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Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces

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We prove that for a suitable class of metric measure spaces the abstract notion of tangent module as defined by the first author can be isometrically identified with the space of L^2-sections of the 'Gromov-Hausdorff tangent bundle'. The key assumption that we make is a form of rectifiability for which the space is 'almost isometrically' rectifiable (up to $\mathfrak{m}$-null sets) via maps that keep under control the reference measure. We point out that $\mathsf{RCD}^*(K,N)$ spaces fit in our framework.

Introduction

In the context of metric geometry there is a well established notion of tangent space at a point: the pointed-Gromov-Hausdorff limit of the blow-up of the space at the chosen point, whenever such limit exists. More recently, the first author proposed in [11] an abstract definition of tangent bundle to a generic metric measure space, such notion being based on the concepts of L^∞-module and Sobolev functions.

It is then natural to ask whether there is any relation between these two notions and pretty easy to realize that without some regularity assumption on the space there is no hope to find any: on the one hand, in general, the study of Sobolev functions might lead to no information about the metric structure of the space under consideration (this is the case, for instance, of spaces admitting no non-constant Lipschitz curves), on the other the pointed-Gromov-Hausdorff limits of the blow-ups can fail to exist at every point.

We restrict the attention to the class of *strongly* $\mathfrak{m}$*-rectifiable* spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$, defined as those spaces whose associated Sobolev space $W^{1,2}(\mathrm{X})$ is reflexive and such that for every $\varepsilon>0$ there exist a sequence of Borel sets (U_i) covering $\mathfrak{m}$-a.e. X and maps $\varphi_i : U_i \to \mathbb{R}^{k_i}$ such that for every i

$$\varphi_i \text{ is } (1+\varepsilon)\text{-biLipschitz with its image} \quad \text{and} \quad (\varphi_i)_*(\mathfrak{m}|_{U_i}) \ll \mathcal{L}^{k_i}|_{\varphi_i(U_i)}.$$

We notice that from this latter assumption only one would expect $W^{1,2}(\mathrm{X})$ to be Hilbert, and thus in particular reflexive. Yet for technical reasons we will need to assume reflexivity a priori in order to make proper use of such charts (see Theorem 2.5). The fact that $W^{1,2}(\mathrm{X})$ is Hilbert will be obtained in Theorem 5.1 as simple consequence of the main result in there. We also recall that a sufficient condition for $W^{1,2}(\mathrm{X})$ to be reflexive is that the metric space (X,d) is doubling (as proved in [1]).

The main results of this paper, namely Theorems 5.1 and 5.3, state that for this class of spaces the two notions of tangent spaces are intimately connected. More specifically, in Theorem 5.1 we show that the abstract tangent module $L^2(T\mathrm{X})$ is canonically and isometrically embedded in the space of sections of the bundle whose fibre at any point x is the pointed-Gromov-Hausdorff limits of rescaled spaces (and it can be proved that such limit is a.e. a Euclidean space, see Theorem 6.7). It is then natural to ask whether such embedding is surjective and simple examples based on fat Cantor sets show that in general this is not the case: see Example 5.2.

Still, in Theorem 5.3 we show that surjectivity is ensured provided for any Lipschitz function $f:\mathrm{X}\to\mathbb{R}$ the local Lipschitz constant $\mathrm{lip}(f)$ equals the minimal weak upper gradient $|\mathrm{D}f|$ at $\mathfrak{m}$-a.e. point (see (5.6)). As it is well known, in Cheeger's celebrated paper [7] it is proved that such condition holds if a local doubling and a local Poincaré inequality are valid in the given space.

The 'analytic' tangent module $L^2(T\mathrm{X})$ and the 'geometric' tangent module $L^2(T_{\mathrm{GH}}\mathrm{X})$ will be introduced in Definitions 1.17 and 4.5, respectively.

Looking for an analogy, one might think at this result as a kind of Rademacher's theorem: in either case when defining a notion of differentiability/tangent space there is on one side a 'concrete' and 'geometric' notion obtained by 'blow-ups' and on the other an 'abstract' and 'analytic' notion obtained by looking at 'weak' derivatives. For general functions/spaces these might be very different, but under appropriate regularity assumptions (Lipschitz/strongly $\mathfrak{m}$-rectifiable) they a.e. coincide.

The motivating example of strongly $\mathfrak{m}$-rectifiable space are $\mathsf{RCD}^*(K,N)$ spaces. Indeed, the local doubling and Poincaré inequality are proved in [19] and [18] respectively. For the rectifiability, the existence of $(1+\varepsilon)$-biLipschitz charts was obtained by Mondino-Naber in [17] and the fact that those maps send the reference measure into something absolutely continuous w.r.t. the Lebesgue one has been independently proved by Kell-Mondino in [16] and by the authors in [14] (in both cases relying on the recent powerful results of De Philippis-Rindler [8]).

Finally, we remark that part of our efforts here are made to give a meaning to the concept of 'measurable sections of the bundle formed by the collection of blow-ups'. Let us illustrate the point with an example.

Suppose that we have a metric space (X,d) such that for every $x\in\mathrm{X}$ the tangent space at x in the sense of pointed-Gromov-Hausdorff limit is the Euclidean space of a certain fixed dimension k. Then obviously all such tangent spaces would be isometric and we might want to identify all of them with a given, fixed $\mathbb{R}^k$. Once this identifications are chosen, given $x\in\mathrm{X}$ and $\mathsf{v}\in\mathbb{R}^k$ we might think at v as an element of the tangent space at x and thus a vector field should be thought of as a map from X to $\mathbb{R}^k$. However, the choice of the identifications/isometries of the abstract tangent spaces with the fixed $\mathbb{R}^k$ is highly arbitrary and affects the structure that one is building: this is better seen if one wonders what it is, say, a Lipschitz vector field, or a continuous, or a measurable one. In fact, in general there is no answer to such questions, in the sense that there is no canonical choice of these identifications: the problem is that, by the very definition, a pointed-Gromov-Hausdorff limit is the isometric class of a metric space, rather than a 'concrete' one.

As we shall see, the situation changes if one works on a strongly $\mathfrak{m}$-rectifiable metric measure space: much like in the smooth setting the charts of a manifold are used to give structure to the tangent bundle, in this case the presence of charts allows for a canonical identification of the tangent spaces while also ensuring existence and uniqueness of a measurable structure on the resulting bundle (and in general nothing more than this, so that we still can't define continuous vector fields). The construction of such measurable bundle, which we call Gromov-Hausdorff tangent bundle and denote by $T_{\rm GH}{\rm X}$, is done in Section 4.2, while in Section 6 we show that its fibres are the pointed-Gromov-Hausdorff limits of the rescaled spaces, thus justifying the terminology. Let us remark that while the initial definition of the Gromov-Hausdorff tangent bundle - and in particular its measurable structure - is simply given by a product, in fact we shall show in Section 6 that such measurable structure is natural, because it is compatible with 'taking all the pGH-limits at the same time', see Theorem 6.6.

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1. Preliminaries

1.1. Metric measure spaces

For the purpose of this paper, a *metric measure space* is a triple $({\rm X},\mathsf{d},\mathfrak{m})$, where

$$\begin{array}{ll} ({\rm X},\mathsf{d}) & \text{is a complete and separable metric space,}\\ \mathfrak{m}\neq 0 & \text{is a non-negative Borel measure on X, finite on balls.}\end{array} \tag{1.1}$$

Given two metric measure spaces $({\rm X},\mathsf{d}_{\rm X},\mathfrak{m}_{\rm X})$ and $({\rm Y},\mathsf{d}_{\rm Y},\mathfrak{m}_{\rm Y})$, we will always implicitly endow the product space ${\rm X}\times{\rm Y}$ with the product distance $\mathsf{d}_{\rm X}\times\mathsf{d}_{\rm Y}$ given by

$$(\mathsf{d}_{\rm X}\times\mathsf{d}_{\rm Y})\big((x_1,y_1),(x_2,y_2)\big)^2 := \mathsf{d}_{\rm X}(x_1,x_2)^2+\mathsf{d}_{\rm Y}(y_1,y_2)^2$$

and the product measure $\mathfrak{m}_{\rm X}\otimes\mathfrak{m}_{\rm Y}$. The notation $\mathscr{B}({\rm X})$ denotes the Borel σ-algebra on X.
Given a metric measure space $({\rm X},\mathsf{d},\mathfrak{m})$ and a point $x\in{\rm spt}(\mathfrak{m})$, we say that the reference measure $\mathfrak{m}$ is *pointwise doubling at x* if

$$\varlimsup_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}<+\infty. \tag{1.2}$$

A metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to be *doubling* provided there exists $C > 0$ such that

$$\mathfrak{m}\big(B_{2r}(x)\big) \leq C\,\mathfrak{m}\big(B_r(x)\big) \quad \text{for every } x \in \mathrm{X} \text{ and } r > 0 \tag{1.3}$$

and the least such constant C is called the *doubling constant* of the space. It is clear that the reference measure of a doubling space is pointwise doubling at all points.

Definition 1.1 (Vitali space). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then* X *is said to be a* Vitali space *provided the following condition is satisfied: given a Borel set* $A \subseteq \mathrm{X}$ *and a family* $\mathcal{F}$ *of closed balls in* X *such that* $\inf\big\{r > 0 \,:\, \overline{B_r(x)} \in \mathcal{F}\big\} = 0$ *holds for* $\mathfrak{m}$*-a.e.* $x \in A$*, there exists a countable family* $\mathcal{G} \subseteq \mathcal{F}$ *of pairwise disjoint balls such that* $\mathfrak{m}\big(A \setminus \bigcup_{B \in \mathcal{G}} B\big) = 0$.

By slightly modifying the arguments contained in the proof of [15, Theorem 1.6], one can readily prove that

$$\mathfrak{m} \text{ is pointwise doubling at } \mathfrak{m}\text{-a.e. } x \in \mathrm{X} \quad \Longrightarrow \quad (\mathrm{X}, \mathsf{d}, \mathfrak{m}) \text{ is a Vitali space.} \tag{1.4}$$

A fundamental property of the Vitali spaces is the Lebesgue differentiation theorem, whose proof can be found e.g. in [15]:

Theorem 1.2 (Lebesgue differentiation theorem). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a Vitali space. Fix a function* $f \in L^1_{\mathrm{loc}}(\mathrm{X})$*. Then*

$$f(x) = \lim_{r \to 0} \frac{1}{\mathfrak{m}\big(B_r(x)\big)} \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m} \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \tag{1.5}$$

Given a point $x \in \mathrm{X}$ and a Borel subset E of X, we say that x is *of density* $\lambda \in [0,1]$ for E if

$$D_E(x) := \lim_{r \to 0} \frac{\mathfrak{m}\big(E \cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} = \lambda. \tag{1.6}$$

By applying Theorem 1.2 to the function χ_E, we deduce that

$$D_E(x) = 1 \quad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{1.7}$$

In the sequel, the following class of metric measure spaces will play a fundamental role:

$$
\begin{aligned}
&(\mathrm{X},\mathsf{d},\mathfrak{m}) \text{ is a metric measure space with the following property:}\\
&\text{for every Borel set } E\subseteq \mathrm{X} \text{ and for } \mathfrak{m}\text{-a.e. } \bar{x}\in E, \text{ it holds that}\\
&\forall\varepsilon>0\ \ \exists r>0:\ \ \forall x\in B_r(\bar{x})\ \ \exists y\in E:\quad \mathsf{d}(x,y)<\varepsilon\,\mathsf{d}(x,\bar{x}).
\end{aligned}
\tag{1.8}
$$

A sufficient condition for satisfying the previous property is given by the next result.

Lemma 1.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Fix* $A\in\mathscr{B}(\mathrm{X})$*. Suppose that there exist constants* $\bar{r},C>0$ *such that* $\mathfrak{m}\big(B_{2r}(x)\big)\leq C\,\mathfrak{m}\big(B_r(x)\big)$ *for every* $0<r<\bar{r}$ *and* $x\in A$*. Then the metric measure space* $\big(A,\mathsf{d}|_{A\times A},\mathfrak{m}|_A\big)$ *satisfies* (1.8)*.*

In particular, any doubling metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *satisfies* (1.8)*.*

Proof. Fix a Borel set $E\subseteq A$ and any point $\bar{x}\in E$ such that $D_E(\bar{x})=1$. We claim that

$$
\forall\varepsilon>0\ \ \exists r>0:\ \ \forall x\in B_r(\bar{x})\cap A\ \ \exists y\in E:\quad \mathsf{d}(x,y)<\varepsilon\,\mathsf{d}(x,\bar{x}).
\tag{1.9}
$$

In light of (1.7), this would be enough to prove the statement (notice that $\big(A,\mathsf{d}|_{A\times A},\mathfrak{m}|_A\big)$ is a Vitali space). We now show the validity of the claim (1.9) arguing by contradiction: assume the existence of $\varepsilon>0$ and of points $\{x_r\}_{r>0}\subseteq A$ with $\mathsf{d}(x_r,\bar{x})<r$ for every $r>0$, such that

$$
E\cap B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)=\emptyset\qquad\text{for every } r>0.
\tag{1.10}
$$

Fix $n\in\mathbb{N}$ such that $2^n\,\varepsilon\geq 2+\varepsilon$. Thus $B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)\subseteq B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\subseteq B_{2^n\,\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)$ for every $r>0$, hence in particular it holds that

$$
\begin{aligned}
\mathfrak{m}\big(B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)\big)&\geq\frac{\mathfrak{m}\big(B_{2^n\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)\big)}{C^n}\\
&\geq\frac{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\big)}{C^n}\qquad\text{if } 0<r<\frac{\bar{r}}{2^{n-1}\,\varepsilon}.
\end{aligned}
\tag{1.11}
$$

Therefore

$$
\begin{aligned}
D_E(\bar{x}) &= \lim_{r\to 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x}) \cap E\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\
\text{(by (1.10))} \qquad &\leq \varliminf_{r\to 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x}) \setminus B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\
&= \varliminf_{r\to 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\big) - \mathfrak{m}\big(B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\,\mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\
\text{(by (1.11))} \qquad &\leq 1 - \frac{1}{C^n} < 1,
\end{aligned}
$$

which contradicts our assumption $D_E(\bar{x}) = 1$. □

1.2. Lipschitz functions

Let $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$ be metric spaces. A function $f : \mathrm{X} \to \mathrm{Y}$ is said to be *Lipschitz* (or, more precisely, λ-*Lipschitz*) if there exists $\lambda \geq 0$ such that $\mathsf{d}_{\mathrm{Y}}\big(f(x), f(y)\big) \leq \lambda\,\mathsf{d}_{\mathrm{X}}(x, y)$ for every $x, y \in \mathrm{X}$. The smallest $\lambda \geq 0$ such that f is λ-Lipschitz is denoted by $\mathrm{Lip}(f)$ and is called *Lipschitz constant* of f. Given any subset E of X, we indicate by $\mathrm{Lip}(f; E)$ the Lipschitz constant of $f|_E$. The family of all the Lipschitz functions from X to Y is denoted by $\mathrm{LIP}(\mathrm{X}, \mathrm{Y})$. For the sake of brevity, we shall write $\mathrm{LIP}(\mathrm{X})$ instead of $\mathrm{LIP}(\mathrm{X}, \mathbb{R})$. We say that a function $f : \mathrm{X} \to \mathrm{Y}$ is λ-*biLipschitz* if it is invertible and f, f^{-1} are λ-Lipschitz.

Definition 1.4 (Local Lipschitz constant). *Let* (X, d) *be a metric space. Let* $f \in \mathrm{LIP}(\mathrm{X})$*. Then the* local Lipschitz constant *of* f *is the function* $\mathrm{lip}(f) : \mathrm{X} \to [0, +\infty)$*, which is defined by* $\mathrm{lip}(f)(x) := 0$ *if* $x \in \mathrm{X}$ *is an isolated point and by*

(1.12)
$$
\mathrm{lip}(f)(x) := \varlimsup_{\substack{y\to x \\ y\in \mathrm{X}\setminus\{x\}}} \frac{\big|f(y) - f(x)\big|}{\mathsf{d}(y, x)} \qquad \text{if } x \in \mathrm{X} \text{ is an accumulation point.}
$$

Definition 1.5 (Asymptotic Lipschitz constant). *Let* (X, d) *be a metric space and let* $f \in \mathrm{LIP}(\mathrm{X})$*. Then the* asymptotic Lipschitz constant *of* f *is the map* $\mathrm{lip}_a(f) : \mathrm{X} \to [0, +\infty)$*, which is defined by*

$$
\mathrm{lip}_a(f)(x) := \inf_{r>0} \mathrm{Lip}\big(f; B_r(x)\big) \qquad \text{for every } x \in \mathrm{X}. \tag{1.13}
$$

One can easily prove that $\mathrm{lip}(f) \leq \mathrm{lip}_a(f) \leq \mathrm{Lip}(f)$ and that

$$\mathrm{lip}(f \circ \varphi) \leq \mathrm{Lip}(\varphi)\, \mathrm{lip}(f) \circ \varphi \tag{1.14}$$

for any couple of metric spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ and functions $\varphi \in \mathrm{LIP}(\mathrm{X}, \mathrm{Y})$ and $f \in \mathrm{LIP}(\mathrm{Y})$.

Given a metric space (X, d), a Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$ and a Borel set $E \in \mathscr{B}(\mathrm{X})$, we have that $\mathrm{lip}\big(f_{|E}\big)(x) \leq \mathrm{lip}(f)(x)$ is satisfied for every $x \in \mathrm{X}$, where $\mathrm{lip}\big(f_{|E}\big)$ is taken in the metric space $\big(E, \mathsf{d}_{|E\times E}\big)$. Simple examples show that in general equality does not hold; however, if we restrict to the case of a doubling metric measure space, then Lemma 1.3 grants that equality holds at least on density points of E:

Proposition 1.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a doubling metric measure space. Fix a Borel set* $E \in \mathscr{B}(\mathrm{X})$ *and a Lipschitz function* $f \in \mathrm{LIP}(\mathrm{X})$. *Then*

$$\mathrm{lip}\big(f_{|E}\big)(x) = \mathrm{lip}(f)(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in E. \tag{1.15}$$

More precisely, the equality above holds at every Lebesgue point $x \in E$ *of density 1.*

Proof. It suffices to prove that $\mathrm{lip}(f)(x) \leq \mathrm{lip}\big(f_{|E}\big)(x)$ for every point $x \in E$ of density 1. Thus fix $x \in E$ with $D_E(x) = 1$. If x is an isolated point in X, then $\mathrm{lip}(f)(x) = \mathrm{lip}\big(f_{|E}\big)(x) = 0$. If x is an accumulation point, then take a sequence $(x_n)_n \subseteq \mathrm{X} \setminus \{x\}$ converging to x. Up to passing to a suitable subsequence, we can assume that $\overline{\lim}_n \big|f(x_n) - f(x)\big|/\mathsf{d}(x_n, x)$ is actually a limit. Moreover, possibly passing to a further subsequence, Lemma 1.3 provides the existence of a sequence $(y_n)_n \subseteq E$ satisfying $\mathsf{d}(x_n, y_n) < \mathsf{d}(x_n, x)/n$ for every $n \geq 1$. In particular, $\lim_n y_n = x$ and $y_n \neq x$ for every $n \geq 1$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \frac{\big|f\big(x_n\big) - f(x)\big|}{\mathsf{d}(x_n, x)} &\leq \varlimsup_{n\to\infty} \frac{\big|f(x_n) - f(y_n)\big|}{\mathsf{d}(x_n, y_n)} \frac{\mathsf{d}(x_n, y_n)}{\mathsf{d}(x_n, x)} \\
&\quad + \varlimsup_{n\to\infty} \frac{\big|f(y_n) - f(x)\big|}{\mathsf{d}(y_n, x)} \frac{\mathsf{d}(y_n, x)}{\mathsf{d}(x_n, x)} \\
&\leq \mathrm{Lip}(f) \lim_{n\to\infty} \frac{1}{n} + \varlimsup_{n\to\infty} \frac{\big|f(y_n) - f(x)\big|}{\mathsf{d}(y_n, x)} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \\
&\leq \mathrm{lip}\big(f_{|E}\big)(x).
\end{aligned}$$

The arbitrariness of $(x_n)_n$ gives the conclusion. $\square$

In what follows we shall frequently use the following fact:

(1.16) Given a metric space (X,d), a subset E of X and $f\in\mathrm{LIP}(E)$, there exists $\bar f\in\mathrm{LIP}(\mathrm{X})$ such that $\bar f|_E=f$ and $\mathrm{Lip}(\bar f)=\mathrm{Lip}(f)$.

An explicit expression - called *McShane extension* - for such a function $\bar f$ is given by the formula $\bar f(x):=\inf\big\{f(y)+\mathrm{Lip}(f)\,\mathsf{d}(x,y)\,\big|\,y\in E\big\}$, $x\in\mathrm{X}$. Arguing componentwise, from this fact we also directly deduce that:

Given a metric space (X,d), a subset E of X and $f\in\mathrm{LIP}(E,\mathbb{R}^n)$, there exists $\bar f\in\mathrm{LIP}(\mathrm{X},\mathbb{R}^n)$ such that $\bar f|_E=f$ and $\mathrm{Lip}(\bar f)\leq\sqrt{n}\,\mathrm{Lip}(f)$.

Let us briefly discuss the case of Lipschitz functions from $\mathbb{R}^k$ into itself. Let $\mathrm{End}(\mathbb{R}^k)$ be the set of linear maps from $\mathbb{R}^k$ to itself, $E\subset\mathbb{R}^k$ be Borel and $f:E\to\mathbb{R}^k$ be a Lipschitz function. Find a Lipschitz extension $\tilde f$ of f to the whole $\mathbb{R}^k$ and use Rademacher theorem to obtain that $\tilde f$ is differentiable $\mathcal{L}^k$-a.e.. Call $\mathrm{d}\tilde f(x)\in\mathrm{End}(\mathbb{R}^k)$ such differential at the point x, whenever it is defined. Then it is well-known (cf., for instance, [9]) that for $\mathcal{L}^k$-a.e. $x\in E$ the value of $\mathrm{d}\tilde f(x)$ does not depend on the chosen extension $\tilde f$, so that the formula

$$\mathrm{d}f(x):=\mathrm{d}\tilde f(x)\qquad\text{for }\mathcal{L}^k\text{-a.e. }x\in E,$$

is well-posed and defines a bounded strongly measurable map from E to $\mathrm{End}(\mathbb{R}^k)$ satisfying

$$\big\|\mathrm{d}f(x)\big\|\leq\mathrm{Lip}(f)\qquad\text{for }\mathcal{L}^k\text{-a.e. }x\in E.$$

1.3. Hausdorff measures

Given a metric space (X,d) and $k\in\mathbb{N}$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure on X. We recall that, taken two metric spaces $(\mathrm{X},\mathsf{d}_\mathrm{X})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$, it holds that

$$\mathcal{H}^k\big(f(A)\big)\leq\mathrm{Lip}(f)^k\,\mathcal{H}^k(A)\quad\text{for every }f\in\mathrm{LIP}(\mathrm{X},\mathrm{Y})\text{ and }A\subseteq\mathrm{X}.\tag{1.17}$$

Another important property of the Hausdorff measures is the following, for whose proof we refer to [6, Theorem 2.4.3].

Proposition 1.7. *Let $(\mathrm{X},\mathsf{d},\mu)$ be a metric measure space and $k\in\mathbb{N}$. Let $A\subseteq\mathrm{X}$ be a Borel set and $\lambda\in(0,+\infty)$. Then*

$$\varlimsup_{r\to 0}\frac{\mu\big(B_r(x)\big)}{\omega_k\,r^k}\geq\lambda\;\text{ for every }x\in A\quad\Longrightarrow\quad\lambda\,\mathcal{H}^k(A)\leq\mu(A), \tag{1.18}$$

where ω_k denotes the Lebesgue measure of the unit ball in $\mathbb{R}^k$.

Given a metric space (X,d), we say that a Borel set $A\subseteq\mathrm{X}$ is *countably $\mathcal{H}^k$-rectifiable* provided there exist a sequence of Borel sets $(B_n)_n\subseteq\mathscr{B}(\mathbb{R}^k)$ and Lipschitz maps $f_n:\,B_n\to\mathrm{X}$ such that $\mathcal{H}^k\big(A\setminus\bigcup_n f_n(B_n)\big)=0$.

We recall a fundamental property of countably $\mathcal{H}^k$-rectifiable sets, see [5, Theorem 5.4]:

Theorem 1.8 (Spherical density). *Let (X,d) be a metric space and $k\in\mathbb{N}$. Let $A\subseteq\mathrm{X}$ be a countably $\mathcal{H}^k$-rectifiable set and $\theta:\,A\to(0,+\infty)$ a Borel map. Define $\mu:=\theta\,\mathcal{H}^k|_A$ and suppose that μ is a finite measure. Then*

$$\lim_{r\to 0}\frac{\mu\big(B_r(x)\big)}{\omega_k\,r^k}=\theta(x)\qquad\text{holds for }\mathcal{H}^k\text{-a.e. }x\in A. \tag{1.19}$$

1.4. Sobolev calculus

The scope of this section is to recall how to build the Sobolev space $W^{1,2}(\mathrm{X})$ on a metric measure space. The following definitions and results are taken from [4] and [12].

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, which will be fixed for the whole section. We say that a curve $\gamma\in C\big([0,1],\mathrm{X}\big)$ is *absolutely continuous* if there exists $f\in L^1(0,1)$ such that

$$\mathsf{d}(\gamma_t,\gamma_s)\leq\int_t^s f(r)\,\mathrm{d}r\qquad\text{for every }t,s\in[0,1]\text{ with }t<s. \tag{1.20}$$

We will denote by $AC\big([0,1],\mathrm{X}\big)$ the set of all the absolutely continuous curves in X. Given any curve $\gamma\in AC\big([0,1],\mathrm{X}\big)$, the limit

$$|\dot\gamma_t|:=\lim_{h\to 0}\frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} \tag{1.21}$$

exists for $\mathcal{L}^1$-a.e. $t\in[0,1]$ and defines an L^1-function. Such map, called *metric speed* of γ, is the minimal (in the a.e. sense) L^1-function which can

be chosen as f in the right hand side of (1.20). For a proof of these results, we refer to Theorem 1.1.2 of [2].
For every $t \in [0,1]$, we denote by $\mathrm{e}_t : C\big([0,1],\mathrm{X}\big) \to \mathrm{X}$ the *evaluation map* at time t, namely

$$\mathrm{e}_t(\gamma) := \gamma_t \quad \text{for every } \gamma \in C\big([0,1],\mathrm{X}\big). \tag{1.22}$$

Recall that $C\big([0,1],\mathrm{X}\big)$ is a metric space, with respect to the sup distance. Hence we can consider a Borel probability measure $\boldsymbol{\pi}$ on $C\big([0,1],\mathrm{X}\big)$. We say that $\boldsymbol{\pi}$ is a *test plan* provided

$$\begin{gathered}(\mathrm{e}_t)_\sharp\boldsymbol{\pi} \leq C\,\mathfrak{m} \ \text{ for every } t \in [0,1], \quad \text{for some constant } C > 0,\\ \iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty, \ \text{ where } \int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t := +\infty \text{ if } \gamma \notin AC\big([0,1],\mathrm{X}\big).\end{gathered} \tag{1.23}$$

In particular, any test plan must be necessarily concentrated on $AC\big([0,1],\mathrm{X}\big)$.

Definition 1.9 (Sobolev class). *The* Sobolev class $\mathrm{S}^2(\mathrm{X})$ *(resp.* $\mathrm{S}^2_{\rm loc}(\mathrm{X})$*) is the space of all the Borel maps* $f : \mathrm{X} \to \mathbb{R}$ *such that there exists* $G \in L^2(\mathfrak{m})$ *(resp.* $G \in L^2_{\rm loc}(\mathfrak{m})$*) satisfying*

$$\int \big|f(\gamma_1) - f(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \quad \textit{for every test plan } \boldsymbol{\pi}. \tag{1.24}$$

Here and in what follows, $L^2_{\rm loc}(\mathfrak{m})$ is the space of functions which for every $x \in \mathrm{X}$ coincide with some function in $L^2(\mathfrak{m})$ on some neighbourhood of x. Similarly for other spaces.

Given $f \in \mathrm{S}^2(\mathrm{X})$, it is possible to prove that there exists a minimal function $|\mathrm{D}f|$ in the $\mathfrak{m}$-a.e. sense which can be chosen as G in (1.24). We call $|\mathrm{D}f|$ the *minimal weak upper gradient* of f.

The main calculus properties of minimal weak upper gradients are the following:
Locality. If $f, g \in \mathrm{S}^2_{\rm loc}(\mathrm{X})$ and $N \in \mathscr{B}(\mathbb{R})$ satisfies $\mathcal{L}^1(N) = 0$, then

$$\begin{aligned} |\mathrm{D}f| &= 0 && \mathfrak{m}\text{-a.e. in } f^{-1}(N),\\ |\mathrm{D}f| &= |\mathrm{D}g| && \mathfrak{m}\text{-a.e. in } \{f = g\}. \end{aligned} \tag{1.25}$$

Lower semicontinuity. Let $(f_n)_n \subseteq \mathrm{S}^2(\mathrm{X})$ satisfy $\lim_n f_n(x) = f(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$, for some $f : \mathrm{X} \to \mathbb{R}$. Assume that $|\mathrm{D}f_n| \rightharpoonup G$ weakly in $L^2(\mathfrak{m})$

as $n \to \infty$, for some $G \in L^2(\mathfrak{m})$. Then $f \in \mathrm{S}^2(\mathrm{X})$ and

$$|\mathrm{D}f| \leq G \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{1.26}$$

Subadditivity. If $f, g \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X})$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X})$ and

$$\big|\mathrm{D}(\alpha f + \beta g)\big| \leq |\alpha||\mathrm{D}f| + |\beta||\mathrm{D}g| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{1.27}$$

Leibniz rule. If $f, g \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}) \cap L^\infty_{\mathrm{loc}}(\mathfrak{m})$, then $fg \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}) \cap L^\infty_{\mathrm{loc}}(\mathfrak{m})$ and

$$\big|\mathrm{D}(fg)\big| \leq |f||\mathrm{D}g| + |g||\mathrm{D}f| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{1.28}$$

Chain rule. Let $f \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X})$ and $\varphi \in \mathrm{LIP}(\mathbb{R})$. Then $\varphi \circ f \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X})$ and

$$\big|\mathrm{D}(\varphi \circ f)\big| = |\varphi'| \circ f \, |\mathrm{D}f| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}, \tag{1.29}$$

where $|\varphi'| \circ f$ is arbitrarily defined at the non-differentiability points of φ. Notice that for $f \in \mathrm{LIP}(\mathrm{X})$, we trivially have that (1.24) is satisfied for $G := \mathrm{lip}(f)$, so that $f \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X})$ and

$$|\mathrm{D}f| \leq \mathrm{lip}(f) \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{1.30}$$

The *Sobolev space* $W^{1,2}(\mathrm{X})$ is defined as

$$W^{1,2}(X) := \mathrm{S}^2(\mathrm{X}) \cap L^2(\mathfrak{m}). \tag{1.31}$$

Whenever ambiguities may arise, we write $W^{1,2}_{\mathfrak{m}}(\mathrm{X})$ and $|Df|_{\mathfrak{m}}$ in place of $W^{1,2}(\mathrm{X})$ and $|Df|$, respectively. It turns out that $W^{1,2}(\mathrm{X})$ is a Banach space if endowed with the norm

$$\|f\|_{W^{1,2}(\mathrm{X})} := \sqrt{\|f\|^2_{L^2(\mathfrak{m})} + \big\||\mathrm{D}f|\big\|^2_{L^2(\mathfrak{m})}} \quad \text{for every } f \in W^{1,2}(\mathrm{X}). \tag{1.32}$$

However, in general $W^{1,2}(\mathrm{X})$ is not a Hilbert space. We then give the following definition:

Definition 1.10 (Infinitesimally Hilbertian). *The metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* infinitesimally Hilbertian *provided* $W^{1,2}(\mathrm{X})$ *is a Hilbert space.*

It has been proved in [3] that Sobolev functions can be approximated by Lipschitz ones:

Theorem 1.11 (Density in energy of Lipschitz functions). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any metric measure space. Then for any function* $f\in W^{1,2}(\mathrm{X})$ *there exists a sequence* $(f_n)_n\subseteq \mathrm{LIP}_c(\mathrm{X})$ *such that* $f_n\to f$ *and* $\mathrm{lip}_a(f_n)\to |Df|$ *in* $L^2(\mathfrak{m})$. *Moreover, if* $W^{1,2}(\mathrm{X})$ *is reflexive then* $(f_n)_n$ *can be chosen so that* $\big|D(f_n-f)\big|\to 0$ *in* $L^2(\mathfrak{m})$, *in other words* $\mathrm{LIP}_c(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})$ *with respect to the* $W^{1,2}(\mathrm{X})$*-norm.*

We conclude recalling that

$$(\mathrm{X},\mathsf{d},\mathfrak{m})\text{ doubling}\quad\Longrightarrow\quad W^{1,2}(\mathrm{X})\text{ is a reflexive space.}\tag{1.33}$$

This non-trivial result, which in fact only requires the doubling property of the distance, has been proved in [1].

1.5. Cotangent and tangent modules

Here we recall some definitions and concepts introduced by the first author in [11], referring to [11] and [10] for a more detailed discussion.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space, which will be fixed throughout the whole section. We first give the definition of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module:

Definition 1.12 ($L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module). *Let* $\mathscr{M}$ *be a Banach space. Then* $\mathscr{M}$ *is said to be an* $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module *provided it is endowed with a bilinear map* $L^\infty(\mathfrak{m})\times\mathscr{M}\ni(f,v)\mapsto fv\in\mathscr{M}$, *called* multiplication, *and a function* $|\cdot|:\mathscr{M}\to L^2(\mathfrak{m})^+$, *called* pointwise norm, *which satisfy the following properties:*

- *(i)* $(fg)v=f(gv)$ *for every* $v\in\mathscr{M}$ *and* $f,g\in L^\infty(\mathfrak{m})$.
- *(ii)* $\mathbf{1}v=v$ *for every* $v\in\mathscr{M}$, *where* $\mathbf{1}\in L^\infty(\mathfrak{m})$ *is the function identically 1.*
- *(iii)* $\big\||v|\big\|_{L^2(\mathfrak{m})}=\|v\|_{\mathscr{M}}$ *for every* $v\in\mathscr{M}$.
- *(iv)* $|fv|=|f|\,|v|$ $\mathfrak{m}$*-a.e. in* X, *for every* $v\in\mathscr{M}$ *and* $f\in L^\infty(\mathfrak{m})$.

Given a Borel set $A\in\mathscr{B}(\mathrm{X})$, we define the 'restriction' $\mathscr{M}_{|A}$ of $\mathscr{M}$ to A as

$$\mathscr{M}_{|A}:=\big\{v\in\mathscr{M}\,\big|\,\chi_{A^c}\cdot v=0\big\}.\tag{1.34}$$

Notice that $\mathscr{M}_{|A}$ inherits the structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Given two $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$, we say that a map $T:\mathscr{M}\to\mathscr{N}$ is a *module morphism* provided it is linear continuous and it satisfies

$$T(fv)=f\,T(v)\quad\text{for every } v\in\mathscr{M}\text{ and } f\in L^\infty(\mathfrak{m}). \tag{1.35}$$

An important class of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules is that of *Hilbert modules*, namely those modules $\mathscr{H}$ that are Hilbert spaces when seen as normed spaces. It turns out that a given normed module $\mathscr{H}$ is a Hilbert module if and only if its pointwise norm satisfies the pointwise parallelogram identity

$$|v+w|^2+|v-w|^2=2\,|v|^2+2\,|w|^2\quad\mathfrak{m}\text{-a.e. in X} \tag{1.36}$$

for any couple of elements $v,w\in\mathscr{H}$.

Definition 1.13 (Dual module). *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then we define the* dual module *$\mathscr{M}^*$ of $\mathscr{M}$ as the family of all linear continuous maps $T:\mathscr{M}\to L^1(\mathfrak{m})$ such that $T(fv)=f\,T(v)$ holds $\mathfrak{m}$-a.e. in X for any $v\in\mathscr{M}$ and $f\in L^\infty(\mathfrak{m})$.*

The space $\mathscr{M}^*$ naturally comes with the structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module: it is a Banach space with respect to the pointwise vector operations and the operator norm, while the multiplication fT between $f\in L^\infty(\mathfrak{m})$ and $T\in\mathscr{M}^*$ is defined as

$$(fT)(v):=f\,T(v)\ \ \mathfrak{m}\text{-a.e. in X},\quad\text{for every } v\in\mathscr{M} \tag{1.37}$$

and the pointwise norm $|T|$ of $T\in\mathscr{M}^*$ is given by

$$|T|:=\operatorname*{ess\,sup}_{\substack{v\in\mathscr{M},\\ |v|\leq 1\ \mathfrak{m}\text{-a.e.}}}\big|T(v)\big|\quad\text{for every } T\in\mathscr{M}^*. \tag{1.38}$$

We recall the notion of local dimension:

Definition 1.14 (Local dimension of normed modules). *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Let $A\in\mathscr{B}(\mathrm{X})$ be such that $\mathfrak{m}(A)>0$. Then:*

(i) Finitely many elements $v_1,\ldots,v_n\in\mathscr{M}$ are said to be independent *on A provided for any $f_1,\ldots,f_n\in L^\infty(\mathfrak{m})$ it holds that*

$$\chi_A\sum_{i=1}^n f_i v_i=0\quad\Longleftrightarrow\quad f_i=0\ \ \mathfrak{m}\text{-a.e. in } A,\quad\text{for every } i=1,\ldots,n. \tag{1.39}$$

(ii) *We say that a set* $S \subset \mathscr{M}$ generates $\mathscr{M}_{|A}$ *provided* $\mathscr{M}_{|A}$ *is the closure of the set of finite sums of objects of the form* $\chi_A f v$ *for* $f \in L^\infty(\mathfrak{m})$ *and* $v \in S$.

(iii) *We say that some elements* $v_1, \ldots, v_n \in \mathscr{M}$ *constitute a* basis *for* $\mathscr{M}_{|A}$ *if they are independent on* A *and generate* $\mathscr{M}_{|A}$.

(iv) *The* local dimension *of* $\mathscr{M}$ *on* A *is defined to be equal to* $n \in \mathbb{N}$ *if* $\mathscr{M}$ *admits a basis of cardinality* n *on* A, *while it is defined to be equal to* ∞ *provided* $\mathscr{M}$ *is not finitely-generated on any Borel subset of* A *having positive* $\mathfrak{m}$*-measure.*

Observe that the notion of local dimension is well-defined, in the sense that two different bases for $\mathscr{M}$ on A must necessarily have the same cardinality.

By using the language of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules described so far, we can now introduce the cotangent module $L^2(T^*\mathrm{X})$ associated to $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Its definition is based upon the following result, whose proof can be found in [10]:

Theorem 1.15. *There exists (up to unique isomorphism) a unique couple* $(\mathscr{M}, \mathrm{d})$, *where* $\mathscr{M}$ *is an* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module and* $\mathrm{d} : \mathrm{W}^{1,2}(\mathrm{X}) \to \mathscr{M}$ *is a linear map, such that*

(i) $|\mathrm{d}f| = |\mathrm{D}f|$ *holds* $\mathfrak{m}$*-a.e. in* X, *for every* $f \in W^{1,2}(\mathrm{X})$,

(ii) $\{\mathrm{d}f \,:\, f \in W^{1,2}(\mathrm{X})\}$ *generates* $\mathscr{M}$ *on* X.

Namely, if two couples $(\mathscr{M}, \mathrm{d})$ *and* $(\mathscr{M}', \mathrm{d}')$ *as above fulfill both (i) and (ii), then there exists a unique module isomorphism* $\Phi : \mathscr{M} \to \mathscr{M}'$ *such that* $\Phi \circ \mathrm{d} = \mathrm{d}'$.

Definition 1.16 (Cotangent module and differential). *The module provided by the previous theorem is called* cotangent module *and denoted by* $L^2(T^*\mathrm{X})$*; its elements are called* 1-forms *on* X. *The map* d *will be called* differential.

The tangent module is then introduced by duality:

Definition 1.17 (Tangent module). *We call* tangent module *the dual of* $L^2(T^*\mathrm{X})$ *and denote it by* $L^2(T\mathrm{X})$. *Its elements are called* vector fields *on* X.

In case of ambiguity, we shall make use of the notation $L^2_{\mathfrak{m}}(T^*\mathrm{X})$, $\mathrm{d}_{\mathfrak{m}} f$ and $L^2_{\mathfrak{m}}(T\mathrm{X})$ instead of $L^2(T^*\mathrm{X})$, $\mathrm{d}f$ and $L^2(T\mathrm{X})$, respectively.

It can be proved that the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if and only if

$$L^2(T^*\mathrm{X}) \text{ and } L^2(T\mathrm{X}) \text{ are Hilbert modules.} \tag{1.40}$$

For this and other equivalent characterizations, we refer to [11, Proposition 2.3.17].

Remark 1.18 (Localisation of the cotangent module). Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix an open set $\Omega \subseteq \mathrm{X}$ and define $\widetilde{\mathfrak{m}} := \mathfrak{m}_{|\Omega}$. Then the cotangent module $L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})$ can be canonically identified with $L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$, in the following sense: there exists a (unique) linear isomorphism $\iota : L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X}) \to L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$ such that

$$\begin{aligned} \big|\iota(v)\big| = |v| \quad \widetilde{\mathfrak{m}}\text{-a.e.} \quad & \text{for every } v \in L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X}), \\ \iota(\mathrm{d}_{\widetilde{\mathfrak{m}}} f) = \mathrm{d}_{\mathfrak{m}} f \quad & \text{for every } f \in W^{1,2}_{\mathfrak{m}}(\mathrm{X}) \\ & \text{with } \mathrm{dist}\big(\mathrm{spt}(f),\mathrm{X}\setminus\Omega\big) > 0. \end{aligned} \tag{1.41}$$

First of all, observe that the second line in (1.41) makes sense, because any map $f \in W^{1,2}_{\mathfrak{m}}(\mathrm{X})$ with $\mathrm{dist}\big(\mathrm{spt}(f),\mathrm{X}\setminus\Omega\big) > 0$ belongs to $W^{1,2}_{\widetilde{\mathfrak{m}}}(\mathrm{X})$ and satisfies $|Df|_{\widetilde{\mathfrak{m}}} = |Df|_{\mathfrak{m}}$ $\widetilde{\mathfrak{m}}$-a.e. (see [4, Theorem 4.19]). Then to check that ι is well defined by the above it is sufficient to verify that the differentials of functions $f \in W^{1,2}_{\widetilde{\mathfrak{m}}}(\mathrm{X})$ with $\mathrm{dist}\big(\mathrm{spt}(f),\mathrm{X}\setminus\Omega\big) > 0$ generate the whole $L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})$. In turn, by definition of 'generate' and the trivial identity $\Omega = \bigcup_{\lambda\in\mathbb{Q}^+}\big\{x\in\Omega\,:\,\mathsf{d}(x,\mathrm{X}\setminus\Omega)>\lambda\big\}$, it is sufficient to check that the same set of differentials generate $L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})$ on $\big\{x\in\Omega\,:\,\mathsf{d}(x,\mathrm{X}\setminus\Omega)>\lambda\big\}$ for any $\lambda > 0$. But this is trivial, indeed let η_λ be a Lipschitz and bounded cut-off function with $\mathrm{dist}\big(\mathrm{spt}(\eta_\lambda),\mathrm{X}\setminus\Omega\big) > 0$ and $\eta_\lambda \equiv 1$ on $\big\{x\in\Omega\,:\,\mathsf{d}(x,\mathrm{X}\setminus\Omega)>\lambda\big\}$. Then for every $f \in W^{1,2}_{\mathfrak{m}}(\mathrm{X})$ we have that $f\eta_\lambda \in W^{1,2}_{\mathfrak{m}}(\mathrm{X})$ satisfies $\mathrm{dist}\big(\mathrm{spt}(f\eta_\lambda),\mathrm{X}\setminus\Omega\big) > 0$ and $\mathrm{d}_{\widetilde{\mathfrak{m}}} f = \mathrm{d}_{\widetilde{\mathfrak{m}}}(f\eta_\lambda)$ on $\big\{x\in\Omega\,:\,\mathsf{d}(x,\mathrm{X}\setminus\Omega)>\lambda\big\}$, whence the claim follows.

Therefore it immediately follows that there exists a (uniquely determined) linear and continuous isomorphism $\iota : L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X}) \to L^2_{\mathfrak{m}}(T\mathrm{X})_{|\Omega}$ such that

$$\begin{aligned} &\iota(\omega)\big(\iota(v)\big) = \omega(v) \quad \text{holds } \widetilde{\mathfrak{m}}\text{-a.e. in X}, \\ &\text{for every } \omega \in L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X}) \text{ and } v \in L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X}). \end{aligned} \tag{1.42}$$

In particular, the equality $\left|\iota(v)\right| = |v|$ is satisfied $\widetilde{\mathfrak{m}}$-a.e. in X for every $v \in L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X})$. ■

We conclude the section discussing the case of $\mathrm{X} = \mathbb{R}^k$. Let us denote by $L^2(\mathbb{R}^k, \mathbb{R}^k)$ the standard space of L^2 vector fields on $\mathbb{R}^k$ and by $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$ its dual, i.e. the space of L^2 1-forms. Notice that the dual of $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$ is $L^2(\mathbb{R}^k, \mathbb{R}^k)$.

We know that the Sobolev space $W^{1,2}(\mathbb{R}^k)$ as defined here coincides with the classically defined one via distributional derivatives and that for $f \in W^{1,2}(\mathbb{R}^k)$ if we consider its distributional differential, which for a moment we denote $\hat{\mathrm{d}} f$ and which naturally belongs to $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$, we have that its norm $|\hat{\mathrm{d}} f|$ coincides with the minimal weak upper gradient $|\mathrm{D}f|$ (see [3]). Also, it is readily verified that 1-forms of the kind $\sum_{i=1}^n \chi_{A_i} \hat{\mathrm{d}} f_i$, for $n \in \mathbb{N}$, (A_i) a partition of $\mathbb{R}^k$ and $(f_i) \subset W^{1,2}(\mathbb{R}^k)$, are dense in $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$. Thanks to Theorem 1.15, these facts are sufficient to conclude that the 'concrete' space of L^2 1-forms $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$ and the abstract cotangent module $L^2(T^*\mathbb{R}^k)$ can be canonically identified by an isomorphism which sends $\hat{\mathrm{d}} f$ to $\mathrm{d}f$.

Once this identification is done, it follows that also the space of L^2 vector fields $L^2(\mathbb{R}^k, \mathbb{R}^k)$ can be canonically identified with the tangent module $L^2(T\mathbb{R}^k)$. Such identification allows us to identify, for a given Borel set $E \subset \mathbb{R}^k$, the restricted module $L^2(T\mathbb{R}^k)_{|E}$ with $L^2(E, \mathbb{R}^k)$.

Finally, we point out that for every function $f \in \mathrm{LIP}(\mathbb{R}^k) \cap W^{1,2}(\mathbb{R}^k)$ it holds that

$$|\mathrm{d}f| = \mathrm{lip}(f) \quad \text{is satisfied } \mathcal{L}^k\text{-a.e. in } \mathbb{R}^k, \tag{1.43}$$

which represents a reinforcement of property (1.30).

2. Maps of bounded deformation

Fix two metric measure spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$. We report here some definitions and results that are taken from [11] and [10], where it is described how the notions of pullback of 1-forms and of differential can be built for a special class of mappings between X and Y, which are said to be of bounded deformation.

We start by recalling what it means for a map $\varphi : \mathrm{X} \to \mathrm{Y}$ to be of bounded deformation:

Definition 2.1 (Map of bounded compression/deformation). *A map* $\varphi : \mathrm{X} \to \mathrm{Y}$ *is said to be* of bounded compression *if it satisfies*

$$\varphi_*\mathfrak{m}_{\mathrm{X}} \leq C\,\mathfrak{m}_{\mathrm{Y}} \quad \textit{for a suitable constant } C > 0. \tag{2.1}$$

The least such C *is called* compression constant *and is denoted by* $\mathrm{Comp}(\varphi)$.

Moreover, the map φ *is said to be* of bounded deformation *provided it is both Lipschitz and of bounded compression.*

For maps of bounded compression/deformation, we have at disposal two notions of *pullback*:

i) Suppose that $\varphi : \mathrm{X} \to \mathrm{Y}$ is a map of bounded compression. Given any $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module $\mathscr{M}$, there exists (up to unique isomorphism) a unique couple $\big(\varphi^*\mathscr{M}, [\varphi^*]\big)$, where $\varphi^*\mathscr{M}$ is an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{X}})$-module and $[\varphi^*] : \mathscr{M} \to \varphi^*\mathscr{M}$ is a linear and continuous operator, such that

$$\begin{aligned} &\big|[\varphi^*]v\big| = |v| \circ \varphi \;\text{ holds } \mathfrak{m}_{\mathrm{X}}\text{-a.e., for every } v \in \mathscr{M}, \\ &\big\{[\varphi^*]v \,:\, v \in \mathscr{M}\big\} \;\text{ generates the whole } \varphi^*\mathscr{M}. \end{aligned} \tag{2.2}$$

We say that $\varphi^*\mathscr{M}$ is the *pullback module* of $\mathscr{M}$ and that $[\varphi^*]$ is the *pullback map*. We shall sometimes write $[\varphi^* v]$ instead of $[\varphi^*]v$.
In general, $\varphi^*\mathscr{M}^*$ is only isometrically embedded into $(\varphi^*\mathscr{M})^*$, but the two modules are actually isomorphic provided, for example, the space $\mathscr{M}^*$ is separable.

ii) Suppose that $\varphi : \mathrm{X} \to \mathrm{Y}$ is a map of bounded deformation. One has that $f \circ \varphi \in W^{1,2}(\mathrm{X})$ whenever $f \in W^{1,2}(\mathrm{Y})$ and that $\big|\mathrm{d}(f \circ \varphi)\big| \leq \mathrm{Lip}(\varphi)\,|\mathrm{d}f| \circ \varphi$ holds $\mathfrak{m}_{\mathrm{X}}$-a.e.. Then there exists a unique linear and continuous operator $\varphi^* : L^2(T^*\mathrm{Y}) \to L^2(T^*\mathrm{X})$, called *pullback of* 1-*forms*, such that

$$\begin{aligned} \varphi^*\mathrm{d}f &= \mathrm{d}(f \circ \varphi) && \text{for every } f \in W^{1,2}(\mathrm{Y}), \\ \varphi^*(h\,\omega) &= h \circ \varphi\,\varphi^*\omega && \text{for every } \omega \in L^2(T^*\mathrm{Y}) \text{ and } h \in L^\infty(\mathfrak{m}_{\mathrm{Y}}). \end{aligned} \tag{2.3}$$

Moreover, it holds that

$$|\varphi^*\omega| \leq \mathrm{Lip}(\varphi)\,|\omega| \circ \varphi \quad \mathfrak{m}_{\mathrm{X}}\text{-a.e.,} \quad \text{for every } \omega \in L^2(T^*\mathrm{Y}). \tag{2.4}$$

We point out that the above two notions of pullback are rather different. The former can be associated to any map of bounded compression, while the latter is tailored for maps of bounded deformation and 1-forms. The most evident discrepancy between them is the following fact: given a map of bounded deformation $\varphi : \mathrm{X} \to \mathrm{Y}$, the pullback map $[\varphi^*]$ defined in i) takes values into $\varphi^* L^2(T^*\mathrm{Y})$, while the pullback map φ^* defined in ii) takes values into $L^2(T^*\mathrm{X})$.

With this said, we are in a position to introduce the differential $\mathrm{d}\varphi$ of a map of bounded deformation $\varphi : \mathrm{X} \to \mathrm{Y}$ by combining the above two concepts of pullback:

Theorem 2.2 (Differential of a map of bounded deformation). *Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation. Assume that $L^2(T\mathrm{Y})$ is separable. Then there exists a unique linear and continuous operator $\mathrm{d}\varphi : L^2(T\mathrm{X}) \to \varphi^* L^2(T\mathrm{Y})$, called* differential *of φ, such that*

$$(2.5) \quad [\varphi^*\omega]\big(\mathrm{d}\varphi(v)\big) = \varphi^*\omega(v) \quad \text{for every } \omega \in L^2(T^*\mathrm{Y}) \text{ and } v \in L^2(T\mathrm{X}).$$

In particular, the map $\mathrm{d}\varphi$ is $L^\infty(\mathfrak{m}_\mathrm{X})$-linear and satisfies

$$(2.6) \qquad \big|\mathrm{d}\varphi(v)\big| \leq \mathrm{Lip}(\varphi)\,|v| \quad \mathfrak{m}_\mathrm{X}\text{-a.e.,} \quad \text{for every } v \in L^2(T\mathrm{X}).$$

In order to continue our analysis, we now need to show that the differential of a map of bounded deformation is a *local* object, as explained in the following two results.

Lemma 2.3. *Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation. Fix a Borel set $E \subseteq \mathrm{X}$. Then*

$$(2.7) \qquad \begin{aligned} &\chi_E \,\big|\mathrm{d}(f \circ \varphi)\big| \leq \mathrm{Lip}(\varphi; E)\,\chi_E\,|\mathrm{d}f| \circ \varphi \quad \text{holds } \mathfrak{m}_\mathrm{X}\text{-a.e.,} \\ &\text{for every } f \in W^{1,2}(\mathrm{Y}). \end{aligned}$$

Proof. Choose a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathrm{Y})$ such that $f_n \to f$ and $\mathrm{lip}_a(f_n) \to |\mathrm{d}f|$ in $L^2(\mathfrak{m}_\mathrm{Y})$. Let $n \in \mathbb{N}$ be fixed. Given any $x \in E$ and $r > 0$, there exists a Lipschitz map $g \in \mathrm{LIP}(\mathrm{X})$ such that $\mathrm{Lip}(g) = \mathrm{Lip}\big(f_n \circ \varphi; E \cap B_r(x)\big)$ and $g|_{E \cap B_r(x)} = f_n \circ \varphi|_{E \cap B_r(x)}$. Since $\varphi\big(E \cap B_r(x)\big)$ is contained in the ball $B_{\mathrm{Lip}(\varphi)r}\big(\varphi(x)\big)$, we have that

$$(2.8) \qquad \begin{aligned} \chi_{E \cap B_r(x)}\,\big|\mathrm{d}(f_n \circ \varphi)\big| &= \chi_{E \cap B_r(x)}\,|\mathrm{d}g| \leq \mathrm{Lip}(g) \\ &\leq \mathrm{Lip}(\varphi; E)\,\mathrm{Lip}\big(f_n; B_{\mathrm{Lip}(\varphi)r}\big(\varphi(x)\big)\big) \end{aligned}$$

holds $\mathfrak{m}_{\rm X}$-a.e.. Given that $B_{{\rm Lip}(\varphi)r}\big(\varphi(x)\big)\subseteq B_{2{\rm Lip}(\varphi)r}\big(\varphi(y)\big)$ is satisfied for every $y\in B_r(x)$, we deduce from (2.8) and Lindelöf lemma that

$$\big|{\rm d}(f_n\circ\varphi)\big|(x)\leq {\rm Lip}(\varphi;E)\,{\rm Lip}\big(f_n;B_{2{\rm Lip}(\varphi)r}\big(\varphi(x)\big)\big)\quad\text{for }\mathfrak{m}_{\rm X}\text{-a.e. }x\in E. \tag{2.9}$$

By letting $r\to 0$ in the above inequality (2.9), we thus obtain that

$$\chi_E\,\big|{\rm d}(f_n\circ\varphi)\big|\leq {\rm Lip}(\varphi;E)\,\chi_E\,{\rm lip}_a(f_n)\circ\varphi\quad\text{holds }\mathfrak{m}_{\rm X}\text{-a.e.,}\quad\text{for every }n\in\mathbb{N}. \tag{2.10}$$

Notice that $\big|{\rm d}(f_n\circ\varphi)\big|\leq{\rm Lip}(\varphi)\,|{\rm d}f_n|\circ\varphi\leq{\rm Lip}(\varphi)\,{\rm lip}_a(f_n)\circ\varphi$ is satisfied $\mathfrak{m}_{\rm X}$-a.e., thus accordingly the set of all functions $\big|{\rm d}(f_n\circ\varphi)\big|$, with $n\in\mathbb{N}$, is norm bounded in $L^2(\mathfrak{m}_{\rm X})$. In particular, possibly passing to a (not relabeled) subsequence, one has that $\big|{\rm d}(f_n\circ\varphi)\big|\rightharpoonup h$ weakly in $L^2(\mathfrak{m}_{\rm X})$ for a suitable map $h\in L^2(\mathfrak{m}_{\rm X})$. By lower semicontinuity of minimal weak upper gradients, we deduce that $\big|{\rm d}(f\circ\varphi)\big|\leq h$ holds $\mathfrak{m}_{\rm X}$-a.e.. Since ${\rm lip}_a(f_n)\circ\varphi\rightharpoonup|{\rm d}f|\circ\varphi$ weakly in $L^2(\mathfrak{m}_{\rm X})$, we finally conclude by recalling (2.10) that

$$\chi_E\,\big|{\rm d}(f\circ\varphi)\big|\leq\chi_E\,h\leq{\rm Lip}(\varphi;E)\,\chi_E\,|{\rm d}f|\circ\varphi\qquad\text{holds }\mathfrak{m}_{\rm X}\text{-a.e.,}$$

yielding (2.7) and accordingly the statement. □

Corollary 2.4. *Let* $({\rm X},{\sf d}_{\rm X},\mathfrak{m}_{\rm X})$, $({\rm Y},{\sf d}_{\rm Y},\mathfrak{m}_{\rm Y})$ *be metric measure spaces such that* $L^2(T{\rm Y})$ *is separable. Let* $\varphi:{\rm X}\to{\rm Y}$ *be a map of bounded deformation. Fix a Borel set* $E\subseteq{\rm X}$*. Then*

$$\big|{\rm d}\varphi(v)\big|\leq{\rm Lip}(\varphi;E)\,|v|\quad\textit{holds }\mathfrak{m}_{\rm X}\textit{-a.e.,}\qquad\textit{for every }v\in L^2(T{\rm X})_{|E}. \tag{2.11}$$

Proof. Given any simple form $\omega=\sum_{i=1}^n\chi_{A_i}\,{\rm d}f_i\in L^2(T^*{\rm Y})$, with $(A_i)_{i=1}^n\subseteq\mathscr{B}({\rm Y})$ disjoint and $(f_i)_{i=1}^n\subseteq W^{1,2}({\rm Y})$, one has $\mathfrak{m}_{\rm X}$-a.e. that

$$\begin{split}\chi_E\,|\varphi^*\omega|&=\sum_{i=1}^n\chi_{\varphi^{-1}(A_i)\cap E}\,\big|{\rm d}(f_i\circ\varphi)\big|\\&\overset{(2.7)}{\leq}{\rm Lip}(\varphi;E)\sum_{i=1}^n\chi_E\,\big(\chi_{A_i}|{\rm d}f_i|\big)\circ\varphi={\rm Lip}(\varphi;E)\,\chi_E\,|\omega|\circ\varphi,\end{split}$$

which grants that $\chi_E\,|\varphi^*\omega| \leq \mathrm{Lip}(\varphi;E)\,\chi_E\,|\omega|\circ\varphi$ $\mathfrak{m}_{\mathrm{X}}$-a.e. for every $\omega \in L^2(T^*\mathrm{Y})$. Hence

$$\begin{aligned}\chi_E\left|[\varphi^*\omega]\big(\mathrm{d}\varphi(v)\big)\right| &= \chi_E\left|\varphi^*\omega(v)\right| \leq \chi_E\,|\varphi^*\omega|\,|v| \leq \mathrm{Lip}(\varphi;E)\,\chi_E\,|\omega|\circ\varphi\,|v| \\ &= \mathrm{Lip}(\varphi;E)\,\chi_E\left|[\varphi^*\omega]\right||v| \qquad \text{holds } \mathfrak{m}_{\mathrm{X}}\text{-a.e.},\end{aligned}$$

which implies that $\left|\mathrm{d}\varphi(v)\right| \leq \mathrm{Lip}(\varphi;E)\,|v|$ is satisfied $\mathfrak{m}_{\mathrm{X}}$-a.e. in E. ☐

In the next section we will deal with functions φ defined on some Borel set $E \subseteq \mathrm{X}$ and taking values into the Euclidean space $\mathbb{R}^k$. In addition, the map $\varphi : E \to \varphi(E)$ under consideration will be of bounded deformation, invertible and with inverse of bounded deformation.

Thanks to the high regularity of the target space $\mathbb{R}^k$ and to the invertibility of φ, it will be possible to associate to any element $v \in L^2(T\mathrm{X})_{|E}$ a 'concrete' vector field $\underline{\mathrm{d}}\varphi(v)$ in $L^2\big(\varphi(E),\mathbb{R}^k\big)$. Such new notion of differential $\underline{\mathrm{d}}\varphi$, tailored for this kind of maps φ, is described in the following result.

Theorem 2.5. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space such that $W^{1,2}(\mathrm{X})$ is a reflexive space. Let $E \subseteq \mathrm{X}$ be a Borel set and let $\varphi : E \to \mathbb{R}^k$ be a Lipschitz map. Suppose that there exist constants $\mathsf{L},\mathsf{C} > 1$ such that*

$$\begin{aligned}&\varphi : E \to \varphi(E) \qquad \textit{is } \mathsf{L}\textit{-biLipschitz}, \\ &\mathsf{C}^{-1}\,\mathcal{L}^k|_{\varphi(E)} \leq \varphi_*\big(\mathfrak{m}|_E\big) \leq \mathsf{C}\,\mathcal{L}^k|_{\varphi(E)}.\end{aligned} \tag{2.12}$$

Then there exists a unique linear and continuous operator $\underline{\mathrm{d}}\varphi : L^2(T\mathrm{X})_{|E} \to L^2\big(\varphi(E),\mathbb{R}^k\big)$, called differential *of φ, which satisfies the following conditions for any $v \in L^2(T\mathrm{X})_{|E}$:*

$$\begin{aligned}\mathrm{d}g\big(\underline{\mathrm{d}}\varphi(v)\big) &= \big(\mathrm{d}(g\circ\overline{\varphi})(v)\big)\circ\varphi^{-1} \qquad \textit{for every } g \in \mathrm{LIP}_c(\mathbb{R}^k), \\ \underline{\mathrm{d}}\varphi(fv) &= f\circ\varphi^{-1}\,\underline{\mathrm{d}}\varphi(v) \qquad \textit{for every } f \in L^\infty(\mathfrak{m}),\end{aligned} \tag{2.13}$$

where $\overline{\varphi} : \mathrm{X} \to \mathbb{R}^k$ is any Lipschitz extension of φ. Moreover, we have that

$$\mathsf{L}^{-1}\,|v|\circ\varphi^{-1} \leq \left|\underline{\mathrm{d}}\varphi(v)\right| \leq \mathsf{L}\,|v|\circ\varphi^{-1} \qquad \textit{holds } \mathcal{L}^k\textit{-a.e. in } \varphi(E), \tag{2.14}$$

for every vector field $v \in L^2(T\mathrm{X})_{|E}$.

Proof. Fix any Lipschitz extension $\overline{\varphi} : \mathrm{X} \to \mathbb{R}^k$ of φ. We divide the proof into several steps:

Step 1. We claim that it is enough to prove the statement for $\mathfrak{m}$ finite.

Indeed, suppose the statement holds for finite measures and consider any (not necessarily finite) reference measure $\mathfrak{m}$ on X. There is a sequence $(K_n)_n$ of disjoint compact subsets of E with $\mathfrak{m}\big(E \setminus \bigcup_n K_n\big) = 0$, by inner regularity of $\mathfrak{m}$. Given that $\mathfrak{m}$ is also outer regular, we can find a sequence $(\Omega_n)_n$ of open subsets of X such that $K_n \subseteq \Omega_n$ and $\mathfrak{m}(\Omega_n) < +\infty$ for every $n \in \mathbb{N}$. Fix any $n \in \mathbb{N}$ and call $\mathfrak{m}_n := \mathfrak{m}_{|\Omega_n}$. Hence we can apply the theorem to the map $\varphi_{|K_n}$, thus obtaining a linear and continuous operator $T_n : L^2_{\mathfrak{m}_n}(T\mathrm{X})_{|K_n} \to L^2\big(\varphi(K_n), \mathbb{R}^k\big)$ such that the following conditions are satisfied $\mathcal{L}^k$-a.e. in $\varphi(K_n)$ for any $v \in L^2_{\mathfrak{m}_n}(T\mathrm{X})_{|K_n}$:

$$
\begin{aligned}
&\mathrm{d}g\big(T_n(v)\big) = \big(\mathrm{d}(g \circ \overline{\varphi})(v)\big) \circ \big(\varphi_{|K_n}\big)^{-1} \quad \text{for every } g \in \mathrm{LIP}_c(\mathbb{R}^k),\\
&T_n(fv) = f \circ \big(\varphi_{|K_n}\big)^{-1} T_n(v) \quad \text{for every } f \in L^\infty(\mathfrak{m}_n),\\
&\mathsf{L}^{-1}\,|v| \circ \big(\varphi_{|K_n}\big)^{-1} \le \big|T_n(v)\big| \le \mathsf{L}\,|v| \circ \big(\varphi_{|K_n}\big)^{-1}.
\end{aligned}
\tag{2.15}
$$

Denote by $\iota_n : L^2_{\mathfrak{m}_n}(T\mathrm{X}) \to L^2_{\mathfrak{m}}(T\mathrm{X})_{|\Omega_n}$ the isomorphism built in Remark 1.18. Therefore we can 'glue' together the functions T_n obtained above (by the third line in (2.15)), in the sense that there exists a unique map $\underline{\mathrm{d}}\varphi : L^2_{\mathfrak{m}}(T\mathrm{X})_{|E} \to L^2\big(\varphi(E), \mathbb{R}^k\big)$ such that

$$
\begin{aligned}
&\chi_{\varphi(K_n)}\, \underline{\mathrm{d}}\varphi(v) = T_n\big(\iota_n^{-1}(\chi_{\Omega_n} v)\big) \text{ holds } \mathcal{L}^k\text{-a.e. in } \varphi(K_n),\\
&\text{for every } v \in L^2_{\mathfrak{m}}(T\mathrm{X})_{|E} \text{ and } n \in \mathbb{N}.
\end{aligned}
$$

We then deduce from (2.15) that $\underline{\mathrm{d}}\varphi$ is a linear and continuous operator satisfying both (2.13) and (2.14), as required.

Step 2. From now on, let us suppose that $\mathfrak{m}$ is a finite measure. Define $\mu := \overline{\varphi}_*\mathfrak{m}$, so that μ is a finite Borel measure on $\mathbb{R}^k$. In particular, we have that $\mathrm{LIP}_c(\mathbb{R}^k) \subseteq W^{1,2}_\mu(\mathbb{R}^k)$. The tangent module $L^2_\mu(T\mathbb{R}^k)$ turns out to be isometrically embedded into the space $L^2(\mathbb{R}^k, \mathbb{R}^k; \mu)$ of all the $L^2(\mu)$-vector fields from $\mathbb{R}^k$ to itself, as proved in [14, Proposition 2.10], thus $L^2_\mu(T\mathbb{R}^k)$ is separable. Since $\overline{\varphi}$ is of bounded deformation when viewed as a function from $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ to $\big(\mathbb{R}^k, |\cdot|, \mu\big)$, we can then consider its differential $\mathrm{d}\overline{\varphi} : L^2(T\mathrm{X}) \to \overline{\varphi}^* L^2_\mu(T\mathbb{R}^k)$. Now fix a vector field $v \in L^2(T\mathrm{X})_{|E}$. The family of all finite sums $\sum_{i=1}^n \chi_{A_i}\, \mathrm{d}g_i$, where $(A_i)_{i=1}^n$ is a Borel partition of $\varphi(E)$ and $(g_i)_{i=1}^n \subseteq \mathrm{LIP}_c(\mathbb{R}^k)$, is a dense vector subspace of $L^2\big(\varphi(E), (\mathbb{R}^k)^*\big)$. Given any such simple 1-form $\omega = \sum_{i=1}^n \chi_{A_i}\, \mathrm{d}g_i \in L^2\big(\varphi(E), (\mathbb{R}^k)^*\big)$, let us define

$$
T_v(\omega) := \sum_{i=1}^n \chi_{A_i}\, [\overline{\varphi}^* \mathrm{d}_\mu g_i]\big(\mathrm{d}\overline{\varphi}(v)\big) \circ \varphi^{-1} \in L^1\big(\varphi(E)\big).
\tag{2.16}
$$

The operator T_v is well-defined, as granted by the following $\mathcal{L}^k|_{\varphi(E)}$-a.e. inequalities:

$$
\begin{aligned}
\big|T_v(\omega)\big| &= \sum_{i=1}^{n} \chi_{A_i} \Big|[\varphi^* \mathrm{d}_\mu g_i]\big(\mathrm{d}\overline{\varphi}(v)\big)\Big| \circ \varphi^{-1} \\
&\leq \big|\mathrm{d}\overline{\varphi}(v)\big| \circ \varphi^{-1} \sum_{i=1}^{n} \chi_{A_i} \, |\mathrm{d}_\mu g_i| \circ \varphi \circ \varphi^{-1} \\
&\leq \big|\mathrm{d}\overline{\varphi}(v)\big| \circ \varphi^{-1} \sum_{i=1}^{n} \chi_{A_i} \, \mathrm{lip}(g_i) = \big|\mathrm{d}\overline{\varphi}(v)\big| \circ \varphi^{-1} \, |\omega|.
\end{aligned}
\tag{2.17}
$$

Another consequence of property (2.17) is that the operator T_v can be uniquely extended to a vector field $\underline{\mathrm{d}}\varphi(v) \in L^2\big(\varphi(E), \mathbb{R}^k\big)$, for which $\big|\underline{\mathrm{d}}\varphi(v)\big| \leq \big|\mathrm{d}\overline{\varphi}(v)\big| \circ \varphi^{-1}$ holds $\mathcal{L}^k$-a.e. in $\varphi(E)$. Furthermore, it can be readily verified that $\underline{\mathrm{d}}\varphi$ is the unique operator satisfying (2.13).

Step 3. In order to conclude the proof, it only remains to show (2.14). Then let $v \in L^2(T\mathrm{X})|_E$ be fixed. It directly follows from Corollary 2.4 that $\big|\underline{\mathrm{d}}\varphi(v)\big| \leq \mathsf{L}\, |v| \circ \varphi^{-1}$ holds $\mathcal{L}^k$-a.e. in $\varphi(E)$. To prove the other inequality in (2.14), we need a more refined argument: fix any $\varepsilon > 0$. Given that $|v| = \operatorname{ess\,sup} \omega(v)$, where the essential supremum is taken among all the $\omega \in L^2(T^*\mathrm{X})$ with $|\omega| \leq 1$ $\mathfrak{m}_\mathrm{X}$-a.e., there exists $\omega \in L^2(T^*\mathrm{X})|_E$ such that $|\omega| = 1$ and $\omega(v) \geq (1-\varepsilon)\,|v|$ are verified $\mathfrak{m}_\mathrm{X}$-a.e. in E. Since the simple forms $\sum_i \chi_{A_i} \, \mathrm{d}f_i \in L^2(T^*\mathrm{X})$ are dense in $L^2(T^*\mathrm{X})$, we can apply Egorov theorem to obtain a partition $(K^n)_{n\in\mathbb{N}}$ of E (up to $\mathfrak{m}_\mathrm{X}$-negligible sets) into compact sets and a sequence $(f^n)_n \subseteq W^{1,2}(\mathrm{X})$ such that $|\mathrm{d}f^n| < 1$ and $\mathrm{d}f^n(v) \geq (1-\varepsilon)^2\,|v|$ hold $\mathfrak{m}_\mathrm{X}$-a.e. in K^n for every $n \in \mathbb{N}$. By using the assumption about reflexivity of $W^{1,2}(\mathrm{X})$, applying Theorem 1.11 and Egorov theorem, we can find a partition $(K^n_m)_{m\in\mathbb{N}}$ of K^n (up to $\mathfrak{m}_\mathrm{X}$-negligible sets) into compact sets and a sequence of maps $(f^n_m)_m \subseteq \mathrm{LIP}(\mathrm{X}) \cap W^{1,2}(\mathrm{X})$ such that $\mathrm{lip}_a(f^n_m) \leq 1$ and $\mathrm{d}f^n_m(v) \geq (1-\varepsilon)^3\,|v|$ are satisfied $\mathfrak{m}_\mathrm{X}$-a.e. in K^n_m for every $m \in \mathbb{N}$. Denote by ψ^n_m the inverse of $\varphi|_{K^n_m} : K^n_m \to \varphi(K^n_m)$ and pick a compactly supported Lipschitz map $h^n_m \in \mathrm{LIP}_c(\mathbb{R}^k)$ such that $h^n_m|_{\varphi(K^n_m)} = f^n_m \circ \psi^n_m$. Observe that the following statement holds $\mathcal{L}^k$-a.e. in $\varphi(K^n_m)$:

$$
\begin{aligned}
|\mathrm{d}h^n_m| &\overset{(1.43)}{=} \mathrm{lip}(h^n_m) \overset{(1.15)}{=} \mathrm{lip}\big(h^n_m|_{\varphi(K^n_m)}\big) \\
&\overset{(1.14)}{\leq} \mathrm{Lip}(\psi^n_m)\, \mathrm{lip}(f^n_m) \circ \psi^n_m \leq \mathsf{L}.
\end{aligned}
\tag{2.18}
$$

Moreover, the fact that $f^n_m|_{K^n_m} = h^n_m \circ \overline{\varphi}|_{K^n_m}$ yields $\chi_{K^n_m}\, \mathrm{d}f^n_m = \chi_{K^n_m}\, \mathrm{d}(h^n_m \circ \overline{\varphi})$, so that

$$\begin{aligned}
&\Big|\big(\chi_{\varphi(K^n_m)}\,\mathrm{d}h^n_m\big)\big(\underline{\mathrm{d}}\varphi(v)\big)\Big| \\
&\qquad = \chi_{\varphi(K^n_m)}\,\Big|[\overline{\varphi}^*\mathrm{d}_\mu h^n_m]\big(\underline{\mathrm{d}}\varphi(v)\big)\Big| \circ \varphi^{-1} \\
&\qquad \geq \chi_{\varphi(K^n_m)}\,\big(\overline{\varphi}^*\mathrm{d}_\mu h^n_m(v)\big) \circ \varphi^{-1} \\
&\qquad = \chi_{\varphi(K^n_m)}\,\big(\mathrm{d}(h^n_m \circ \overline{\varphi})(v)\big) \circ \varphi^{-1} \\
&\qquad = \chi_{\varphi(K^n_m)}\,\big(\mathrm{d}f^n_m(v)\big) \circ \varphi^{-1} \\
&\qquad \geq (1-\varepsilon)^3\,\chi_{\varphi(K^n_m)}\,|v| \circ \varphi^{-1} \qquad \text{holds } \mathcal{L}^k\text{-a.e. in } \varphi(K^n_m).
\end{aligned}$$

In particular, (2.18) grants that $\big|\underline{\mathrm{d}}\varphi(v)\big| \geq (1-\varepsilon)^3\,\mathsf{L}^{-1}\,|v| \circ \varphi^{-1}$ is satisfied $\mathscr{L}^k$-a.e. in $\varphi(K^n_m)$ for any $n, m \in \mathbb{N}$, hence also $\mathscr{L}^k$-a.e. in all of $\varphi(E)$. By letting $\varepsilon \searrow 0$, we finally obtain that the inequality $\big|\underline{\mathrm{d}}\varphi(v)\big| \geq \mathsf{L}^{-1}\,|v| \circ \varphi^{-1}$ holds $\mathscr{L}^k$-a.e. in $\varphi(E)$, concluding the proof of (2.14). Therefore the statement is achieved. □

3. Strongly $\mathfrak{m}$-rectifiable spaces

We introduce a new class of metric measure spaces, called strongly $\mathfrak{m}$-rectifiable spaces. Roughly speaking, these spaces can be partitioned (up to negligible sets) into countably many Borel sets, which are biLipschitz equivalent to suitable subsets of the Euclidean space, by means of maps that also keep under control the measure.

For the sake of simplicity, it is convenient to use the following notation: given a measured space $(S, \mathcal{M}, \mu)$, we say that $(E_i)_{i\in\mathbb{N}} \subseteq \mathcal{M}$ is a *μ-partition* of $E \in \mathcal{M}$ provided it is a partition of some $F \in \mathcal{M}$ such that $F \subseteq E$ and $\mu(E \setminus F) = 0$. Moreover, given two μ-partitions $(E_i)_i$ and $(F_j)_j$ of E, we say that $(F_j)_j$ is a *refinement* of $(E_i)_i$ if for every $j \in \mathbb{N}$ with $F_j \neq \emptyset$ there exists (a unique) $i \in \mathbb{N}$ such that $F_j \subseteq E_i$.

Definition 3.1 (Strongly $\mathfrak{m}$-rectifiable space). *A metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* $\mathfrak{m}$-rectifiable *provided it is a disjoint union* $\mathrm{X} = \bigcup_{k\in\mathbb{N}} A_k$ *of suitable* $(A_k)_k \subset \mathscr{B}(\mathrm{X})$*, such that the following condition is satisfied: given any* $k \in \mathbb{N}$*, there exists an* $\mathfrak{m}$*-partition* $(U_i)_{i\in\mathbb{N}} \subseteq \mathscr{B}(\mathrm{X})$ *of* A_k *and a sequence* $(\varphi_i)_{i\in\mathbb{N}}$ *of maps* $\varphi_i : U_i \to \mathbb{R}^k$ *such that*

(3.1) $\varphi_i : U_i \to \varphi_i(U_i)$ *is biLipschitz and* $(\varphi_i)_*(\mathfrak{m}|_{U_i}) \ll \mathcal{L}^k$ *for every* $i \in \mathbb{N}$.

The partition $\mathrm{X} = \bigcup_{k\in\mathbb{N}} A_k$ *- which is clearly unique up to modification of negligible sets - is called* dimensional decomposition *of* X.

The space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is said to be* strongly $\mathfrak{m}$-rectifiable *provided it satisfies the following property: given any* $\varepsilon > 0$, *there exists a family of couples* (U_i,φ_i) *as above so that all the maps* φ_i *are* $(1+\varepsilon)$*-biLipschitz.*

Remark 3.2. Given an $\mathfrak{m}$-rectifiable space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ with dimensional decomposition $(A_k)_k$, we have that each set A_k is countably $\mathcal{H}^k$-rectifiable. Moreover, observe that each measure $\mathcal{H}^k|_{A_k}$ is σ-finite, therefore it follows from (1.17), (3.1) and the Radon-Nikodým theorem (cf. also the discussion preceding Theorem 5.4 in [5]) that there exists a sequence $(N_k)_k$ of Borel sets $N_k \subseteq A_k$ with $\mathfrak{m}(N_k) = 0$ such that

$$\mathfrak{m}|_{A_k\setminus N_k} = \theta_k\,\mathcal{H}^k|_{A_k\setminus N_k} \quad \text{for every } k\in\mathbb{N}, \tag{3.2}$$

where the density θ_k is a suitable Borel map $\theta_k : A_k \setminus N_k \to (0,+\infty)$. ■

When working on $\mathfrak{m}$-rectifiable spaces, it is natural to adopt the following terminology, which is inspired by the language of differential geometry:

Definition 3.3 (Charts and atlases). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathfrak{m}$*-rectifiable metric measure space. A* chart *on* X *is a couple* (U,φ), *where* $U \in \mathscr{B}(A_k)$ *for some* $k\in\mathbb{N}$ *and* $\varphi : U \to \mathbb{R}^k$ *satisfies*

$$\begin{aligned} &\varphi : U \to \varphi(U) \quad \text{is biLipschitz,}\\ &C^{-1}\,\mathcal{L}^k|_{\varphi(U)} \leq \varphi_*\big(\mathfrak{m}|_U\big) \leq C\,\mathcal{L}^k|_{\varphi(U)}, \end{aligned} \tag{3.3}$$

for a suitable constant $C \geq 1$. *An* atlas *on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a family* $\mathscr{A} = \bigcup_{k\in\mathbb{N}} \big\{(U_i^k,\varphi_i^k)\big\}_{i\in\mathbb{N}}$ *of charts on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *such that* $(U_i^k)_{i\in\mathbb{N}}$ *is an* $\mathfrak{m}$*-partition of* A_k *for every* $k\in\mathbb{N}$.

The chart (U,φ) *is said to be an* ε-chart *provided* $\varphi : U \to \varphi(U)$ *is* $(1+\varepsilon)$*-biLipschitz and an atlas is said to be an* ε-atlas *provided all of its charts are* ε*-charts.*

We collect few simple facts about atlases which we shall frequently use in what follows:

i) Any $\mathfrak{m}$-rectifiable space admits an atlas and any strongly $\mathfrak{m}$-rectifiable space admits an ε-atlas for every $\varepsilon > 0$. Indeed, for (U_i,φ_i) as in (3.1) we can consider the density ρ_i of $\varphi_*\big(\mathfrak{m}|_{U_i}\big)$ w.r.t. the Lebesgue measure and the sets $U_{i,j} := \varphi_i^{-1}\big(\{2^j \leq \rho_i < 2^{j+1}\}\big)$, $j\in\mathbb{Z}$. It is clear that

$\big(U_{i,j},\varphi_i|_{U_{i,j}}\big)$ is a chart for every j and that the $U_{i,j}$'s provide an $\mathfrak{m}$-partition of U_i, so that repeating the construction for every i yields the desired atlas.

ii) Let $(U_i,\varphi_i)_{i\in\mathbb{N}}$ be an atlas and, for every i, let $(U_{i,j})_{j\in\mathbb{N}}$ an $\mathfrak{m}$-partition of U_i. Then $\big(U_{i,j},\varphi_i|_{U_{i,j}}\big)_{i,j\in\mathbb{N}}$ is also an atlas. In particular, by inner regularity of $\mathfrak{m}$, every $\mathfrak{m}$-rectifiable space admits an atlas whose charts are defined on compact sets.

Remark 3.4. Using the finite dimensionality results obtained by Cheeger in [7] it is not hard to see that the dimensional decomposition $(A_k)_k$ of a PI space (i.e. a doubling metric measure space supporting a weak $(1,2)$-Poincaré inequality) which is also $\mathfrak{m}$-rectifiable must be so that $\mathfrak{m}(A_k)=0$ for all k sufficiently large. Yet, our discussion is independent on this specific result and thus we won't insist on this point. ■

Proposition 3.5. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathfrak{m}$*-rectifiable space. Then* X *is a Vitali space. In particular, given any Borel subset* E *of* X *it holds that* $\mathfrak{m}$*-a.e.* $x\in E$ *is of density* 1 *for* E*.*

Proof. By recalling (1.4), it is sufficient to prove that $\mathfrak{m}$ is pointwise doubling at $\mathfrak{m}$-almost every point of X. To this aim, call $(A_k)_k$ the dimensional decomposition of X and fix $k\in\mathbb{N}$. Let $(N_k)_k$ be as in Remark 3.2 and call $A'_k:=A_k\setminus N_k$ for all $k\in\mathbb{N}$. We claim that

$$\lim_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)}{\omega_k\,2^k\,r^k}=0\quad\text{holds for }\mathcal{H}^k\text{-a.e. }x\in A'_k. \tag{3.4}$$

We argue by contradiction: if not, there exist a Borel set $P\subseteq A'_k$ with $\mathcal{H}^k(P)>0$ and a constant $\lambda>0$ such that

$$\varlimsup_{r\to 0}\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)/(\omega_k\,2^k\,r^k)\geq\lambda$$

holds for any point $x\in P$. Hence (1.18) with $\mu:=\mathfrak{m}|_{\mathrm{X}\setminus A'_k}$ yields $\lambda\,\mathcal{H}^k(P)\leq\mathfrak{m}(P\setminus A'_k)=0$, which leads to a contradiction.

Therefore (1.19) and (3.4) grant that for $\mathcal{H}^k$-a.e. (thus also $\mathfrak{m}$-a.e.) point $x\in A'_k$ it holds

$$\varlimsup_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}\leq\lim_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\cap A'_k\big)}{\mathfrak{m}\big(B_r(x)\cap A'_k\big)}+\lim_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)}{\mathfrak{m}\big(B_r(x)\cap A'_k\big)}=2^k,$$

getting the statement. □

When we restrict our attention to the smaller class of strongly $\mathfrak{m}$-rectifiable spaces, we have a geometric characterization of the tangent module. Section 5 will be entirely devoted to describe such result. In order to further develop our theory in that direction, we need to provide any strongly $\mathfrak{m}$-rectifiable space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ with a special sequence of atlases, which are aligned in a suitable sense.

Definition 3.6 (Aligned family of atlases). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable space. Let* $\varepsilon_n \downarrow 0$ *and* $\delta_n \downarrow 0$*. Let* $(\mathscr{A}_n)_{n\in\mathbb{N}}$ *be a sequence of atlases on* X*. Then we say that* $(\mathscr{A}_n)_n$ *is an* aligned family of atlases *of parameters* ε_n *and* δ_n *provided the following conditions are satisfied:*

(i) Each $\mathscr{A}_n = \big\{(U_i^{k,n},\varphi_i^{k,n})\big\}_{k,i}$ *is an* ε_n*-atlas and the domains* $U_i^{k,n}$ *are compact.*

(ii) The family $(U_i^{k,n})_{k,i}$ *is a refinement of* $(U_j^{k,n-1})_{k,j}$ *for any* $n\in\mathbb{N}^+$.

(iii) If $n\in\mathbb{N}^+$, $k\in\mathbb{N}$ *and* $i,j\in\mathbb{N}$ *satisfy* $U_i^{k,n}\subseteq U_j^{k,n-1}$*, then*

$$\left\| \mathrm{d}\Big(\mathrm{id}_{\mathbb{R}^k} - \varphi_j^{k,n-1}\circ\big(\varphi_i^{k,n}\big)^{-1}\Big)(y)\right\| \leq \delta_n \quad \textit{for } \mathcal{L}^k\textit{-a.e. } y\in\varphi_i^{k,n}(U_i^{k,n}). \tag{3.5}$$

The discussions made before grant that any strongly $\mathfrak{m}$-rectifiable space admits atlases satisfying $(i),(ii)$ above. In fact, as we shall see in a moment, also (iii) can be fulfilled by an appropriate choice of atlases, but in order to show this we need a small digression.

Recall that $O(\mathbb{R}^k)$ denotes the group of linear isometries of $\mathbb{R}^k$ and for $\varepsilon>0$ let us introduce

$$O^\varepsilon(\mathbb{R}^k) := \Big\{T:\mathbb{R}^k\to\mathbb{R}^k \text{ linear, invertible} \\ \text{and such that } \|T\|,\|T^{-1}\|\leq 1+\varepsilon\Big\}.$$

Notice that $O^\varepsilon(\mathbb{R}^k)$ - being closed and bounded - is compact for every $\varepsilon>0$ and that $O(\mathbb{R}^k)=\bigcap_{\varepsilon>0}O^\varepsilon(\mathbb{R}^k)$. Then we have the following simple result:

Proposition 3.7. *Let* $k\in\mathbb{N}$ *and* $\delta>0$*. Then there exist* $\varepsilon>0$ *and a Borel function* $R:O^\varepsilon(\mathbb{R}^k)\to O(\mathbb{R}^k)$ *whose image has finite cardinality such that*

$$\big\|T-R(T)\big\|\leq\delta \quad \textit{for every } T\in O^\varepsilon(\mathbb{R}^k).$$

Proof. From the compactness of $O(\mathbb{R}^k)$ we know that there are $T_1,\dots,T_n\in O(\mathbb{R}^k)$ such that $O(\mathbb{R}^k)\subset U_\delta:=\bigcup_i B_\delta(T_i)$. We claim that there exists $\varepsilon>0$

such that $O^\varepsilon(\mathbb{R}^k) \subset U_\delta$ and argue by contradiction. If not, the compact set $K^\varepsilon := O^\varepsilon(\mathbb{R}^k) \setminus U_\delta$ would be not empty for every $\varepsilon > 0$. Since clearly $K^\varepsilon \subset K^{\varepsilon'}$ for $\varepsilon \le \varepsilon'$, the family K^ε has the finite intersection property, but on the other hand the identity $O(\mathbb{R}^k) = \bigcap_{\varepsilon>0} O^\varepsilon(\mathbb{R}^k)$ yields $\bigcap_{\varepsilon>0} K^\varepsilon = \emptyset$, which is a contradiction. Thus there exists $\varepsilon > 0$ such that $O^\varepsilon(\mathbb{R}^k) \subset U_\delta$. For such ε we define $R : O^\varepsilon(\mathbb{R}^k) \to O(\mathbb{R}^k)$ to be equal to T_1 on $B_\delta(T_1)$ and then recursively to be equal to T_n on $B_\delta(T_n) \setminus \bigcup_{i<n} B_\delta(T_i)$. □

Using Proposition 3.7 it is possible to show that any strongly $\mathfrak{m}$-rectifiable space admits an aligned family of atlases:

Theorem 3.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable metric measure space. Let* $\varepsilon_n \downarrow 0$ *and* $\delta_n \downarrow 0$ *be two given sequences. Then* X *admits an aligned family* $(\mathscr{A}_n)_n$ *of atlases of parameters* ε_n *and* δ_n.

Proof. Let $(A_k)_k$ be the dimensional decomposition of X and notice that to conclude it is sufficient to build, for every $k \in \mathbb{N}$, aligned charts as in (*iii*) of Definition 3.6 covering $\mathfrak{m}$-almost all A_k. For $k, n \in \mathbb{N}$, let $\varepsilon'_{n,k}$ be associated to δ_n and k as in Proposition 3.7 and choose $\bar\varepsilon_{n,k} > 0$ such that

$$\bar\varepsilon_{n,k} \le \varepsilon_n \quad \text{and} \quad (1 + \bar\varepsilon_{n-1,k})(1 + \bar\varepsilon_{n,k}) \le 1 + \varepsilon'_{n,k} \quad \text{for every } k, n \in \mathbb{N}. \tag{3.6}$$

We now construct the required aligned family $(\mathscr{A}_n)_n$ of atlases by recursion: start observing that since $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is strongly $\mathfrak{m}$-rectifiable, there exists an atlas $\mathscr{A}_0$ such that the charts with domain included in A_k are $\bar\varepsilon_{0,k}$-biLipschitz. Now assume that for some $n \in \mathbb{N}$ we have already defined $\mathscr{A}_0, \ldots, \mathscr{A}_{n-1}$ satisfying the alignment conditions and say that $\mathscr{A}_{n-1} = \big\{(U_i^k, \varphi_i^k)\big\}_{k,i}$. Again using the strong $\mathfrak{m}$-rectifiability of X, find an atlas $\big\{(V_j^k, \psi_j^k)\big\}_{k,j}$ whose domains $(V_j^k)_{k,j}$ constitute a refinement of the domains $(U_i^k)_{k,i}$ of $\mathscr{A}_{n-1}$ and such that those charts with domain included in A_k are $\bar\varepsilon_{n,k}$-biLipschitz.

Fix $k, j \in \mathbb{N}$ and let $i \in \mathbb{N}$ be the unique index such that $V_j^k \subseteq U_i^k$. For the sake of brevity, let us denote by τ the transition map $\varphi_i^k \circ (\psi_j^k)^{-1} : \psi_j^k(V_j^k) \to \varphi_i^k(V_j^k)$ and observe that it is $(1 + \varepsilon'_{n,k})$-biLipschitz by (3.6). Hence its differential $\mathrm{d}\tau$ satisfies $\big\|\mathrm{d}\tau(y)\big\|, \big\|\mathrm{d}\tau(y)^{-1}\big\| \le 1 + \varepsilon'_{n,k}$, or equivalently $\mathrm{d}\tau(y) \in O^{\varepsilon'_{n,k}}(\mathbb{R}^k)$, for $\mathcal{L}^k$-a.e. $y \in \psi_j^k(V_j^k)$.

Let $R : O^{\varepsilon'_{n,k}}(\mathbb{R}^k) \to O(\mathbb{R}^k)$ be given by Proposition 3.7 with $\delta := \delta_n$ and denote by $F_j^k \subset O(\mathbb{R}^k)$ its finite image. For $T \in F_j^k$ let $P_T := (R \circ \mathrm{d}\tau)^{-1}(T) \subset \mathbb{R}^k$, so that $(P_T)_{T \in F_j^k}$ is a $\mathcal{L}^k$-partition of $\psi_j^k(V_j^k)$. For $\mathcal{L}^k$-a.e.

$y \in T(P_T) \subset \mathbb{R}^k$ we have

$$
\begin{aligned}
\Big\|\mathrm{d}\big(\varphi_i^k \circ \big(T \circ \psi_j^k\big)^{-1} - \mathrm{id}_{\mathbb{R}^k}\big)(y)\Big\| &= \Big\|\mathrm{d}\big(\tau \circ T^{-1} - \mathrm{id}_{\mathbb{R}^k}\big)(y)\Big\| \\
&= \Big\|\mathrm{d}\big((\tau - T) \circ T^{-1}\big)(y)\Big\| \\
&\le \big\|\mathrm{d}\tau(T^{-1}(y)) - T\big\| \, \big\|T^{-1}\big\| \qquad (3.7)\\
&= \big\|\mathrm{d}\tau(T^{-1}(y)) - T\big\| \\
\text{(because } T^{-1}(y) \in P_T) \quad &= \big\|\mathrm{d}\tau(T^{-1}(y)) - R\big(\mathrm{d}\tau(T^{-1}(y))\big)\big\| \\
\text{(by definition of } R) \quad &\le \delta_n.
\end{aligned}
$$

We therefore define

$$
\bar U_{j,T}^k := (\psi_j^k)^{-1}(P_T) \quad \text{and} \quad \bar\varphi_{j,T}^k := T \circ \psi_j^k|_{\bar U_{j,T}^k} \quad \text{for every } T \in F_j^k, \tag{3.8}
$$

so that accordingly

$$
\mathscr{A}_n := \Big\{\big(\bar U_{j,T}^k, \bar\varphi_{j,T}^k\big) \,:\, k, j \in \mathbb{N},\, T \in F_j^k\Big\} \tag{3.9}
$$

is an atlas on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, which fulfills (ii), (iii) of Definition 3.6 and such that the charts with domain included in A_k are $\bar\varepsilon_{n,k}$-biLipschitz.

Up to a further refining we can assume that the charts in $\mathscr{A}_n$ have compact domains and, since $\bar\varepsilon_{n,k} \le \varepsilon_n$ for every $k, n \in \mathbb{N}$, the statement is proved. □

4. Gromov-Hausdorff tangent module

4.1. Measurable banach bundle

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a fixed metric measure space. We propose a notion of measurable Banach bundle:

Definition 4.1 (Measurable Banach bundle). *The quadruplet* $\mathbb{T} := \big(T, \mathcal{M}, \pi, \mathsf{n}\big)$ *is said to be a* measurable Banach bundle *over* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *provided:*

i) $\mathcal{M}$ *is a* σ*-algebra over the set* T*.*

ii) π *is a measurable map from* $(T, \mathcal{M})$ *to* $\big(\mathrm{X}, \mathscr{B}(\mathrm{X})\big)$ *which we shall call* projection *and*

$$
T_x := \pi^{-1}\big(\{x\}\big) \quad \textit{is an } \mathbb{R}\textit{-vector space for } \mathfrak{m}\textit{-a.e. } x \in \mathrm{X}. \tag{4.1}
$$

iii) $\mathsf{n} : T \to [0,+\infty)$ *is a measurable map which we shall call* norm *such that for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *it holds:*

$$
\begin{aligned}
&\mathsf{n}|_{T_x} \text{ is a norm on } T_x,\\
&\big(T_x, \mathsf{n}|_{T_x}\big) \text{ is a Banach space},\\
&\mathscr{B}(T_x) = \mathcal{M}|_{T_x} := \big\{E \cap T_x \,:\, E \in \mathcal{M}\big\}.
\end{aligned}
\tag{4.2}
$$

iv) *The* measurable sections *of* $\mathbb{T}$*, i.e. those measurable maps* $\mathsf{v} : \mathrm{X} \to T$ *for which the identity* $\pi \circ \mathsf{v} = \mathrm{id}_{\mathrm{X}}$ *holds* $\mathfrak{m}$*-a.e. in* X*, satisfy the following properties:*
   a) *The* null section $\mathrm{X} \ni x \mapsto 0 \in T_x$ *is a measurable section of* $\mathbb{T}$*.*
   b) *Let* v, w *be measurable sections of* $\mathbb{T}$ *and let* $\alpha, \beta \in \mathbb{R}$*. Then the pointwise linear combination* $\alpha\,\mathsf{v} + \beta\,\mathsf{w} : \mathrm{X} \to T$*, given by* $(\alpha\,\mathsf{v} + \beta\,\mathsf{w})(x) := \alpha\,\mathsf{v}(x) + \beta\,\mathsf{w}(x) \in T_x$ *for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$*, is a measurable section of* $\mathbb{T}$ *as well.*
   c) *Let* $(\mathsf{v}_n)_n$ *be a sequence of measurable sections of* $\mathbb{T}$*. Suppose that the limit*

$$\mathsf{v}(x) := \lim_n \mathsf{v}_n(x) \in T_x$$

exists for $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$*. Then* $\mathsf{v} : \mathrm{X} \to T$ *is a measurable section of* $\mathbb{T}$ *as well.*

Remark 4.2. Intuitively speaking, item iv) is what describes 'how the various fibres are glued to each other' and as such is not a consequence of i), ii) and iii).

Concerning the relation among iv-a) , iv-b) , iv-c) we notice that iv-a) follows from iv-b) by taking $\alpha = \beta = 0$ provided the collection of measurable sections is not empty and that it is not hard to build examples satisfying iv-a) and iv-c) but not iv-b).

We believe that it is also possible to build examples for which iv-a) and iv-b) hold while iv-c) does not, but constructing such example is outside the scope of this paper. ■

Given two measurable Banach bundles $\mathbb{T}_i = \big(T_i, \mathcal{M}_i, \pi_i, \mathsf{n}_i\big)$, $i = 1, 2$, a *bundle morphism* is a measurable map $\varphi : T_1 \to T_2$ such that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ it holds

$$
\begin{aligned}
&\varphi \text{ maps } (T_1)_x \text{ into } (T_2)_x,\\
&\varphi|_{(T_1)_x} \text{ is linear and 1-Lipschitz from } \big((T_1)_x, \mathsf{n}_1|_{(T_1)_x}\big) \text{ to } \big((T_2)_x, \mathsf{n}_2|_{(T_2)_x}\big).
\end{aligned}
\tag{4.3}
$$

Two bundle morphisms $\varphi, \psi : T_1 \to T_2$ are declared to be equivalent provided

$$\varphi|_{(T_1)_x} = \psi|_{(T_1)_x} \text{ for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X} \tag{4.4}$$

and accordingly two measurable Banach bundles $\mathbb{T}_i = \big(T_i, \mathcal{M}_i, \pi_i, \mathbf{n}_i\big)$, $i = 1,2$ are declared to be isomorphic provided there are bundle morphisms $\varphi : T_1 \to T_2$ and $\psi : T_2 \to T_1$ such that $\varphi \circ \psi \sim \mathrm{id}_{T_2}$ and $\psi \circ \varphi \sim \mathrm{id}_{T_1}$, which is the same as to say that

$$\begin{aligned} &\psi \circ \varphi|_{(T_1)_x} = \mathrm{id}_{(T_1)_x} \text{ and } \varphi \circ \psi|_{(T_2)_x} = \mathrm{id}_{(T_2)_x} \text{ for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}, \\ &\varphi|_{(T_1)_x} : (T_1)_x \to (T_2)_x \text{ is an isometric isomorphism for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \end{aligned}$$

Let $\mathbb{T} = \big(T, \mathcal{M}, \pi, \mathbf{n}\big)$ be a measurable Banach bundle over $\mathbb{X}$. We denote by $[\mathsf{v}]$ the equivalence class of any measurable section v of $\mathbb{T}$ with respect to $\mathfrak{m}$-a.e. equality. We define $L^2(\mathbb{T})$ as

$$L^2(\mathbb{T}) := \left\{ [\mathsf{v}] \;\middle|\; \mathsf{v} \text{ is a section of } \mathbb{T} \text{ with } \int_{\mathrm{X}} \mathbf{n}\big(\mathsf{v}(x)\big)^2 \,\mathrm{d}\mathfrak{m}(x) < +\infty \right\}. \tag{4.5}$$

With a (common) slight abuse of notation, the elements of $L^2(\mathbb{T})$ will be tipically denoted by v instead of $[\mathsf{v}]$. Notice that $L^2(\mathbb{T})$ has a canonical structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module on X: for $\mathsf{v} \in L^2(\mathbb{T})$ and $h \in L^\infty(\mathfrak{m})$ define

$$\begin{aligned} (h\mathsf{v})(x) &:= h(x)\,\mathsf{v}(x) \in T_x, \\ |\mathsf{v}|(x) &:= \mathbf{n}\big(\mathsf{v}(x)\big), \end{aligned} \tag{4.6}$$

for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. The fact that $h\mathsf{v} \in L^2(\mathbb{T})$ follows from these observations: given any $A \subseteq \mathrm{X}$ Borel, it holds that $\chi_A \mathsf{v}$ is a measurable section of $\mathbb{T}$, as item a) of Definition 4.1 gives that

$$\begin{aligned} &(\chi_A \mathsf{v})^{-1}(B) = \big(A \cap \mathsf{v}^{-1}(B)\big) \cup \{x \in \mathrm{X} \setminus A \,:\, 0_{T_x} \in B\} \\ &\text{is a Borel subset of X} \end{aligned}$$

for all $B \in \mathcal{M}$; this fact, together with item b) of Definition 4.1, grant that $h\mathsf{v}$ is a measurable section of $\mathbb{T}$ whenever h is a simple function, whence also for any other $h \in L^\infty(\mathfrak{m})$ by an approximation argument together with item c) of Definition 4.1; finally, we have $h\mathsf{v} \in L^2(\mathbb{T})$ since $\int_{\mathrm{X}} \mathbf{n}\big((h\mathsf{v})(x)\big)^2 \,\mathrm{d}\mathfrak{m}(x) \leq \|h\|^2_{L^\infty(\mathfrak{m})} \int_{\mathrm{X}} \mathbf{n}\big(\mathsf{v}(x)\big)^2 \,\mathrm{d}\mathfrak{m}(x) < +\infty$.

Remark 4.3. The collection of measurable Banach bundles on X and of isomorphism classes of bundle morphisms form a category, which we shall denote by $\mathbf{MBB}(\mathrm{X})$.

Similarly, the collection of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules on X and of 1-Lipschitz module morphisms between them form a category, which we denote by $\mathbf{Mod}_{2-L^\infty}(\mathrm{X})$.

The map which sends each measurable Banach bundle $\mathbb{T}$ to the space of its L^2-sections $L^2(\mathbb{T})$ and each bundle morphism $\varphi : T_1 \to T_2$ to the map $L^2(\mathbb{T}_1) \ni \mathsf{v} \mapsto \varphi \circ \mathsf{v} \in L^2(\mathbb{T}_2)$, is easily seen to be a fully faithful functor, so that $\mathbf{MBB}(\mathrm{X})$ can be thought of as a full subcategory of $\mathbf{Mod}_{2-L^\infty}(\mathrm{X})$. ■

Remark 4.4. Consider measurable Banach bundles where the fibres are Hilbert spaces. This object is closely related to the notion of direct integral of Hilbert spaces (we refer the interested reader to [20] for a detailed account on this topic). The difference between the two concepts is in how measurable sections are specified: here we impose an a priori set of compatibility conditions for them (in item iv) of the definition) while for direct integrals this is instead required by coupling a section with a given set of 'measurable test sections' specified in advance (and typically countable).

In fact, under appropriate separability assumptions the two notions can be seen to fully coincide. A way to see this is by observing if one considers only separable Hilbert bundles (where separability is intended at the level of L^2 sections), then the tensor described in the previous remark provides in fact an equivalence with the category of Hilbert modules: this is a consequence of the characterization given in [11, Theorem 1.4.11]. Then recalling that separable Hilbert modules and direct integral of Hilbert spaces provide different descriptions of the same mathematical object (see [11, Remark 1.4.12]), we obtain the claimed identification of the concepts of Hilbert bundles and direct integral of Hilbert spaces. We omit the details. ■

4.2. Gromov-Hausdorff tangent bundle

Recall that given a measurable space $(S, \mathcal{M})$, a set S' and a function $f : S \to S'$, the *push-forward* $f_*\mathcal{M}$ of $\mathcal{M}$ via f is the σ-algebra on S' defined by

$$f_*\mathcal{M} := \big\{ E \subseteq S' \,:\, f^{-1}(E) \in \mathcal{M} \big\}. \tag{4.7}$$

Notice that $f_*\mathcal{M}$ is the greatest σ-algebra $\mathcal{M}'$ on S' for which the function f is measurable from $(S, \mathcal{M})$ to $(S', \mathcal{M}')$.

With this said, let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a strongly $\mathfrak{m}$-rectifiable metric measure space, (A_k) its dimensional decomposition and define the following objects:

i) The set $T_{\mathrm{GH}}\mathrm{X}$ is defined as

$$T_{\mathrm{GH}}\mathrm{X} := \bigsqcup_{k\in\mathbb{N}} A_k \times \mathbb{R}^k \tag{4.8}$$

and the σ-algebra $\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ is given by

$$\mathcal{M}_{\mathrm{GH}}(\mathrm{X}) := \bigcap_{k\in\mathbb{N}} (\iota_k)_*\mathscr{B}(A_k \times \mathbb{R}^k), \tag{4.9}$$

where $\iota_k : A_k \times \mathbb{R}^k \hookrightarrow T_{\mathrm{GH}}\mathrm{X}$ is the natural inclusion, for every $k \in \mathbb{N}$.
In other words, a subset E of $T_{\mathrm{GH}}\mathrm{X}$ belongs to $\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ if and only if $E \cap (A_k \times \mathbb{R}^k)$ is a Borel subset of $A_k \times \mathbb{R}^k$ for every $k \in \mathbb{N}$.

ii) The projection $\pi : T_{\mathrm{GH}}\mathrm{X} \to \mathrm{X}$ of $T_{\mathrm{GH}}\mathrm{X}$ is given by

$$\pi(x,v) := x \quad \text{for every } (x,v) \in T_{\mathrm{GH}}\mathrm{X}. \tag{4.10}$$

iii) The norm $\mathsf{n} : T_{\mathrm{GH}}\mathrm{X} \to [0,+\infty)$ on $T_{\mathrm{GH}}\mathrm{X}$ is given by

$$\mathsf{n}(x,v) := |v|_{\mathbb{R}^k} \quad \text{for every } k \in \mathbb{N} \text{ and } (x,v) \in A_k \times \mathbb{R}^k \subseteq T_{\mathrm{GH}}\mathrm{X}. \tag{4.11}$$

Definition 4.5 (Gromov-Hausdorff tangent bundle). *The* Gromov-Hausdorff tangent bundle *of* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is the measurable Banach bundle*

$$\big(T_{\mathrm{GH}}\mathrm{X}, \mathcal{M}_{\mathrm{GH}}(\mathrm{X}), \pi, \mathsf{n}\big). \tag{4.12}$$

The space of the L^2-sections of such bundle is called Gromov-Hausdorff tangent module *and is denoted by $L^2(T_{\mathrm{GH}}\mathrm{X})$.*

Since all fibers are assumed to be Euclidean, we could have called $T_{\mathrm{GH}}\mathrm{X}$ the *Hilbertian Gromov-Hausdorff tangent bundle*. We preferred to omit the adjective 'Hilbertian' from our terminology for the sake of simplicity, since this is the only notion of GH bundle we are actually concerned with. Nevertheless, in other metric contexts it would be surely interesting to consider Gromov-Hausdorff tangent bundles whose fibers are not necessarily Euclidean.

The choice of this measurable structure on $T_{\mathrm{GH}}\mathrm{X}$ could seem to be naïve, but we now prove that it is the only one coherent with some (thus any) atlas on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, in the sense which we now describe.

Let us fix an ε-atlas $\mathscr{A}=\big\{(U_i^k,\varphi_i^k)\big\}_{k,i}$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. For every $k,i\in\mathbb{N}$, choose a constant $C_i^k\geq 1$ such that

$$(C_i^k)^{-1}\,\mathcal{L}^k|_{\varphi_i^k(U_i^k)}\leq(\varphi_i^k)_*\big(\mathfrak{m}|_{U_i^k}\big)\leq C_i^k\,\mathcal{L}^k|_{\varphi_i^k(U_i^k)}. \tag{4.13}$$

Fix a sequence of radii $r_j\downarrow 0$ and define $\widehat{\varphi}_{ij}^k:U_i^k\times U_i^k\to A_k\times\mathbb{R}^k$ as

$$\widehat{\varphi}_{ij}^k(\bar{x},x):=\left(\bar{x}\,,\,\frac{\varphi_i^k(x)-\varphi_i^k(\bar{x})}{r_j}\right)\qquad\text{for every }(\bar{x},x)\in U_i^k\times U_i^k. \tag{4.14}$$

For the sake of brevity, for $k,i,j\in\mathbb{N}$ let us call

$$\begin{aligned}W_{ij}^k&:=\widehat{\varphi}_{ij}^k(U_i^k\times U_i^k),\\ W^k&:=\textstyle\bigcup_{i,j\in\mathbb{N}}W_{ij}^k\end{aligned} \tag{4.15}$$

and notice that simple computations yield

(4.16)
$$\begin{aligned}&\widehat{\varphi}_{ij}^k:U_i^k\times U_i^k\to W_{ij}^k\quad\text{ is }\sqrt{1+(1+\varepsilon)^2/(r_j)^2}\text{-biLipschitz},\\ &\frac{(r_j)^k}{C_i^k}\,(\mathfrak{m}\otimes\mathcal{L}^k)|_{W_{ij}^k}\leq\big(\widehat{\varphi}_{ij}^k\big)_*\big((\mathfrak{m}\otimes\mathfrak{m})|_{U_i^k\times U_i^k}\big)\leq(r_j)^kC_i^k\,(\mathfrak{m}\otimes\mathcal{L}^k)|_{W_{ij}^k}.\end{aligned}$$

In particular, $W_{ij}^k\in\mathscr{B}(A_k\times\mathbb{R}^k)$ for every k,i,j, thus accordingly also $W^k\in\mathscr{B}(A_k\times\mathbb{R}^k)$. Put $N_k:=(A_k\times\mathbb{R}^k)\setminus W^k$.

Lemma 4.6. *With the notation just introduced, for every $k\in\mathbb{N}$ we have*

$$(\mathfrak{m}\otimes\mathcal{L}^k)(N_k)=0.$$

Proof. For $k\in\mathbb{N}$ put

$$D_k:=\bigcup_{i\in\mathbb{N}}\Big\{x\in U_i^k\;:\;\varphi_i^k(x)\text{ is a point of density 1 for }\varphi_i^k(U_i^k)\Big\}.$$

From (4.13) and (1.7) we see that $\mathfrak{m}(A_k\setminus D_k)=0$, therefore for every $i,m,h\in\mathbb{N}$ and $\bar{x}\in D_k$, there is $j\in\mathbb{N}$ such that

$$1\geq\frac{\mathcal{L}^k\left(\frac{\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})}{r_j}\cap B_m(0)\right)}{\mathcal{L}^k\big(B_m(0)\big)}=\frac{\mathcal{L}^k\Big(\varphi_i^k(U_i^k)\cap B_{mr_j}\big(\varphi_i^k(\bar{x})\big)\Big)}{\mathcal{L}^k\Big(B_{mr_j}\big(\varphi_i^k(\bar{x})\big)\Big)}>1-\frac{1}{h},$$

whence $\mathcal{L}^k\Big(B_m(0)\setminus\bigcup_j\big(\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})\big)/r_j\Big)=0$ for all $i,m\in\mathbb{N}$ and $\bar{x}\in D_k$. Therefore by Fubini's theorem we deduce

$$
\begin{aligned}
(\mathfrak{m}\otimes\mathcal{L}^k)&\Big(\big(A_k\times B_m(0)\big)\setminus W^k\Big)\\
&=\sum_{i\in\mathbb{N}}(\mathfrak{m}\otimes\mathcal{L}^k)\Big(\big(U_i^k\times B_m(0)\big)\setminus W^k\Big)\\
&\leq\sum_{i\in\mathbb{N}}\int_{D_k}\mathcal{L}^k\Big(B_m(0)\setminus\bigcup\nolimits_j\big(\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})\big)/r_j\Big)\,\mathrm{d}\mathfrak{m}(\bar{x})=0,
\end{aligned}
$$

so that $(\mathfrak{m}\otimes\mathcal{L}^k)(N_k)=\lim_m(\mathfrak{m}\otimes\mathcal{L}^k)\Big(\big(A_k\times B_m(0)\big)\setminus W^k\Big)=0.$ □

We now endow $T_{\mathrm{GH}}\mathrm{X}$ with a new σ-algebra $\mathcal{M}\big(\mathscr{A},(r_j)\big)$, depending on the atlas $\mathscr{A}$ and the sequence (r_j). Let $\bar{\iota}_k: N^k\hookrightarrow T_{\mathrm{GH}}\mathrm{X}$ be the inclusion maps, then define

$$
\mathcal{M}\big(\mathscr{A},(r_j)\big):=\bigcap_{k\in\mathbb{N}}\Bigg((\bar{\iota}_k)_*\mathscr{B}(N^k)\cap\bigcap_{i,j\in\mathbb{N}}(\iota_k\circ\widehat{\varphi}_{ij}^k)_*\mathscr{B}(U_i^k\times U_i^k)\Bigg). \tag{4.17}
$$

Equivalently, a subset E of $T_{\mathrm{GH}}\mathrm{X}$ belongs to $\mathcal{M}\big(\mathscr{A},(r_j)\big)$ if and only if $E\cap N^k\in\mathscr{B}(N^k)$ for every $k\in\mathbb{N}$ and $(\widehat{\varphi}_{ij}^k)^{-1}\big(E\cap(A_k\times\mathbb{R}^k)\big)\in\mathscr{B}(U_i^k\times U_i^k)$ for every k,i,j.

The fact that our choice of the σ-algebra $\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ on $T_{\mathrm{GH}}\mathrm{X}$ is canonical is encoded in the following proposition:

Proposition 4.7. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a strongly $\mathfrak{m}$-rectifiable metric measure space, $\mathscr{A}$ an ε-atlas and $r_j\downarrow 0$ a given sequence. Then*

$$
\mathcal{M}_{\mathrm{GH}}(\mathrm{X})=\mathcal{M}\big(\mathscr{A},(r_j)\big). \tag{4.18}
$$

Proof. If $E\in\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ then $\iota_k^{-1}(E)\in\mathscr{B}(A_k\times\mathbb{R}^k)$ for every $k\in\mathbb{N}$, so accordingly $E\cap N^k$ belongs to $\mathscr{B}(N^k)$ and $(\widehat{\varphi}_{ij}^k)^{-1}\big(\iota_k^{-1}(E)\big)$ belongs to $\mathscr{B}(U_i^k\times U_i^k)$ for every k,i,j, which proves that $E\in\mathcal{M}\big(\mathscr{A},(r_j)\big)$.

Conversely, let $E\in\mathcal{M}\big(\mathscr{A},(r_j)\big)$. Hence $E\cap N^k\in\mathscr{B}(N^k)\subseteq\mathscr{B}(A_k\times\mathbb{R}^k)$, while $F_{ij}^k:=(\widehat{\varphi}_{ij}^k)^{-1}\big(\iota_k^{-1}(E)\big)\in\mathscr{B}(U_i^k\times U_i^k)$ implies that $E\cap W_{ij}^k=\widehat{\varphi}_{ij}^k(F_{ij}^k)\in\mathscr{B}(A_k\times\mathbb{R}^k)$. Thus $\iota_k^{-1}(E)=(E\cap N^k)\cup\bigcup_{i,j}(E\cap W_{ij}^k)\in\mathscr{B}(A_k\times\mathbb{R}^k)$ for every $k\in\mathbb{N}$, which is equivalent to saying that $E\in\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$. □

Remark 4.8. This last proposition does not use the strong $\mathfrak{m}$-rectifiability of the space but only the $\mathfrak{m}$-rectifiability, as seen by the fact that we did not consider a sequence of ε_n-atlases. We chose this presentation because the reason for the introduction of the Gromov-Hausdorff tangent module is in the statement contained in the next section, which grants that the space of its sections is isometric to the abstract tangent module $L^2(T\mathrm{X})$, a result which we have only for strongly $\mathfrak{m}$-rectifiable spaces (under some additional assumptions on X). ■

5. Equivalence of $L^2(T\mathrm{X})$ and $L^2(T_{\mathrm{GH}}\mathrm{X})$

The main result of this article is the following: the two different notions of tangent modules described so far, namely the "analytic" tangent module $L^2(T\mathrm{X})$ and the "geometric" Gromov-Hausdorff tangent module $L^2(T_{\mathrm{GH}}\mathrm{X})$, can be actually identified (under some additional assumptions on X). More precisely, given a strongly $\mathfrak{m}$-rectifiable space X whose associated Sobolev space is reflexive, there exists an embedding of $L^2(T\mathrm{X})$ into $L^2(T_{\mathrm{GH}}\mathrm{X})$ which preserves the pointwise norm and, as the construction, such embedding can be canonically chosen once an aligned sequence of atlases is given. Furthermore, if the space X under consideration additionally satisfies (5.6), then such embedding is actually an isometric isomorphism.

Theorem 5.1 (Embedding of $L^2(T\mathrm{X})$ in $L^2(T_{\mathrm{GH}}\mathrm{X})$). *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a strongly $\mathfrak{m}$-rectifiable space such that $W^{1,2}(\mathrm{X})$ is reflexive. Then there exists an isometric embedding of modules $\mathscr{I}: L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$, so that in particular it holds*

$$\big|\mathscr{I}(\mathsf{v})\big| = |\mathsf{v}| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}, \qquad \text{for every } \mathsf{v} \in L^2(T\mathrm{X}). \tag{5.1}$$

As a consequence, $L^2(T\mathrm{X})$ is a Hilbert module and thus $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian.

Proof. Consider an aligned family $(\mathscr{A}_n)_n$ of atlases $\mathscr{A}_n = \big\{(U_i^{k,n},\varphi_i^{k,n})\big\}_{k,i}$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, of parameters $\varepsilon_n := 1/2^n$ and $\delta_n := 1/2^n$, whose existence is guaranteed by Theorem 3.8. Now let $\mathsf{v} \in L^2(T\mathrm{X})$ and $n \in \mathbb{N}$ be fixed. For $k,i \in \mathbb{N}$ put $V_i^{k,n} := \varphi_i^{k,n}\big(U_i^{k,n}\big) \in \mathscr{B}(\mathbb{R}^k)$ and recall that $\varphi_i^{k,n}: U_i^{k,n} \to V_i^{k,n}$ and its inverse are maps of bounded deformation. Thus it makes sense to consider $\underline{\mathrm{d}}\varphi_i^{k,n}\big(\chi_{U_i^{k,n}}\mathsf{v}\big) \in L^2(V_i^{k,n},\mathbb{R}^k)$ and we can define

$$\mathsf{w}_i^{k,n}(x) := \begin{cases} \big(\underline{\mathrm{d}}\varphi_i^{k,n}\big(\chi_{U_i^{k,n}}\mathsf{v}\big)\big)\big(\varphi_i^{k,n}(x)\big) & \text{for } \mathfrak{m}\text{-a.e. } x \in U_i^{k,n}, \\ 0 & \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X} \setminus U_i^{k,n}. \end{cases}$$

The bound (2.14) gives

$$\big|\mathsf{w}_i^{k,n}\big|(x) \le \mathrm{Lip}(\varphi_i^{k,n})\,|\mathsf{v}|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in U_i^{k,n}, \tag{5.2}$$

so that $\big\|\mathsf{w}_i^{k,n}\big\|_{L^2(T_{\mathrm{GH}}\mathrm{X})} \le (1+2^{-n})\big\||\mathsf{v}|\big\|_{L^2(U_i^{k,n})}$. In particular, the series $\sum_{i,k}\mathsf{w}_i^{k,n}$ converges in $L^2(T_{\mathrm{GH}}\mathrm{X})$ to some vector field $\mathscr{I}_n(\mathsf{v})$ whose norm is bounded by $(1+2^{-n})\big\||\mathsf{v}|\big\|_{L^2(\mathrm{X})}$ and which satisfies

$$\chi_{U_i^{k,n}}\mathscr{I}_n(\mathsf{v}) = \mathsf{w}_i^{k,n} \qquad \text{for every } k,i \in \mathbb{N}. \tag{5.3}$$

It is then clear that $\mathscr{I}_n : L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$ is L^∞-linear, continuous and satisfying $\big|\mathscr{I}_n(\mathsf{v})\big| \le \big(1+2^{-n}\big)\,|\mathsf{v}|$ $\mathfrak{m}$-a.e. for every $\mathsf{v} \in L^2(T\mathrm{X})$. We now claim that

(5.4) the sequence $(\mathscr{I}_n)_n$ is Cauchy w.r.t. the operator norm.

To prove this, let $\mathsf{v} \in L^2(T\mathrm{X})$, $k,i,j \in \mathbb{N}$ with $U_i^{k,n+1} \subseteq U_j^{k,n}$. For $\mathfrak{m}$-a.e. point $x \in U_i^{k,n+1}$, putting for brevity $y := \varphi_i^{k,n+1}(x)$, it holds that

$$\begin{aligned}
\big|\mathscr{I}_{n+1}(\mathsf{v}) - \mathscr{I}_n(\mathsf{v})\big|(x) &= \Big|\Big(\underline{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big)\big(\varphi_i^{k,n+1}(x)\big) \\
&\qquad - \Big(\underline{\mathrm{d}}\varphi_j^{k,n}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big)\big(\varphi_j^{k,n}(x)\big)\Big| \\
((2.13),(2.14)) \qquad &\le \Big\|\,\mathrm{d}\Big(\mathrm{id}_{V_i^{k,n+1}} - \varphi_j^{k,n}\circ\big(\varphi_i^{k,n+1}\big)^{-1}\Big)(y)\Big\| \\
&\qquad \times \Big|\underline{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big|(y) \\
(\delta_{n+1} = 2^{-n-1}) \qquad &\le \frac{1}{2^{n+1}}\,\Big|\underline{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big|\big(\varphi_i^{k,n+1}(x)\big) \\
(\varepsilon_{n+1} = 2^{-n-1}) \qquad &\le \frac{1}{2^{n+1}}\,\Big(1+\frac{1}{2^{n+1}}\Big)\,|\mathsf{v}|(x) \le \frac{1}{2^n}\,|\mathsf{v}|(x).
\end{aligned}$$

It follows that $\big\|\mathscr{I}_{n+1}(\mathsf{v}) - \mathscr{I}_n(\mathsf{v})\big\|_{L^2(T_{\mathrm{GH}}\mathrm{X})} \le 2^{-n}\,\|\mathsf{v}\|_{L^2(T\mathrm{X})}$ which by the arbitrariness of v means that

$$\big\|\mathscr{I}_{n+1} - \mathscr{I}_n\big\| \le \frac{1}{2^n},$$

where the norm in the left hand side is the operator one. Hence $\sum_{n=0}^\infty \big\|\mathscr{I}_{n+1} - \mathscr{I}_n\big\| < +\infty$ and the claim (5.4) is proved.

Let $\mathscr{I} : L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$ be the limit of $(\mathscr{I}_n)_n$ and notice that being the limit of L^∞-linear maps, it is also L^∞-linear. Moreover, the fact that

$\mathscr{I}_n(\mathsf{v}) \to \mathscr{I}(\mathsf{v})$ in $L^2(T_{\mathrm{GH}}\mathrm{X})$ implies that $\big|\mathscr{I}_n(\mathsf{v})\big| \to \big|\mathscr{I}(\mathsf{v})\big|$ in $L^2(\mathrm{X})$, hence - up to subsequences - we have

$$
\begin{aligned}
(5.5) \qquad \big|\mathscr{I}(\mathsf{v})\big|(x) &= \lim_{n\to\infty} \big|\mathscr{I}_n(\mathsf{v})\big|(x) \\
&\leq \lim_{n\to\infty} \left(1+\frac{1}{2^n}\right)|\mathsf{v}|(x) = |\mathsf{v}|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in\mathrm{X}.
\end{aligned}
$$

Moreover, it follows from the lower bound in (2.14) that

$$
|\mathsf{w}_i^{k,n}|(x) \geq \frac{|\mathsf{v}|(x)}{\mathrm{Lip}\big((\varphi_i^{k,n})^{-1}\big)} \qquad \text{for every } n,k,i\in\mathbb{N} \text{ and } \mathfrak{m}\text{-a.e. } x\in U_i^{k,n},
$$

thus accordingly (up to subsequences) it holds that

$$
\big|\mathscr{I}(\mathsf{v})\big|(x) = \lim_{n\to\infty}\big|\mathscr{I}_n(\mathsf{v})\big|(x) \geq \lim_{n\to\infty}\frac{|\mathsf{v}|(x)}{1+2^{-n}} = |\mathsf{v}|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in\mathrm{X}.
$$

This proves that the map $\mathscr{I}$ is an isometric embedding.

For the last claim we simply observe that any couple of vector fields $v,w\in L^2(T_{\mathrm{GH}}\mathrm{X})$ trivially satisfies (1.36). Thus (5.1) ensures that the same holds for elements of $L^2(T\mathrm{X})$, which accordingly is a Hilbert module. By duality $L^2(T^*\mathrm{X})$ is a Hilbert module and therefore $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian by (1.40). □

It is natural to wonder whether the embedding $\mathscr{I}$ built in the previous theorem is surjective or not. The next simple example shows that this is not always the case:

Example 5.2. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a fat Cantor subset of $[0,1]$ equipped with the Euclidean distance and (the restriction of) the Lebesgue measure. Then the canonical inclusion φ of X in $[0,1]$ is 1-biLipschitz with its image and satisfies $\varphi_*\mathfrak{m}\leq\mathcal{L}^1$, showing that X is strongly $\mathfrak{m}$-rectifiable. Also, since X is totally disconnected, any continuous curve in X must be constant and thus any test plan must be concentrated on constant curves. Then a direct verification of Definition 1.9 shows that any Borel function belongs to $\mathrm{S}^2(\mathrm{X})$ with 0 minimal weak upper gradient. In particular $W^{1,2}(\mathrm{X})\sim L^2(\mathrm{X})$ and thus the space is infinitesimally Hilbertian and Theorem 5.1 above applies.

Now notice that since the minimal weak upper gradient of any Borel function is 0, the cotangent module - and thus also the tangent one - reduces to the 0 module. On the other hand, the very definition of $T_{\mathrm{GH}}\mathrm{X}$ gives $T_{\mathrm{GH}}\mathrm{X}=\mathrm{X}\times\mathbb{R}$ and thus $L^2(T_{\mathrm{GH}}\mathrm{X})=L^2(\mathrm{X})\neq 0$, showing that $\mathscr{I}: L^2(T\mathrm{X})\to L^2(T_{\mathrm{GH}}\mathrm{X})$ is not surjective. ■

We can prove surjectivity of $\mathscr{I}$ on spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ such that

$$\mathrm{lip}(f) = |\mathrm{D}f| \quad \mathfrak{m} - a.e. \qquad \text{for every } f : \mathrm{X} \to \mathbb{R} \text{ Lipschitz.} \tag{5.6}$$

We recall that the seminal paper of Cheeger [7] ensures that (5.6) holds on locally doubling spaces supporting a local, weak 1-2 Poincaré inequality.

We then have the following result:

Theorem 5.3 (Surjectivity of the embedding of $L^2(T\mathrm{X})$ into $L^2(T_{\mathrm{GH}}\mathrm{X})$). *With the same assumptions and notation of Theorem 5.1 above, assume furthermore that* (5.6) *holds.*

Then the embedding $\mathscr{I} : L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$ *given by Theorem 5.1 is surjective.*

Proof. Since clearly the dimension of $L^2(T_{\mathrm{GH}}\mathrm{X})$ on A_k is k, to conclude it is enough to prove that the dimension of $L^2(T\mathrm{X})$ is $\geq k$ on the same set. Thus fix $\varepsilon > 0$, let $U \subset A_k$ and $(\varphi_1, \ldots, \varphi_k) = \varphi : U \to \varphi(U) \subset \mathbb{R}^k$ be as in Definition 3.1. By the arbitrariness of such U the theorem will be proved if we show that $\mathrm{d}\varphi_1 \ldots, \mathrm{d}\varphi_k$ are independent on U.

Let $\ell : \mathbb{R}^k \to \mathbb{R}$ be linear and y a Lebesgue point of $\varphi(U)$. Then Proposition 1.6 gives that

$$\begin{aligned} \|\ell\| = \mathrm{lip}(\ell)(y) = \mathrm{lip}(\ell|_{\varphi(U)})(y) &= \mathrm{lip}\big((\ell \circ \varphi) \circ \varphi^{-1}\big)(y) \\ &\leq \mathrm{Lip}(\varphi^{-1})\, \mathrm{lip}(\ell \circ \varphi)(\varphi^{-1}(y)) \end{aligned}$$

and thus (5.6) and $\varphi_*(\mathfrak{m}|_U) \ll \mathcal{L}^k$ give $(1+\varepsilon)^{-1}\|\ell\| \leq |\mathrm{D}(\ell \circ \varphi)|$ $\mathfrak{m}$-a.e. on U (here the quantity $|\mathrm{D}(\ell \circ \varphi)|$ is well defined $\mathfrak{m}|_U$-a.e. - thanks to the locality of the minimal weak upper gradient - as $|\mathrm{D}f|$ for any f Lipschitz which coincides with $\ell \circ \varphi$ on U). Writing $\ell(z) = \sum_i a_i z_i$ we have $\mathrm{d}(\ell \circ \varphi) = \sum_i a_i \mathrm{d}\varphi_i$ and thus the last inequality can be written as

$$(1+\varepsilon)^{-1} \sqrt{\sum_i |a_i|^2} \leq \Big| \sum_i a_i \mathrm{d}\varphi_i \Big| \qquad \mathfrak{m} - a.e. \text{ on } U. \tag{5.7}$$

It is then trivial to notice that the above also holds for any choice of $a_i : U \to \mathbb{R}$ attaining only a finite number of values. Since these functions are dense in $L^0(U)$ and since both sides of (5.7) are continuous in a_i as functions from $L^0(U)$ to itself, we conclude that (5.7) holds for any $a_1, \ldots, a_k \in L^0(U)$. The independence of $\mathrm{d}\varphi_1 \ldots, \mathrm{d}\varphi_k$ is now easily obtained: suppose that $\sum_i a_i \mathrm{d}\varphi_i = 0$ on U. Then (5.7) shows that $a_1 = \ldots = a_k = 0$ $\mathfrak{m}$-a.e. on U, as desired. □

6. Geometric interpretation of $T_{\rm GH}{\rm X}$

6.1. Gromov-Hausdorff Convergence

The aim of this conclusive section is to discuss in which sense for strongly $\mathfrak{m}$-rectifiable spaces the space $T_{\rm GH}{\rm X}$ can be obtained by looking at the pointed measured Gromov limits of the rescalings of X around (almost) all of its points.

Since we are dealing with possibly non-compact and non-doubling spaces, we shall work with the notion of pointed measured Gromov convergence that had been proposed in [13]. In order to introduce it, we first need to give some preliminary definitions.

We say that $({\rm X},\mathsf{d},\mathfrak{m},\bar{x})$ is a *pointed metric measure space* provided $({\rm X},\mathsf{d},\mathfrak{m})$ is a metric measure space and the reference point $\bar{x}\in{\rm X}$ belongs to ${\rm spt}(\mathfrak{m})$. Two pointed metric measure spaces $({\rm X},\mathsf{d}_{\rm X},\mathfrak{m}_{\rm X},\bar{y})$ and $({\rm Y},\mathsf{d}_{\rm Y},\mathfrak{m}_{\rm Y},\bar{y})$ are said to be *isomorphic* if there exists an isometric embedding $\iota:{\rm spt}(\mathfrak{m}_{\rm X})\to{\rm Y}$ such that $\iota_*\mathfrak{m}_{\rm X}=\mathfrak{m}_{\rm Y}$ and $\iota(\bar{x})=\bar{y}$. The equivalence class of a given space $({\rm X},\mathsf{d},\mathfrak{m},\bar{x})$ under this isomorphism relation will be denoted by $[{\rm X},\mathsf{d},\mathfrak{m},\bar{x}]$.

Finally, given a complete and separable metric space $({\rm X},\mathsf{d})$ and a sequence $(\mu_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of non-negative Borel measures on X that are finite on bounded sets, we say that μ_n *weakly converges to* μ_∞ as $n\to\infty$, briefly $\mu_n\rightharpoonup\mu_\infty$, provided

$$\lim_{n\to\infty}\int f\,{\rm d}\mu_n=\int f\,{\rm d}\mu_\infty\qquad\text{for every } f\in C_{\rm bs}({\rm X}),\tag{6.1}$$

where $C_{\rm bs}({\rm X})$ denotes the space of all bounded continuous maps on X having bounded support.

We can now give the definition of pointed measured Gromov convergence. It is convenient for our purposes to follow the so-called 'extrinsic approach', cf. [13, Definition 3.9]:

Definition 6.1 (Pointed measured Gromov convergence). *Fix a sequence of pointed metric measure spaces* $({\rm X}_n,\mathsf{d}_n,\mathfrak{m}_n,\bar{x}_n)$*,* $n\in\mathbb{N}\cup\{\infty\}$*. Then we say that the sequence of classes* $[{\rm X}_n,\mathsf{d}_n,\mathfrak{m}_n,\bar{x}_n]$ *converges to* $[{\rm X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty,\bar{x}_\infty]$ *in the* pointed measured Gromov sense*, or briefly* pmG-sense*, provided there exist a complete and separable metric space* $({\rm W},\mathsf{d}_{\rm W})$ *and a sequence of isometric embeddings* $\iota_n:{\rm X}_n\to{\rm W}$*, for* $n\in\mathbb{N}\cup\{\infty\}$*,* *such that*

$$\begin{aligned}\lim_{n\to\infty} \iota_n(\bar{x}_n) &= \iota_\infty(\bar{x}_\infty) \in \operatorname{spt}\big((\iota_\infty)_*\mathfrak{m}_\infty\big),\\ (\iota_n)_*\mathfrak{m}_n &\rightharpoonup (\iota_\infty)_*\mathfrak{m}_\infty \quad \textit{as } n\to\infty.\end{aligned} \tag{6.2}$$

Let us fix a shorthand notation: given a pointed metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x})$ and a radius $r>0$, we define the *normalized* measure $\mathfrak{m}_r^{\bar{x}}$ on X as

$$\mathfrak{m}_r^{\bar{x}} := \frac{\mathfrak{m}}{\mathfrak{m}\big(B_r(\bar{x})\big)}. \tag{6.3}$$

We can now introduce the notion of tangent cone to a pointed metric measure space:

Definition 6.2 (Tangent cone). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x})$ *be a pointed metric measure space. Then we denote by* $\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x}]$ *the family of all the classes* $[\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y},\bar{y}]$ *that are obtained as pmG-limits of* $\big[\mathrm{X},\mathsf{d}/r_n,\mathfrak{m}_{r_n}^{\bar{x}},\bar{x}\big]$*, for a suitable sequence* $r_n\searrow 0$*. We will call* $\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x}]$ *the* tangent cone *of* $[\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x}]$*.*

Proposition 6.3 (Locality of the tangent cone). *Fix a metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *and a Borel set* $A\subseteq\mathrm{X}$*. Let* $\bar{x}\in A$ *be a point of density* 1 *for* A *such that* $\mathfrak{m}$ *is pointwise doubling at* $\bar{x}$*. Then*

$$\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x}] = \mathrm{Tan}\big[A,\mathsf{d}|_{A\times A},\mathfrak{m}|_A,\bar{x}\big]. \tag{6.4}$$

Proof. For the sake of simplicity, let us denote $\mathsf{d}' := \mathsf{d}|_{A\times A}$ and $\mathfrak{m}' := \mathfrak{m}|_A$. Suppose that the class $[\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y},\bar{y}]$ is the pmG-limit of $\big[\mathrm{X},\mathsf{d}/r_n,\mathfrak{m}_{r_n}^{\bar{x}},\bar{x}\big]$ for some $r_n\searrow 0$. Then there exist a complete and separable metric space $(\mathrm{Z},\mathsf{d}_\mathrm{Z})$, an isometric embedding $\iota_\mathrm{Y}:\mathrm{Y}\to\mathrm{Z}$ and a sequence $(\iota_n)_n$ of isometries $\iota_n:\big(\mathrm{X},\mathsf{d}/r_n\big)\to(\mathrm{Z},\mathsf{d}_\mathrm{Z})$ such that $\iota_n(\bar{x})\to\iota_\mathrm{Y}(\bar{y})\in\operatorname{spt}\big((\iota_\mathrm{Y})_*\mathfrak{m}_\mathrm{Y}\big)$ and $(\iota_n)_*\mathfrak{m}_{r_n}^{\bar{x}}\rightharpoonup(\iota_\mathrm{Y})_*\mathfrak{m}_\mathrm{Y}$. Hence let us define $\iota_n' := \iota_n|_A$ for every $n\in\mathbb{N}$. Clearly each map ι_n' is an isometry from $\big(A,\mathsf{d}'/r_n\big)$ to $(\mathrm{Z},\mathsf{d}_\mathrm{Z})$. To conclude that $[\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y},\bar{y}]\in\mathrm{Tan}[A,\mathsf{d}',\mathfrak{m}',\bar{x}]$, it is enough to show that $(\iota_n')_*(\mathfrak{m}')_{r_n}^{\bar{x}}\rightharpoonup(\iota_\mathrm{Y})_*\mathfrak{m}_\mathrm{Y}$. Thus fix $f\in C_{\mathrm{bs}}(\mathrm{Z})$. Choose $R>0$ such that $\operatorname{spt}(f)\subseteq B_R\big(\iota_\mathrm{Y}(\bar{y})\big)$, whence $\operatorname{spt}(f\circ\iota_n)\subseteq B_{2Rr_n}(\bar{x})$ for n big enough. Then

$$\begin{aligned}\int f\,\mathrm{d}(\iota_n')_*(\mathfrak{m}')_{r_n}^{\bar{x}} = \frac{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}&\int f\,\mathrm{d}(\iota_n)_*\mathfrak{m}_{r_n}^{\bar{x}}\\ &-\frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}\int_{B_{2Rr_n}(\bar{x})\setminus A} f\circ\iota_n\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Since $D_A(\bar{x}) = 1$ and $\mathfrak{m}$ is pointwise doubling at $\bar{x}$, one has

$$\frac{\left|\int_{B_{2Rr_n}(\bar{x})\setminus A} f\circ\iota_n\,\mathrm{d}\mathfrak{m}\right|}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)} \leq \frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\setminus A\big)}{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)} \times \frac{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}\,\frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\max_{\mathrm{Z}}|f| \overset{n}{\longrightarrow} 0,$$

which grants that $\int f\,\mathrm{d}(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} \to \int f\,\mathrm{d}(\iota_n)_*\mathfrak{m}_{\mathrm{Y}}$, as required.

Conversely, let $[\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}},\bar{y}]$ be the pmG-limit of $\big[A,\mathsf{d}'/r_n,(\mathfrak{m}')^{\bar{x}}_{r_n},\bar{x}\big]$ for some $r_n\searrow 0$. Then take a complete separable metric space $(\mathrm{W},\mathsf{d}_{\mathrm{W}})$, an isometric embedding $\iota'_{\mathrm{Y}}:\mathrm{Y}\to\mathrm{W}$ and a sequence of maps $\iota'_n:A\to\mathrm{W}$, which are isometries from $\big(A,\mathrm{d}'/r_n\big)$ to $(\mathrm{W},\mathsf{d}_{\mathrm{W}})$, such that $\iota'_n(\bar{x}_n)\to\iota'_{\mathrm{Y}}(\bar{y})\in\mathrm{spt}\big((\iota'_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}\big)$ and $(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n}\rightharpoonup(\iota'_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}$. Hence there exist a complete separable metric space $(\mathrm{Z},\mathfrak{m}_{\mathrm{Z}})$, an isometric embedding $\iota_{\mathrm{W}}:\mathrm{W}\to\mathrm{Z}$ and a sequence of maps $\iota_n:\mathrm{X}\to\mathrm{Z}$, which are isometries from $\big(\mathrm{X},\mathsf{d}/r_n\big)$ to $(\mathrm{Z},\mathsf{d}_{\mathrm{Z}})$, such that $\iota_n|_A=\iota_{\mathrm{W}}\circ\iota'_n$ holds for every $n\in\mathbb{N}$, see for instance [13, Proposition 3.10]. Denote $\iota_{\mathrm{Y}}:=\iota_{\mathrm{W}}\circ\iota'_{\mathrm{Y}}$. We clearly have that $\iota_n(\bar{x})=\iota_{\mathrm{W}}\big(\iota'_n(\bar{x})\big)\to\iota_{\mathrm{Y}}(\bar{y})\in\mathrm{spt}\big((\iota_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}\big)$ as $n\to\infty$, thus it only remains to prove that $(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n}\rightharpoonup(\iota_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}$ as $n\to\infty$. To this aim, fix $f\in C_{\mathrm{bs}}(\mathrm{Z})$. Observe that

$$\int f\,\mathrm{d}(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n} = \frac{\mathfrak{m}\big(A\cap B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int f\circ\iota_{\mathrm{W}}\,\mathrm{d}(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} + \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int_{\mathrm{X}\setminus A} f\circ\iota_n\,\mathrm{d}\mathfrak{m}.$$

The first addendum in the right hand side of the previous equation tends to $\int f\circ\iota_{\mathrm{W}}\,\mathrm{d}(\iota'_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}$, because $D_A(\bar{x})=1$ and $f\circ\iota_{\mathrm{W}}\in C_{\mathrm{bs}}(\mathrm{W})$. To estimate the second one, take any $R>0$ such that $\mathrm{spt}(f)\subseteq B_R\big(\iota_{\mathrm{Y}}(\bar{y})\big)$, so that $\mathrm{spt}(f\circ\iota_n)\subseteq B_{2Rr_n}(\bar{x})$ for n sufficiently big. Then

$$\left|\frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int_{\mathrm{X}\setminus A} f\circ\iota_n\,\mathrm{d}\mathfrak{m}\right| \leq \frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\setminus A\big)}{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}\,\frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\max_{\mathrm{Z}}|f|\longrightarrow 0.$$

Therefore $\int f\,\mathrm{d}(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n}\to\int f\,\mathrm{d}(\iota_{\mathrm{Y}})_*\mathfrak{m}_{\mathrm{Y}}$, proving that $[\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}},\bar{y}]\in\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x}]$ and accordingly the statement. □

The previous result will allow us to concentrate our attention only on spaces that satisfy (1.8).

In such context, it is easier to study the blow-ups of the space by means of a different notion of convergence (see [13, Definition 3.24]):

Definition 6.4 (Pointed measured Gromov-Hausdorff convergence). *Fix a sequence of pointed metric measure spaces* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$, $n \in \mathbb{N} \cup \{\infty\}$. *Then we say that the sequence of spaces* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ *converges to* $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the* pointed measured Gromov-Hausdorff sense, *or briefly* pmGH-sense, *provided for any fixed* $\varepsilon, R > 0$ *with* $\varepsilon < R$ *there exist* $\bar{n} \in \mathbb{N}$ *and a sequence of Borel maps* $f_n : B_R(\bar{x}_n) \to \mathrm{X}_\infty$, *for* $n \geq \bar{n}$, *such that*

i) $f_n(\bar{x}_n) = \bar{x}_\infty$ *for every* $n \geq \bar{n}$,

ii) $\left|\mathsf{d}_\infty\big(f_n(x), f_n(y)\big) - \mathsf{d}_n(x,y)\right| \leq \varepsilon$ *for every* $n \geq \bar{n}$ *and* $x, y \in B_R(\bar{x}_n)$,

iii) *the* ε*-neighbourhood of* $f_n\big(B_R(\bar{x}_n)\big)$ *contains* $B_{R-\varepsilon}(\bar{x}_\infty)$ *for every* $n \geq \bar{n}$,

iv) $(f_n)_*\big(\mathfrak{m}_n|_{B_R(\bar{x}_n)}\big) \rightharpoonup \mathfrak{m}_\infty|_{B_R(\bar{x}_\infty)}$ *as* $n \to \infty$, *for a.e.* $R > 0$.

As shown in [13, Proposition 3.30], the relation between the two notions of convergence for pointed metric measure spaces introduced so far is the following:

Proposition 6.5 (From pmGH to pmG). *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ *be a sequence of pointed metric measure spaces that converges to some* $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the pmGH-sense. Then the sequence of classes* $[\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$ *pmG-converges to* $[\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$.

6.2. Limits of the rescaled spaces

Let us now focus on metric measure spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ satisfying the following properties:

$$
\begin{aligned}
&(\mathrm{X}, \mathsf{d}, \mathfrak{m}) \text{ is a strongly } \mathfrak{m}\text{-rectifiable space which satisfies (1.8),}\\
&\text{having constant dimension } k \in \mathbb{N} \text{ and whose reference measure is}\\
&\text{given by } \mathfrak{m} = \theta\, \mathcal{H}^k, \text{ for some continuous density } \theta : \mathrm{X} \to (0, +\infty).
\end{aligned}
\tag{6.5}
$$

Consider a family $\mathscr{A}_n = \big\{(U_i^n, \varphi_i^n)\big\}_{i \in \mathbb{N}}$ of ε_n-atlases on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, with compact domains U_i^n. We can use the atlases to build Borel maps $\Psi_n : \mathrm{X} \times$

$\left(\frac{1}{r_n}\mathrm{X}\right) \to T_{\mathrm{GH}}\mathrm{X}$ which are 'bundle maps', i.e. which fix the first coordinate, and that are approximate isometries as maps on the second variable in the following way. We first recall that for any closed subset U of X there exists a Borel map $P_U : \mathrm{X} \to U$ such that

$$\mathsf{d}\big(x, P_U(x)\big) \leq 2\mathsf{d}(x, U) \qquad \text{for every } x \in \mathrm{X}.$$

This can be built by first considering a countable dense subset $(x_n)_n$ of U and then by declaring $P_U(x) := x$ for $x \in U$ and for $x \notin U$ defining

$$P_U(x) := x_n, \qquad \text{where } n \text{ is the least number such that } \mathsf{d}(x, x_n) \leq 2\mathsf{d}(x, U).$$

Then given a sequence $r_n \downarrow 0$ we put

$$\Phi_n(x, y) := \frac{\varphi_i^n\big(P_{U_i^n}(y)\big) - \varphi_i^n(x)}{r_n} \in \mathbb{R}^k \quad \text{for every } x \in U_i^n \text{ and } y \in \mathrm{X}, \tag{6.6}$$

while $\Phi_n(x, y) := 0_{\mathbb{R}^k}$ if $x \notin \bigcup_i U_i^n$. Finally we define

$$\Psi_n(x, y) := \big(x, \Phi_n(x, y)\big) \qquad \text{for every } x, y \in \mathrm{X}.$$

Notice that the function Ψ_n is Borel for every $n \in \mathbb{N}$. In the next theorem we show that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the maps $y \mapsto \Phi_n(x, y)$ provide approximate measured isometries from X rescaled by a factor $\frac{1}{r_n}$ to $\mathbb{R}^k$, thus showing not only that the tangent space of X at x is $\mathbb{R}^k$, but also that there is a 'compatible' choice of approximate isometries making the resulting global maps, i.e. Ψ_n, Borel.

Theorem 6.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a space satisfying* (6.5). *Let* $\varepsilon_n \downarrow 0$ *and let* $\mathscr{A}_n = \big\{(U_i^n, \varphi_i^n)\big\}_i$ *be a family of* ε_n*-atlases with compact domains* U_i^n. *Then there exists a sequence* $r_n \downarrow 0$ *such that, defining* Φ_n *as in* (6.6), *for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *the following holds: for every* $R > \varepsilon > 0$ *there is* $\bar{n} \in \mathbb{N}$ *so that for every* $n \geq \bar{n}$ *we have*

$$\begin{gathered}\left|\big|\Phi_n(x, y_0) - \Phi_n(x, y_1)\big|_{\mathbb{R}^k} - \frac{\mathsf{d}(y_0, y_1)}{r_n}\right| \leq \varepsilon \qquad \textit{for every } y_0, y_1 \in B_{r_n R}(x), \\ B_{R-\varepsilon}(0_{\mathbb{R}^k}) \subset \varepsilon\textit{-neighbourhood of } \big\{\Phi_n(x, y) \,:\, y \in B_{r_n R}(x)\big\}, \\ \Phi_n(x, \cdot)_*\big(\mathfrak{m}_{r_n}^x|_{B_{r_n R}(x)}\big) \rightharpoonup \omega_k^{-1}\,\mathcal{L}^k|_{B_R(0)} \qquad \textit{as } n \to \infty.\end{gathered} \tag{6.7}$$

In particular, the space $\big(\mathrm{X}, \mathsf{d}/r_n, \mathfrak{m}_{r_n}^x, x\big)$ *pmGH-converges to* $\big(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathcal{L}^k/\omega_k, 0\big)$ *as* $n \to \infty$.

Proof. For any $i,n \in \mathbb{N}$ put $V_i^n := \varphi_i^n(U_i^n)$. Note that from (3.3) we see that for $\mathfrak{m}$-a.e. $x \in U_i^n$ the point $\varphi_i^n(x)$ is of density 1 for V_i^n. Let us call D' the set of all the points $x \in \mathrm{X}$ that satisfy $\mathcal{H}^k\big(B_r(x) \cap U_{i(n)}^n\big)/(\omega_k\, r^k) \to 1$ as $r \searrow 0$ for every $n \in \mathbb{N}$, where $i(n) \in \mathbb{N}$ is chosen so that $x \in U_{i(n)}^n$. Since each domain U_i^n is countably $\mathcal{H}^k$-rectifiable, we thus deduce from Theorem 1.8 that $\mathcal{H}^k(\mathrm{X} \setminus D') = 0$. Hence the set

$$D := D' \cap \bigcap_n \bigcup_i \Big\{ x \in U_i^n \ : \ x,\, \varphi_i^n(x) \text{ are points of density 1 for } U_i^n,\, V_i^n, \text{ respectively} \Big\}$$

is Borel and $\mathfrak{m}(\mathrm{X} \setminus D) = 0$. Fix $\bar{x} \in D$ and $R > \varepsilon > 0$. Let $i(n) \in \mathbb{N}$ be such that $\bar{x} \in U_{i(n)}^n$. For brevity, call $B_n := B_{r_n R}(\bar{x})$, $U_n := U_{i(n)}^n$, $V_n := V_{i(n)}^n$ and $\varphi_n := \varphi_{i(n)}^n$. Let us denote

$$\mathrm{avg}_n := \frac{1}{\mathcal{H}^k(B_n \cap U_n)} \int_{B_n \cap U_n} \theta \, \mathrm{d}\mathcal{H}^k \qquad \text{for every } n \in \mathbb{N}.$$

Step 1. Fix $\bar{\varepsilon} < \varepsilon / \max\{4R, R - \varepsilon\}$ positive and repeatedly apply property (1.8) to $\bar{x}$, U_n and to $\varphi_n(\bar{x})$, V_n, with $\bar{\varepsilon}$ in place of ε, to find a sequence $r_n \downarrow 0$ such that for $n \in \mathbb{N}$ it holds

$$\begin{aligned} \mathsf{d}\big(y, P_{U_n}(y)\big) &\le 2\,\bar{\varepsilon}\, r_n R &&\text{for every } y \in B_n, \\ \mathsf{d}_{\mathbb{R}^k}(z, V_n) &\le \bar{\varepsilon}\, \big|z - \varphi_n(\bar{x})\big| &&\text{for every } z \in B_{r_n R}\big(\varphi_n(\bar{x})\big). \end{aligned} \tag{6.8}$$

Furthermore, since $\bar{x} \in D$ and the map θ is continuous, we can also require that

(6.9)

$$\lim_{n\to\infty} \frac{\mathcal{H}^k(B_n \cap U_n)}{\omega_k\, r_n^k\, R^k} = \lim_{n\to\infty} \frac{(1+\varepsilon_n)^k\, \mathcal{L}^k\Big(V_n \cap B_{r_n R/(1+\varepsilon_n)}\big(\varphi_n(\bar{x})\big)\Big)}{\omega_k\, r_n^k\, R^k} = 1,$$

$$\big|\theta(x) - \mathrm{avg}_n\big| \le \frac{1}{n} \qquad \text{for every } n \in \mathbb{N}^+ \text{ and } x \in B_n \cap U_n,$$

$$\lim_{n\to\infty} \frac{\mathfrak{m}(B_n \cap U_n)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} = R^k.$$

From the fact that φ_n is $(1+\varepsilon_n)$-biLipschitz we see that for any $y_0, y_1 \in B_n$ it holds that

$$\begin{aligned}\big|\Phi_n(\bar x, y_0) - \Phi_n(\bar x, y_1)\big|_{\mathbb{R}^k} &\leq \frac{1+\varepsilon_n}{r_n}\,\mathsf{d}\big(P_{U_n}(y_0), P_{U_n}(y_1)\big)\\ \text{(by (6.8))}\qquad &\leq \frac{1+\varepsilon_n}{r_n}\,\big(\mathsf{d}(y_0,y_1) + 4\,\bar\varepsilon\, r_n R\big).\end{aligned}$$

Similarly we get $\big|\Phi_n(\bar x, y_0) - \Phi_n(\bar x, y_1)\big|_{\mathbb{R}^k} \geq \frac{1}{(1+\varepsilon_n) r_n}\big(\mathsf{d}(y_0,y_1) - 4\,\bar\varepsilon\, r_n R\big)$, thus

$$\left|\big|\Phi_n(\bar x, y_0) - \Phi_n(\bar x, y_1)\big| - \frac{\mathsf{d}(y_0,y_1)}{r_n}\right| \leq 2R\max\left\{2(1+\varepsilon_n)\bar\varepsilon + \varepsilon_n, \frac{2\bar\varepsilon + \varepsilon_n}{1+\varepsilon_n}\right\}$$

for every $y_0, y_1 \in B_{r_n R}(\bar x)$. Since $\bar\varepsilon < \varepsilon/(4R)$, this is sufficient to show that the first in (6.7) is fulfilled for n large enough.

Step 2. For the second in (6.7), let $w \in \mathbb{R}^k$ be with $|w| < R - \varepsilon$ and put $z_n := \varphi_n(\bar x) + r_n w$. Thus the point z_n belongs to $B_{r_n R}\big(\varphi_n(\bar x)\big)$. From the second in (6.8) and the compactness of U_n, we deduce that there exists $y_n \in U_n$ such that

$$\big|z_n - \varphi_n(y_n)\big| \leq \bar\varepsilon\, r_n |w|. \tag{6.10}$$

Since the right hand side is bounded from above by $\bar\varepsilon\, r_n R$, for n sufficiently large it is bounded above by ε, so that to conclude it suffices to show that, independently on the choice of w, for all n sufficiently large it holds that $y_n \in B_n$. To see this, recall that the inverse of the function φ_n is $(1+\varepsilon_n)$-Lipschitz to get that

$$\begin{aligned}\mathsf{d}(\bar x, y_n) &\leq (1+\varepsilon_n)\,\big|\varphi_n(\bar x) - \varphi_n(y_n)\big|\\ &\leq (1+\varepsilon_n)\Big(\big|\varphi_n(\bar x) - z_n\big| + \big|z_n - \varphi_n(y_n)\big|\Big)\\ \text{by (6.10)}\qquad &\leq r_n(1+\varepsilon_n)(1+\bar\varepsilon)\,|w| \leq r_n(1+\varepsilon_n)(1+\bar\varepsilon)(R-\varepsilon).\end{aligned}$$

Since $\bar\varepsilon < \varepsilon/(R-\varepsilon)$ we have that $(1+\bar\varepsilon)(R-\varepsilon) < R$, therefore for n sufficiently large we have that $r_n(1+\varepsilon_n)(1+\bar\varepsilon)(R-\varepsilon) < r_n R$, which concludes the proof of the second in (6.7).

Step 3. Let us now denote $\psi_n := \varphi_n \circ P_{U_n} - \varphi_n(\bar x)$, so that $\Phi_n(\bar x, \cdot) = \psi_n / r_n$. We have that

$$\mathcal{L}^k\left(\frac{\psi_n(B_n \cap U_n)}{r_n}\Delta B_R(0)\right) \longrightarrow 0 \quad \text{when } n \to \infty, \tag{6.11}$$

as one can easily prove by using (6.9), which grants that

$$\mathcal{H}^k(B_n \cap U_n)/\mathcal{H}^k(B_n) \to 1.$$

To prove the third in (6.7), fix $f \in C_c(\mathbb{R}^k)$. Observe that

$$\int f \,\mathrm{d}\Phi_n(\bar{x},\cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big)$$

can be written as $Q_1(n)\,\mathfrak{m}(B_n \cap U_n)/\mathfrak{m}\big(B_{r_n}(\bar{x})\big) + Q_2(n) + Q_3(n)$, where

$$\begin{aligned} Q_1(n) &:= \frac{1}{\mathcal{H}^k(B_n \cap U_n)} \int f(\,\cdot\,/r_n)\,\mathrm{d}(\psi_n)_*\big(\mathcal{H}^k|_{B_n\cap U_n}\big),\\ Q_2(n) &:= \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \int_{B_n\cap U_n} f\circ\Phi_n(\bar{x},\cdot)\,\big(\theta - \mathrm{avg}_n\big)\,\mathrm{d}\mathcal{H}^k,\\ Q_3(n) &:= \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \int_{B_n\setminus U_n} f\circ\Phi_n(\bar{x},\cdot)\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

First of all, it directly follows from the last two statements in (6.9) that

$$\begin{aligned} \big|Q_2(n)\big| &\le \frac{1}{n}\,\frac{\mathfrak{m}(B_n\cap U_n)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\,\max_{\mathbb{R}^k}|f| \longrightarrow 0,\\ \big|Q_3(n)\big| &\le \frac{\mathfrak{m}(B_n\setminus U_n)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\,\max_{\mathbb{R}^k}|f| \longrightarrow 0. \end{aligned} \tag{6.12}$$

Moreover, (1.17) yields $(1+\varepsilon_n)^{-k}\,\mathcal{L}^k|_{\psi_n(B_n\cap U_n)} \le (\psi_n)_*\big(\mathcal{H}^k|_{B_n\cap U_n}\big) \le (1+\varepsilon_n)^k\,\mathcal{L}^k|_{\psi_n(B_n\cap U_n)}$, thus accordingly it holds that

$$\begin{aligned} &\frac{(1+\varepsilon_n)^{-k}\,r_n^k}{\mathcal{H}^k(B_n\cap U_n)} \int_{\frac{\psi_n(B_n\cap U_n)}{r_n}} f\,\mathrm{d}\mathcal{L}^k\\ &\qquad \le Q_1(n) \le \frac{(1+\varepsilon_n)^k\,r_n^k}{\mathcal{H}^k(B_n\cap U_n)} \int_{\frac{\psi_n(B_n\cap U_n)}{r_n}} f\,\mathrm{d}\mathcal{L}^k. \end{aligned} \tag{6.13}$$

Finally, by recalling (6.11) we can immediately deduce that

$$\begin{aligned} &\left|\int_{\frac{\psi_n(B_n\cap U_n)}{r_n}} f\,\mathrm{d}\mathcal{L}^k - \int_{B_R(0)} f\,\mathrm{d}\mathcal{L}^k\right|\\ &\qquad \le \mathcal{L}^k\left(\frac{\psi_n(B_n\cap U_n)}{r_n}\Delta B_R(0)\right)\,\max_{\mathbb{R}^k}|f| \longrightarrow 0. \end{aligned}$$

Therefore the first in (6.9) gives $Q_1(n) \to (\omega_k\, R^k)^{-1} \int_{B_R(0)} f \, \mathrm{d}\mathcal{L}^k$, which together with (6.12) and the third in (6.9) grant that $\omega_k^{-1} \int_{B_R(0)} f \, \mathrm{d}\mathcal{L}^k = \lim_n \int f \, \mathrm{d}\Phi_n(\bar{x},\cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big)$. This means that $\Phi_n(x,\cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big) \rightharpoonup \omega_k^{-1}\, \mathcal{L}^k|_{B_R(0)}$, which proves the statement. □

By combining several results obtained so far, it is then easy to prove the following:

Theorem 6.7 (Euclidean tangent cone). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable space, whose dimensional decomposition is denoted by* $(A_k)_k$*. Then for every* $k \in \mathbb{N}$ *it holds that*

$$\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},x] = \Big\{\big[\mathbb{R}^k,\mathsf{d}_{\mathbb{R}^k},\mathcal{L}^k/\omega_k,0\big]\Big\} \quad \textit{for } \mathfrak{m}\textit{-a.e. } x \in A_k. \tag{6.14}$$

Proof. Let the sequence $(N_k)_k$ be as in Remark 3.2 and define $A_k' := A_k \setminus N_k$ for every $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and write $\mathfrak{m}|_{A_k'} = \theta_k\, \mathcal{H}^k|_{A_k'}$ for a suitable Borel density $\theta_k : A_k' \to (0,+\infty)$. Let

$$A_k^i := \big\{x \in A_k' \;:\; 2^i \leq \theta_k(x) < 2^{i+1}\big\} \qquad \text{for every } i \in \mathbb{Z},$$

then $(A_k^i)_i$ constitutes a Borel partition of A_k'. Thus fix $i \in \mathbb{Z}$. By arguing as in the proof of Proposition 3.5, one can see that $\lim_{r\to 0} \mathfrak{m}\big(B_r(x)\big)/(\omega_k\, r^k) = \theta_k(x)$ for $\mathfrak{m}$-a.e. $x \in A_k^i$. By applying Lusin theorem and Egorov theorem, we can cover $\mathfrak{m}$-a.a. of A_k^i with countably many compact sets $A_k^{ij} \subseteq A_k^i$, where $j \in \mathbb{N}$, in such a way that the maps $\theta_k|_{A_k^{ij}}$ are continuous and

$$\left|\frac{\mathfrak{m}\big(B_r(x)\big)}{\omega_k\, r^k} - \theta_k(x)\right| < 2^{i-1} \quad \text{for every } x \in A_k^{ij} \text{ and } r > 0$$
$$\text{smaller than some } r_k^{ij} > 0.$$

In particular, it holds that

$$\omega_k\, r^k\, 2^{i-1} < \mathfrak{m}\big(B_r(x)\big) < 5\, \omega_k\, r^k\, 2^{i-1} \qquad \text{for every } x \in A_k^{ij} \text{ and } r < r_k^{ij}.$$

Therefore A_k^{ij} fulfills the hypotheses of Lemma 1.3, so accordingly each space A_k^{ij} (with the restricted distance and measure) satisfies (6.5). Hence Theorem 6.6 and Proposition 6.5 give

$$\mathrm{Tan}\Big[A_k^{ij},\mathsf{d}|_{A_k^{ij}\times A_k^{ij}},\mathfrak{m}|_{A_k^{ij}},x\Big] = \Big\{\big[\mathbb{R}^k,\mathsf{d}_{\mathbb{R}^k},\mathcal{L}^k/\omega_k,0\big]\Big\} \quad \text{for } \mathfrak{m}\text{-a.e. } x \in A_k^{ij},$$

since $r_n \downarrow 0$ in Theorem 6.6 can be actually chosen among the subsequences of any fixed sequence converging to 0 and the pmG topology is metrizable, cf. [13, Theorem 3.15]. Given that $\mathfrak{m}$-a.e. point of A_k^{ij} is of density 1 for A_k^{ij} itself and $\mathfrak{m}$ is pointwise doubling at $\mathfrak{m}$-a.e point by Proposition 3.5, we deduce from Proposition 6.3 that $\left[\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathcal{L}^k/\omega_k, 0\right]$ is the unique element of $\mathrm{Tan}[\mathrm{X}, \mathsf{d}, \mathfrak{m}, x]$ for $\mathfrak{m}$-a.e. $x \in A_k^{ij}$. By arbitrariness of i and j, we finally conclude that (6.14) is satisfied, proving the statement. □

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