f(z)|.$$

Note the boundary $\partial D$ of a bounded symmetric domain $D$ is non-smooth and contains complex varieties, unless $D$ is biholomorphic to the unit ball.

We now introduce our main result. Let $\rho$ and $E_k'$s be as above which are associated to an irreducible bounded symmetric domain $D$. Write $\rho \circ F = \rho(F, \overline{F})$ for a holomorphic map $F$ into $D$.

**Theorem 1.** *Let $\Omega \subset \mathbb{C}^n$ be a domain and $D$ an (irreducible) classical domain in $\mathbb{C}^m$ with rank $r$. Fix $1 \leq k \leq r$. Let $F$ be a holomorphic map from $\Omega$ to $D$ and extends $C^1-$smoothly to a smooth boundary point $a \in \partial\Omega$. Assume $F(a) \in E_k$. Then*

(1) *The following limit exists (as a finite number) and satisfies the sign condition:*

$$(-1)^k \lim_{t \to 0^-} \frac{\rho \circ F(a + t\nu)}{t^k} > 0. \tag{1.1}$$

*Here $\nu$ is the outward pointing unit normal vector of $\Omega$ at $a$.*

(2) *If in addition $F$ is $C^k$ near $a$, then all derivatives of $\rho \circ F$ of order $\leq k-1$ must vanish at $a$. Moreover,*

$$(1.2)\qquad (-1)^k \frac{\partial^k(\rho \circ F)}{\partial \nu^k}\Big|_a > 0;$$

$$(1.3)\qquad L^k(\rho \circ F)\big|_a \neq 0.$$

*Here $L$ is the complex normal direction of $\partial\Omega$ at $a$ with $\frac{\partial}{\partial\nu} = 2\mathrm{Re}L$.*

**Remark 1.1.** (i). Note $\rho \circ F$ is always positive in $\Omega$, and thus part (1) in Theorem 1 is equivalent to say the limit in (1.1) is finite and nonzero. In this case, we will say $\rho \circ F$ vanishes to the $k^{\mathrm{th}}$ order at the point $a$ along the normal direction. This conclusion may fail if in Theorem 1 we merely assume the map $F$ has a Hölder continuous extension to the point $a$. See Example 1.1.

(ii). In part (2) since all derivatives of $\rho \circ F$ of order $\leq k-1$ vanish at $a$, the quantities on the left hand side of (1.2) and (1.3) do not depend on the extensions of $\frac{\partial}{\partial\nu}$ and $L$ to a neighborhood of $a$.

(iii). Note the equation (1.2) follows from (1.1) when $F$ is $C^k$ smooth at $a$. In the proof, however, we indeed establish part (2) of Theorem 1 first.

**Example 1.1.** Let $\Delta \subset \mathbb{C}$ be the unit disc and $D_m^{IV}$ the classical domain of type IV (See Section 3.4 for the definition of the latter). Let $F : \Delta \to D_m^{IV}$ be defined by

$$F(\xi) = (\xi, 0, \cdots, 0, 1 - \sqrt{1-\xi^2}).$$

Then $F$ is holomorphic in $\Delta$ and admits a Hölder continuous extension up to the circle $\partial\Delta$. Set $a = 1 \in \partial\Delta$. Then $F(a) = (1, 0, \cdots, 0, 1) \in E_2$ (thus $k = 2$, see Section 3.4 for the explicit defining formula of $E_2$ for type IV domains) and $F$ does not have a $C^1-$extension to the point $a$. In this case, the outward pointing normal unit vector of $\partial\Delta$ at $a$ is $\nu = (1,0) \in \mathbb{R}^2 = \mathbb{C}$. We compute $\lim_{t\to 0^-} \frac{\rho\circ F(a+t\nu)}{t} = -2$. Thus $\rho \circ F$ vanishes only to the first order at $a$ along the normal direction.

We next recall the following definition.

**Definition 1.1.** Let $k \geq 1$. Let $U$ be an open subset in $\mathbb{R}^N$ and $\psi \in C^k(U)$. Denote by $x = (x_1, \cdots, x_N)$ the coordinates in $\mathbb{R}^N$ and for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_N)$, write $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We say $\psi$ vanishes to the $k^{\mathrm{th}}$ order at $p \in U$ if $\frac{\partial^{|\alpha|}\psi}{\partial x^\alpha}(p) = 0$ for all $|\alpha| \leq k-1$ and $\frac{\partial^{|\beta_0|}\psi}{\partial x^{\beta_0}}(p) \neq 0$ for some multi-index $\beta_0$ with $|\beta_0| = k$.

To prove the main theorem, the first step is to establish the following characterization of the boundary orbits of classical domains. Note when $k = 1$, Theorem 2 follows from Lemma 2 in [M5].

**Theorem 2.** *Let $D$ be an (irreducible) classical domain in $\mathbb{C}^m$ with rank $r$. Let $1 \le k \le r$. Then*

$$E_k = \{b \in \partial D : \rho(Z, \overline{Z}) \text{ vanishes to the } k^{th} \text{ order at } b\}.$$

**Remark 1.2.** From Theorem 1 and 2, one immediately sees that under the setting of Theorem 1, there exists some component of the map $F$ whose normal derivative is nonzero at $a$.

We next discuss applications of our Hopf lemma in studying holomorphic isometric maps into classical domains. The study of metric-preserving maps dates back to the work of Calabi [Ca] and Bochner [Bo]. More recently, the isometric mappings between bounded symmetric domains attract lots of attention from many researchers. One particular motivation comes from arithmetic geometry in the work of Clozel-Ullmo [CU] in understanding the modular correspondence problem, which was reduced to a rigidity problem for isometric mappings in purely complex geometric settings (See [CU] for more details). Mok (cf. [M2-5]) then led an extensive study on such holomorphic isometric mapping problem. Let $F : (\Omega, \lambda ds^2_\Omega) \to (D, ds^2_D)$ be an isometric mapping between bounded symmetric domains $\Omega$ and $D$ which we equip with the Bergman metrics $ds^2_\Omega$ and $ds^2_D$. When $D$ is irreducible and of rank at least 2, Clozel-Ullmo [CU] observed the proof of Hermitian metric rigidity of Mok (See [M1] and [M3]) yields already the total geodesy of $F$. When $\Omega$ is a complex unit ball $\mathbb{B}^n (n \ge 2)$, and $D$ is a product of some unit balls, Yuan-Zhang ([YZ], See also [M3], [Ng1] for related work) showed $F$ must be totally geodesic. Hence, if we assume $\Omega$ is irreducible and has dimension at least two, then the simplest case that one can expect a non-totally geodesic map $F$ is when $\Omega$ is the unit ball $\mathbb{B}^n$ and $D$ is irreducible and of rank $\ge 2$. Mok [M5] initiated the study of isometric maps from the unit ball to bounded symmetric domains of higher rank, and in partiular proved the existence of such non-totally geodesic maps. Since the work of Mok [M5], various authors contributed along this direction of understanding isometric maps from the unit ball to bounded symmetric domains of higher rank, including [CM], [Ch1], [UWZ], [XY1-2], etc. More related results on metric-preserving or volume-preserving mappings between Hermitian symmetric spaces can be found in [HY], [MN], [Ng2-3], [Y], [FHX], [Ch2], [CY], [X] and references therein.

Let $D$ be an irreducible bounded symmetric domain. Denote by $\omega_D$ and $\omega_{\mathbb{B}^n}$ the normalized Bergman metric (Kähler-Einstein metric) such that the minimal disc is of constant Gaussian curvature $-2$ for $D$ and $\mathbb{B}^n$, respectively. One can verify the normalized Bergman metric $\omega_D$ of $D$ is given by $\omega_D = \sqrt{-1}\partial\overline{\partial}\log\rho^{-1}(Z,\overline{Z})$, where $\rho$ is the generic norm associated to $D$. Let $F : \mathbb{B}^n \to D$ be an isometric map satisfying $F^*(\omega_D) = \lambda\omega_{\mathbb{B}^n}$ for some positive constant $\lambda$. Chan-Mok [CM] showed that $\lambda$ must be an integer and $1 \leq \lambda \leq r$. We have the following consequence of Theorem 1 on the boundary behavior of such isometric maps.

**Theorem 3.** *Let $F$ be a holomorphic isometric map from $\mathbb{B}^n (n \geq 1)$ to an (irreducible) classical domain $D$ of rank $r : F^*(\omega_D) = \lambda\omega_{\mathbb{B}^n}$ for some positive constant $\lambda$. Fix $1 \leq k \leq r$. Then the following statements are equivalent:*

(1) $\lambda = k$.

(2) *There is a point $a \in \partial\mathbb{B}^n$ such that $F$ has a $C^1$ smooth extension to $a$ and $F(a) \in E_k$.*

(3) *For every $\xi \in \partial\mathbb{B}^n$ to which $F$ has a $C^1$ smooth extension, it must hold that $F(\xi) \in E_k$.*

**Remark 1.3.** (i). By Mok [M1], every (local) holomorphic isometry between bounded symmetric domains must be algebraic and thus extends holomorphically across a generic boundary point. Hence Theorem 3 implies that $\lambda = k$ if and only if $F$ maps a generic boundary point to $E_k$. Note this fact was known to Mok [M5] when $k = 1$.

(ii). In particular, if $F$ is a holomorphic polynomial isometric map from $\mathbb{B}^n$ to an (irreducible) classical domain $D$ of rank $r$, i.e., $F^*(\omega_D) = k\omega_{\mathbb{B}^n}$ for some $1 \leq k \leq r$, then $F$ maps every point on $\partial\mathbb{B}^n$ to $E_k$.

We have the following consequences of Theorem 3.

**Corollary 1.1.** *Let $F$ be a holomorphic isometric map from $\mathbb{B}^n (n \geq 1)$ to an (irreducible) classical domain of with rank $r : F^*(\omega_D) = k\omega_{\mathbb{B}^n}$ for some $1 \leq k \leq r$. Assume $a \in \partial\mathbb{B}^n$ and there is a sequence $\{a_i\}_{i=1}^{\infty} \subset \mathbb{B}^n$ converging to $a$ such that $\lim_{i\to\infty} F(a_i) \notin E_k$. Then $F$ cannot be extended $C^1-$smoothly to $a$.*

**Corollary 1.2.** *Let $F$ be a holomorphic isometric map from $\mathbb{B}^n$ to an (irreducible) classical domain $D$ of rank $r$, i.e., $F^*(\omega_D) = k\omega_{\mathbb{B}^n}$ for some $1 \leq k \leq r$. Assume $F$ extends continuously to a point $\xi \in \partial\mathbb{B}^n$. Then $F(\xi) \in \cup_{l=k}^{r} E_l$.*

**Remark 1.4.** (i). Under the assumption of Corollary 1.2, where $F$ only extends continuously to $\xi$, one cannot expect $F(\xi) \in E_k$ in general. See Example 1.2.

(ii). When $n = 1$, i.e., the source domain is the unit disc $\Delta$, it follows from [M3] that $F$ must extend continuously to the closed disc $\overline{\Delta}$. Thus the conclusion in Corollary 1.2 holds for every boundary point of $\Delta$. We remark, however, for larger $n$, there exists a holomorphic isometric map from $\mathbb{B}^n$ to a classical domain which fails to extend continuously to the whole boundary $\partial\mathbb{B}^n$. See Example 1.3.

The following result shows that the image of a non-totally geodesic isometric map $F : \mathbb{B}^n \to D$ cannot touch the Shilov boundary of $D$ at a high order.

**Corollary 1.3.** *Let $F$ be a holomorphic isometric map from $\mathbb{B}^n$ to an (irreducible) classical domain $D$. If $F$ extends $C^1-$smoothly to a point $\xi \in \partial\mathbb{B}^n$ and maps $\xi$ to the Shilov boundary of $D$, then $F$ must be totally geodesic.*

**Remark 1.5.** The conclusion in Corollary 1.3 fails if we only assume $F$ has a Hölder continuous extension to the point $\xi$. See Example 1.2.

In the following two examples, let $D_m^{IV}$ and $D_{p,q}^{I}$ be classical domains of type IV and I, respectively. (See Section 3 for their definitions).

**Example 1.2.** Let $F$ be a holomorphic map from $\mathbb{B}^n$ to $D_m^{IV} (m \geq n+1)$ given by

$$\left(z_1, \cdots, z_{n-1}, z_n, 0, \cdots, 0, 1 - \sqrt{1 - \sum_{j=1}^{n} z_j^2}\right).$$

One can verify that $F$ is an isometric map satisfying $F^*(\omega_D) = \omega_{\mathbb{B}^n}$ and $F$ is not totally geodesic. Let $V$ be the real subvariety of $\partial\mathbb{B}^n$ defined by

$$V = \{z \in \partial\mathbb{B}^n : \sum_{i=1}^{n} z_j^2 = 1\}.$$

Then for every point $a \in V$, $F$ admits a Hölder continuous extension to $a$, and $F(a) \in E_2$, the Shilov boundary of $D_m^{IV}$. However, $F$ has no $C^1-$extension to $a$. This supports the assertions in Corollary 1.1 and Remark 1.5.

**Example 1.3.** Let $q \geq p \geq 2$. Write the coordinates in $\mathbb{B}^{p+q-1}$ as $\xi = (\xi_1, \cdots, \xi_p, \eta_2, \cdots, \eta_q)$. Let $F : \mathbb{B}^{p+q-1} \to D_{p,q}^{I}$ be the following isometric map with respect to the normalized Bergman metrics (This example is taken from equation (42) in [XY2]):

$$F(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_q \\ \eta_2 & f_{22} & \cdots & f_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ \eta_p & f_{p2} & \cdots & f_{pq} \end{pmatrix},$$

where $f_{ij} = \frac{\eta_i \xi_j}{\xi_1 - 1}, 2 \leq i \leq p, 2 \leq j \leq q$. Then $F$ has no continuous extension to the point $a := (1, 0, \cdots, 0)$. Indeed, one can verify $F$ has different limits when $\xi$ goes to $a$ along the two paths: $(1-t, t, t, 0, \cdots, 0)$ and $(1-t^2, \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, 0, \cdots, 0)$ where $0 < t << 1$.

More generally, we can use Theorem 1 to study isometric maps between reducible bounded symmetric domains. Let $\Omega = \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_N}$ be a bounded symmetric domain, where $\Omega_i$'s are the irreducible factors of $\Omega$, and $D = D_1 \times \cdots \times D_{N'} \subset \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_{N'}}$ a product of irreducible classical domains. Assume $\mathrm{rank}(\Omega_i) = r_i$ for $1 \leq i \leq N$ so that $\mathrm{rank}(\Omega) = r := \sum_{i=1}^{N} r_i$. Similarly, assume $\mathrm{rank}(D_j) = t_j$ for $1 \leq j \leq N'$ so that $\mathrm{rank}(D) = t = \sum_{j=1}^{N'} t_j$. Write the decomposition of the boundaries of $\Omega_i$ and $D_j$ as $\partial\Omega_i = \cup_{s=1}^{r_i} E_s^i$ and $\partial D_j = \cup_{s=1}^{t_j} \widetilde{E}_s^j$ respectively, for $1 \leq i \leq N, 1 \leq j \leq N'$. We will also write $E_0^i = \Omega_i, 1 \leq i \leq N$ and $\widetilde{E}_0^j = D_j, 1 \leq j \leq N'$ so that $\overline{\Omega_i} = \cup_{s=0}^{r_i} E_s^i$ and $\overline{D_j} = \cup_{s=0}^{r_j} \widetilde{E}_s^j$. Let $F = (F_1, \cdots, F_{N'})$ be a holomorphic isometric map from $\Omega = (\Omega_1, \lambda_1 \omega_{\Omega_1}) \times \cdots \times (\Omega_N, \lambda_N \omega_{\Omega_N})$ to $D = (D_1, \mu_1 \omega_{D_1}) \times \cdots \times (D_{N'}, \mu_{N'} \omega_{D_{N'}})$ for some positive constants $\lambda_i$'s and $\mu_j$'s in the sense that

$$\sum_{j=1}^{N'} \mu_j F_j^*(\omega_{D_j}) = \bigoplus_{i=1}^{N} \lambda_i \omega_{\Omega_i}. \tag{1.4}$$

Then we have the following result for the boundary behavior of $F$.

**Theorem 4.** *Let $\Omega, D$ be as above and $F : \Omega \to D$ a holomorphic isometric map satisfying (1.4). Assume $F$ extends $C^1-$smoothly to a neighborhood of a boundary point $a$ of $\Omega$. Write $a = (a_1, \cdots, a_N)$, where $a_i \in \overline{\Omega_i}$ and $F(a) = (b_1, \cdots, b_{N'})$, where $b_j \in \overline{D_j}$. Assume for each $1 \leq i \leq N$, $a_i \in E_{l_i}^i$ for some $0 \leq l_i \leq r_i$. Similarly, for each $1 \leq j \leq N'$, assume $b_j \in \widetilde{E}_{k_j}^j$ for some $0 \leq k_j \leq t_j$. Then*

$$\sum_{i=1}^{N} \lambda_i l_i = \sum_{j=1}^{N'} \mu_j k_j. \tag{1.5}$$

**Corollary 1.4.** *Let $\Omega, D$ be as above, $U$ a connected open subset of $\Omega$, and $F : U \to D$ a holomorphic isometric map satisfying (1.4) in $U$. Then $(\lambda_1, \cdots, \lambda_N)$ and $(\mu_1, \cdots, \mu_{N'})$ satisfy the following number theoretic condition: For any $N-$tuple of integers $(\sigma_1, \cdots, \sigma_N)$ with $0 \le \sigma_i \le r_i$, there exists an $N'-$tuple of integers $(\eta_1, \cdots, \eta_{N'})$ with $0 \le \eta_j \le t_j$ such that*

$$\sum_{i=1}^{N} \lambda_i \sigma_i = \sum_{j=1}^{N'} \mu_j \eta_j. \tag{1.6}$$

*In particular, each $\lambda_i, 1 \le i \le N$, is a linear combination of $\mu_j{}'s$ with integer coefficients:*

$$\lambda_i = \sum_{j=1}^{N'} c^i_j \mu_j, \tag{1.7}$$

*where each $c^i_j$ is an integer and $0 \le c^i_j \le t_j$.*

Theorem 1 can be also applied to study the transversality of proper maps into classical domains. Recall a map $F : M \to M'$ between real hypersurfaces is said to be CR-transversal at $p \in M$ if $T^{(1,0)}_{F(p)} M' + T^{(0,1)}_{F(p)} M' + dF(\mathbb{C}T_p M) = \mathbb{C}T_{F(p)} M'$(See [BER]). The following proposition generalizes a transversality result of Mok (Lemma 7 in [M4]) when the target is a classical domain.

**Proposition 1.1.** *Let $F$ be a holomorphic map from a domain $\Omega$ in $\mathbb{C}^n$ to an (irreducible) classical domain $D$ in $\mathbb{C}^m$ with rank $r$. Let $p \in \partial\Omega$ be a smooth boundary point of $\Omega$. Assume $F$ extends $C^1-$smoothly across $p$ and $F$ maps an open piece of $\partial\Omega$ near $p$ to $E_k$ for some $1 \le k \le r$. Then there exists a germ of real algebraic smooth real hypersurface $M'$ in $\mathbb{C}^m$ containing $E_k$ near $F(p)$ such that $F$ is CR transversal to $M'$ at $p$. In particular, it holds that $dF\left(\mathbb{C}T_p(\partial\Omega)\right) \not\subset T^{(1,0)}_{F(p)} E_k + T^{(0,1)}_{F(p)} E_k$.*

The paper is organized as follows. Section 2 discusses a typical application of the classical Hopf lemma(i.e. Proposition 2.1), from which Theorem 1 follows in the special case $k = 1$. We prove the general case of Theorem 1 and Theorem 2 in Section 3. As applications of our high order Hopf lemma, we prove in Section 4 Theorem 3, Theorem 4, and Proposition 1.1, as well as their corollaries.

To end this section, we explain our strategy to prove our main theorem (Theorem 1). The proof heavily relies on the invariance property of the generic norm $\rho$ under the action of the isotropy group $K \subset \mathrm{Aut}_0(D)$. This allows us to apply the action of the group $K$ to normalize the map $F$. Under a special normalization, to prove part (2) of Theorem 1 we show the $k^{\text{th}}$ derivatives $\frac{\partial^k(\rho\circ F)}{\partial\nu^k}\big|_a$ and $L^k(\rho\circ F)\big|_a$ only depend on the first jet of $F$ at $a$. This also explains why the $C^1-$smoothness assumption is essential in part (1). Furthermore, under this normalization, the hypothesis that $F(a)\in E_k$ can be used to construct some holomorphic functions (from the map $F$) which attain the maximal modulus at $a$. In this way, we can apply the classical Hopf lemma to obtain certain non-degeneracy in the first jet of $F$ at $a$. Although we stick to this basic strategy, due to the distinct structures of $K$ and expressions of $\rho$, we have to treat the four types of classical domains case by case.

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## 2. A transversality result

We first observe in the case $k=1$, Theorem 1 is just a consequence of the classical Hopf Lemma. For completeness, we sketch a proof for the following folklore transversality result (See [Fo]). For a function $h$ in $U\subset\mathbb{C}^n$ and $a\in U$, we will often switch between the notations $h|_a$ and $h(a)$ which both denote the value of $h$ at $a$.

**Proposition 2.1.** *Let $\Omega\subset\mathbb{C}^n$ be a domain and $D$ a convex domain in $\mathbb{C}^m(m\geq n\geq 1)$. Assume $b$ is a smooth point of $\partial D$ and $r$ a local defining function of $D$ in a small neighborhood $V$ of $b$ :*

$$D\cap V=\{W\in V: r(W,\overline{W})<0\},\quad dr\neq 0 \text{ in } V.$$

*Let $F$ be a holomorphic map from $\Omega$ to $D$ and extends $C^1-$smoothly up to a smooth point $a\in\partial\Omega$. If $F(a)=b$, then $\frac{\partial(r\circ F)}{\partial\nu}|_a>0$ and $L(r\circ F)|_a\neq 0$. Here $\nu$ is the outward pointing normal vector of $\Omega$ at $a$, and $L$ is complex normal direction of $\partial\Omega$ at $a$ such that $\frac{\partial}{\partial\nu}=2\mathrm{Re}L$.*

**Remark 2.1.** Let $D$ be an irreducible bounded symmetric domain. Then $D$ is convex, and recall $b\in\partial D$ is smooth if and only if $b\in E_1$ (See Lemma 2.2.3 in [MN]). Moreover, the function $-\rho$, where $\rho$ was introduced in Section 1, is a local defining function of $D$ (See Lemma 2 in [M5]) at every smooth point. Then Theorem 1 follows from Proposition 2.1 in the special case $k=1$.

**Proof of Proposition 2.1:** We first recall the following well-known fact (cf. [Ho], [Kr]) about convex sets.

**Lemma 2.1.** *If $b$ is some smooth boundary point of a convex open set $D \subset \mathbb{R}^m$, then there exists a neighborhood $\hat{V}$ of $b$ and some smooth defining function $\phi$ of $D$ in $\hat{V}$ such that $\phi$ is convex in $\hat{V}$. That is,*

$$\sum_{j,k=1}^{m} \frac{\partial^2 \phi(x)}{\partial x_j \partial x_k} u_j u_k \geq 0 \tag{2.1}$$

*for every $x \in \hat{V}$ and $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$.*

Lemma 2.1 yields that there exists a smooth local defining function $\phi$ of $D$ in some neighborhood $\hat{V}$ of $b$ such that $\phi$ is convex in $\hat{V}$ (in the sense of (2.1)). This implies in particular $\phi$ is plurisubharmonic in $\hat{V}$. Pick a small neighborhood $U$ of $a$, such that $F$ maps $\Omega \cap U$ to $\hat{V}$. As $F$ is holomorphic in $\Omega$, we conclude $\phi \circ F := \phi(F, \overline{F})$ is plurisubharmonic and thus subharmonic in $\Omega \cap U$. Note $\phi(F, \overline{F}) < 0$ in $\Omega \cap U$ and $\phi(F(a), \overline{F}(a)) = 0$. It follows from the classical Hopf lemma that $\frac{\partial(\phi \circ F)}{\partial \nu}|_a > 0$. Recall $L = \frac{1}{2}(\frac{\partial}{\partial \nu} + \sqrt{-1}\frac{\partial}{\partial \mu})$ for some unit vector $\mu$ tangent to $\partial\Omega$ at $a$. Note $\phi \circ F$, when restricted to $\partial\Omega$, attains its local maximum at $a$. It follows that $\frac{\partial(\phi \circ F)}{\partial \mu}|_a = 0$, and thus $L(\phi \circ F)|_a \neq 0$. Since $r$ and $\phi$ are both defining functions of $D$ at the smooth point $b$, we conclude that $r = h\phi$ near $b$ for some positive smooth function $h$ in a small neighborhood of $b$. This implies $\frac{\partial(r \circ F)}{\partial \nu}|_a = h(b)\frac{\partial(\phi \circ F)}{\partial \nu}|_a > 0$ and $L(r \circ F)|_a = h(b)L(\phi \circ F)|_a \neq 0$. We have thus established Proposition 2.1. ∎

## 3. Proof of Theorem 1 and 2

The classical domains are classified into four different types, which are called Cartan's four types of classical domains. Due to their distinct structures, we will prove Theorem 1 (as well as Theorem 2) separately for the four types of domains.

### 3.1. Hopf lemma for type I case

Assume $p \leq q$ and write $\mathbb{C}^{p \times q}$ for the space of $p \times q$ matrices with entries of complex numbers. The classical domain of type I is defined by:

$$D^I_{p,q} = \{Z \in \mathbb{C}^{p \times q} : I_p - Z\overline{Z}^t > 0\}.$$

The Bergman kernel of $D^I_{p,q}$ is given by

$$K_{D^I_{p,q}}(Z,\overline{Z}) = c_I \left(\det(I_p - Z\overline{Z}^t)\right)^{-(p+q)},$$

where $c_I$ is some positive constant. The generic norm $\rho$ for $D^I_{p,q}$ is given by

$$\rho(Z,\overline{Z}) = \det(I_p - Z\overline{Z}^t).$$

The boundary of $D^I_{p,q}$ is given by

$$\partial D^I_{p,q} = \{Z \in \mathbb{C}^{p\times q} : I_p - Z\overline{Z}^t \geq 0;\ \ \det(I_p - Z\overline{Z}^t) = 0\}.$$

Note the rank of $D^I_{p,q}$ is of rank $p$ and the boundary $\partial D^I_{p,q}$ decomposes into $p$ orbits under the action of the identity component of $\mathrm{Aut}(D^I_{p,q}) : \partial D^I_{p,q} = \cup_{i=1}^p E_i$. Here $E_k$ lies in the closure of $E_l$ when $k > l; E_1$ is the smooth part of $\partial D^I_{p,q}$, and $E_p$ is the Shilov boundary. More explicitly in this type I case,

$$E_k = \{Z \in \partial D^I_{p,q} :\ \text{the corank of } I_p - Z\overline{Z}^t \text{ equals } k\}, \quad 1 \leq k \leq p.$$

To better illustrate the boundary, we recall the following basic fact from linear algebra. Let $Z$ be a $p \times q (p \leq q)$ matrix. Then there exist a $p \times p$ unitary matrix $U$ and a $q \times q$ unitary matrix $V$ which normalize $Z$ into the following form:

$$Z = U \begin{pmatrix} r_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_p & 0 & \cdots & 0 \end{pmatrix} V \tag{3.1}$$

with $r_1 \geq r_2 \geq \cdots \geq r_p \geq 0$. The above equation is often called the singular value decomposition of $Z$. The $r_i$'s are called the singular values of $Z$ and their squares give the eigenvalues of $Z\overline{Z}^t$. The strata $E_k{}'$s are then equivalently given by

$$E_k = \{Z \in \mathbb{C}^{p\times q} : 1 = r_1 = \cdots = r_k > r_{k+1} \geq \cdots \geq r_p \geq 0\}$$

where $r_i$'s are singular values of $Z$ as above.

We recall the following well known Laplacian expansion for the determinant of a square matrix in linear algebra. Let $B$ be an $m \times m$ matrix. Let $1 \leq i_1 < \cdots < i_s \leq m$ and $1 \leq j_1 < \cdots < j_s \leq m$. We denote by $B\binom{i_1 \ \cdots \ i_s}{j_1 \ \cdots \ j_s}$

the determinant of the submatrix of $B$ formed by its ${i_1}^{\text{th}}, \cdots, {i_s}^{\text{th}}$ rows and ${j_1}^{\text{th}}, \cdots, {j_s}^{\text{th}}$ columns.

**Proposition 3.1.** *Let $B = (B_{ij})_{1\le i,j\le m}$ be an $m\times m$ matrix. Fix $1\le s< m$. Then*

$$\det B = \sum_{1\le j_1<\cdots<j_s\le m} \mathcal{E}_{j_1\cdots j_s} B\binom{1 \quad \cdots \quad s}{j_1 \quad \cdots \quad j_s} B\binom{s+1 \quad \cdots \quad m}{j_{s+1} \quad \cdots \quad j_m} \tag{3.2}$$

*where $j_1,\cdots,j_s,j_{s+1},\cdots,j_m$ is a permutation of $1,2,\cdots,m$ with $j_{s+1}<\cdots<j_m$. Here we write $\mathcal{E}_{j_1\cdots j_s} = (-1)^{1+2+\cdots+s+j_1+\cdots+j_s}$.*

We first give a proof for Theorem 2 in the type I case. The proof for this case is inspired by [LT].

**Proof of Theorem 2 for the type I case:** Fix $1\le k\le p$ and $Z_0\in E_k$. As discussed above, there exist a $p\times p$ unitary matrix $U$ and a $q\times q$ unitary matrix $V$ such that the single value decomposition in (3.1) holds for $Z_0$ with $1=r_1=\cdots=r_k<r_{k+1}\le\cdots\le r_p$. Note $\rho$ is invariant under the action of $U$ and $V$, we can thus assume $Z_0$ takes the normalized form in (3.1), i.e.,

$$Z_0 = \begin{pmatrix}\mathbf{D}(Z_0) & \mathbf{0}_{p\times(q-p)}\end{pmatrix}, \tag{3.3}$$

where $\mathbf{D}(Z_0) = \mathrm{diag}(1,\cdots,1,r_{k+1},\cdots,r_p)$ is the $p\times p$ diagonal matrix and $\mathbf{0}_{s\times t}$ denotes the $s\times t$ zero matrix. Write the $p\times p$ matrix

$$A(Z) = A(Z,\overline{Z}) = I_p - Z\overline{Z}^t. \tag{3.4}$$

In particular,

$$A(Z_0,\overline{Z_0}) = \mathrm{diag}(0,\cdots,0,1-r_{k+1}^2,\cdots,1-r_p^2) \tag{3.5}$$

where the first $k$ entries on the diagonal are all zero.

In the following, we will denote by $I$ a $k-$tuple of integers from 1 to $p$ in the increasing order:

$$I = (i_1,\cdots,i_k), 1\le i_1<i_2<\cdots<i_k\le p.$$

The length of $I$ is defined to be $k$ and will be denoted by $[I]=k$. For each $I=(i_1,\cdots,i_k)$, we will write

$$A_I(Z) = A\binom{1 \quad \cdots \quad k}{i_1 \quad \cdots \quad i_k}$$

for the determinant of the submatrix of $A(Z)$ (defined in (3.4)) formed by its $1^{\text{th}}, \cdots, k^{\text{th}}$ rows and $i_1^{\text{th}}, \cdots, i_k^{\text{th}}$ columns. It follows from (3.5) that all entries of $A(Z)$ in the first $k$ rows vanish at $Z = Z_0$. Consequently, $A_I(Z)$ vanishes at least to the $k^{\text{th}}$ order at $Z = Z_0$ for any $I$ with length $[I] = k$.

For $I = (i_1, \cdots, i_k)$, we write $J_I(Z)$ for the complement minor of $A_I(Z)$, i.e., the determinant of the submatrix of $A(Z)$ obtained by deleting the $1^{\text{st}}, \cdots, k^{\text{th}}$ rows and $i_1^{\text{th}}, \cdots, i_k^{\text{th}}$ columns. In the special case $k = p$, $J_I(Z)$ is understood to be identically 1. Again it follows from (3.5) that when $k < p$,

$$J_I(Z_0) = \begin{cases} 0, & \text{if } I \neq (1, \cdots, k) \\ \prod_{i=k+1}^{p}(1 - r_i^2), & \text{if } I = (1, \cdots, k) \end{cases} \tag{3.6}$$

By Proposition 3.1, we have

$$\rho(Z, \overline{Z}) = \det(A(Z)) = \sum_{|I|=k} \mathcal{E}_I A_I(Z) J_I(Z). \tag{3.7}$$

Here $\mathcal{E}_I$ is the sign defined in Proposition 3.1. As $A_I(Z)$ vanishes at least to the $k^{\text{th}}$ order at $Z_0$ for each $I$, it follows that $\rho(Z, \overline{Z})$ vanishes at least to the $k^{\text{th}}$ order as well. To prove it vanishes precisely to the $k^{\text{th}}$ order, we establish the following lemma. Write the coordinates $Z$ of $\mathbb{C}^{p\times q}$ in the matrix form $Z = (z_{ij})_{1\leq i\leq p, 1\leq j\leq q}$.

**Lemma 3.1.**

$$\frac{\partial^k \rho(Z, \overline{Z})}{\partial z_{11} \partial z_{22} \cdots \partial z_{kk}}\Big|_{Z_0} \neq 0.$$

*Proof.* Note $A_I(Z)$ vanishes at least to the $k^{\text{th}}$ order at $Z_0$ for all $I$ with length $[I] = k$ and $J_I(Z)$ vanishes at $Z_0$ unless $I = I_0 := (1, \cdots, k)$. Thus $A_I(Z)J_I(Z)$ vanishes at least to $(k+1)^{\text{th}}$ order at $Z_0$ if $I \neq I_0$. It then follows from (3.7) that

$$\begin{aligned} \frac{\partial^k \rho}{\partial z_{11} \partial z_{22} \cdots \partial z_{kk}}\Big|_{Z_0} &= \mathcal{E}_{I_0} \left( \frac{\partial^k A_{I_0}}{\partial z_{11} \partial z_{22} \cdots \partial z_{kk}}\Big|_{Z_0} \right) J_{I_0}(Z_0) \\ &= \left( \frac{\partial^k A_{I_0}}{\partial z_{11} \partial z_{22} \cdots \partial z_{kk}}\Big|_{Z_0} \right) J_{I_0}(Z_0). \end{aligned} \tag{3.8}$$

**Claim:**

$$\frac{\partial^k A_{I_0}}{\partial z_{11} \partial z_{22} \cdots \partial z_{kk}}\Big|_{Z_0} \neq 0.$$

**Proof of Claim:** Write $C(Z)$ for the submatrix of $A(Z)$ formed by its $1^{\rm st}, \cdots, k^{\rm th}$ rows and $1^{\rm st}, \cdots, k^{\rm th}$ columns and thus $A_{I_0}(Z) = \det\ C(Z)$. It follows from (3.5) that $C(Z_0) = \mathbf{0}_{k\times k}$. We write

$$C(Z) = \begin{pmatrix} \mathbf{c}_1(Z) \\ \cdots \\ \cdots \\ \mathbf{c}_k(Z) \end{pmatrix} = (c_{st}(Z))_{1\le s,t\le k},$$

where $\mathbf{c}_j, 1 \le j \le k$, is the $j^{\rm th}$ row of $C(Z)$. Note that

$$c_{st}(Z) = \begin{cases} -\sum_{l=1}^q z_{sl}\overline{z}_{tl}, & \text{if } s \ne t; \\ 1 - \sum_{l=1}^q z_{sl}\overline{z}_{sl}, & \text{if } s = t. \end{cases} \tag{3.9}$$

One can easily verify

$$\frac{\partial c_{st}(Z)}{\partial z_{jj}}\Big|_{Z_0} = -\left(\delta_{sj}\overline{z}_{tj}\right)|_{Z_0} = 0, \quad \text{if } 1 \le s \ne t \le k.$$

Here $\delta_{sj}$ is the Kronecker symbol, i.e., $\delta_{sj}$ takes the value 1 if $s = j$ and 0 otherwise. Similarly,

$$\frac{\partial c_{ss}(Z)}{\partial z_{jj}}\Big|_{Z_0} = -\left(\delta_{sj}\overline{z}_{sj}\right)|_{Z_0} = -\delta_{sj}, \quad \text{for } 1 \le s \le k.$$

Consequently,

$$\frac{\partial \mathbf{c}_s(Z)}{\partial z_{jj}}\Big|_{Z_0} = \begin{cases} (0, \cdots, 0), & \text{if } j \ne s, \\ (0, \cdots, 0, -1, 0, \cdots, 0), & \text{if } j = s, \end{cases} \tag{3.10}$$

where "$-1$" is at the $j^{\rm th}$ position.

Note for any first order differential operator $\Lambda$ in $\mathbb{C}^{p\times q}$, we have

$$\Lambda A_{I_0}(Z) = \Lambda\left(\det\ C(Z)\right) = \begin{vmatrix} \Lambda\mathbf{c}_1 \\ \mathbf{c}_2 \\ \cdots \\ \mathbf{c}_k \end{vmatrix} + \cdots + \begin{vmatrix} \mathbf{c}_1 \\ \cdots \\ \mathbf{c}_{k-1} \\ \Lambda\mathbf{c}_k \end{vmatrix}.$$

Using this property, we successively apply $\frac{\partial}{\partial z_{11}}, \cdots, \frac{\partial}{\partial z_{kk}}$ to $A_{I_0}$. Then we evaluate at $Z_0$ and use the fact that $C(Z_0) = \mathbf{0}_{k\times k}$ to obtain

$$\frac{\partial^k A_{I_0}}{\partial z_{11}\partial z_{22}\cdots\partial z_{kk}}\Big|_{Z_0} = \sum_{i_1,\cdots,i_k} \begin{vmatrix} \frac{\partial \mathbf{c}_1(Z_0)}{\partial z_{i_1 i_1}} \\ \cdots \\ \frac{\partial \mathbf{c}_k(Z_0)}{\partial z_{i_k i_k}} \end{vmatrix} \tag{3.11}$$

where $\sum_{i_1\cdots i_k}$ is the sum over all possible permutations $i_1, \cdots, i_k$ of $1, 2, \cdots, k$. It then follows from (3.10) that

$$\frac{\partial^k A_{I_0}}{\partial z_{11}\partial z_{22}\cdots\partial z_{kk}}\Big|_{Z_0} = \begin{vmatrix} \frac{\partial \mathbf{c}_1(Z_0)}{\partial z_{11}} \\ \cdots \\ \frac{\partial \mathbf{c}_k(Z_0)}{\partial z_{kk}} \end{vmatrix} = \begin{vmatrix} -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -1 \end{vmatrix} = (-1)^k \neq 0.$$

This establishes the claim. ∎

Now Lemma 3.1 follows from the claim and (3.8). □

Lemma 3.1 yields that $\rho$ vanishes to the $k^{\text{th}}$ order at $Z_0$. We thus establish Theorem 2 for the type I case. ∎

**Proof of Theorem 1 in type I case:** We will prove part (2) and (1) of Theorem 1 separately. Write the coordinates in $\mathbb{C}^n$ as $\xi = (\xi_1, \cdots, \xi_n)$.

**1. Proof of part (2):** We will first prove part (2) of Theorem 1 and now assume $F$ extends $C^k$−smoothly to $\xi = a$. Write $Z_0 = F(a)$. By assumption, $Z_0 \in E_k$. By Theorem 2, $\rho(Z, \overline{Z})$ vanishes to the $k^{\text{th}}$ order at $Z_0$. It follows that $\rho(F(\xi), \overline{F(\xi)})$ vanishes at least to $k^{\text{th}}$ order at $\xi = a$. Thus it suffices to prove equations (1.2) and (1.3).

Again as $\rho$ is invariant under the actions of unitary matrices: $\rho(F, \overline{F}) = \rho(UFV, \overline{UFV})$. Replacing $F$ by $UFV$ with suitable unitary matrices $U, V$ if necessary, we can assume $Z_0$ takes the form as in (3.3). Write $A(F) = A(F, \overline{F}) = I_p - F\overline{F}^t$, where $F$ is in the matrix form: $F = (f_{ij})_{1\leq i\leq p, 1\leq j\leq q}$. By (3.7), we have

$$\rho(F, \overline{F}) = \det(A(F)) = \sum_{|I|=k} \mathcal{E}_I A_I(F) J_I(F) \tag{3.12}$$

Again as $A_I(Z)$ vanishes at least to the $k^{\text{th}}$ order at $Z_0$ for all $I$ with length $[I] = k$ and $J_I(Z)$ vanishes at $Z_0$ unless $I = I_0 := (1, \cdots, k)$. We have all $k^{\text{th}}$ order derivative of $A_I(F)J_I(F)$ vanishes at $a$ if $I \neq I_0$. Then (3.12) yields that

(3.13)
$$\begin{aligned} L^k\rho(F,\overline{F})\big|_a =&\mathcal{E}_{I_0}\left(L^kA_{I_0}(F)\big|_a\right)J_{I_0}(F)\big|_a = \left(L^kA_{I_0}(F)\big|_a\right)J_{I_0}(F)\big|_a;\\ \frac{\partial^k\rho(F,\overline{F})}{\partial\nu^k}\Big|_a =&\mathcal{E}_{I_0}\left(\frac{\partial^kA_{I_0}(F)}{\partial\nu^k}\Big|_a\right)J_{I_0}(F)|_a = \left(\frac{\partial^kA_{I_0}(F)}{\partial\nu^k}\Big|_a\right)J_{I_0}(F)\big|_a. \end{aligned}$$

We pause to prove the following lemma. Recall $F$ is written in the matrix form: $F=(f_{ij})_{1\le i\le p,1\le j\le q}$.

**Lemma 3.2.** (i) *The $k\times k$ matrix $(Lf_{ij}(a))_{1\le i,j\le k}$ is nondegenerate.*

(ii) *The determinant of the $k\times k$ matrix $\left(\frac{\partial(f_{ij}+\overline{f_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}$ is positive.*

*Proof.* Write the matrices

$$M=(Lf_{ij}(a))_{1\le i,j\le k}\,;N=\left(\frac{\partial(f_{ij}+\overline{f_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}.$$

By a well-known fact in linear algebra, there is a $k\times k$ unitary matrix $T$ such that $TM\overline{T}^t$ is lower-triangular. Note $N$ is Hermitian, thus there exists a $k\times k$ unitary matrix $R$ such that $RN\overline{R}^t$ is diagonal. Now let

$$U_1=\begin{pmatrix} T & \mathbf{0}_{k\times(p-k)}\\ \mathbf{0}_{(p-k)\times k} & I_{p-k}\end{pmatrix},V_1=\begin{pmatrix} \overline{T}^t & \mathbf{0}_{k\times(q-k)}\\ \mathbf{0}_{(q-k)\times k} & I_{q-k}\end{pmatrix};$$

Similarly, let

$$U_2=\begin{pmatrix} R & \mathbf{0}_{k\times(p-k)}\\ \mathbf{0}_{(p-k)\times k} & I_{p-k}\end{pmatrix};V_2=\begin{pmatrix} \overline{R}^t & \mathbf{0}_{k\times(q-k)}\\ \mathbf{0}_{(q-k)\times k} & I_{q-k}\end{pmatrix}.$$

Clearly $U_l'$s, $V_l'$s are $p\times p$ and $q\times q$ unitary matrices, respectively. Set $\widetilde{F}=U_1FV_1$ and $\hat{F}=U_2FV_2$. Write $\widetilde{F}=(\widetilde{f_{ij}})_{1\le i\le p,1\le j\le q}$ and $\hat{F}=(\hat{f}_{ij})_{1\le i\le p,1\le j\le q}$, respectively. Noting that $Z\to U_lZV_l$ is an automorphism of $D^I_{p,q}$ for each $l=1,2$, one can readily check the following facts hold:

(1) $\widetilde{F}$ and $\hat{F}$ both map $\Omega$ to $D^I_{p,q}$;

(2) $\widetilde{F}(a)=U_1F(a)V_1=U_1Z_0V_1=Z_0$. Similarly, $\hat{F}(a)=U_2F(a)V_2=U_2Z_0V_2=Z_0$;

(3) $\Big(L\tilde{f}_{ij}(a)\Big)_{1\le i,j\le k} = T\big(Lf_{ij}(a)\big)_{1\le i,j\le k}\overline{T}^t = TM\overline{T}^t$ is lower-triangular; $\left(\frac{\partial(\hat{f}_{ij}+\overline{\hat{f}_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k} = R\left(\frac{\partial(f_{ij}+\overline{f_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}\overline{R}^t = RN\overline{R}^t$ is diagonal.

By (1), we have $I_p - \widetilde{F}\overline{\widetilde{F}}^t > 0$, and $I_p - \hat{F}\overline{\hat{F}}^t > 0$ in $\Omega$. In particular, we must have for each $1\le i\le p, |\widetilde{f}_{ii}|^2 < 1$ and $|\hat{f}_{ii}|^2 < 1$ in $\Omega$. Note by (2), $\widetilde{f}_{ii}(a) = 1$ and $\hat{f}_{ii}(a) = 1$ for every $1\le i\le k$. By the classical Hopf lemma, $\frac{\partial|\tilde{f}_{ii}|^2}{\partial\nu}\big|_a > 0$ and $\frac{\partial|\hat{f}_{ii}|^2}{\partial\nu}\big|_a > 0$ for $1\le i\le k$. Consequently, $\frac{\partial(\mathrm{Re}\hat{f}_{ii})}{\partial\nu}\big|_a > 0$ and $\frac{\partial(\mathrm{Re}\widetilde{f}_{ii})}{\partial\nu}\big|_a > 0$. The latter in particular implies $L\tilde{f}_{ii}(a)\neq 0$.

Now as $\Big(L\widetilde{f}_{ij}(a)\Big)_{1\le i,j\le k}$ is lower-triangular and each of its diagonal entries $L\tilde{f}_{ii}(a)$ is nonzero, the matrix must be nondegenerate. By the fact (3) above, $\big(Lf_{ij}(a)\big)_{1\le i,j\le k}$ is also nondegenerate. This proves part (i) of Lemma 3.2.

Similarly, as $\frac{\partial(\mathrm{Re}\hat{f}_{ii})}{\partial\nu}\big|_a > 0$, i.e., $\frac{\partial(\hat{f}_{ii}+\overline{\hat{f}_{ii}})}{\partial\nu}\big|_a > 0$, we conclude each of its diagonal entries and thus the determinant of $\left(\frac{\partial(\hat{f}_{ij}+\overline{\hat{f}_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}$ are positive. By the fact (3), the determinant of $\left(\frac{\partial(f_{ij}+\overline{f_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}$ is positive as well. This proves part (ii) of the lemma. □

We now continue to prove the quantity in (3.13) is nonzero. For that we establish the following lemma.

**Lemma 3.3.** *$L^kA_{I_0}(F)|_a\neq 0$ and $(-1)^k\frac{\partial^k A_{I_0}(F)}{\partial\nu^k}\big|_a > 0$.*

*Proof.* Let $C(Z)$ be as in the proof of Lemma 3.1. Then $A_{I_0}(F(\xi)) = \det C(F(\xi))$, and

$$C(F) = \begin{pmatrix}\mathbf{c}_1(F)\\ \cdots\\ \cdots\\ \mathbf{c}_k(F)\end{pmatrix} = (c_{st}(F))_{1\le s,t\le k}.$$

Moreover, $C(F(a)) = \mathbf{0}_{k\times k}$. It follows from (3.9) that

$$c_{st}(F) = \begin{cases} -\sum_{l=1}^q f_{sl}\overline{f}_{tl}, & \text{if } s\neq t;\\ 1-\sum_{l=1}^q f_{sl}\overline{f}_{sl}, & \text{if } s=t.\end{cases}.$$

It follows that for each $1 \le s, t \le k$,

$$Lc_{st}(F) = -\sum_{l=1}^{q} Lf_{sl}\overline{f}_{tl}; \tag{3.14}$$

and

$$\frac{\partial c_{st}(F)}{\partial \nu} = -\left(\sum_{l=1}^{q} \frac{\partial f_{sl}}{\partial \nu}\overline{f}_{tl} + \sum_{l=1}^{q} f_{sl}\frac{\partial \overline{f}_{tl}}{\partial \nu}\right). \tag{3.15}$$

Similarly as in (3.11), we apply $L$ (or $\frac{\partial}{\partial \nu}$) to $A_{I_0}(F)$ for $k$ times, evaluate at $\xi = a$ and use the fact that $C(F(a)) = \mathbf{0}_{k\times k}$ to obtain

$$L^k A_{I_0}(F)\big|_a = k! \begin{vmatrix} L\,\mathbf{c}_1(F)|_a \\ \cdots \\ L\,\mathbf{c}_k(F)|_a \end{vmatrix} = k! \det\big(L\,C(F)|_a\big); \tag{3.16}$$

$$\frac{\partial^k A_{I_0}(F)}{\partial \nu^k}\Big|_a = k! \begin{vmatrix} \frac{\partial \mathbf{c}_1(F)}{\partial \nu}|_a \\ \cdots \\ \frac{\partial \mathbf{c}_k(F)}{\partial \nu}|_a \end{vmatrix} = k! \det\left(\frac{\partial C(F)}{\partial \nu}|_a\right). \tag{3.17}$$

Note (3.14) yields that

$$\big(L\,C(F)|_a\big) = \left(-\sum_{l=1}^{q} Lf_{sl}\overline{f}_{tl}\big|_a\right)_{1\le s,t\le k} = -\left(Lf_{ij}|_a\right)_{1\le i\le k, 1\le j\le q}\cdot \overline{B}^t \tag{3.18}$$

where the $k\times q$ matrix $B = (f_{ij}|_a)_{1\le i\le k, 1\le j\le q}$. Similarly,

$$\left(\frac{\partial C(F)}{\partial \nu}|_a\right) = \left(\sum_{l=1}^{q} \frac{\partial f_{sl}}{\partial \nu}|_a\overline{f}_{tl}|_a + \sum_{l=1}^{q} f_{sl}|_a\frac{\partial \overline{f}_{tl}}{\partial \nu}|_a\right)_{1\le s,t\le k} = -A\overline{B}^t - B\overline{A}^t. \tag{3.19}$$

Here $A = \left(\frac{\partial f_{ij}}{\partial \nu}|_a\right)_{1\le i\le k, 1\le j\le q}$. As $F(a) = Z_0$, we have $B = \begin{pmatrix} I_k & \mathbf{0}_{k\times(q-k)}\end{pmatrix}$. We substitute this into (3.18) and (3.19) to obtain

$$\begin{aligned} \big(L\,C(F)|_a\big) &= -\left(Lf_{ij}\big|_a\right)_{1\le i,j\le k}; \\ \left(\frac{\partial C(F)}{\partial \nu}\Big|_a\right) &= -\left(\frac{\partial (f_{ij}+\overline{f_{ji}})}{\partial \nu}\Big|_a\right)_{1\le i,j\le k}. \end{aligned} \tag{3.20}$$

With equations (3.16), (3.17), and (3.20), Lemma 3.3 becomes a consequence of Lemma 3.2. □

Now part (2) of Theorem 1 in the type I case follows from equation (3.13) and Lemma 3.3 (Note $J_{I_0}(F)|_a$ is positive by (3.6)). ■

**2. Proof of Part (1):** By assumption, $F$ extends $C^1-$smoothly to $a$. Denote by $F^*$ the first order truncation of $F$ at $a$. That is, writing $a = (a_1, \cdots, a_n) \in \mathbb{C}^n$,

$$F^*(\xi) = F(a) + \sum_{i=1}^{n} \frac{\partial F(a)}{\partial \xi_i}(\xi_i - a_i). \tag{3.21}$$

Here $F^*$ is also understood as a matrix-valued function: $F^* = (f^*_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$. By Taylor's theorem,

$$F(\xi) = F^*(\xi) + o_a(1). \tag{3.22}$$

Here we say a (vector-valued) function $h(\xi)$ is $o_a(l)$ for some positive integer if $\lim_{\xi \to a} \frac{||h||}{||\xi - a||^l} = 0$. Recall by Theorem 2, $\rho(Z, \overline{Z})$ vanishes to the $k^{\text{th}}$ order at $Z_0$. We write

$$\rho(Z, \overline{Z}) = P_k(Z - Z_0, \overline{Z - Z_0}) + o_{Z_0}(k), \tag{3.23}$$

where we say a function $\phi(Z, \overline{Z})$ is $o_{Z_0}(k)$ if $\lim_{Z \to Z_0} \frac{|\phi|}{||Z - Z_0||^k} = 0$. Here $P_k(\eta, \overline{\eta})$ is a homogeneous polynomial in $\eta$ and $\overline{\eta}$ of degree $k$, where $\eta \in \mathbb{C}^{p \times q}$. Recall $F(a) = Z_0$. Since $F$ is $C^1$ at $a$, we have

$$F(\xi) - Z_0 = F(\xi) - F(a) = O_a(1).$$

Here we say a vector-valued function $\psi(\xi)$ is $O_a(l)$ if $\frac{||\psi(\xi)||}{||\xi - a||^l} \leq C$ for some positive constant $C$ when $\xi$ is close to $a$. Consequently, if $\phi(Z, \overline{Z})$ is $o_{Z_0}(k)$, then $\phi(F, \overline{F})$ and $\phi(F^*, \overline{F^*})$ are both $o_a(k)$. We now substitute $Z = F$ and $Z = F^*$ respectively into (3.23) to obtain,

$$\rho(F, \overline{F}) = P_k(F(\xi) - F(a), \overline{F(\xi) - F(a)}) + o_a(k); \tag{3.24}$$

$$\rho(F^*, \overline{F^*}) = P_k(F^*(\xi) - F(a), \overline{F^*(\xi) - F(a)}) + o_a(k). \tag{3.25}$$

Combining (3.22) and (3.24) and using the fact that $F^*(\xi) - F(a) = O_a(1)$, we have

$$
\begin{aligned}
\rho(F,\overline{F}) =& P_k\left(F^*(\xi) - F(a) + o_a(1), \overline{F^*(\xi) - F(a) + o_a(1)}\right) + o_a(k) \\
=& P_k\left(F^*(\xi) - F(a), \overline{F^*(\xi) - F(a)}\right) + o_a(k).
\end{aligned} \tag{3.26}
$$

Now it follow from (3.25) and (3.26) that

$$
\rho(F,\overline{F}) = \rho(F^*,\overline{F^*}) + o_a(k), \qquad \text{when } \xi \text{ is close to } a. \tag{3.27}
$$

We next prove the following fact.

**Lemma 3.4.**

$$
(-1)^k \frac{\partial^k \rho(F^*,\overline{F^*})}{\partial \nu^k}\Big|_a > 0 \quad \textit{and} \quad L^k\rho(F^*,\overline{F^*})\Big|_a \neq 0.
$$

**Proof of Lemma 3.4:** The argument that we applied to derive (3.13) also leads to

(3.28)

$$
\frac{\partial^k \rho(F^*,\overline{F^*})}{\partial \nu^k}\Big|_a = \mathcal{E}_{I_0}\left(\frac{\partial^k A_{I_0}(F^*)}{\partial \nu^k}\Big|_a\right) J_{I_0}(F^*)|_a = \left(\frac{\partial^k A_{I_0}(F^*)}{\partial \nu^k}\Big|_a\right) J_{I_0}(F)|_a;
$$

(3.29)

$$
L^k\rho(F^*,\overline{F^*})\Big|_a = \mathcal{E}_{I_0}\left(L^k A_{I_0}(F^*)\Big|_a\right) J_{I_0}(F^*)\Big|_a = \left(L^k A_{I_0}(F^*)\Big|_a\right) J_{I_0}(F)\Big|_a.
$$

Writing $F^*$ in the matrix form $F^* = (f^*_{ij})_{1\le i\le p, 1\le j\le q}$, we claim the following facts hold.

(I). The $k\times k$ matrix $\left(\frac{\partial(f^*_{ij}+\overline{f^*_{ji}})}{\partial\nu}(a)\right)_{1\le i,j\le k}$ has positive determinant.

(II). $(-1)^k\frac{\partial^k A_{I_0}(F^*)}{\partial\nu^k}\Big|_a > 0$

Note part (I) follows from part (ii) of Lemma 3.2. Part (II) can be proved similarly as Lemma 3.3 by applying part (I). Indeed, we only used the information of the first jet of $F$ at $a$ in the proof of Lemma 3.3. Then the first equation of Lemma 3.4 follows from part (II) and equations (3.28) and (3.6). The second equation of Lemma 3.4 can be proved similarly using (3.29), Lemma 3.2, and (the proof of) Lemma 3.3. ∎

We apply an appropriate linear holomorphic change of coordinates in $\mathbb{C}^n$ to make $a = 0 \in \mathbb{C}^n$ and the outward pointing unit normal vector $\frac{\partial}{\partial\nu} = \frac{\partial}{\partial x}$, where $\xi_n = x + \sqrt{-1}y$. Set the real line $H := \{(0,\cdots,0,x) \in \mathbb{C}^n : x \in \mathbb{R}\}$. As $F(a) \in E_k$, Theorem 2 implies $\rho(F^*,\overline{F^*})$ vanishes to at least $k^{\text{th}}$ order at $a$. It then follows from Lemma 3.4 that $\rho(F^*,\overline{F^*})|_H = cx^k + o_0(k)$ near $x = 0$ for some constant $c$ satisfying $(-1)^k c > 0$. We substitute this into (3.27) to conclude part (1) of Theorem 1 for type I domains. ∎

We have thus established Theorem 1 for type I domains. ∎

### 3.2. Hopf Lemma for type III domain

We postpone the proof for type II case and now discuss the type III case. Denote by $\mathbb{C}_{III}^{\frac{m(m+1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = Z^t\}$ the set of all symmetric square matrices of size $m \times m$. Recall the classical domain of type III is defined by

$$D_m^{III} = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t > 0\}.$$

The Bergman kernel of $D_m^{III}$ is given by

$$K_{D_m^{III}} = c_{III}\left(\det(I_m - Z\overline{Z}^t)\right)^{-(n+1)}$$

for some positive integer $c_{III}$. The generic norm for $D_m^{III}$ is given by $\rho(Z,\overline{Z}) = \det(I_m - Z\overline{Z}^t)$. The boundary of $D_m^{III}$ is given by

$$\partial D_m^{III} = \{Z \in \mathbb{C}_{III}^{\frac{m(m+1)}{2}} : I_m - Z\overline{Z}^t \geq 0;\ \ \det(I_m - Z\overline{Z}^t) = 0\}.$$

Note $D_m^{III}$ is of rank $m$ and the boundary $\partial D_m^{III}$ decomposes into $m$ orbits under the action of the identity component $G_0$ of $\mathrm{Aut}(D_m^{III}) : \partial D_m^{III} = \cup_{i=1}^m E_i^{III}$, where we use the superscript "$III$" to distinguish the notations from other types. Here $E_k^{III}$ lies in the closure of $E_l^{III}$ when $k > l$. More explicitly in this type III case,

$$E_k^{III} = \{Z \in \partial D_m^{III} :\ \text{the corank of } I_p - Z\overline{Z}^t \text{ equals } k\}, \quad 1 \leq k \leq m.$$

The type III classical domain $D_m^{III}$ can be realized as a submanifold of the type I domain $D_{m,m}^I$ by the canonical embedding from $D_m^{III}$ to $D_{m,m}^I : i(Z) = Z$. Recall the boundary of $D_{m,m}^I$ is decomposed into $m$ orbits: $\partial D_{m,m}^I = \cup_{i=1}^m E_i^I$ as in Section 3.1. Under the canonical embedding $i$, the boundary orbit $E_k^{III}$ is embedded into $E_k^I$ for each $1 \leq k \leq m$. Moreover, if we write $\rho^I$ and $\rho^{III}$ respectively for the function $\rho$ in the type case $D_{m,m}^I$

and the type III case $D_m^{III}$, then

$$\rho^I(i(Z),\overline{i(Z)}) = \rho^{III}(Z,\overline{Z}) = \det(I_m - Z\overline{Z}^t).$$

**Proof of Theorem 1 and 2 for type III domains:** By the facts mentioned above, Theorem 1 and 2 in the type III case for $D_m^{III}$ follows from the theorems in the type I case for $D_{m,m}^I$. ∎

### 3.3. Hopf Lemma for type II domain

Denote by $\mathbb{C}_{II}^{\frac{m(m-1)}{2}} = \{Z \in \mathbb{C}^{m\times m} : Z = -Z^t\}$ the set of all skew-symmetric square matrices of size $m \times m$. Recall the classical domain of type II is defined by

$$D_m^{II} = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t > 0\}.$$

The boundary of $D_m^{III}$ is given by

$$\partial D_m^{II} = \{Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}} : I_m - Z\overline{Z}^t \geq 0;\ \ \det(I_m - Z\overline{Z}^t) = 0\}.$$

Note the rank of $D_m^{II}$ is of rank $r = \lfloor \frac{1}{2}m \rfloor$. Here $\lfloor \cdot \rfloor$ denotes the floor function, i.e., $2r = m$ if $m$ is even and $2r + 1 = m$ if $m$ is odd. The boundary $\partial D_m^{II}$ decomposes into $r$ orbits under the action of the identity component $G_0$ of $\mathrm{Aut}(D_m^{II}) : \partial D_m^{II} = \cup_{i=1}^r E_i^{II}$, where we use the superscript "$II$" to distinguish the notations from other types. Here $E_k^{II}$ lies in the closure of $E_l^{II}$ if $k > l$. In particular, $E_1^{II}$ is the smooth part of $\partial\Omega, E_r^{II}$ is the Shilov boundary. More explicitly in this type II case (See [Wo] for more details),

$$E_k^{II} = \{Z \in \partial D_m^{II} :\ \text{the corank of } I_p - Z\overline{Z}^t \text{ equals } 2k\}, \quad 1 \leq k \leq r.$$

The type II domain $D_m^{II}$ can be realized as a submanifold of the type I domain $D_{m,m}^I$ by the canonical embedding from $D_m^{II}$ to $D_{m,m}^I : i(Z) = Z$. Under the canonical embedding $i$, the boundary orbit $E_k^{II}$ is embedded into $E_{2k}^I$ for each $k$, where $\partial D_{m,m}^I = \cup_{i=1}^m E_i^I$. If we write $\rho^I(Z,\overline{Z})$ and $\rho^{II}(Z,\overline{Z})$ respectively for the function $\rho$ in the type case $D_{m,m}^I$ and the type II case $D_m^{II}$, then

$$\rho^I(i(Z),\overline{i(Z)}) = \left(\rho^{II}(Z,\overline{Z})\right)^2 = \det(I_m - Z\overline{Z}^t). \tag{3.30}$$

Indeed, we have if $Z \in \mathbb{C}_{II}^{\frac{m(m-1)}{2}}$, then (cf. Lemma 4.3 in [X])

$$\det(I_n - Z\overline{Z}^t) = \left(1 + \sum_{1 \le k \le n, 2|k} (-1)^{\frac{k}{2}} \left(\sum_{1 \le i_1 < \ldots < i_k \le n} \left| Z \begin{pmatrix} i_1 & \ldots & i_k \\ i_1 & \ldots & i_k \end{pmatrix} \right| \right)\right)^2.$$

Here "$2|k$" means that $k$ is divisible by 2. The expression in the big parentheses on the right hand side gives the formula for $\rho^{II}(Z, \overline{Z})$.

**Proof of Theorem 2 for type II domains:** Fix $1 \le k \le r$. Let $Z_0 \in E_k^{II}$. By the discussion above, under the canonical embedding $i$, $Z_0 \in E_{2k}^I$. By Theorem 2 for type I case,we have $\rho^I(Z, \overline{Z})$ vanishes to the $(2k)^{\text{th}}$ order. Then it follows from (3.30) that $\rho^{II}(Z, \overline{Z})$ vanishes to $k^{\text{th}}$ order. This establishes Theorem 2. ■

**Proof of Theorem 1 for type II domains:** By assumption, $F$ is $C^1$ at $a$. As in the type I case, we set $F^*$ to be the first order truncation of $F$ as in (3.21), and thus (3.22) holds. By the same argument as in Section 3.1, we can obtain similarly as in (3.27):

$$\rho^{II}(F, \overline{F}) = \rho^{II}(F^*, \overline{F^*}) + o_a(k), \quad \text{when } \xi \text{ is close to } a. \tag{3.31}$$

Note $\rho^{II}(F^*, \overline{F^*})$ is smooth (Indeed it is a real polynomial). Theorem 2 yields that $\rho^{II}(F^*, \overline{F^*})$ vanishes to at least $k^{\text{th}}$ order at $a$. It follows from (3.30) that

$$\rho^I(F^*, \overline{F^*}) = \left(\rho^{II}(F^*, \overline{F^*})\right)^2. \tag{3.32}$$

By Lemma 3.4, as $F^*(a) = F(a) \in E_{2k}^I$,

$$L^{2k} \rho^I(F^*, \overline{F^*})\big|_a \neq 0; \quad \frac{\partial^{2k} \rho^I(F^*, \overline{F^*})}{\partial \nu^{2k}}\Big|_a > 0.$$

Combining this with (3.32), we conclude

$$L^k \rho^{II}(F^*, \overline{F^*})\big|_a \neq 0; \quad \frac{\partial^k \rho^{II}(F^*, \overline{F^*})}{\partial \nu^k}\Big|_a \neq 0. \tag{3.33}$$

As before, we apply an appropriate linear holomorphic change of coordinates in $\mathbb{C}^n$ to make $a = 0 \in \mathbb{C}^n$ and the outward pointing unit normal vector $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial x}$, where $\xi_n = x + \sqrt{-1}y$. Set the real line $H := \{(0, \cdots, 0, x) \in \mathbb{C}^n : x \in \mathbb{R}\}$. It follows from (3.33) that $\rho^{II}(F^*, \overline{F^*})|_H = cx^k + o_0(k)$ near $x = 0$ for some $c \neq 0$. Substituting this into (3.31), we get $\rho^{II}(F, \overline{F})|_H = cx^k + o_0(k)$ for $x$ near 0. This implies

$$\lim_{t \to 0^-} \frac{\rho^{II}\left(F(a+t\nu), \overline{F(a+t\nu)}\right)}{(-t)^k} = \widetilde{c} := (-1)^k c \neq 0.$$

As $\rho^{II}(F, \overline{F}) > 0$ in $\Omega$, we conclude $\widetilde{c} > 0$. This establishes part (1) of Theorem 1. Moreover, when $F$ is $C^k$ smooth at $a$, it follows that

$$(-1)^k \frac{\partial^k(\rho^{II} \circ F)}{\partial \nu^k}\Big|_a > 0;$$

and by equations (3.31), (3.33), we conclude $L^k(\rho^{II} \circ F)\big|_a \neq 0$. This establishes part (2) of Theorem 1 for the type II case. ■

### 3.4. Hopf lemma for the type IV case:

Recall the type IV classical domain $D_m^{IV} (m \geq 2)$, often called the Lie ball, is defined by

$$D_m^{IV} = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t < 2, \quad 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 > 0\}.$$

When $m = 2, D_2^{IV}$ is biholomorphic to the bidisc. The Bergman metric of $D_m^{IV}$ is given by

$$K_{D_m^{IV}} = c_{IV}\left(1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2\right)^{-m}$$

for some positive constant $c_{IV}$. The generic norm $\rho$ for $D_m^{IV}$ is given by

$$\rho(Z, \overline{Z}) = 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2.$$

The boundary of $D_m^{IV}$ is given by

$$\partial D_m^{IV} = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t \leq 2, 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0\}.$$

Since the type IV domain $D_m^{IV}$ is always of rank two, its boundary is stratified into two orbits: $\partial D_m^{IV} = E_1 \cup E_2$, where $E_1$ is the smooth boundary and $E_2$ is the Shilov boundary of $D_m^{IV}$. Note

$$E_1 = \{Z = (z_1, \cdots, z_m) \in \mathbb{C}^m : Z\overline{Z}^t < 2, 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0\}; \tag{3.34}$$

$$\begin{aligned} E_2 &= \{Z \in \mathbb{C}^m : Z\overline{Z}^t = 2, 1 - Z\overline{Z}^t + \frac{1}{4}|ZZ^t|^2 = 0\} \\ &= \{Z \in \mathbb{C}^m : ||Z||^2 = |ZZ^t| = 2\}. \end{aligned} \tag{3.35}$$

We first observe the following characterization of points on the Shilov boundary of $D_m^{IV}$. Let $\mathbb{S}^{m-1} := \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : ||x|| = 1\}$ be the unit sphere in $\mathbb{R}^m$.

**Proposition 3.2.** *Let $Z = (z_1, \cdots, z_m) \in \mathbb{C}^m$. Then $Z \in E_2$ if and only if there exists some $\theta \in (-\pi, 0]$ such that $\frac{1}{\sqrt{2}}e^{i\theta}Z \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$.*

*Proof.* Note $Z \in E_2$ if and only if $||Z||^2 = |ZZ^t| = 2$. This is equivalent to the existence of some $\alpha \in [0, 2\pi)$ such that for each $1 \leq i \leq n$, we have $z_i^2 = e^{i\alpha}r_i$. Here $r_i \geq 0$ and $\sum_{i=1}^m r_i = 2$. Consequently, if we take $\theta = -\frac{\alpha}{2}$, then $\frac{1}{\sqrt{2}}e^{i\theta}z_i = \pm\sqrt{\frac{r_i}{2}}$. The conclusion thus follows. $\square$

Theorem 2 can be easily proved for type IV domains in which case the rank $r$ always equals 2.

**Proof of Theorem 2 for type IV case:** When $k = 1$, the result follows from [M5] or one can directly compute that $d\rho(Z, \overline{Z}) \neq 0$ when $||Z||^2 < 2$. Once we have the result for $k = 1$, since $\partial\Omega = E_1 \cup E_2$, we must have

$$E_2 \supset \{q \in \partial D_m^{IV} : \rho(Z, \overline{Z}) \text{ vanishes to the second order at } q\}$$

To prove the other inclusion for $k = 2$, we note first if $Z_0 \in E_2$, then by Proposition 3.2, $Z_0 = \sqrt{2}e^{i\alpha}x$ for some $\alpha \in [0, \pi)$ and $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. This implies for each $1 \leq j \leq m$,

$$\frac{\partial \rho}{\partial z_j}(Z_0) = (-\overline{z}_j + \frac{1}{2}z_j \sum_{i=1}^m \overline{z_i}^2)|_{Z_0} = 0.$$

Moreover,

$$\frac{\partial^2 \rho}{\partial z_j^2}(Z_0) = \frac{1}{2}\sum_{i=1}^n \overline{z}_i^2|_{Z_0} \neq 0, \quad 1 \leq j \leq m.$$

Hence $\rho(Z, \overline{Z})$ vanishes to the second order at $Z_0$. This proves Theorem 2 for the case $k = 2$. ■

To establish Theorem 1, we first note the following fact for mappings into type IV domain.

**Lemma 3.5.** *Let $\Omega$ be a domain in $\mathbb{C}^n$ and $F$ be a holomorphic map from $\Omega$ to $D_m^{IV}$ and extends $C^1-$smoothly up to a smooth point $a \in \partial\Omega$. Assume $F(a) \in E_2$. Then*

$$\sum_{j=1}^{m} F_j|_a LF_j|_a \neq 0, \quad \sum_{j=1}^{m} F_j|_a \frac{\partial F_j}{\partial \nu}|_a \neq 0.$$

*Here $\nu$ and $L$ are the same as in Theorem 1.*

*Proof.* Set $h = FF^t = \sum_{i=1}^m F_i^2$. Note $|h|^2$ is subharmonic in $\Omega$ and achieves a maximal value at $a$. Indeed, $|h| < 2$ in $\Omega$ and $|h(a)| = 2$ by the defining equation (3.35) of $E_2$. Then it follows from the classical Hopf lemma that $\frac{\partial |h|^2}{\partial \nu}|_a \neq 0$. Consequently, $\frac{\partial h}{\partial \nu}|_a \neq 0$, and thus $Lh|_a \neq 0$. The conclusion then follows easily by a direct computation. $\square$

**Proof of Theorem 1 for type IV domains:** Again we will prove part (2) of Theorem 1 first.

**1. Proof of Part (2):** The case $k = 1$ is covered by Proposition 2.1. We now assume $k = 2$, and $F$ extends $C^2-$smoothly to $a$ and $F(a) \in E_2$. It follows from Theorem 2 that all first order derivatives of $\rho(F, \overline{F})$ vanish at $a$. It remains to prove (1.2) and (1.3), i.e., $\frac{\partial^2 \rho(F,\overline{F})}{\partial \nu^2}\big|_a > 0$ and $L^2\rho(F, \overline{F})\big|_a \neq 0$. We will prove them separately.

**Proposition 3.3.** $\frac{\partial^2 \rho(F,\overline{F})}{\partial \nu^2}\big|_a > 0$.

*Proof.* We assume $F$ is merely $C^1-$smooth at the point $a$ momentarily. Write $Z_0 = F(a) \in E_2$. By Proposition 3.2, there exists $\theta \in (-\pi, 0]$ and a (real) orthogonal matrix $T$ such that $e^{i\theta} Z_0 T = (\sqrt{2}, 0, \cdots, 0)$. As $\rho$ is invariant under the action $Z \to e^{i\theta} ZT$, i.e., $\rho(Z, \overline{Z}) = \rho(e^{i\theta} ZT, \overline{e^{i\theta} ZT})$, we can thus assume $Z_0 = F(a) = (\sqrt{2}, 0, \cdots, 0)$. We write $\frac{\partial F}{\partial \nu}|_a = (\tau_1, \cdots, \tau_m)$ and prove the following lemma. We emphasize that in Lemma 3.6, we only need to assume $F$ is $C^1-$smooth at $a$(This fact will be used later).

**Lemma 3.6.** $(\mathrm{Re}\tau_1)^2 - \sum_{j=2}^m (\mathrm{Im}\tau_j)^2 > 0$.

**Proof of Lemma 3.6:** Suppose not, i.e., suppose $(\mathrm{Re}\tau_1)^2 - \sum_{i=2}^m (\mathrm{Im}\tau_i)^2 \leq 0$. Then there exists $(c_2, \cdots, c_m) \in \mathbb{S}^{m-2} \subset \mathbb{R}^{m-1}$ such that

$$\mathrm{Re}\tau_1 = \sum_{j=2}^m c_j(\mathrm{Im}\tau_j). \tag{3.36}$$

We next set $\psi(Z) = Z_1 + i\sum_{j=2}^m (c_j Z_j)$ and make the following claim.

**Claim:** $|\psi(Z)| < \sqrt{2}$ for $Z \in D_m^{IV}$.

**Proof of Claim:** We first note

$$\begin{aligned}\sup_{Z \in D_m^{IV}} \big|Z_1 + i\sum_{j=2}^m (c_j Z_j)\big| &\leq \sup_{Z \in \overline{D_m^{IV}}} \big|Z_1 + i\sum_{j=2}^m (c_j Z_j)\big| \\ &\leq \sup_{Z \in E_2} \big|Z_1 + i\sum_{j=2}^m (c_j Z_j)\big|.\end{aligned}$$

The last inequality holds due to the definition of the Shilov boundary. Now fix $Z \in E_2$. By Proposition 3.2, there exists some $\beta \in [0, \pi)$ such that $Z = \sqrt{2}e^{i\beta}\mathbf{x}$, where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. Then

$$r(Z) := |\psi(Z)| = \big|Z_1 + i\sum_{j=2}^m (c_j Z_j)\big| = \sqrt{2}\big|x_1 + i\sum_{j=2}^m (c_j x_j)\big|.$$

We further conclude, by the Cauchy-Schwarz inequality,

$$\frac{r^2(Z)}{2} = (x_1)^2 + \left(\sum_{j=2}^m c_j x_j\right)^2 \leq (x_1)^2 + \left(\sum_{j=2}^m c_j^2\right)\left(\sum_{j=2}^m x_j^2\right) \leq \sum_{j=1}^m x_j^2 \leq 1.$$

Hence $r(Z) \leq \sqrt{2}$ for $Z \in D_m^{IV}$. As $\psi(Z) = Z_1 + i\sum_{j=2}^m (c_j Z_j)$ is non-constant, we must have $r(Z) < \sqrt{2}$ for $Z \in D_m^{IV}$. This proves the claim. ∎

Now set $h(\xi) = F_1 + i\sum_{j=2}^m (c_j F_j)$, where $F = (F_1, \cdots, F_m)$. As $F$ maps $\Omega$ to $D_m^{IV}$, it follows from the above claim that $|h(\xi)| < \sqrt{2}$ for $\xi \in \Omega$. On the other hand, by the normalization of $F$, $h(a) = F_1(a) = \sqrt{2}$. Thus $|h(\xi)|$ attains its maximal values at $\xi = a$. By the classical Hopf lemma, we conclude $\frac{\partial |h|^2}{\partial \nu}|_a > 0$. Consequently, $\frac{\partial \mathrm{Re} h}{\partial \nu}|_a > 0$. This implies, by the definition of $h$,

$$\mathrm{Re}\tau_1 - \sum_{j=2}^{m} (c_j \mathrm{Im}\tau_j) > 0.$$

This is a contradiction to (3.36). We have thus established Lemma 3.6. ∎

We now assume $F$ is $C^2-$smooth at $a$ and compute under the normalization $F(a) = (\sqrt{2}, 0, \cdots, 0)$ :

$$\begin{aligned}\frac{\partial^2 \rho(F, \overline{F})}{\partial \nu^2}|_a &= \sum_{i=1}^{m} \left( \frac{\partial F_i}{\partial \nu}|_a - \frac{\partial \overline{F}_i}{\partial \nu}|_a \right)^2 + 2 \left| \sum_{i=1}^{m} F_i|_a \frac{\partial F_i}{\partial \nu}|_a \right|^2 \\ &= \sum_{j=1}^{m} (\tau_j - \overline{\tau_j})^2 + 4|\tau_1|^2 = 4 \left( (\mathrm{Re}\tau_1)^2 - \sum_{j=2}^{m} (\mathrm{Im}\tau_j)^2 \right).\end{aligned} \tag{3.37}$$

This is nonzero by Lemma 3.6. This establishes Proposition 3.3. □

**Proposition 3.4.** $L^2 \rho(F, \overline{F})|_a \neq 0.$

*Proof.* Write $\mathbf{v} = LF|_a$. Note there exists $\alpha \in [0, 2\pi)$ such that $\mathrm{Re}(e^{i\alpha}\mathbf{v})$ and $\mathrm{Im}(e^{i\alpha}\mathbf{v})$ are orthogonal. Then there exists a (real) orthogonal matrix $T$ such that $e^{i\alpha}\mathbf{v}T = (c_1, c_2 i, 0, \cdots, 0)$ with $c_1, c_2 \in \mathbb{R}$. Note $e^{i\alpha}T$ is an automorphism of $D_m^{IV}$ and $\rho(Z, \overline{Z})$ is invariant under the action of $e^{i\alpha}T$. Thus, by applying this automorphism, we can assume $\mathbf{v} = LF|_a = (c_1, c_2 i, 0, \cdots, 0)$.

We next set $\tilde{F} = F\tilde{T}$, where $\tilde{T}$ is a (real) orthogonal matrix of form:

$$\tilde{T} = \begin{pmatrix} T_1 & \mathbf{0}_{2\times(m-2)} \\ \mathbf{0}_{(m-2)\times 2} & T_2 \end{pmatrix}$$

where $T_1, T_2$ are $2 \times 2$ and $(m-2) \times (m-2)$ orthogonal matrices, respectively. Here $\mathbf{0}_{k\times l}$ denotes a $k \times l$ zero matrix. In light of Proposition 3.2, we can choose appropriate orthogonal matrices $T_1$ and $T_2$ such that

$$\tilde{F}|_a = e^{-i\theta}(A, 0, B, 0, \cdots, 0), \quad L\tilde{F}|_a = (\tilde{\lambda}, \tilde{\mu}, 0, \cdots, 0).$$

Here $\theta \in [0, 2\pi), A, B \in \mathbb{R}, A \geq 0$, and $\tilde{\lambda}, \tilde{\mu} \in \mathbb{C}$. In this way, replacing $F$ by $e^{i\theta}\tilde{F}$ if necessary, we can assume $F$ satisfies the following normalization:

$$F|_a = (A, 0, B, 0, \cdots, 0) \in E_2, \quad LF|_a = (\lambda, \mu, 0, \cdots, 0), \tag{3.38}$$

where $A, B \in \mathbb{R}, A \geq 0$ and $\lambda, \mu \in \mathbb{C}$. Since $F(a) \in E_2$, by (3.35) we have $A^2 + B^2 = 2$. It follows from Lemma 3.5 that $A\lambda \neq 0$. Consequently, $A > 0$.

We will need the following lemma.

**Lemma 3.7.** $\lambda^2 + \mu^2 \neq 0$.

**Proof of Lemma 3.7:** We will prove by contradiction. Suppose $\lambda^2 + \mu^2 = 0$. Then $\mu = \pm\lambda i$. Replacing $F$ by $\hat{F} := (F_1, -F_2, F_3, \cdots F_m)$ if necessary, we can assume $\mu = \lambda i$. Set the vector $\mathbf{u} = (1, i, \frac{B}{A}, 0, \cdots, 0) \in \mathbb{C}^m$. Then we have $LF|_a \cdot \mathbf{u}^t = \lambda + \mu i = 0$. Set $h := F \cdot \mathbf{u}^t$. Note $h$ is holomorphic in $\Omega$ and extends $C^2-$smoothly to $a$. Moreover, $Lh|_a = 0$. We make the following claim:

**Claim:** $|h|$ attains its maximal value at $a$. More precisely, $|h(a)| > |h(\xi)|$ for all $\xi \in \Omega$.
**Proof of Claim:** we first note

$$\sup_{Z \in D_m^{IV}} |Z \cdot \mathbf{u}^t| \leq \sup_{Z \in \overline{D_m^{IV}}} |Z \cdot \mathbf{u}^t| = \sup_{Z \in E_2} |Z \cdot \mathbf{u}^t|.$$

The last equality holds as $E_2$ is the Shilov boundary of $D_m^{IV}$. Now fix $Z \in E_2$. By Proposition 3.2, there exists some $\beta \in [0, \pi)$ such that $Z = \sqrt{2}e^{i\beta}\mathbf{x}$, where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$. Consequently,

$$r(Z) := |Z \cdot \mathbf{u}^t| = \sqrt{2}|\mathbf{x} \cdot \mathbf{u}^t| = \sqrt{2}|x_1 + ix_2 + \frac{B}{A}x_3|.$$

Then

$$\begin{aligned} r^2 = 2(x_1 + \frac{B}{A}x_3)^2 + 2x_2^2 &\leq 2\left((1 + \frac{B^2}{A^2})(x_1^2 + x_3^2) + x_2^2\right) \\ &\leq 2\left((1 + \frac{B^2}{A^2})(1 - x_2^2) + x_2^2\right). \end{aligned}$$

Here we have used the Cauchy-Schwarz inequality. Recall $A^2 + B^2 = 2$. We thus have

$$r^2 \leq 2\left(\frac{2}{A^2}(1 - x_2^2) + x_2^2\right) = \frac{4}{A^2} + 2(1 - \frac{2}{A^2})x_2^2 \leq \frac{4}{A^2}.$$

Hence $r = |Z \cdot \mathbf{u}^t| \leq \frac{2}{A}$ for $Z \in D_m^{IV}$. Note the function $\phi(Z) := Z \cdot \mathbf{u}^t$ is non-constant. We must have, for every $Z \in D_m^{IV}$, $|Z \cdot \mathbf{u}^t| < \frac{2}{A}$. As $F$ maps $\Omega$ to $D_m^{IV}$, we have $|h(z)| < \frac{2}{A}$ in $\Omega$. On the other hand,

$$h(a) = F|_a \cdot \mathbf{u}^t = A + \frac{B^2}{A} = \frac{2}{A}.$$

This establishes the claim. ∎

Note $|h|^2$ is subharmonic and by the claim, it attains the maximal value at $a$. By the classical Hopf lemma, $\frac{\partial |h|^2}{\partial \nu}|_a > 0$. Consequently, $\frac{\partial h}{\partial \nu}|_a \neq 0$ and thus $Lh|_a \neq 0$. This is a contradiction. We have thus established Lemma 3.7. ■

Now we compute, by using Lemma 3.7 and the fact that $F|_a \in \mathbb{R}^m$,

$$
\begin{aligned}
L^2(\rho \circ F)|_a &= -\sum_{i=1}^m L^2 F_i\big|_a \overline{F}_i\big|_a \\
&\quad + \frac{1}{2}\left(\sum_{i=1}^m (LF_i\big|_a)^2 + \sum_{i=1}^m F_i\big|_a L^2 F_i\big|_a\right) \sum_{i=1}^m \overline{F}_i^2\big|_a \\
&= \lambda^2 + \mu^2 \neq 0.
\end{aligned}
$$

This proves Proposition 3.4. □

Part (2) of Theorem 1 thus follows from Propositions 3.3 and 3.4. ■

**2. Proof of Part (1):** The statement follows from Proposition 2.1 in the case $k = 1$. It remains to prove the case $k = 2$. Assume $F$ extends $C^1-$smoothly to $a$ and $F(a) \in E_2$. As before, we can assume $F(a) = (\sqrt{2}, 0, \cdots, 0)$. Again write $\frac{\partial F}{\partial \nu}|_a = (\tau_1, \cdots, \tau_m)$. Recall Lemma 3.6 only requires $C^1-$smoothness of $F$ at $a$ and thus still holds in this setting. Now let $F^*$ be the first order truncation of $F$ at $a$. Using the same argument as in Section 3.1, we conclude equation (3.27) also holds in this case:

$$
\rho(F, \overline{F}) = \rho(F^*, \overline{F^*}) + o_a(2), \quad \text{when } \xi \text{ is close to } a. \tag{3.39}
$$

Write $F^* = (F_1^*, \cdots, F_m^*)$. Then one can compute, similarly as in (3.37),

$$
\begin{aligned}
\frac{\partial^2 \rho(F^*, \overline{F^*})}{\partial \nu^2}|_a &= \sum_{i=1}^m \left(\frac{\partial F_i^*}{\partial \nu}|_a - \frac{\partial \overline{F_i^*}}{\partial \nu}|_a\right)^2 + 2\left|\sum_{i=1}^m F_i^*|_a \frac{\partial F_i^*}{\partial \nu}|_a\right|^2 \\
&= 4\left((\mathrm{Re}\tau_1)^2 - \sum_{j=2}^m (\mathrm{Im}\tau_j)^2\right) > 0.
\end{aligned} \tag{3.40}
$$

Then equation (1.1) follows from (3.40) and (3.39) by the same argument as we did for the type I case. This proves part (1) of Theorem 1. ■

Theorem 1 is thus established for type IV case. ■

## 4. Applications of Theorem 1

In this section, as consequences of Theorem 1, we prove Theorem 3, 4, and their corollaries, as well as Proposition 1.1.

**Proof of Theorem 3:** We first establish the following lemma.

**Lemma 4.1.** *Let $F$ be as in Theorem 3, a holomorphic isometric map from $\mathbb{B}^n$ to $D$ with $F^*(\omega_D) = \lambda\omega_{\mathbb{B}^n}$. If $F$ extends $C^1-$smoothly across $a \in \partial\mathbb{B}^n$ and $F(a) \in E_k$, then $\lambda = k$.*

**Proof of Lemma 4.1:** By Chan-Mok [CM], we must have $\lambda = \hat{k}$ for some $1 \leq \hat{k} \leq r$. By the metric-preserving assumption, we have

$$\partial\overline{\partial}\log\rho(F, \overline{F}) = \hat{k}\partial\overline{\partial}\log(1 - ||\xi||^2). \tag{4.1}$$

By composing $F$ with an automorphism of $D$ if necessary, we assume $F(0) = 0$. Moreover, Since the automorphisms of $D$ preserve each $E_k$, we still have $F(a) \in E_k$. Now by standard reduction, as $\rho(Z, \overline{Z}) = 1$ when $Z = 0$ and $\rho(Z, \overline{Z}) - 1$ consists of only mixed terms in $Z$ and $\overline{Z}$ (cf. [CXY]), we derive from (4.1) that

$$\rho(F(\xi), \overline{F(\xi)}) = (1 - ||\xi||^2)^{\hat{k}}. \tag{4.2}$$

We compare the vanishing order of both sides of (4.2) at $a$ along the normal direction $\nu$. Note

$$\lim_{t\to 0^-} \frac{(1 - ||a + t\nu||^2)^{\hat{k}}}{(-t)^l} = \lim_{t\to 0^-} \frac{\big(1 - (1 + t)^2\big)^{\hat{k}}}{(-t)^l} = \begin{cases} 0, & \text{if } l < \hat{k} \\ c_1, & \text{if } l = \hat{k} \end{cases} \tag{4.3}$$

for some constant $c_1 \neq 0$. On the other hand, as $F(a) \in E_k$, it follows from Theorem 1 that

$$\lim_{t\to 0^-} \frac{\rho\big(F(a + t\nu), \overline{F(a + t\nu)}\big)}{(-t)^l} = \begin{cases} 0, & \text{if } l < k \\ c_2, & \text{if } l = k \end{cases} \tag{4.4}$$

for some constant $c_2 \neq 0$. By comparing (4.3) and (4.4), we conclude $k = \hat{k}$. ∎

By Mok's algebraicity theorem [M3], the isometric map $F$ must be algebraic, and thus extends holomorphically across a generic boundary point.

Combining this fact with Lemma 4.1, one easily sees "(1) $\Leftrightarrow$ (2)" and "(1) $\Leftrightarrow$ (3)". This establishes Theorem 3. ∎

We next prove the corollaries.

**Proof of Corollary 1.1:** Suppose not, i.e., suppose $F$ has a $C^1-$extension to $\xi$. Then by assumption, $F(\xi) = \lim_{i\to\infty} F(\xi_i) \in E_l$ for some $l \neq k$. This contradicts Theorem 3. We have thus proved Corollary 1.1.

**Proof of Corollary 1.2:** Again as $F$ is an isometric map, by Mok's algebraicity theorem [M3], $F$ is algebraic and extends holomorphically across a dense open subset of the boundary $\partial\mathbb{B}^n$. In particular, there exists a sequence $\{\xi_i\}_{i=1}^{\infty} \subset \partial\mathbb{B}^n$ converging to $\xi$ such that $F$ extends holomorphically across every $\xi_i$. By Theorem 3, $F(\xi_i) \in E_k$. By continuity of $F$ at $\xi$, we conclude $F(\xi) = \lim_{i\to\infty} F(\xi_i) \in \overline{E_k} = \cup_{l=k}^{r} E_l$.

**Proof of Corollary 1.3:** By assumption, $F$ is $C^1$ at $\xi$ and $F(\xi) \in E_r$, the Shilov boundary of $D$. It follows from Theorem 3 that $F^*(\omega_D) = r\omega_{\mathbb{B}^n}$. Then by a result of Chan and Mok (Proposition 1 in [CM]), $F$ is totally geodesic.

**Proof of Theorem 4:** Let $K_i$ be the isotropy group of $\Omega_i \subset \mathbb{C}^{m_i}$ at 0 for each $1 \leq i \leq N$. By the polydisc theorem(cf. [M1], [Wo]), there exists an automorphism $\gamma_i \in K_i$, such that $\gamma_i(a_i)$ takes the following form in the Harish-Chandra coordinates:

$$(1, \cdots, 1, \eta_{l_i+1}, \cdots, \eta_{r_i}, 0, \cdots, 0), \tag{4.5}$$

where the first $l_i$ components all equal to 1, and all $|\eta_s| < 1$ for $l_i + 1 \leq s \leq r_i$. Hence, without loss of generality, we assume each $a_i$ takes the form in (4.5). We define a holomorphic isometric map $J$ from the unit disc $\Delta$ to $\Omega$ :

$$J(\xi) = (J_1(\xi), \cdots, J_N(\xi)), \ \xi \in \Delta.$$

Here each $J_i$ maps $\Delta$ to $\Omega_i$ and takes the following form in the Harish-Chandra coordinates of $\Omega_i$: $J_i(\xi) = (\xi, \cdots, \xi, \eta_{l_i+1}, \cdots, \eta_{r_i}, 0, \cdots, 0)$ with the first $l_i$ components all equal to $\xi$. Note $J_i(1) = a_i$ and thus $J(1) = a$. Moreover, we have $J_i^*(\omega_{\Omega_i}) = l_i\omega_\Delta$. It then follows that

$$J^*\left(\bigoplus_{i=1}^{N} \lambda_i\omega_{\Omega_i}\right) = \sum_{i=1}^{N} \lambda_i J_i^*(\omega_{\Omega_i}) = (\sum_{i=1}^{N} \lambda_i l_i)\omega_\Delta.$$

We now consider the map $H := F \circ J$ from $\Delta$ to $D$. It follows from the assumption (1.4) that

$$H^*(\omega_D) = J^*(F^*(\omega_D)) = (\sum_{i=1}^{N} \lambda_i l_i)\omega_\Delta. \tag{4.6}$$

Moreover, note $H(1) = F(a) = (b_1, \cdots, b_{N'})$ and $H$ extends $C^1-$smoothly across $\xi^* = 1$. Pick an appropriate automorphism $\beta \in \mathrm{Aut}(D_1) \times \cdots \times \mathrm{Aut}(D_{N'})$ such that $\beta \circ H(0) = 0$. For simplicity, we still denote the new map $\beta \circ H$ by $H$ and the image of $\xi^* = 1$ under the new map by $(b_1, \cdots, b_{N'})$. Note, as the boundary orbit $\widetilde{E}^j_{k_j}$ is preserved by $\mathrm{Aut}(D_j)$, we still have $b_j \in \widetilde{E}^j_{k_j}$ for each $j$. Moreover, the new $H$ has a $C^1-$extension to $\xi^*$ as well. By (4.6),

$$(\sum_{i=1}^{N} \lambda_i l_i)\partial\overline{\partial}\log(1 - |\xi|^2) = \partial\overline{\partial}\log\prod_{j=1}^{N'}(\rho_j(H_j, \overline{H_j}))^{\mu_j}, \tag{4.7}$$

where $\rho_j$ is the generic norm associated to $\Omega_j$ and $H = (H_1, \cdots, H_{N'})$ where each $H_j, 1 \leq j \leq N'$, maps $\Delta$ to $D_j$. Note the generic norm $\rho$ of a classical domain $D$ has the property (cf. [CXY]) that

$$\rho(Z, \overline{Z}) = 1 + Q(Z, \overline{Z}),$$

where $Q$, with $Q(0) = 0$, is a real polynomial that only has mixed terms in $Z$ and $\overline{Z}$. Using this fact, we conclude from (4.7) by a standard reduction that

$$(1 - |\xi|^2)^{\sum_{i=1}^{N}\lambda_i l_i} = \prod_{j=1}^{N'}(\rho_j(H_j, \overline{H_j}))^{\mu_j}, \quad \xi \in \Delta. \tag{4.8}$$

Write $\Phi_1(\xi)$ and $\Phi_2(\xi)$ for the functions on the left and right hand side of (4.8), respectively. Let $\nu$ be the outward pointing unit normal vector of $\Delta$ at $\xi^*$. Then

$$\lim_{t\to 0^-} \frac{\Phi_1(\xi^* + t\nu)}{(-t)^s} = \begin{cases} 0, \text{ if } s < \sum_{i=1}^{N}\lambda_i l_i \\ C, \text{ if } s = \sum_{i=1}^{N}\lambda_i l_i \end{cases} \tag{4.9}$$

for some constant $C \neq 0$. On the other hand, by Theorem 1, we have

$$\lim_{t\to 0^-} \frac{\rho_j\left(H_j(\xi^*+t\nu), \overline{H_j(\xi^*+t\nu)}\right)}{(-t)^s} = \begin{cases} 0, \text{ if } s<k_j \\ C_j, \text{ if } s=k_j \end{cases}$$

for some $C_j \neq 0$. Consequently,

$$\lim_{t\to 0^-} \frac{\Phi_2(\xi^*+t\nu)}{(-t)^s} = \begin{cases} 0, \text{ if } s<\sum_{j=1}^{N'}\mu_j k_j \\ \hat{C}, \text{ if } s=\sum_{j=1}^{N'}\mu_j k_j \end{cases} \tag{4.10}$$

for some $\hat{C} \neq 0$. We compare equations (4.9) and (4.10) to obtain $\sum_{i=1}^{N}\lambda_i l_i = \sum_{j=1}^{N'}\mu_j k_j$. This proves Theorem 4. ∎

**Proof of Corollary 1.4:** We now prove the first part (i.e., equation (1.6)) of Corollary 1.4. By applying the automorphisms of $\Omega$ and $D$ if necessary, we can assume $0 \in U$ and $F(0)=0$.

Fix $(\sigma_1, \cdots, \sigma_N)$ with $0 \leq \sigma_i \leq r_i$. Note (1.6) is trivially true if all $\sigma_i$'s are zero. Now assume $\sigma_i$'s are not all zero. We define a holomorphic isometric map $I$ from $\Delta$ to $\Omega$ :

$$I(\xi) = (I_1(\xi), \cdots, I_N(\xi)).$$

Here $I_i(\xi) = (\xi, \cdots, \xi, 0, \cdots, 0)$ in the Harish-Chandra coordinates of $D_i$, with the first $\sigma_i$ components all equal to $\xi$. Note $I_i^*(\omega_{\Omega_i}) = \sigma_i\omega_\Delta$. Set $G = F \circ I$. Then $G$ is a holomorphic isometric map from a small neighborhood $V \subset \Delta$ of 0 to $D$ :

(4.11)
$$G^*(\omega_D) = I^*(\sum_{j=1}^{N'}\mu_j F^*(\omega_{D_j})) = I^*\left(\bigoplus_{i=1}^{N}\lambda_i\omega_{\Omega_i}\right) = \left(\sum_{i=1}^{N}\lambda_i\sigma_i\right)\omega_\Delta \quad \text{in } V.$$

By Theorem 4.25 in [CXY], $G$ is algebraic and extends to a proper map from $\Delta$ to $D$. Consequently, $G$ extends holomorphically across some boundary point $\xi^* \in \partial\Delta$. Then we apply Theorem 4 to the map $G$ at the point $\xi^*$ to conclude (1.6).

The second part (equation (1.7)) of Corollary 1.4 then follows from (1.6) if we take $(\sigma_1, \cdots, \sigma_N) = (0, \cdots, 0, 1, 0, \cdots, 0)$, where "1" is at the $i^{\text{th}}$ position. ∎

**Proof of Proposition 1.1:** When $k=1, E_1$ itself is a real algebraic hypersurface and the conclusion follows from Proposition 2.1. Now assume $k \geq 2$. Write $x = (x_1, \cdots, x_{2m})$ for the underlying real coordinates of $\mathbb{C}^m$. It follows from Theorem 2 that for every $b \in E_k$ and every multi-index $\beta$ with $|\beta| = k-1$ it holds

$$\frac{\partial^{|\beta|}\rho}{\partial x^{\beta}}\Big|_b = 0. \tag{4.12}$$

As before, write $F^*$ for the first order truncation of $F$ at $a$. We make the following claim.

**Claim:** There is a multi-index $\beta_0$ with $|\beta_0| = k-1$ such that

$$\frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta_0|}\rho}{\partial x^{\beta_0}}(F^*, \overline{F^*})\right)\Big|_a \neq 0. \tag{4.13}$$

Consequently, there is some $x_{j_0}, 1 \leq j_0 \leq 2m$, such that $\frac{\partial}{\partial x_{j_0}}\left(\frac{\partial^{|\beta_0|}\rho}{\partial x^{\beta_0}}\right)\Big|_{F(a)} \neq 0$.

**Proof of Claim:** Suppose not, that is, suppose for every multi-index $\beta$ with $|\beta| = k-1$, it holds that

$$\frac{\partial}{\partial \nu}\left(\frac{\partial^{|\beta|}\rho}{\partial x^{\beta}}(F^*, \overline{F^*})\right)\Big|_a = 0. \tag{4.14}$$

Note by Theorem 2, (4.14) also holds when $|\beta| < k-1$. On the other hand, since $F^*$ is a linear polynomial, by chain rule we have

$$\frac{\partial^{k-1}\rho(F^*, \overline{F^*})}{\partial \nu^{k-1}} = P\left(\left(\frac{\partial^{|\alpha|}\rho(F^*, \overline{F^*})}{\partial x^{\alpha}}\right)_{1\leq|\alpha|\leq k-1}, \left(\frac{\partial \mathrm{Re} F^*}{\partial \nu}\right), \left(\frac{\partial \mathrm{Im} F^*}{\partial \nu}\right)\right)$$

for some real polynomial $P$. Combining this with (4.12) and (4.14), we conclude

$$\frac{\partial^k \rho(F^*, \overline{F^*})}{\partial \nu^k}\Big|_a = \frac{\partial}{\partial \nu}\left(\frac{\partial^{k-1}\rho(F^*, \overline{F^*})}{\partial \nu^{k-1}}\right)\Big|_a = 0.$$

This, however, contradicts with (the proof of) Theorem 1(See Lemma 3.4 for type I case and equation (3.40) for type IV case). We have thus established the claim. ∎

We now set $\hat{\rho} = \frac{\partial^{|\beta_0|}\rho}{\partial x^{\beta_0}}$. By the claim, $d\hat{\rho} \neq 0$ near $F(a)$. Thus $M' = \{Z \approx F(a) : \hat{\rho}(Z) = 0\}$ is a germ of real algebraic smooth real hypersurface at $F(a)$. Note $M'$ contains $E_k$ near $F(a)$ by Theorem 2. Moreover, it follows from (4.13) that

$$\frac{\partial\hat{\rho}(F,\overline{F})}{\partial\nu}|_a = \frac{\partial}{\partial\nu}\left(\frac{\partial^{|\beta_0|}\rho}{\partial x^{\beta_0}}(F,\overline{F})\right)\Big|_a \neq 0.$$

Hence $F$ is CR transversal to $M'$ at $a$.

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