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Regularity of Lie groups

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We solve the regularity problem for Milnor's infinite dimensional Lie groups in the C^0-topological context, and provide necessary and sufficient regularity conditions for the (standard) C^k-topological setting. Specifically, we prove that if G is an infinite dimensional Lie group in Milnor's sense, then the evolution map is C^0-continuous on its domain *iff* G is locally μ-convex – This is a continuity condition imposed on the Lie group multiplication that generalizes the triangle inequality for locally convex vector spaces. We furthermore show that if the evolution map is defined on all smooth curves, then G is Mackey complete – This is a completeness condition formulated in terms of the Lie group operations that generalizes Mackey completeness as defined for locally convex vector spaces; so that we generalize the well known fact that a locally convex vector space is Mackey complete if each smooth (compactly supported) curve is Riemann integrable. Then, under the presumption that G is locally μ-convex, we show that each C^k-curve, for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$, is integrable (contained in the domain of the evolution map) *iff* G is Mackey complete and k-confined. The latter condition states that each C^k-curve in the Lie algebra $\mathfrak{g}$ of G can be uniformly approximated by a special type of sequence consisting of piecewise integrable curves – A similar result is proven for the case $k \equiv 0$; and we provide several mild conditions that ensure that G is k-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. We finally discuss the differentiation of parameter-dependent integrals in the standard topological context (C^k-topology). In particular, we show that if the evolution map is defined and continuous on $C^k([0,1], \mathfrak{g})$ for $k \in \mathbb{N} \sqcup \{\infty\}$, then it is smooth thereon:

- For $k = 0$: *iff* if it is differentiable at zero
  iff $\mathfrak{g}$ is integral complete.
- For $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$: *iff* if it is differentiable at zero
  iff $\mathfrak{g}$ is Mackey complete.

This result is obtained by calculating the directional derivatives explicitly – recovering the standard formulas (Duhamel) that hold, e.g., in the Banach (finite dimensional) case.

1. Introduction

The right logarithmic derivative and its inverse – the evolution map – play a central role in Lie theory. For instance, the existence of the exponential map – indispensable for structure theory of Lie groups – is based on the integrability of each constant curve (each such curve is contained in the domain of the evolution map). Moreover, given a principal fibre bundle, the existence of holonomies – essential for gauge field theories – is based on the integrability of curves that are pairings of a smooth connection with the derivative of a smooth curve in the base manifold. In this paper, we discuss the evolution map in the infinite dimensional setting introduced by Milnor

[2, 6, 8, 9]. Specifically, we consider an infinite dimensional Lie group G as defined in [2] that is modeled over a Hausdorff locally convex vector space E, with system of continuous seminorms $\mathfrak{P}$. We denote the Lie algebra of G by $(\mathfrak{g}, [\cdot,\cdot])$, the inversion of G by $\mathrm{inv}\colon G \ni g \mapsto g^{-1} \in G$, the Lie group multiplication by $\mathrm{m}\colon G \times G \to G$; and define $\mathrm{L}_g := \mathrm{m}(g,\cdot)$ as well as $\mathrm{R}_g := \mathrm{m}(\cdot, g)$ for each $g \in G$. We furthermore let $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ denote the adjoint action; and fix a chart $\Xi\colon G \supseteq \mathcal{U} \to \mathcal{V} \in E$ with $\mathcal{V}$ convex, $e \in \mathcal{U}$, and $\Xi(e) = 0$. The right logarithmic derivative is defined by

$$\delta^r\colon C^1(D, G) \to C^0(D, \mathfrak{g}), \qquad \mu \mapsto \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot{\mu})$$

for $D \subseteq \mathbb{R}$ a non-singleton interval and $\mu^{-1} \equiv \mathrm{inv} \circ \mu$; and, the evolution maps by

$$\begin{aligned} \mathrm{Evol}\colon \mathrm{D} &\to C^1([0,1], G), \qquad && \delta^r(\mu) \mapsto \mu \cdot \mu^{-1}(0) \\ \mathrm{evol}\colon \mathrm{D} &\to G, && \delta^r(\mu) \mapsto \mu(1) \cdot \mu^{-1}(0) \end{aligned}$$

for $\mu \in \mathrm{D} := \delta^r(C^1([0,1], G))$. Then, the differential equation to be investigated is

$$\phi = \delta^r(\mu) \qquad \text{for} \qquad \phi \in C^0(D, \mathfrak{g}),\ \ \mu \in C^1(D, G); \tag{1}$$

whereby, in contrast to the Banach case, no theory of ODE's is available in the generic locally convex case – The core of this problem is rather the "infinite dimensionality" of the locally convex topology than the infinite dimensionality of the vector space E itself. More specifically, in the context of a given continuous (linear) map $\phi\colon E \to E$, continuous seminorms can usually only be estimated against each other but not against themselves – In general, this prevents the Banach fixed-point theorem (Picard-Lindelöf) and the Grönwall lemma to work.[1] Thus, given a specific differential equation, one has to use its particular "symmetries" in order to prove existence and uniqueness of solutions for arbitrary initial values. The "symmetries" hidden

[1]Even if E is metrizable via $\mathrm{d}\colon E \times E \to \mathbb{R}_{\geq 0}$, this metric usually fails to have the important property that $\mathrm{d}(\lambda \cdot X + \lambda' \cdot X', 0) \leq |\lambda| \cdot \mathrm{d}(X, 0) + |\lambda'| \cdot \mathrm{d}(X', 0)$ holds for all $\lambda, \lambda' \in \mathbb{R}$ and $X, X' \in E$ [13] – making it incompatible with the Riemann integral (mean values).

in (1) are

$$
(2)\qquad
\begin{gathered}
\delta^r(\mu\cdot g)=\delta^r(\mu)\qquad\text{and}\qquad \delta^r(\mu|_{D'})=\delta^r(\mu)|_{D'}\\
\delta^r(\mu\circ\varrho)=\dot{\varrho}\cdot(\delta^r(\mu)\circ\varrho)\\
\delta^r(\mu\cdot\nu)=\delta^r(\mu)+\mathrm{Ad}_\mu(\delta^r(\nu))\qquad\text{implying}\\
\delta^r(\mu^{-1}\nu)=\mathrm{Ad}_{\mu^{-1}}(\delta^r(\nu)-\delta^r(\mu))
\end{gathered}
$$

for all $\mu,\nu\in C^1(D,G)$, $g\in G$, $\mathfrak{I}\ni D'\subseteq D\in\mathfrak{I}$, and each $\rho\colon\mathfrak{I}\ni D''\to D$ of class C^1; where $\mathfrak{I}$ denotes the set of all non-singleton intervals in $\mathbb{R}$.

For instance, already in the Banach (finite dimensional) case, the first line in (2) is used to glue together local solutions that are provided by the Picard-Lindelöf theorem in this context. Following this philosophy, we will apply the second line in (2) to Riemann integrals of suitable bump functions to prove that (cf. Theorem 2):

Theorem. *G is Mackey complete if $C^\infty([0,1],\mathfrak{g})\subseteq \mathrm{D}$ holds; i.e., if each smooth curve is integrable.*

Here, *Mackey completeness* is a condition formulated in terms of the Lie group operations that generalizes Mackey completeness as defined for locally convex vector spaces. The above theorem thus generalizes the well-known fact that (cf., e.g., Theorem 2.14 in [7]) a Hausdorff locally convex vector space E is Mackey complete if the Riemann integral of each (compactly supported) smooth curve (in E) exists in E.

Now, there is a further property of the evolution map that can be encoded in a topological condition imposed on the Lie group operations: We consider the restriction $\mathrm{evol}_k\colon \mathrm{D}\cap C^k([0,1],\mathfrak{g})\to G$ for each $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$; and say that evol_k is C^p-continuous for $p\leq k$ ($p=0$ for $k\equiv\mathrm{lip}$) *iff* it is continuous w.r.t. the subspace topology that is inherited by the C^p-topology on $C^k([0,1],\mathfrak{g})$. We furthermore say that G is *locally μ-convex iff* for each $\mathfrak{u}\in\mathfrak{P}$, there exists some $\mathfrak{u}\leq\mathfrak{o}\in\mathfrak{P}$ with

$$
(3)\qquad (\mathfrak{u}\circ\Xi)(\Xi^{-1}(X_1)\cdot\ldots\cdot\Xi^{-1}(X_n))\leq\mathfrak{o}(X_1)+\ldots+\mathfrak{o}(X_n)
$$

for all $X_1,\ldots,X_n\in E$, with $\mathfrak{o}(X_1)+\ldots+\mathfrak{o}(X_n)\leq 1$.[2] Then, using the second line in (2), we will show that (cf. Theorem 1):

[2] This notion was originally introduced in [3] as a tool to investigate regularity properties of weak direct products of Lie groups.

Theorem. evol_0 *is* C^0*-continuous iff* G *is locally* μ*-convex iff* evol_∞ *is* C^0*-continuous.*

Evidently, (3) *generalizes the triangle inequality for locally convex vector spaces; and, due to the above theorem, it is independent of the explicit choice of the chart* Ξ*.*

Then, using the above two theorems, we will be able to partially answer the question under which circumstances G is C^k-semiregular [3] for some given $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$; i.e., under which circumstances $C^k([0,1], \mathfrak{g}) \subseteq \mathrm{D}$ holds (cf. Theorem 3):

Theorem. *Suppose that* G *is locally* μ*-convex. Then,* G *is* C^k*-semiregular for* $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ *iff* G *is Mackey complete and* k*-confined. Moreover,* G *is* C^0*-semiregular if* G *is sequentially complete and* 0*-confined.*

Here, k-confinedness is an approximation property for C^k-curves that is automatically fulfilled, e.g., if $(\mathfrak{g}, [\cdot,\cdot])$ is submultiplicative; or, if G admits an exponential map, and $(\mathfrak{g}, [\cdot,\cdot])$ is constricted. The precise definitions, and more conditions can be found in Sect. 7.2 or in Sect. 2.4.

In the last part of this paper, we will discuss the differentiation of parameter-dependent integrals in the standard topological setting. We first show that if G is C^k-semiregular and evol_k is C^k-continuous for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, then the directional derivative (w.r.t. the C^k-topology) of evol_k at zero along some $\phi \in C^k([0,1], \mathfrak{g})$ exists in the completion $\overline{\mathfrak{g}}$ of $\mathfrak{g}$, as explicitly given by

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \mathrm{evol}_k(h \cdot \phi) = \textstyle\int \phi(s)\, \mathrm{d}s \in \overline{\mathfrak{g}}.$$

More generally: Recall that $\mathfrak{g}$ is said to be integral complete [3] *iff* $\int \phi(s)\, \mathrm{d}s \in \mathfrak{g}$ exists for each $\phi \in C^0([0,1], \mathfrak{g})$; and let

$$\textstyle\int^s \phi := \mathrm{Evol}(\phi)(s) \quad \text{as well as} \quad \int \phi := \int^1 \phi \qquad \forall\, \phi \in \mathrm{D},\ \ s \in [0,1].$$

Then, the above statement generalizes to (cf. Theorem 4):

Theorem.

1) *Suppose that* G *is* C^0*-semiregular and that* evol_0 *is* C^0*-continuous. Then,* evol_0 *is of class* C^1 *iff* $\mathfrak{g}$ *is integral complete iff* evol_0 *is differentiable at zero.*
2) *Suppose that* G *is* C^k*-semiregular and that* evol_k *is* C^k*-continuous, for* $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$*. Then,* evol_k *is of class* C^1 *iff* $\mathfrak{g}$ *is Mackey complete iff* evol_k *is differentiable at zero.*

Here, for $k = 0$ in the first case, and $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ in the second case, we have

$$\big(\mathrm{d}_\phi \mathrm{evol}_k\big)(\psi) = \mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int \mathrm{Ad}_{[\int^s\phi]^{-1}}(\psi(s))\, \mathrm{d}s\big) \qquad \forall\, \phi, \psi \in C^k([0,1], \mathfrak{g}).$$

For $k \in \mathbb{N} \sqcup \{\infty\}$, Theorem E in [3] additionally shows that evol_k *is even smooth.*

Recall that G is said to be *C^k-regular* for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ *iff* G is C^k-semiregular and evol_k is smooth (w.r.t. the C^k-topology). Then,

- the first point in the above theorem generalizes Theorem C.(a) in [3], stating that each C^0-regular Lie group has an integral complete Lie algebra (modeling space). It furthermore generalizes Theorem F in [3], as it drops the presumption that there exists a point-separating family $(\alpha_j)_{j\in J}$ of smooth Lie group homomorphisms $\alpha_j \colon G \to H_j$ to C^0-regular Lie groups H_j.
- since C^∞-regular Lie groups are C^∞-semiregular, the second point in the above theorem generalizes the result announced in Remark II.5.3.(b) in [11], stating that each C^∞-regular Lie group has a Mackey complete Lie algebra.

Actually, the last theorem is a consequence of a more general theorem concerning differentiation of parameter-dependent integrals. We write $\mathrm{s} \preceq k$ for $\mathrm{s} \in \mathbb{N}$ and

- $k \in \mathbb{N}$ *iff* $\mathrm{s} \leq k$ holds,
- $k \equiv \mathrm{lip}$ *iff* $\mathrm{s} = 0$ holds,
- $k \equiv \infty$ *iff* $\mathrm{s} \in \mathbb{N}$ holds.

We furthermore recall that the C^k-topology on $C^k([r, r'], \mathfrak{g})$ for $r < r'$ is generated by the seminorms

(4) $$\mathfrak{p}^{\mathrm{s}}_\infty(\phi) := \sup\{(\mathfrak{p} \circ \mathrm{d}_e\Xi)\big(\phi^{(m)}(t)\big) \mid 0 \leq m \leq \mathrm{s},\ t \in [r, r']\} \qquad \forall\, \phi \in C^k([r, r'], \mathfrak{g})$$

for $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$; and let $\mathfrak{p}_\infty \equiv \mathfrak{p}^0_\infty$ for each $\mathfrak{p} \in \mathfrak{P}$. Then, we will show that (cf. Theorem 5):

Theorem. *Suppose that G is C^k-semiregular and that* evol_k *is C^k-continuous, for some $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Let furthermore $\Phi \colon I \times [0, 1] \to \mathfrak{g}$*

($I \subseteq \mathbb{R}$ open) be given with $\Phi(z,\cdot) \in C^k([0,1],\mathfrak{g})$ for each $z \in I$. Then,

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\big([\textstyle\int \Phi(x,\cdot)]^{-1}[\int \Phi(x+h,\cdot)]\big) = \int \mathrm{Ad}_{[\int^s \Phi(x,\cdot)]^{-1}}(\partial_1\Phi(x,s))\,\mathrm{d}s \in \overline{\mathfrak{g}}$$

holds for $x \in I$, provided that

a) We have $(\partial_1\Phi)(x,\cdot) \in C^k([0,1],\mathfrak{g})$.
b) For each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \leq k$, there exists $L_{\mathfrak{p},\mathrm{s}} \geq 0$, as well as $I_{\mathfrak{p},\mathrm{s}} \subseteq I$ open with $x \in I_{\mathfrak{p},\mathrm{s}}$, such that

$$\tfrac{1}{|h|} \cdot \mathfrak{p}^{\mathrm{s}}_\infty(\Phi(x+h,\cdot) - \Phi(x,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, h \in \mathbb{R}_{\neq 0} \;\textit{ with }\; x+h \in I_{\mathfrak{p},\mathrm{s}}.$$

In particular, in Sect. 8.3, we will derive Duhamel's formula from the above theorem. This paper is organized as follows:

- In Sect. 2, we give a precise synopsis of the results obtained in this paper, and compare them to the results obtained in the literature so far.
- In Sect. 3, we provide the basic definitions, and prove the most elementary properties of the core mathematical objects of this paper.
- In Sect. 4, we prove certain continuity properties of the evolution map; and discuss piecewise integrable curves.
- In Sect. 5, we show equivalence of C^0-continuity and local μ-convexity.
- In Sect. 6, we show that each C^∞-semiregular Lie group is Mackey complete; and prove certain approximation statements that are relevant for our discussion in Sect. 7.2.
- In Sect. 7, we show that, under the presumption that G is locally μ-convex, G is C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ *iff* G is Mackey complete and k-confined. Similar statements are proven for the case $k \equiv 0$.
- In Sect. 8, we discuss the differentiation of parameter-dependent integrals.

2. Precise synopsis of the results

In this section, we give a precise synopsis of the most important results obtained in this paper, and compare them to the results obtained in the literature so far, primarily in [2].

2.1. Setting the stage

We are concerned with the following situation in this paper. We are given a Lie group G in the sense of [2] (cf. Definition 3.1 and Definition 3.3 in [2]) that is modeled over a Hausdorff locally convex vector space E, with system of continuous seminorms $\mathfrak{P}$. We denote Lie algebra of G by $\mathfrak{g}$, fix a chart $\Xi\colon G \supseteq \mathcal{U} \to \mathcal{V} \subseteq E$ with $\mathcal{V}$ convex, $e \in \mathcal{U}$, $\Xi(e) = 0$; and identify $\mathfrak{g}$ with E via $\mathrm{d}_e\Xi\colon \mathfrak{g} \to E$ – more specifically, this means that we define the seminorms $\{\boldsymbol{.}\mathfrak{p} := \mathfrak{p} \circ \mathrm{d}_e\Xi \mid \mathfrak{p} \in \mathfrak{P}\}$ on $\mathfrak{g}$. We denote the inversion in G by $\mathrm{inv}\colon G \ni g \mapsto g^{-1} \in G$, the Lie group multiplication by $\mathrm{m}\colon G \times G \to G$; and let $\mathrm{L}_g := \mathrm{m}(g, \cdot)$ as well as $\mathrm{R}_g := \mathrm{m}(\cdot, g)$ for each $g \in G$. The adjoint action is denoted by $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$; i.e., we have

$$\mathrm{Ad}(g, X) \equiv \mathrm{Ad}_g(X) := \mathrm{d}_e\mathrm{Conj}_g(X) \qquad \text{with} \qquad \mathrm{Conj}_g\colon G \ni h \mapsto g \cdot h \cdot g^{-1}$$

for each $X \in \mathfrak{g}$ and $g \in G$. The differential equation under consideration then is

$$(5) \quad \phi = \delta^r(\mu) \equiv \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot{\mu}) \quad \text{for} \quad \phi \in C^0(D, \mathfrak{g}),\ \ \mu \in C^1(D, G),\ \ D \in \mathfrak{I},$$

where $\mathfrak{I}$ denotes the set of all non-singleton intervals $D \subseteq \mathbb{R}$. It is immediate from the definitions that

$$(6) \qquad \delta^r(\mu \cdot g) = \delta^r(\mu) \qquad \text{and} \qquad \delta^r(\mu|_{D'}) = \delta^r(\mu)|_{D'}$$

$$(7) \qquad \delta^r(\mu \circ \varrho) = \dot{\varrho} \cdot (\delta^r(\mu) \circ \varrho)$$

$$(8) \qquad \delta^r(\mu \cdot \nu) = \delta^r(\mu) + \mathrm{Ad}_\mu(\delta^r(\nu)) \qquad \text{implying}$$
$$\delta^r(\mu^{-1}\nu) = \mathrm{Ad}_{\mu^{-1}}(\delta^r(\nu) - \delta^r(\mu))$$

holds, for all $\mu, \nu \in C^1(D, G)$, $g \in G$, $\mathfrak{I} \ni D' \subseteq D \in \mathfrak{I}$, and each $\rho\colon \mathfrak{I} \ni D'' \to D$ of class C^1 (we write $\mu^{-1} \equiv \mathrm{inv} \circ \mu$). Together with smoothness of the Lie group operations, and

$$(9) \qquad \gamma(t) - \gamma(r) = \textstyle\int_r^t \dot{\gamma}(s)\, \mathrm{d}s \in F \qquad \forall\, t \in [r, r'],\ \ \gamma \in C^1([r, r'], F)$$

for F a Hausdorff locally convex vector space, these are the only properties we have in hand to investigate Equation (5). Let now $\mathfrak{K} \subseteq \mathfrak{I}$ denote the set of all non-singleton compact intervals $[r, r'] \subseteq \mathbb{R}$. Then,

- It follows from (6) that (cf. Lemma 10) for $k \in \mathbb{N}$, we have
  - $\delta^r\colon C^{k+1}([r, r'], G) \to C^k([r, r'], \mathfrak{g})$.
  - $\mu \in C^{k+1}([r, r'], G)$ for each $\mu \in C^1([r, r'], G)$ with $\delta^r(\mu) \in C^k([r, r'], \mathfrak{g})$.

- It is now immediate from (9) and the right side of (8) that (cf. Lemma 9)
$$\delta^r\colon C^{k+1}_*([r,r'],G)\to C^k([r,r'],\mathfrak{g})$$
is injective for $k\geq 0$, with
$$C^{k+1}_*([r,r'],G):=\{\mu\in C^{k+1}([r,r'],G)\mid \mu(r)=e\}.$$

We let $\mathfrak{D}_{[r,r']}:=\delta^r(C^1([r,r'],G))$ for each $[r,r']\in\mathfrak{K}$, as well as $\mathfrak{D}^k_{[r,r']}:=\mathfrak{D}_{[r,r']}\cap C^k([r,r'],\mathfrak{g})$ for each $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$. Then, (we let $\mathrm{lip}+1:=1$, $\infty+1:=\infty$)
$$\mathrm{Evol}^k_{[r,r']}\colon \mathfrak{D}^k_{[r,r']}\to C^{k+1}_*([r,r'],G),\qquad \delta^r(\mu)\mapsto \mu\cdot\mu^{-1}(r)$$
is well defined for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$, as well as surjective for $k\in\mathbb{N}\sqcup\{\infty\}$. We define
$$\mathrm{evol}^k_{[r,r']}\colon \mathfrak{D}^k_{[r,r']}\to G,\qquad \phi\mapsto \mathrm{Evol}_{[r,r']}(\phi)(r') \tag{10}$$
with $\mathrm{Evol}_{[r,r']}\equiv\mathrm{Evol}^0_{[r,r']}$, for each $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$; and denote
$$\textstyle\int_a^b\phi:=\mathrm{Evol}_{[a,b]}(\phi|_{[a,b]})(b),\qquad \int\phi:=\int_r^{r'}\phi,\qquad \int_c^c\phi:=e,$$
$$\textstyle\int_r^\bullet\phi\colon [r,r']\ni t\mapsto\int_r^t\phi$$
for each $\phi\in\mathfrak{D}_{[r,r']}$, with $r\leq a<b\leq r'$ and $c\in[r,r']$. There are now several issues to be clarified. We first discuss

2.2. Semiregularity and Mackey completeness

We say that G is C^k-*semiregular* for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ *iff* $\mathfrak{D}^k_{[0,1]}=C^k([0,1],\mathfrak{g})$ holds. In this case,

- G is C^p-semiregular for each $p\geq k$ (we let $1\geq\mathrm{lip}\geq\mathrm{lip}\geq 0$).
- it is straightforward from (7) that $\mathfrak{D}^k_{[r,r']}=C^k([r,r'],\mathfrak{g})$ holds for each $[r,r']\in\mathfrak{K}$ (cf. Lemma 12).

One then clearly wants to have criteria in hand for G to be C^k-semiregular for some given $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$. We provide the following necessary condition (cf. Theorem 2):

Theorem I. *G is Mackey complete if G is C^∞-semiregular.*

Here, G is said to be *Mackey complete iff* each *Mackey-Cauchy sequence* converges in G; i.e., each sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq G$, such that

$$(\mathfrak{p}\circ\Xi)(g_m^{-1}\cdot g_n) \leq \mathfrak{c}_\mathfrak{p}\cdot\lambda_{m,n} \qquad \forall\, m,n\geq \mathfrak{l}_\mathfrak{p},\ \ \mathfrak{p}\in\mathfrak{P}$$

holds for certain $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.

- This definition is independent of the explicit choice of the chart Ξ (cf. Remark 3).
- This definition specializes to Mackey completeness as defined for locally convex vector spaces; i.e., the case where $(G,\cdot)\equiv(E,+)$ equals the additive group of a locally convex vector space E.

  Theorem I thus generalizes the well-known fact (cf. Theorem 2.14 in [7]) that a locally convex vector space is Mackey complete if each smooth (compactly supported) curve is Riemann integrable.
- Mackey completeness is exemplarily verified in Example 3 for Banach Lie groups; and the setting considered in [5].

Remark I. *The idea of the proof of Theorem I is to construct some* $\phi\in C^\infty([0,1],\mathfrak{g})$ *whose integral* $\int\phi$ *is the limit of a (subsequence of a) given Mackey-Cauchy sequence* $\{g_n\}_{n\in\mathbb{N}}\subseteq G$*. Roughly speaking, we will use* (7) *to glue together smooth curves whose integrals equal* $g_n^{-1}\cdot g_{n-1}$ *via suitable bump functions. Here, it is important that (1.) a Mackey-Cauchy sequence converges iff one of its subsequences converges, and (2.) passing to a subsequence if necessary, we can achieve that* $(\mathfrak{p}\circ\Xi)(g_n^{-1}\cdot g_{n-1})$ *decreases suitably fast – namely, (up to a factor* $\mathfrak{c}_\mathfrak{p}$*) in the same way for all seminorms* $\mathfrak{p}\in\mathfrak{P}$*: This ensures that the so-constructed* ϕ *is defined and smooth at* 1 *(where all of its derivatives must necessarily be zero).* ‡

2.3. Topologies and continuity

We say that $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous for $p\leq k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ and $[r,r']\in\mathfrak{K}$ *iff* it is continuous w.r.t. the seminorms (4), for $\mathrm{s}\preceq p$. We say that G is

- p.k-continuous for $p\preceq k$ *iff* $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous for each $[r,r']\in\mathfrak{K}$,
- k-continuous *iff* G is k.k-continuous.

It is straightforward from (7) and the right side of (8) that (cf. Lemma 15)

Lemma I. *G is* p.k*-continuous iff* $\mathrm{evol}^k_{[0,1]}$ *is* C^p*-continuous at zero.*

Under the presumtion that G is C^k-semiregular (for $k \in \mathbb{N} \sqcup \{\infty\}$), it had already been shown in Theorem D in [3] that $\mathrm{evol}^k_{[0,1]}$ *is C^k-continuous iff it is C^k-continuous at zero.*

Clearly, for $k \geq 1$, the C^0-topology is strictly coarser than the C^k-topology; so that 0.k-continuity implies k-continuity but usually not vice versa. Anyhow, it is straightforward from (7) and (8) that (cf. Lemma 16):

Lemma II. *If G is abelian, then G is* k*-continuous for $k \in \mathbb{N} \sqcup \{\infty\}$ iff G is* 0.k*-continuous.*

The important feature of 0.k-continuity is that it can be encoded in a continuity property of the Lie group multiplication: Recall that G is said to be locally μ-convex *iff* (3) holds. We will show that (cf. Theorem 1):

Theorem II. *G is 0-continuous iff G is locally μ-convex iff G is $0.\infty$-continuous.*

Specifically, for $k \in \mathbb{N} \sqcup \{\infty\}$ and $[r, r'] \in \mathfrak{K}$, let $\mathfrak{D}\mathrm{P}^k([r, r'], \mathfrak{g})$ denote the set of all maps $\phi\colon [r, r'] \to \mathfrak{g}$ such that there exist $r = t_0 < \ldots < t_n = r'$ and $\phi[p] \in \mathfrak{D}^k_{[t_p,t_{p+1}]}$ for $p = 0, \ldots, n-1$ with

$$\phi|_{(t_p,t_{p+1})} = \phi[p]|_{(t_p,t_{p+1})} \qquad \forall\, p = 0, \ldots, n-1.$$

Moreover, define the integral of ϕ by (well-definedness is straightforward from (6))

$$\begin{aligned} &\textstyle\int_r^t \phi := \int_{t_p}^t \phi[p] \cdot \int_{t_{p-1}}^{t_p} \phi[p-1] \cdot \ldots \cdot \int_{t_0}^{t_1} \phi[0] \\ &\forall\, t \in (t_p, t_{p+1}],\;\; p = 0, \ldots, n-1. \end{aligned} \tag{11}$$

Then, the one direction in Theorem II is covered by (cf. Proposition 2):

Proposition I. *Suppose that G is locally μ-convex. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$, such that*

$$\begin{aligned} &\textstyle\int \mathfrak{q}(\phi(s))\,\mathrm{d}s \leq 1 \quad \textit{for} \quad \phi \in \mathfrak{D}\mathrm{P}^0([r, r'], \mathfrak{g}) \\ &\Longrightarrow \qquad (\mathfrak{p} \circ \Xi)\big(\textstyle\int_r^\bullet \phi\big) \leq \int_r^\bullet \mathfrak{q}(\phi(s))\,\mathrm{d}s, \end{aligned}$$

for each $[r, r'] \in \mathfrak{K}$.

Remark II. *Apart from Proposition I, the set $\mathfrak{D}\mathrm{P}^k([r, r'], \mathfrak{g})$ plays an important role in the proof of the other direction in Theorem II. Here, the* *key observation is that* $\phi \in \mathfrak{DP}^k([r,r'],\mathfrak{g})$ *given with* $\mathfrak{q}_\infty(\phi) \leq 1/2$ *for some* $\mathfrak{q} \in \mathfrak{P}$*, it is possible to construct* $\varrho\colon [r,r'] \to [r,r']$ *smooth with* $|\dot{\varrho}| \leq 2$*, such that*

$$\dot{\varrho} \cdot (\phi \circ \varrho) \in \mathfrak{D}^k_{[r,r']} \qquad \textit{as well as} \qquad \textstyle\int \phi = \int \dot{\varrho} \cdot (\phi \circ \varrho)$$

holds; i.e., $\mathfrak{q}_\infty(\dot{\varrho} \cdot (\phi \circ \varrho)) \leq 1$ *(cf. Lemma 24). Continuity of* $\mathrm{evol}^k_{[r,r']}$ *w.r.t. to the seminorms* $\{\mathfrak{p}_\infty\}_{\mathfrak{p}\in\mathfrak{P}}$ *thus carries over to the set* $\mathfrak{DP}^k([r,r'],\mathfrak{g})$*; and, then* (3) *is a straightforward consequence of* (11)*. Here,* ϱ *is obtained by glueing together (and then integrating) suitable bump functions; so that the argument fails on the level of the* C^k*-topology for* $k \geq 1$*, just because the higher derivatives of a so-constructed* ϱ *become that larger that finer the decomposition of* $[r,r']$ *is made.* ‡

Finally, we say that G is L^1-continuous *iff* $\mathrm{evol}^0_{[r,r']}$ is continuous w.r.t. the L^1-seminorms

$$\mathfrak{p}_\int(\phi) := \textstyle\int \mathfrak{p}(\phi(s))\,\mathrm{d}s \qquad \forall\, \mathfrak{p} \in \mathfrak{P},\ \phi \in C^0([r,r'],\mathfrak{g}) \tag{12}$$

for each $[r,r'] \in \mathfrak{K}$. Then, Theorem II and Proposition I show that G is L^1-continuous *iff* G is locally μ-convex *iff* G is $0.\infty$-continuous; which generalizes Lemma 14.9 in [3]. Here, the equivalence of L^1-continuity and 0-continuity is already straightforward from (7) (cf. cf. Lemma 17).

2.4. Integrability

We now come back to the question under which circumstances a given $\phi \in C^0([0,1],\mathfrak{g})$ is integrale, i.e., contained in $\mathfrak{D}^0_{[0,1]}$. A sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^0([0,1],\mathfrak{g})$ is said to be *tame iff* for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that

$$\mathfrak{v} \circ \mathrm{Ad}_{[\int_0^\bullet \phi_n]^{-1}} \leq \mathfrak{w} \qquad \forall\, n \in \mathbb{N}$$

holds. Moreover, $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^0([0,1],\mathfrak{g})$ is said to be a

- Cauchy sequence *iff* to each $\mathfrak{p} \in \mathfrak{P}$ and $\varepsilon > 0$, there exists some $p \in \mathbb{N}$ with $\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \varepsilon$ for all $m, n \geq p$.
- Mackey-Cauchy sequence *iff* there exists a net $\mathbb{R}_{\geq 0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$, as well as constants $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$ and $\{\mathfrak{l}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, such that for

each $\mathfrak{p} \in \mathfrak{P}$, we have

$$\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \mathfrak{c}_\mathfrak{p} \cdot \lambda_{m,n} \qquad \forall\, m, n \geq \mathfrak{l}_\mathfrak{p}.$$

Then, $\phi \in C^0([0,1], \mathfrak{g})$ is said to be

- $\mathfrak{s}$-*integrable iff* there exists a tame Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0([0,1],\mathfrak{g})$ with $\{\phi_n\}_{n\in\mathbb{N}} \to \phi$ uniformly (i.e., w.r.t the seminorms $\{\mathfrak{p}_\infty\}_{\mathfrak{p}\in\mathfrak{P}}$).
- $\mathfrak{m}$-*integrable iff* there exists a tame Mackey-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0([0,1],\mathfrak{g})$ with $\{\phi_n\}_{n\in\mathbb{N}} \to \phi$ uniformly.

We will show that (cf. Lemma 31 and Proposition 3):

Proposition II. *Suppose that G is locally μ-convex.*

1) If G is sequentially complete, then $\phi \in \mathfrak{D}^0_{[0,1]}$ holds for $\phi \in C^0([0,1],\mathfrak{g})$ iff ϕ is $\mathfrak{s}$-integrable.
2) If G is Mackey complete, then $\phi \in \mathfrak{D}^{\mathrm{lip}}_{[0,1]}$ holds for $\phi \in C^{\mathrm{lip}}([0,1],\mathfrak{g})$ iff ϕ is $\mathfrak{m}$-integrable.

Remark III. *The one direction in Proposition II is immediate from the fact that $\mu\colon t \mapsto \int_0^t \phi$ has compact image, for each $\phi \in \mathfrak{D}^0_{[0,1]}$. For the other direction (in analogy to the Riemann integral) one defines*

$$\mu(t) := \lim_n \textstyle\int_0^t \phi_n \qquad \forall\, t \in [0,1];$$

and then has to verify (1.) that the limit exists pointwise, i.e., that μ is defined, (2.) that μ is continuous, (3.) that $\{\int_0^\bullet \phi_n\}_{n\in\mathbb{N}} \to \mu$ converges uniformly, and (4.) that μ is of class C^1 with $\delta^r(\mu) = \phi$. ‡

We say that G is k-*confined*

- for $k \equiv 0$: *iff* each $\phi \in C^0([0,1],\mathfrak{g})$ is $\mathfrak{s}$-integrable,
- for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$: *iff* each $\phi \in C^k([0,1],\mathfrak{g})$ is $\mathfrak{m}$-integrable;

and obtain from Theorem I that (cf. Theorem 3):

Theorem III. *Suppose that G is locally μ-convex. Then, G is C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ iff G is Mackey complete and* k*-confined. Moreover, G is C^0-semiregular if G is sequentially complete and* 0*-confined.*

For instance, G is k-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ (cf. Sect. 7.2):

- If G is abelian; or, more generally, if $(\mathfrak{g}, [\cdot,\cdot])$ is submultiplicative.
- If G is locally μ-convex and *reliable*; i.e., if for each $\mathfrak{v} \in \mathfrak{P}$, there exists a symmetric neighbourhood $V \subseteq G$ of e, and a sequence $\{\mathfrak{w}_n\}_{n\in\mathbb{N}_{\geq 1}} \subseteq \mathfrak{P}$, such that

$$\mathfrak{v} \circ \mathrm{Ad}_{g_1} \circ \ldots \circ \mathrm{Ad}_{g_n} \leq \mathfrak{w}_n \qquad \forall\, g_1,\ldots,g_n \in V,\;\; n \geq 1.$$

  This is the case, e.g., for the unit group $\mathcal{A}^\times$ of a continuous inverse algebra $\mathcal{A}$ fulfilling the condition $(*)$ (cf. (59)) formulated in the theorem proven in [5].
- If G admits an exponential map, is constricted, and has a sequentially complete Lie algebra.

  Here, the first condition means that $\phi_X|_{[0,1]} \in \mathfrak{D}_{[0,1]}$ holds, for each constant curve $\phi_X \colon \mathbb{R} \ni t \mapsto X \in \mathfrak{g}$; i.e., that

$$\exp\colon \mathfrak{g} \ni X \mapsto \textstyle\int_0^1 \phi_X \in G \tag{13}$$

  is defined. Moreover, constrictedness states that for each bounded subset $\mathrm{B} \subseteq \mathfrak{g}$, and each $\mathfrak{v} \in \mathfrak{P}$, there exist $C \geq 0$ and $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that

$$\mathfrak{v} \circ \mathrm{ad}\, X_1 \circ \ldots \circ \mathrm{ad}\, X_n \leq C^n \cdot \mathfrak{w} \qquad \forall\, X_1,\ldots,X_n \in \mathrm{B},\;\; n \geq 1$$

  holds, with $\mathrm{ad}\, X \colon \mathfrak{g} \ni Y \mapsto [X,Y] \in \mathfrak{g}$ for each $X \in \mathfrak{g}$.

In particular,

Corollary I. *If G is abelian, then G is C^∞-semiregular and ∞-continuous iff G is Mackey complete and locally μ-convex iff G is C^k-semiregular and* k*-continuous for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$.*

Proof. If G is C^∞-semiregular and ∞-continuous, then G is Mackey complete by Theorem I, as well as $0.\infty$-continuous by Lemma II; thus, locally μ-convex by Theorem II. Conversely, if G is locally μ-convex, then G is (even 0.)k-continuous for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ by Theorem II. Since G is lip-confined, Theorem III shows that G is C^k-semiregular for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ if G is additionally Mackey complete. □

2.5. Smoothness and differentiation

In Sect. 8, we will discuss the differentiation of parameter-dependent integrals in the standard setting; i.e., w.r.t. the C^k-topology. Our key observation there is (cf. Proposition 7):

Proposition III. *Suppose that G is* k-continuous *for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$; and that $(-\delta, \delta) \cdot \phi \subseteq \mathfrak{D}^k_{[r,r']}$ holds for some $\phi \in C^k([r,r'],\mathfrak{g})$ for $[r,r'] \in \mathfrak{K}$ and $\delta > 0$. Then, we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\, \mathrm{evol}^k_{[r,r']}(h \cdot \phi) = \textstyle\int \phi(s)\, \mathrm{d}s \in \overline{\mathfrak{g}}.$$

Thus, the directional derivative of $\mathrm{evol}^k_{[r,r']}$ at zero along such a $\phi \in C^k([r,r'],\mathfrak{g})$ always exists; namely, in the completion $\overline{\mathfrak{g}}$ of $\mathfrak{g}$.

We say that $\mathfrak{g}$ is *integral complete* [3] *iff* $\int \phi(s)\, \mathrm{d}s \in \mathfrak{g}$ exists for each $\phi \in C^0([0,1],\mathfrak{g})$; and recall that $\mathfrak{g}$ is Mackey complete *iff* $\int \phi(s)\, \mathrm{d}s \in \mathfrak{g}$ exists for each $\phi \in C^\infty([0,1],\mathfrak{g})$. Then, the above proposition immediately shows that (cf. Corollary 9):

Corollary II.

1) *Suppose that G is* 0*-continuous and C^0-semiregular. Then, $\mathrm{evol}^0_{[0,1]}$ is differentiable at zero iff $\mathfrak{g}$ is* integral complete.
2) *Suppose that G is* k*-continuous for some $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$, as well as C^∞-semiregular. Then, $\mathrm{evol}^k_{[0,1]}\big|_{C^\infty([0,1],\mathfrak{g})}$ is differentiable at zero iff $\mathfrak{g}$ is Mackey complete.*

Here, the first point generalizes Theorem C.(a) in [3] stating that each C^0-regular Lie group has an integral complete Lie algebra (modeling space). The second point generalizes the analogous result announced in Remark II.5.3.(b) in [11] stating that each C^∞-regular Lie group has a Mackey complete Lie algebra – Recall that G is said to be *C^k-regular* for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ *iff* G is C^k-semiregular and $\mathrm{evol}^k_{[0,1]}$ is smooth w.r.t. the C^k-topology.

Next, using the above proposition, we show that, cf. Theorem 5:

Theorem IV. *Suppose that G is* k*-continuous and C^k-semiregular for some $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$; and let $\Phi\colon I \times [r,r'] \to \mathfrak{g}$ ($I \subseteq \mathbb{R}$ open) be fixed with $\Phi(z,\cdot) \in C^k([r,r'],\mathfrak{g})$ for each $z \in I$. Then,*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\big([\textstyle\int \Phi(x,\cdot)]^{-1}[\int \Phi(x+h,\cdot)]\big) = \int \mathrm{Ad}_{[\int_r^s \Phi(x,\cdot)]^{-1}}(\partial_1\Phi(x,s))\, \mathrm{d}s \in \overline{\mathfrak{g}}$$

holds for $x \in I$, provided that

a) *We have $(\partial_1\Phi)(x,\cdot) \in C^k([r,r'],\mathfrak{g})$.*

b) For each $\mathfrak{p}\in\mathfrak{P}$ *and* $\mathrm{s}\preceq k$*, there exists* $L_{\mathfrak{p},\mathrm{s}}\geq 0$*, as well as* $I_{\mathfrak{p},\mathrm{s}}\subseteq I$ *open with* $x\in I_{\mathfrak{p},\mathrm{s}}$*, such that*

$$\tfrac{1}{|h|}\cdot \mathfrak{p}^{\mathrm{s}}_\infty(\Phi(x+h,\cdot)-\Phi(x,\cdot))\leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, h\in\mathbb{R}_{\neq 0} \ \textit{ with } \ x+h\in I_{\mathfrak{p},\mathrm{s}}.$$

For instance, we obtain (cf. Corollary 11):

Corollary III. *Suppose that G is ∞-continuous, and C^∞-semiregular; and that $\mathfrak{g}$ is Mackey complete. Then, for $\mathfrak{X}\colon I\to\mathfrak{g}$ of class C^1, we have*

$$\partial_z \exp(\mathfrak{X}(x)) = \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big(\textstyle\int \mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\partial_z\mathfrak{X}(x))\,\mathrm{d}s\big) \qquad \forall\, x\in I.$$

Imposing further presumptions, this specializes to *Duhamel's formula* (cf. Proposition 8):
(Actually, in Proposition 8, a slightly more general situation is considered.)

Proposition IV. *Suppose that G is ∞-continuous, C^∞-semiregular, and constricted; and that $\mathfrak{g}$ is sequentially complete. Then, for each $\mathfrak{X}\colon I\to\mathfrak{g}$ of class C^1, we have*[3]

$$\partial_z \exp(\mathfrak{X}(x)) = \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\Big(\tfrac{\mathrm{id}_{\mathfrak{g}}-\exp(-\mathrm{ad}\,\mathfrak{X}(x))}{\mathrm{ad}\,\mathfrak{X}(x)}(\partial_z\mathfrak{X}(x))\Big) \qquad \forall\, x\in I.$$

Now, Theorem E in [3] states that $\mathrm{evol}^k_{[0,1]}$ is smooth if G is C^k-semiregular, and $\mathrm{evol}^k_{[0,1]}$ is of class C^1. Combining this with Corollary II and Theorem IV, we obtain (cf. Theorem 4):

Theorem V.

1) If G is 0-continuous and C^0-semiregular, then $\mathrm{evol}^0_{[r,r']}$ is smooth for each $[r,r']\in\mathfrak{K}$ iff $\mathfrak{g}$ is integral complete iff $\mathrm{evol}^0_{[0,1]}$ is differentiable at zero.

2) If G is k-continuous and C^k-semiregular for $k\in\mathbb{N}_{\geq 1}\sqcup\{\infty\}$, then $\mathrm{evol}^k_{[r,r']}$ is smooth for each $[r,r']\in\mathfrak{K}$ iff $\mathfrak{g}$ is Mackey complete iff $\mathrm{evol}^k_{[0,1]}$ is differentiable at zero.

Here, for $k=0$ in the first-, and $k\in\mathbb{N}_{\geq 1}\sqcup\{\infty\}$ in the second case, we have

$$\begin{aligned}(14)\qquad &\big(\mathrm{d}_\phi\,\mathrm{evol}^k_{[r,r']}\big)(\psi) = \mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s\phi]^{-1}}(\psi(s))\,\mathrm{d}s\big)\\ &\forall\,\phi,\psi\in C^k([r,r'],\mathfrak{g}),\ \ [r,r']\in\mathfrak{K}.\end{aligned}$$

[3] The precise definition of the expression in the parentheses on the left side can be found in Sect. 8.3 (cf. Equation (95)).

Sketch of the Proof given in Sect. 8.4. By Corollary II, it suffices to show that (under the given presumptions) $\mathrm{evol}^k_{[r,r']}$ is smooth and fulfills (14), namely

- for $k \equiv 0$ if $\mathfrak{g}$ is integral complete,
- for $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$ if $\mathfrak{g}$ is Mackey complete.

Now, in both cases, formula (14) is immediate from Theorem IV applied to

$$\Phi[\phi, \psi]\colon (0,1) \times [r,r'] \ni (h,t) \mapsto \phi(t) + h \cdot \psi(t) \qquad \forall\, \phi, \psi \in C^k([r,r'], \mathfrak{g}),$$

whereby the right side of (14) is easily seen to be continuous (cf. Lemma 41). It thus follows from Theorem E in [3] that $\mathrm{evol}^k_{[0,1]}$ is smooth. Then, smoothness of $\mathrm{evol}^k_{[r,r']}$ for $[r,r'] \in \mathfrak{K}$ is clear from

$$\mathrm{evol}^k_{[r,r']} \stackrel{(7)}{=} \mathrm{evol}^k_{[0,1]} \circ \eta,$$

for $\eta\colon C^k([r,r'], \mathfrak{g}) \to C^k([0,1], \mathfrak{g})$ given by

$$\begin{aligned} &\eta(\phi) \mapsto \dot{\varrho} \cdot (\phi \circ \varrho) \equiv |r' - r| \cdot (\phi \circ \varrho) \\ &\text{with} \quad \varrho\colon [0,1] \to [r,r'], \quad t \mapsto r + t \cdot |r' - r|, \end{aligned}$$

as η is evidently smooth. □

Remark IV.

- *Up to the point where Theorem E from [3] is applied, the above argument also works for the Lipschitz case (cf. Corollary 13); i.e., we have:*
  2') *If G is* lip*-continuous as well as C^{lip}-semiregular, then* $\mathrm{evol}^{\mathrm{lip}}_{[r,r']}$ *is of class C^1 for each $[r,r'] \in \mathfrak{K}$ iff $\mathfrak{g}$ is Mackey complete iff* $\mathrm{evol}^{\mathrm{lip}}_{[0,1]}$ *is differentiable at zero.*
- *Then, instead of using Theorem E from [3] in the above argument, one might use the explicit formula* (14) *to prove smoothness of* $\mathrm{evol}^k_{[0,1]}$ *inductively for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, which would strengthen the statement in the previous point of course. The details, however, seem to be quite elaborate and technical; so that we leave this issue to another paper.*

Now, Theorem V shows:

A) G is C^0-regular *iff* G is C^0-semiregular and 0-continuous, with $\mathfrak{g}$ integral complete.

B) G is C^k-regular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$ *iff* G is C^k-semiregular and k-continuous, with $\mathfrak{g}$ Mackey complete.

Here, A) generalizes Theorem F in [3] stating that G is C^0-regular if G is C^0-semiregular and 0-continuous with integral complete Lie algebra, such that there exists a point-separating family $(\alpha_j)_{j \in J}$ of smooth Lie group homomorphisms $\alpha_j \colon G \to H_j$ to C^0-regular Lie groups H_j.

Moreover, let us say that G admits a C^1-exponential map *iff* exp as defined in (13) is of class C^1. We then have

Lemma III. *Suppose that G is abelian. Then,*

1) G is C^0-regular iff G admits a C^1-exponential map, and $\mathfrak{g}$ is integral complete.
2) G is C^∞-regular iff G admits a C^1-exponential map, and $\mathfrak{g}$ is Mackey complete.
iff G is C^k-regular for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$.

Proof. If G is abelian and $\exp\colon \mathfrak{g} \to G$ is of class C^1, then we have (cf. Remark 2.3))

$$\textstyle\int \phi = \exp(\int \phi(s)\,\mathrm{d}s) \quad \text{for each} \quad \phi \in C^0([0,1], \mathfrak{g}) \quad \text{with} \quad \int_0^t \phi(s)\,\mathrm{d}s \in \mathfrak{g} \qquad \forall\, t \in [0,1];$$

which is obviously continuous w.r.t. the seminorms $\{\mathfrak{p}_\infty\}_{\mathfrak{p} \in \mathfrak{P}}$. Thus,

- G is C^0-semiregular and 0-continuous if $\mathfrak{g}$ is integral complete; thus, C^0-regular by A).
- G is C^k-semiregular and k-continuous for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ if $\mathfrak{g}$ is Mackey complete; thus, C^k-regular for each $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ by B).

Since exp is of class C^1 if G is C^∞-regular (cf. Remark 2.2)), the rest is clear from A) and B). □

Then, using Proposition V.1.9 in [10], we obtain:

Proposition V. *Suppose that G is connected and abelian. Then,*

*1) G is C^0-*regular *iff $G \cong E/\Gamma$ holds for a discrete subgroup $\Gamma \subseteq E$, with E integral complete*
iff G admits a C^1-exponential map, and E is integral complete.

*2) G is C^∞-*regular *iff* $G \cong E/\Gamma$ *holds for a discrete subgroup* $\Gamma \subseteq E$, *with* E Mackey complete
iff G *admits a* C^1*-exponential map, and* E *is Mackey complete*
iff G *is* C^k*-regular for each* $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$.

Proof. Observe that E is integral/Mackey complete *iff* $\mathfrak{g}$ is integral/Mackey complete; and that, by Lemma III, it suffices to show the equivalences in the first line of 1) and 2):

- If G is C^k-regular for $k \in \{0, \infty\}$, then G is C^∞-regular. Then, exp is smooth (cf. Remark 2.2)); and, by A) and B), E is Mackey complete – even integral complete for $k \equiv 0$. Consequently, $G \cong E/\Gamma$ holds for a discrete subgroup $\Gamma \subseteq E$, by Proposition V.1.9 in [10].
- Suppose that $G = E/\Gamma$ holds for a discrete subgroup $\Gamma \subseteq E = \mathfrak{g}$; and let $k \in \{0, \infty\}$ be fixed. Suppose furthermore that E is Mackey complete for $k \equiv \infty$, and integral complete for $k \equiv 0$. Then, the evolution map (10) (for $[r, r'] \equiv [0, 1]$) of $(E, +)$ is given by

$$\textstyle\int_E^k \colon C^k([0,1], E) \to E, \qquad \phi \mapsto \int \phi(s)\,\mathrm{d}s,$$

which is obviously smooth w.r.t. the C^0-topology (as it is linear and continuous therein); and the canonical projection $\pi\colon E \to E/\Gamma$ is a smooth Lie group homomorphism – confer Example 2.1) for more details concerning the Lie group structure on E/Γ. We thus have (confer, e.g., statement *f)* in Sect. 3.5.2)

$$\mathrm{evol}^k_{[0,1]}(\phi) = \big(\pi \circ \textstyle\int_E^k\big)(\phi) \qquad \forall\, \phi \in C^k([0,1], E);$$

which is evidently smooth w.r.t. the C^0-topology. It is thus clear that G is C^k-regular.

The claim now follows from Lemma III. □

Evidently, Proposition V generalizes Theorem C.(b),(c) in [3] stating that $(E, +)$ is C^0-regular *iff* E is integral complete; and that $(E, +)$ is C^1-regular *iff* E is Mackey complete.

3. Preliminaries

In this section, we fix the notations; and recall the most important facts concerning locally convex vector spaces, differentiable maps, and Lie groups that we will need in the main text.

3.1. Conventions

Intervals are non-empty, non-singleton, connected subsets of $\mathbb{R}$. In the following, D always denotes an arbitrary-, I an open-, and K a compact interval. The set of all intervals is denoted by $\mathfrak{I}$, and the set of all compact ones by $\mathfrak{K}$. Let F be a (Hausdorff) locally convex vector space with corresponding system of continuous seminorms $\mathfrak{Q}$. We recall that $\mathfrak{Q}$ is filtrating, i.e., that for $\mathfrak{q}_1, \dots, \mathfrak{q}_n \in \mathfrak{Q}$ with $n \geq 1$ given, there exists some $\mathfrak{q} \in \mathfrak{Q}$ with $\mathfrak{q}_1, \dots, \mathfrak{q}_n \leq \mathfrak{q}$. For $\varepsilon > 0$ and $\mathfrak{q} \in \mathfrak{Q}$, we define

$$\mathrm{B}_{\mathfrak{q},\varepsilon} := \{X \in F \mid \mathfrak{q}(X) < \varepsilon\} \qquad\qquad \overline{\mathrm{B}}_{\mathfrak{q},\varepsilon} := \{X \in F \mid \mathfrak{q}(X) \leq \varepsilon\};$$

and write $\mathfrak{q} \prec V$ (or $V \succ \mathfrak{q}$) for $\mathfrak{q} \in \mathfrak{Q}$ and $V \subseteq F$ *iff* $\overline{\mathrm{B}}_{\mathfrak{q},1} \subseteq V$ holds. We say that $\mathrm{B} \subseteq F$ is bounded *iff* it is von Neumann bounded, i.e., *iff* we have

$$\sup\{\mathfrak{q}(X) \mid X \in \mathrm{B}\} < \infty \qquad \forall\, \mathfrak{q} \in \mathfrak{Q}.$$

We let $\overline{F}$ denote the completion of F; as well as $\overline{\mathfrak{q}}$ the (unique) extension of $\mathfrak{q} \in \mathfrak{Q}$ to $\overline{F}$.

Manifolds and Lie groups are always assumed to be in the sense of [2] (cf. Definition 3.1 and Definition 3.3 in [2]); i.e., smooth, Hausdorff, and modeled over a Hausdorff locally convex vector space: The corresponding differential calculus is reviewed in Sect. 3.3. If $f\colon M \to N$ is a C^1-map between the manifolds M and N, then $\mathrm{d}f\colon TM \to TN$ denotes the corresponding tangent map between their tangent manifolds; and we write $\mathrm{d}_x f \equiv \mathrm{d}f(x,\cdot)\colon T_xM \to T_{f(x)}N$ for each $x \in M$. A curve is a continuous map $\gamma\colon D \to M$, where M is a manifold and $D \in \mathfrak{I}$ an interval. If $D \equiv I$ is open, then γ is said to be of class C^k for $k \in \mathbb{N} \sqcup \{\infty\}$ *iff* it is of class C^k when considered as a map between the manifolds I and M. We say that $\gamma\colon D \to M$ is of class C^k for $k \in \mathbb{N} \sqcup \{\infty\}$ – and write $\gamma \in C^k(D, M)$ – *iff* $\gamma = \widetilde{\gamma}|_D$ holds for some $\widetilde{\gamma}\colon I \to M$ of class C^k with $D \subseteq I$. If $\gamma\colon D \to M$ is of class C^1 (or differentiable), we let $\dot{\gamma}(t) \in T_{\gamma(t)}M$ denote the corresponding tangent vector at $\gamma(t) \in M$. The same conventions also hold if $M \equiv F$ is a

Hausdorff locally convex vector space – In this case, we let $C^{\mathrm{lip}}([r,r'],F)$ denote the set of all Lipschitz curves on $[r,r'] \in \mathfrak{K}$; i.e., all curves $\gamma\colon [r,r'] \to F$ with

$$\mathfrak{q}(\gamma(t)-\gamma(t')) \leq L_{\mathfrak{q}} \cdot |t-t'| \qquad \forall\, t,t' \in [r,r'],\ \ \mathfrak{q} \in \mathfrak{Q}$$

for certain Lipschitz constants $\{L_{\mathfrak{q}}\}_{\mathfrak{q}\in\mathfrak{Q}} \subseteq \mathbb{R}_{\geq 0}$. We let $\infty + 1 := \infty$ as well as $\mathrm{lip} + 1 := 1$; and, for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ and $[r,r'] \in \mathfrak{K}$, we define

$$\begin{aligned} \mathfrak{q}^{\mathrm{s}}(\gamma) &:= \mathfrak{q}\big(\gamma^{(\mathrm{s})}\big), \\ \mathfrak{q}^{\mathrm{s}}_{\infty}(\gamma) &:= \sup\big\{\mathfrak{q}\big(\gamma^{(m)}(t)\big) \,\big|\, 0 \leq m \leq \mathrm{s},\ \ t \in [r,r']\big\}, \\ \mathfrak{q}_{\infty} &:= \mathfrak{q}^{0}_{\infty} \end{aligned}$$

for each $s \preceq k$ and $\gamma \in C^k([r,r'],F)$ – Here, $s \preceq k$ means

- $\mathrm{s} \leq k$ for $k \in \mathbb{N}$,
- $\mathrm{s} = 0$ for $k \equiv \mathrm{lip}$,
- $\mathrm{s} \in \mathbb{N}$ for $k \equiv \infty$.

The C^k-topology on $C^k([r,r'],F)$ is the Hausdorff locally convex topology that is generated by the seminorms $\mathfrak{q}^{\mathrm{s}}_{\infty}$, for each $\mathfrak{q} \in \mathfrak{Q}$ and $\mathrm{s} \preceq k$.

In this paper, G will always denote an infinite dimensional Lie group (in Milnor's sense) that is modeled over a Hausdorff locally convex vector space E, with system of continuous seminorms $\mathfrak{P}$. We denote the Lie algebra of G by $(\mathfrak{g}, [\cdot,\cdot])$, fix a chart $\Xi\colon G \supseteq \mathcal{U} \to \mathcal{V} \subseteq E$ with $\mathcal{V}$ convex, $e \in \mathcal{U}$, $\Xi(e) = 0$; and identify $\mathfrak{g}$ with E via $\mathrm{d}_e\Xi\colon E \to \mathfrak{g}$ – specifically, we define

$$\centerdot\mathfrak{P} := \{\centerdot\mathfrak{p} \equiv \mathfrak{p} \circ \mathrm{d}_e\Xi\colon \mathfrak{g} \to \mathbb{R}_{\geq 0} \mid \mathfrak{p} \in \mathfrak{P}\}.$$

We denote the inversion and the Lie group multiplication by

$$\mathrm{inv}\colon G \to G, \quad g \mapsto g^{-1} \qquad \text{and} \qquad \mathrm{m}\colon G \times G \to G, \quad (g,g') \mapsto g \cdot g',$$

respectively, say that $\mathrm{A} \subseteq G$ is symmetric *iff* $\mathrm{inv}(\mathrm{A}) = \mathrm{A}$ holds; and recall the product rule[4]

$$(15)\quad \mathrm{d}_{(g,h)}\mathrm{m}(v,w) = \mathrm{d}_g\mathrm{R}_h(v) + \mathrm{d}_h\mathrm{L}_g(w) \qquad \forall\, g,h \in G,\, v \in T_gG,\, w \in T_hG.$$

[4] Confer, e.g., **e**) in Sect. 3.3.1.

We let $\mathrm{Conj}\colon G\times G\ni(g,h)\mapsto \mathrm{Conj}_g(h)\in G$ with

$$\mathrm{Conj}_g:=\mathrm{L}_g\circ\mathrm{R}_{g^{-1}}\quad\text{for}\quad \mathrm{R}_g:=\mathrm{m}(\cdot,g)\quad\text{and}\quad \mathrm{L}_g:=\mathrm{m}(g,\cdot)\qquad\forall\, g\in G,$$

define $\mathrm{Ad}_g:=\mathrm{d}_e\mathrm{Conj}_g\colon\mathfrak{g}\to\mathfrak{g}$ for each $g\in G$; and let $\mathrm{Ad}\colon G\times\mathfrak{g}\ni(g,X)\mapsto \mathrm{Ad}_g(X)\in\mathfrak{g}$ denote the adjoint action. We furthermore let

$$\begin{aligned}&\mathrm{ad}\,X(Y):=\mathrm{d}_e\mathrm{Ad}[Y](X)\qquad\forall\,X\in\mathfrak{g}\\ &\text{for}\qquad \mathrm{Ad}[Y]\colon G\ni g\mapsto\mathrm{Ad}_g(Y)\in\mathfrak{g};\end{aligned}$$

and recall that $\mathrm{ad}\,X(Y)=[X,Y]$ holds for each $X,Y\in\mathfrak{g}$.

3.2. Locally convex vector spaces

Let $F_1,\dots,F_n$ be (Hausdorff) locally convex vector spaces with corresponding system of continuous seminorms $\mathfrak{Q}_1,\dots,\mathfrak{Q}_n$. Obviously, the Tychonoff topology on $F:=F_1\times\ldots\times F_n$ is the (Hausdorff) locally convex topology that is generated by the seminorms

$$\mathfrak{m}[\mathfrak{q}_1,\dots,\mathfrak{q}_n]\colon F\ni(X_1,\dots,X_n)\mapsto\max\{\mathfrak{q}_k(X_k)\,|\,k=1,\dots,n\},\tag{16}$$

with $\mathfrak{q}_k\in\mathfrak{Q}_k$ for $k=1,\dots,n$. Let E be a further locally convex vector space with system of continuous seminorms $\mathfrak{P}$. We then have

Lemma 1. *Let X be a topological space; and let $\Phi\colon X\times F_1\times\ldots\times F_n\to E$ be continuous with $\Phi(x,\cdot)$ n-multilinear for each $x\in X$. Then, for each $x\in X$ and $\mathfrak{p}\in\mathfrak{P}$, there exist seminorms $\mathfrak{q}_1\in\mathfrak{Q}_1,\dots,\mathfrak{q}_n\in\mathfrak{Q}_n$ as well as $V\subseteq X$ open with $x\in V$, such that*

$$(\mathfrak{p}\circ\Phi)(y,X_1,\dots,X_n)\leq\mathfrak{q}_1(X_1)\cdot\ldots\cdot\mathfrak{q}_n(X_n)\qquad\forall\,y\in V$$

holds for all $X_1\in F_1,\dots,X_n\in F_n$.

Proof. The proof is elementary, and can be found in Appendix A.1. □

Corollary 1. *Let X be a topological space; and let $\Phi\colon X\times F_1\times\ldots\times F_n\to E$ be continuous with $\Phi(x,\cdot)$ n-multilinear for each $x\in X$. Then, for each compact $\mathrm{K}\subseteq X$ and each $\mathfrak{p}\in\mathfrak{P}$, there exist seminorms $\mathfrak{q}_1\in\mathfrak{Q}_1,\dots,\mathfrak{q}_n\in\mathfrak{Q}_n$ as well as $O\subseteq X$ open with $\mathrm{K}\subseteq O$, such that*

$$(\mathfrak{p}\circ\Phi)(y,X_1,\dots,X_n)\leq\mathfrak{q}_1(X_1)\cdot\ldots\cdot\mathfrak{q}_n(X_n)\qquad\forall\,y\in O\tag{17}$$

holds for all $X_1\in F_1,\dots,X_n\in F_n$.

Proof. The proof is elementary, and can be found in Appendix A.2. □

Let us finally recall the following standard result concerning completions.

Lemma 2. *Let $F_1, \dots, F_n, E$ be Hausdorff locally convex vector spaces; and let $\Phi\colon F_1 \times \dots \times F_n \to E$ be continuous and n-multilinear. Then, Φ extends uniquely to a continuous n-multilinear map $\overline{\Phi}\colon \overline{F}_1 \times \dots \times \overline{F}_n \to \overline{E}$.*

3.3. Differentiation and integrals

In this subsection, we recall the differential calculus from [2, 6, 8, 9]; and provide some facts that we will need to work efficiently in the main text.

3.3.1. Differentiable maps. Let E and F be Hausdorff locally convex vector spaces with systems of continuous seminorms $\mathfrak{P}$ and $\mathfrak{Q}$, respectively. Let $U \subseteq F$ be open, and $f\colon U \to E$ be a map.

We say that f is differentiable at $x \in U$ *iff*

$$(D_v f)(x) := \lim_{t\to 0} \tfrac{1}{t} \cdot (f(x + t \cdot v) - f(x)) \in E$$

exists for each $v \in F$. Moreover,

- f is said to be differentiable *iff* it is differentiable at each $x \in U$; i.e., *iff* $D_v f\colon U \to E$ is defined for each $v \in F$.
- f is said to be k-times differentiable for $k \geq 1$ *iff*

  $$D_{v_k,\dots,v_1} f \equiv D_{v_k}(D_{v_{k-1}}(\dots(D_{v_1}(f))\dots))\colon U \to E$$

  is defined for each $v_1, \dots, v_k \in F$; implicitly meaning that f is p-times differentiable for each $1 \leq p \leq k$. In this case, we define

  $$\begin{aligned}&\mathrm{d}^p_x f(v_1, \dots, v_p) \equiv \mathrm{d}^p f(x, v_1, \dots, v_p) := D_{v_p,\dots,v_1} f(x)\\ &\forall\, x \in U,\ \ v_1, \dots, v_p \in F,\end{aligned}$$

  for $p = 1, \dots, k$; and let $\mathrm{d}f \equiv \mathrm{d}^1 f$, as well as $\mathrm{d}_x f \equiv \mathrm{d}^1_x f$ for each $x \in U$.

Then,

- f is said to be of class C^0 *iff* it is continuous; and we let $\mathrm{d}^0 f \equiv f$ in this case.

- f is said to be of class C^k for $k \geq 1$ *iff* it is k-times differentiable, such that

$$\mathrm{d}^p f\colon U \times F^p \to E, \qquad (x, v_1, \dots, v_p) \mapsto D_{v_p,\dots,v_1} f(x)$$

  is continuous for $p = 0, \dots, k$.

  In this case, $\mathrm{d}^p_x f$ is symmetric and p-multilinear for each $x \in U$ and $p = 1, \dots, k$, cf. [2].
- f is said to be of class C^∞ *iff* it is of class C^k for each $k \in \mathbb{N}$.

We have the following differentiation rules, cf. [2]:

a) A map $f\colon F \supseteq U \to E$ is of class C^k for $k \geq 1$ *iff* $\mathrm{d}f$ is of class C^{k-1} when considered as a map $F' \supseteq U' \to E$ for $F' \equiv F \times F$ and $U' \equiv U \times F$.

b) If $f\colon F \to E$ is linear and continuous, then f is smooth; with $\mathrm{d}^1_x f = f$ for each $x \in F$, as well as $\mathrm{d}^k f = 0$ for each $k \geq 2$.

c) Let $E_1, \dots, E_m$ be Hausdorff locally convex vector spaces; and $f_u\colon F \supseteq U \to E_u$ be of class C^k for $k \geq 1$ and $u = 1, \dots, m$. Then,

$$f = f_1 \times \ldots \times f_m\colon U \to E_1 \times \ldots \times E_m, \qquad x \mapsto (f_1(x), \dots, f_m(x))$$

if of class C^k with $\mathrm{d}^p f = \mathrm{d}^p f_1 \times \cdots \times \mathrm{d}^p f_m$ for $p = 1, \dots, k$.

d) Suppose that $f\colon F \supseteq U \to U' \subseteq F'$ and $f'\colon F' \supseteq U' \to F''$ are of class C^k for $k \geq 1$, for Hausdorff locally convex vector spaces F, F', F''. Then, $f' \circ f\colon U \to F''$ is of class C^k with

$$\mathrm{d}_x(f' \circ f) = \mathrm{d}_{f(x)} f' \circ \mathrm{d}_x f \qquad \forall\, x \in U.$$

e) Let $F_1, \dots, F_m, E$ be Hausdorff locally convex vector spaces, and $f\colon F_1 \times \ldots \times F_m \supseteq U \to E$ be of class C^0. Then, f is of class C^1 *iff* the "partial derivatives"

$$\begin{aligned}\partial_u f\colon U \times F_u \ni{}& ((x_1, \dots, x_m), v_u)\\ &\mapsto \lim_{t \to 0} \tfrac{1}{t} \cdot (f(x_1, \dots, x_u + t \cdot v_u, \dots, x_m) - f(x_1, \dots, x_m))\end{aligned}$$

exist in E and are continuous, for $u = 1, \dots, m$. In this case, we have

$$\begin{aligned}\mathrm{d}_{(x_1,\dots,x_m)} f(v_1, \dots, v_m) &= \textstyle\sum_{u=1}^m \partial_u f((x_1, \dots, x_m), v_u)\\ &= \textstyle\sum_{u=1}^m \mathrm{d}f((x_1, \dots, x_m), (0, \dots, 0, v_u, 0, \dots, 0))\end{aligned}$$

for all $(x_1, \dots, x_m) \in U$, and $v_u \in F_u$ for $u = 1, \dots, m$.

Finally, for $f\colon F \supseteq U \to E$ of class C^k for $k \geq 1$, we have Taylor's formula, cf. [2]

$$
\begin{aligned}
f(x+\Delta) &= f(x) + \mathrm{d}^1_x f(\Delta) + \ldots + \tfrac{1}{(k-1)!}\cdot \mathrm{d}^{k-1}_x f(\Delta,\ldots,\Delta) \\
&\quad + \tfrac{1}{(k-1)!}\cdot \textstyle\int_0^1 (1-s)^{k-1}\cdot \mathrm{d}^k_{x+s\cdot\Delta} f(\Delta,\ldots,\Delta)\,\mathrm{d}s
\end{aligned} \tag{18}
$$

for each $x \in U$ and $\Delta \in F$ with $x + [0,1]\cdot\Delta \subseteq U$. Here, $\int \mathrm{d}s$ denotes the Riemann integral, discussed in Sect. 3.3.3 below.

3.3.2. Differentiable curves. We now consider the situation where $f \equiv \gamma\colon I \to E$ holds – i.e., we have $F \equiv \mathbb{R}$, and $U \equiv I$ is an open interval. It is then not hard to see that γ is of class C^k for $k \geq 1$ *iff* $\gamma^{(p)}$, inductively defined by $\gamma^{(0)} := \gamma$ and[5]

$$\gamma^{(p)}(t) := \lim_{h\to 0} \tfrac{1}{h}\cdot(\gamma^{(p-1)}(t+h) - \gamma^{(p-1)}(t)) \qquad \forall\, t \in I,\ p = 1,\ldots,k,$$

exists and is continuous for $p = 0,\ldots,k$. Then, for $\gamma \in C^k(D,E)$ with extension $\widetilde{\gamma}\colon D \supseteq I \to E$, we define $\gamma^{(p)} := \widetilde{\gamma}^{(p)}|_D$ for $p = 0,\ldots,k$, and let $\dot{\gamma} \equiv \widetilde{\gamma}^{(1)}|_D$.

Lemma 3. *Suppose that $\gamma \in C^k(D,E)$ holds for $k \geq 1$ and $D \in \mathfrak{I}$. Then, γ is of class C^{k+1} iff γ is of class C^k with $\gamma^{(k)}$ of class C^1.*

Proof. The proof is elementary, and can be found in Appendix A.3. □

Lemma 4. *Let F_1, F_2, E be Hausdorff locally convex vector spaces; and $\gamma_i\colon D \to W_i \subseteq F_i$ be of class C^k for $i = 1,2$, for some $k \geq 1$. Suppose furthermore that $\Omega\colon W_1 \times W_2 \to E$ is smooth. Then, $\delta\colon D \ni t \mapsto \Omega(\gamma_1(t),\gamma_2(t))$ is of class C^k; and $\delta^{(p)}$, for $0 \leq p \leq k$, can be written as a finite sum of terms of the form*

$$
\begin{aligned}
&\alpha = \Psi\big(\gamma^{(z_1)}_{i_1},\ldots,\gamma^{(z_m)}_{i_m}\big) \\
&\textit{for some}\quad 0 \leq z_1,\ldots,z_m \leq p,\ 1 \leq i_1,\ldots,i_m \leq 2,\ m \geq 2,
\end{aligned} \tag{19}
$$

where $\Psi\colon V \equiv V_{i_1} \times \ldots \times V_{i_m} \to E$ is smooth with open neighbourhoods $V_{i_u} \subseteq F_{i_u}$ for $u = 1,\ldots,m$.

Proof. The proof is elementary, and can be found in Appendix A.4. □

---

[5]We have $\gamma^{(p)}(t) = \mathrm{d}^p_t\gamma(1,\ldots,1)$ for $p = 1,\ldots,k$, $t \in I$.

Corollary 2. *Let F, E be Hausdorff locally convex vector spaces; and suppose that $\gamma_1\colon D \to W_1 \subseteq E$ is of class C^1, $\gamma_2\colon D \to W_2 \subseteq F$ is of class C^k for some $k \geq 1$, and that $\dot{\gamma}_1 = \Omega(\gamma_1, \gamma_2)$ holds for a smooth map $\Omega\colon W_1 \times W_2 \to E$. Then, γ_1 is of class C^{k+1}.*

Proof. This follows inductively from Lemma 3 and Lemma 4. □

Corollary 3. *Let F_1, F_2, E be Hausdorff locally convex vector spaces; and $\gamma_i\colon D \to W_i \subseteq F_i$ be of class C^k for $i = 1, 2$, for some $k \geq 1$. Suppose furthermore that $\Omega\colon W_1 \times F_2 \to E$ is smooth, as well as linear in the second argument. Then, $\delta\colon D \ni t \mapsto \Omega(\gamma_1, \gamma_2)$ is of class C^k; and $\delta^{(p)}$, for $1 \leq p \leq k$, can be written as a finite sum of terms of the form*[6]

$$([\partial_1]^m\Omega)\big(\gamma_1, \gamma_1^{(z_1)}, \dots, \gamma_1^{(z_m)}, \gamma_2^{(q)}\big)$$
$$\textit{for certain} \qquad 0 \leq z_1, \dots, z_m, q \leq p, \;\; m \geq 1.$$

Proof. Lemma 4 shows that δ is of class C^k; and the rest follows inductively from **b**), **d**), **e**). □

Lemma 5. *Let F_1, F_2, E be Hausdorff locally convex vector spaces with systems of continuous seminorms $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{P}$. Suppose that $W_1 \subseteq F_1$ is open; and that $\Omega\colon W_1 \times F_2 \to E$ is smooth, as well as linear in the second argument. Then, the following statements hold:*

1) *For $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{u} \in \mathbb{N}$ fixed, there exist $\mathfrak{m} \in \mathfrak{Q}_1$ and $\mathfrak{q} \in \mathfrak{Q}_2$, such that for each $[r, r'] \in \mathfrak{K}$ and $\gamma \in C^{\mathrm{u}}([r, r'], W_1)$ with $\mathfrak{m}^{\mathrm{u}}_\infty(\gamma) \leq 1$, we have*

$$\mathfrak{p}^{\mathrm{p}}(\Omega(\gamma, \psi)) \leq \mathfrak{q}^{\mathrm{p}}(\psi) \qquad \forall\, \psi \in C^{\mathrm{u}}([r, r'], F_2), \;\; 0 \leq \mathrm{p} \leq \mathrm{u}.$$

2) *For $\mathfrak{p} \in \mathfrak{P}$, $\mathrm{u} \in \mathbb{N}$, and $\gamma \in C^{\mathrm{u}}([r, r'], W_1)$ fixed, there exists some $\mathfrak{q} \in \mathfrak{Q}_2$ with*

$$\mathfrak{p}^{\mathrm{p}}(\Omega(\gamma, \psi)) \leq \mathfrak{q}^{\mathrm{p}}(\psi) \qquad \forall\, \psi \in C^{\mathrm{u}}([r, r'], F_2), \;\; 0 \leq \mathrm{p} \leq \mathrm{u}.$$

Proof. The proof is elementary, and can be found in Appendix A.5. □

3.3.3. The Riemann integral. Let F be a Hausdorff locally convex vector space with system of continuous seminorms $\mathfrak{Q}$, and completion $\overline{F}$. We

---

[6] Evidently, $[\partial_1]^m\Omega$ is continuous, as well as multilinear in the last $m + 1$ arguments.

denote the Riemann integral of $\gamma \in C^0([r,r'],F)$ by $\int \gamma(s)\,\mathrm{d}s \in \overline{F}$. We furthermore define

$$
\begin{aligned}
&\textstyle\int_a^b \gamma(s)\,\mathrm{d}s := \int \gamma|_{[a,b]}(s)\,\mathrm{d}s,\\
&\textstyle\int_b^a \gamma(s)\,\mathrm{d}s := -\int_a^b \gamma(s)\,\mathrm{d}s,\\
&\textstyle\int_c^c \gamma(s)\,\mathrm{d}s := 0
\end{aligned}
\tag{20}
$$

for $r \leq a < b \leq r'$, $c \in [r,r'] \in \mathfrak{K}$. Clearly, the Riemann integral is linear, with

$$
\textstyle\int_a^c \gamma(s)\,\mathrm{d}s = \int_a^b \gamma(s)\,\mathrm{d}s + \int_b^c \gamma(s)\,\mathrm{d}s \qquad \forall\, r \leq a < b < c \leq r', \tag{21}
$$

$$
\overline{\mathfrak{q}}\big(\textstyle\int_r^t \gamma(s)\,\mathrm{d}s\big) \leq \int_r^t \mathfrak{q}(\gamma(s))\,\mathrm{d}s \qquad \forall\, t \in [r,r'],\ \mathfrak{q} \in \mathfrak{Q}. \tag{22}
$$

It is furthermore not hard to see that

$$
\begin{aligned}
&\Gamma \in C^1([r,r'],\overline{F}) \quad \text{with} \quad \dot{\Gamma} = \gamma\\
&\text{holds for} \quad \Gamma\colon [r,r'] \ni t \mapsto \textstyle\int_r^t \gamma(s)\,\mathrm{d}s.
\end{aligned}
\tag{23}
$$

More importantly, we have, cf. [2]

$$
\gamma(t) - \gamma(r) = \textstyle\int_r^t \dot{\gamma}(s)\,\mathrm{d}s \in F \qquad \forall\, t \in [r,r'],\ \gamma \in C^1([r,r'],F). \tag{24}
$$

From this, we obtain

$$
\begin{aligned}
\textstyle\int \gamma(s)\,\mathrm{d}s = \Gamma(\varrho(\ell')) - \Gamma(\varrho(\ell)) &\overset{(24)}{=} \textstyle\int \partial_t(\Gamma \circ \varrho)(s)\,\mathrm{d}s\\
&\overset{\mathbf{d)}}{=} \textstyle\int \dot{\varrho}(s) \cdot \gamma(\varrho(s))\,\mathrm{d}s
\end{aligned}
\tag{25}
$$

for each $\gamma \in C^0([r,r'],F)$, and each $\varrho\colon [\ell,\ell'] \to [r,r']$ of class C^1 with $\varrho(\ell) = r$ and $\varrho(\ell') = r'$. Moreover,

Lemma 6. *For each $\gamma \in C^1([r,r'],F)$, we have*

$$
\mathfrak{q}(\gamma(t) - \gamma(r)) \leq \textstyle\int_r^t \mathfrak{q}(\dot{\gamma}(s))\,\mathrm{d}s \qquad \forall\, t \in [r,r'],\ \mathfrak{q} \in \mathfrak{Q}.
$$

Proof. Combine (22) with (24). □

Remark 1 (Banach Spaces). *Suppose that E, F are Banach spaces; and that $f\colon F \supseteq U \to E$ is of class C^{n+1} for some $n \geq 1$. Then, using Lemma 1 and Lemma 6, one can show that f is of class C^n in the Fréchet sense, cf. also [9]. In particular, if f is of class C^∞, then f is smooth in the Fréchet sense.* ‡

Lemma 7. *Let F, E be Hausdorff locally convex vector spaces; and let $f\colon F \supseteq U \to E$ be of class C^2. Suppose that $\gamma\colon D \to F \subseteq \overline{F}$ is continuous at $t \in D$, such that $\lim_{h\to 0} \frac{1}{h} \cdot (\gamma(t+h) - \gamma(t)) =: X \in \overline{F}$ exists. Then,*

$$\lim_{h\to 0} \tfrac{1}{h} \cdot (f(\gamma(t+h)) - f(\gamma(t))) = \overline{\mathrm{d}_{\gamma(t)} f}\,(X).$$

Proof. The proof is elementary, and can be found in Appendix A.6. □

We finally need to discuss the Riemann integral for piecewise continuous curves:

- We let $\mathrm{CP}^0([r,r'],F)$ denote the set of piecewise C^0-curves on $[r,r'] \in \mathfrak{K}$; i.e., all $\gamma\colon [r,r'] \to F$ such that there exist $r = t_0 < \ldots < t_n = r'$ as well as $\gamma[p] \in C^0([t_p, t_{p+1}], F)$ for $p = 0, \ldots, n-1$ with

$$\gamma|_{(t_p,t_{p+1})} = \gamma[p]|_{(t_p,t_{p+1})} \qquad \forall\, p = 0, \ldots, n-1.$$

- We let $\mathrm{CoP}([r,r'],F)$ denote the set of piecewise constant curves on $[r,r'] \in \mathfrak{K}$; i.e., all $\gamma\colon [r,r'] \to F$, such that there exist $r = t_0 < \ldots < t_n = r'$ as well as $X_0, \ldots, X_{n-1} \in F$ with

$$\gamma|_{(t_p,t_{p+1})} = X_p \qquad \forall\, p = 0, \ldots, n-1.$$

We clearly have $\mathrm{CoP}([r,r'],F) \subseteq \mathrm{CP}^0([r,r'],F)$; and for $\gamma \in \mathrm{CP}^0([r,r'],F)$ as above, we define

$$\textstyle\int \gamma(s)\,\mathrm{d}s := \sum_{p=0}^{n-1} \int \gamma[p](s)\,\mathrm{d}s. \tag{26}$$

A standard refinement argument in combination with (21) then shows that this is well defined; i.e., independent of any choices we have made. We define $\int_a^b \gamma(s)\,\mathrm{d}s$ and $\int_c^c \gamma(s)\,\mathrm{d}s$ as in (20); and observe that (26) is linear and fulfills (21).

3.4. Some estimates for Lie groups

In this subsection, we collect some elementary estimates concerning Lie group operations and coordinate changes that will be relevant for our argumentation in the main text. Let thus G be an infinite dimensional Lie group (in Milnor's sense) that is modeled over the Hausdorff locally convex vector space E, with system of continuous seminorms $\mathfrak{P}$ in the following.

3.4.1. Lie group operations. We observe that $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ is smooth (continuous) by **a**), because Conj smooth with

$$\mathrm{Ad}_g(X) = \mathrm{d}_{(g,e)}\mathrm{Conj}(0, X) \qquad \forall\, g \in G,\;\; X \in \mathfrak{g}.$$

We thus obtain from Lemma 1 and Corollary 1 that:

- For each $\mathfrak{q} \in \mathfrak{P}$, there exists some $\mathfrak{q} \leq \mathfrak{n} \in \mathfrak{P}$, as well as $V \subseteq G$ symmetric open with $e \in V$, such that

$$(27) \qquad {}_{\bullet}\mathfrak{q}(\mathrm{Ad}_g(X)) \leq {}_{\bullet}\mathfrak{n}(X) \qquad \forall\, g \in V,\;\; X \in \mathfrak{g}.$$

- For each $\mathfrak{n} \in \mathfrak{P}$, and each compact $\mathrm{C} \subseteq G$, there exists some $\mathfrak{n} \leq \mathfrak{m} \in \mathfrak{P}$, as well as $O \subseteq G$ open with $\mathrm{C} \subseteq O$, such that

$$(28) \qquad {}_{\bullet}\mathfrak{n} \circ \mathrm{Ad}_g \leq {}_{\bullet}\mathfrak{m} \qquad \forall\, g \in O.$$

Similarly, the maps

$$(29) \qquad \omega\colon \mathcal{V} \times E \to \mathfrak{g}, \qquad (x, X) \mapsto \mathrm{d}_{\Xi^{-1}(x)}\mathrm{R}_{[\Xi^{-1}(x)]^{-1}}(\mathrm{d}_x\Xi^{-1}(X))$$
$$(30) \qquad \upsilon\colon \mathcal{V} \times \mathfrak{g} \to E, \qquad (x, X) \mapsto \big(\mathrm{d}_{\Xi^{-1}(x)}\Xi \circ \mathrm{d}_e\mathrm{R}_{\Xi^{-1}(x)}\big)(X)$$

are smooth, as they can be written as

$$(31) \qquad \omega(x, X) = \mathrm{d}_{(x,x)}\Omega(0, X) \qquad \text{and} \qquad \upsilon(x, X) = \mathrm{d}_{(x,e)}\Upsilon(0, X)$$

for the smooth maps

$$\Omega\colon \mathcal{V} \times \mathcal{V} \to G, \qquad (x, y) \mapsto \mathrm{m}(\Xi^{-1}(y), [\Xi^{-1}(x)]^{-1})$$
$$\Upsilon\colon \mathcal{V} \times \mathcal{U} \to E, \qquad (x, g) \mapsto (\Xi \circ \mathrm{m})\big(g, \Xi^{-1}(x)\big).$$

Thus, by Lemma 1, for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathcal{V} \succ \mathfrak{w} \in \mathfrak{P}$ with $\mathfrak{v} \leq \mathfrak{w}$, such that

$$(32) \qquad {}_{\bullet}\mathfrak{v}(\omega(x, X)) \leq \mathfrak{w}(X) \qquad \forall\, x \in \overline{\mathrm{B}}_{\mathfrak{w},1}\,,\;\; X \in E$$
$$(33) \qquad \mathfrak{v}(\upsilon(x, X)) \leq {}_{\bullet}\mathfrak{w}(X) \qquad \forall\, x \in \overline{\mathrm{B}}_{{}_{\bullet}\mathfrak{w},1},\;\; X \in \mathfrak{g}.$$

More generally, we obtain from **a**) and **e**) that ω is smooth with

$$(34) \qquad \omega[n] := [\partial_1]^n\omega\colon \mathcal{V} \times E^{n+1} \to \mathfrak{g}$$

continuous as well as multilinear in the last $n+1$ arguments, for each $n \in \mathbb{N}$. For each $p \in \mathbb{N}$ and $\mathfrak{v} \in \mathfrak{P}$, there thus exists some $\mathcal{V} \prec \mathfrak{w} \in \mathfrak{P}$ with $\mathfrak{v} \leq \mathfrak{w}$, such that

$$(35)\qquad (\mathfrak{v} \circ \omega[q])(x, X_1, \dots, X_{q+1}) \leq \mathfrak{w}(X_1) \cdot \ldots \cdot \mathfrak{w}(X_{q+1})$$

holds for all $x \in \overline{\mathrm{B}}_{\mathfrak{w},1}$, $X_1, \dots, X_{q+1} \in E$, and $0 \leq q \leq p$.

Finally, since $\mathrm{inv}\colon G \to G$ is smooth, for each $\mathfrak{m} \in \mathfrak{P}$, there exists some $\mathcal{V} \prec \mathfrak{n} \in \mathfrak{P}$ with $\mathfrak{m} \leq \mathfrak{n}$, such that $\mathfrak{m} \circ \mathrm{d}_x(\Xi \circ \mathrm{inv} \circ \Xi^{-1}) \leq \mathfrak{n}$ holds for each $x \in \overline{\mathrm{B}}_{\mathfrak{n},1}$. We thus obtain from Lemma 6 that

$$(36)\qquad \mathfrak{m} \circ \Xi \circ \mathrm{inv} \circ \Xi^{-1} \leq \mathfrak{n} \qquad \text{holds on} \qquad \overline{\mathrm{B}}_{\mathfrak{n},1},$$

just by considering the curve $\gamma_X\colon [0,1] \ni t \mapsto t \cdot X$ for each $X \in \overline{\mathrm{B}}_{\mathfrak{n},1}$.

3.4.2. Coordinate changes. For $h \in G$, we define $\Xi_h(g) := \Xi(h^{-1} \cdot g)$ for each $g \in h \cdot \mathcal{U}$; i.e.,

$$[\Xi_h]^{-1}(x) = h \cdot \Xi^{-1}(x) \qquad \forall\, x \in \mathcal{V}.$$

Let now $\mathrm{C} \subseteq \mathcal{U}$ be a fixed compact:

- We choose $\widetilde{\mathrm{C}}, U \subseteq \mathcal{U}$ open with $\mathrm{C} \subseteq \widetilde{\mathrm{C}}$ and $e \in U$, such that $\widetilde{\mathrm{C}} \cdot U \subseteq \mathcal{U}$ holds.
- We let $U' := \Xi(U)$, and observe that

  $$\xi\colon \widetilde{\mathrm{C}} \times U' \to \mathcal{V}, \qquad (\widetilde{c}, u') \mapsto (\Xi \circ \mathrm{m})(\widetilde{c}, \Xi^{-1}(u'))$$

  is defined and smooth; i.e., that $\Theta \equiv \partial_2 \xi\colon \widetilde{C} \times U' \times E \to E$ is continuous, and linear in E.

Let now $\mathfrak{p} \in \mathfrak{P}$ be fixed:

- Corollary 1, applied to $\Phi \equiv \Theta$, $X \equiv \widetilde{C} \times U'$, $F_1 \equiv E$, and $\mathrm{K} \equiv \mathrm{C} \times \{0\}$, provides us with an open subset $O \subseteq \widetilde{C} \times U'$ containing $\mathrm{C} \times \{0\}$, as well as $\mathfrak{u} \equiv \mathfrak{q}_1 \in \mathfrak{P}$, such that

  $$(\mathfrak{p} \circ \Theta)(z, X) \leq \mathfrak{u}(X) \qquad \forall\, z \in O,\ \ X \in E$$

  holds. We fix an open neighbourhood $W \subseteq \mathcal{U}$ of e with $\mathrm{C} \cdot W \times \Xi(W) \subseteq O$, and obtain

$$(37)\quad (\mathfrak{p} \circ \Theta)(g \cdot h, \Xi(q), X) \leq \mathfrak{u}(X) \qquad \forall\, g \in \mathrm{C},\ \ h \in W,\ \ q \in W,\ \ X \in E.$$

- Here, we can assume that $\Xi(W)$ is convex; and choose $V \subseteq W$ symmetric open with $e \in V$ and $V \cdot V \subseteq W$. Moreover, since $\mathfrak{P}$ is filtrating, we can additionally assume that $\overline{\mathrm{B}}_{\mathfrak{u},1} \subseteq \Xi(V)$ holds.

We obtain

Lemma 8. *Let* $\mathrm{C} \subseteq \mathcal{U}$ *be compact. Then, for each* $\mathfrak{p} \in \mathfrak{P}$*, there exists some* $\mathfrak{p} \leq \mathfrak{u} \in \mathfrak{P}$*, and a symmetric open neighbourhood* $V \subseteq \mathcal{U}$ *of* e *with* $\mathrm{C} \cdot V \subseteq \mathcal{U}$ *and* $\overline{\mathrm{B}}_{\mathfrak{u},1} \subseteq \Xi(V)$*, such that*

$$\mathfrak{p}(\Xi(q) - \Xi(q')) \leq \mathfrak{u}(\Xi_{g \cdot h}(q) - \Xi_{g \cdot h}(q')) \qquad \forall\, q, q' \in g \cdot V,\;\; h \in V$$

holds for each $g \in \mathrm{C}$.

Proof. We choose $V, W, \mathfrak{u}$ as above. Then, for $g \in \mathrm{C}$, $q, q' \in g \cdot V$, and $h \in V$ fixed, we define

- $x := \Xi_{g \cdot h}(q)$, $x' := \Xi_{g \cdot h}(q') \;\in\; \Xi(V \cdot V) \subseteq \Xi(W)$,
- $\delta\colon [0,1] \to \Xi(W), \quad t \mapsto x' + t \cdot (x - x')$,
- $\gamma := \xi(g \cdot h, \delta)$.

We conclude from (37) and Lemma 6 that

$$\begin{aligned}
\mathfrak{p}(\Xi(q) - \Xi(q')) &= \mathfrak{p}(\xi(g \cdot h, \delta(1)) - \xi(g \cdot h, \delta(0))) \\
&= \mathfrak{p}(\gamma(1) - \gamma(0)) = \mathfrak{p}\big(\textstyle\int \dot{\gamma}(s)\, \mathrm{d}s\big) \\
&= \mathfrak{p}\big(\textstyle\int \Theta(g \cdot h, \delta(s), \dot{\delta}(s))\, \mathrm{d}s\big) \\
&\leq \textstyle\int \mathfrak{u}(\dot{\delta}(s))\, \mathrm{d}s = \int \mathfrak{u}(x - x')\, \mathrm{d}s \\
&= \mathfrak{u}(\Xi_{g \cdot h}(q) - \Xi_{g \cdot h}(q'))
\end{aligned}$$

holds, which shows the claim. □

3.5. The evolution map

We now introduce the central object of this paper – the evolution map – and discuss its most important properties.

3.5.1. The right logarithmic derivative. The right logarithmic derivative is defined by

$$\delta^r\colon C^1(D, G) \to C^0(D, \mathfrak{g}), \qquad \mu \mapsto \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}(\dot{\mu}) \qquad\quad \forall\, D \in \mathfrak{I}.$$

Then, for each $\mu \in C^1(D,G)$, $g \in G$, $\mathfrak{I} \ni D' \subseteq D$, and each $\varrho\colon \mathfrak{I} \ni D'' \to D$ of class C^1, we have

$$\begin{aligned} \delta^r(\mu \cdot g) &= \delta^r(\mu) \\ \delta^r(\mu|_{D'}) &= \delta^r(\mu)|_{D'} \\ \delta^r(\mu \circ \varrho) &= \dot{\varrho} \cdot (\delta^r(\mu) \circ \varrho). \end{aligned} \tag{38}$$

Moreover, for $\mu, \nu \in C^1(D,G)$, we conclude from the product rule (15) that

$$\delta^r(\mu \cdot \nu) = \delta^r(\mu) + \mathrm{Ad}_\mu(\delta^r(\nu)) \tag{39}$$

holds; thus,

$$0 = \delta^r(\mu^{-1}\mu) = \delta^r(\mu^{-1}) + \mathrm{Ad}_{\mu^{-1}}(\delta^r(\mu)) \tag{40}$$

$$\implies \quad \delta^r(\mu^{-1}) = -\mathrm{Ad}_{\mu^{-1}}(\delta^r(\mu))$$

$$\delta^r(\mu^{-1}\nu) = \delta^r(\mu^{-1}) + \mathrm{Ad}_{\mu^{-1}}(\delta^r(\nu)). \tag{41}$$

Here, we denote $\mu^{-1} := \mathrm{inv} \circ \mu$ for each $\mu \in C^0(D,\mathfrak{g})$ in the following. Then, combining (41) with the second line in (40), we obtain

$$\delta^r(\mu^{-1}\nu) = \mathrm{Ad}_{\mu^{-1}}(\delta^r(\nu) - \delta^r(\mu)) \qquad \forall\, \mu, \nu \in C^1(D,G). \tag{42}$$

We conclude that

Lemma 9. *Let $\mu, \nu \in C^1(D,G)$ for $D \in \mathfrak{I}$ be given. Then, we have*

$$\delta^r(\mu) = \delta^r(\nu) \qquad \Longleftrightarrow \qquad \nu = \mu \cdot g \quad \text{holds for some} \quad g \in G.$$

Proof. By (38), we have $\delta^r(\mu) = \delta^r(\mu \cdot g)$ for each $g \in G$; which shows the one direction. For the other direction, we fix $\tau \in D$, define $\alpha := \mu^{-1}\nu \cdot g$ for $g := \nu^{-1}(\tau) \cdot \mu(\tau)$, and obtain

$$\delta^r(\alpha) \overset{(38)}{=} \delta^r(\mu^{-1}\nu) \overset{(42)}{=} 0;$$

thus, $\dot{\alpha} = 0$ as $\mathrm{d}_q\mathrm{R}_{q^{-1}}$ is bijective for each $q \in G$. For each $[r,r'] \subseteq D$ with $\tau \in [r,r']$ and $\alpha([r,r']) \subseteq \mathfrak{U}$, we thus obtain from (24) that $(\Xi \circ \alpha)|_{[r,r']} = 0$ holds; so that the claim follows from a standard supremum-contradiction argument. □

We furthermore obtain

Lemma 10. *Let $D \in \mathfrak{I}$ and $k \in \mathbb{N}$ be fixed. Then,*

1) $\delta^r(\mu) \in C^k(D,\mathfrak{g})$ *holds for each* $\mu \in C^{k+1}(D,G)$.
2) $\mu \in C^{k+1}(D,G)$ *holds for each* $\mu \in C^1(D,G)$ *with* $\delta^r(\mu) \in C^k(D,\mathfrak{g})$.

Proof. By the second identity in (38), in both situations it suffices to show that for each $t \in D$ there exists an open interval $J \subseteq \mathbb{R}$ containing t, such that the claim holds for $\nu := \mu|_{D\cap J}$. Moreover, by the first identity in (38), we can additionally assume that $\mathrm{im}[\nu] \subseteq \mathcal{U}$ holds, just by shrinking J if necessary. We let $\gamma := \Xi \circ \nu$, and obtain

- $\delta^r(\nu) = \omega(\gamma, \dot\gamma)$ for ω defined by (29); so that 1) is clear from Lemma 4.
- $\dot\gamma = \upsilon(\gamma, \delta^r(\nu))$ for υ defined by (30); so that 2) is clear from Corollary 2.

This establishes the proof. □

Finally, if H is a Lie group, and $\Psi\colon G \to H$ a C^1-Lie group homomorphism, we immediately obtain

$$\delta^r(\Psi \circ \mu) = \mathrm{d}_e\Psi \circ \delta^r(\mu) \qquad \forall\, \mu \in C^1(D,H),\ \ D \in \mathfrak{I}. \tag{43}$$

3.5.2. The product integral. We define

$$\mathfrak{D} := \bigsqcup\nolimits_{[r,r']\in\mathfrak{K}} \mathfrak{D}_{[r,r']} \quad \text{with} \quad \mathfrak{D}_{[r,r']} := \delta^r(C^1([r,r'],G)) \quad \text{for each} \quad [r,r'] \in \mathfrak{K}.$$

Then, Lemma 9 shows that $\mathrm{Evol}_{[r,r']}\colon \mathfrak{D}_{[r,r']} \to C^1([r,r'],G)$ given by

$$\mathrm{Evol}_{[r,r']}(\delta^r(\mu)) := \mu \cdot \mu^{-1}(r) \qquad \forall\, \mu \in C^1([r,r'],G),\ \ [r,r'] \in \mathfrak{K}$$

is well defined; and we let

$$\mathrm{Evol}^k_{[r,r']} \equiv \mathrm{Evol}|_{\mathfrak{D}^k_{[r,r']}} \qquad \text{for} \qquad \mathfrak{D}^k_{[r,r']} := \mathfrak{D}_{[r,r']} \cap C^k([r,r'],\mathfrak{g})$$
$$\mathrm{evol}^k_{[r,r']}\colon \mathfrak{D}^k_{[r,r']} \ni \phi \mapsto \mathrm{Evol}_{[r,r']}(\phi)(r') \in G$$

for $[r,r'] \in \mathfrak{K}$ and $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty\}$. Moreover, for $k \in \mathbb{N} \sqcup \{\infty\}$ and $[r,r'] \in \mathfrak{K}$, we define

$$C^k_*([r,r'],G) := \{\mu \in C^k([r,r'],G) \mid \mu(r) = e\};$$

and obtain that

Corollary 4. *For each* $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty\}$ *and* $[r,r'] \in \mathfrak{K}$*, we have*

$$\mathrm{Evol}^k_{[r,r']}\colon \mathfrak{D}^k_{[r,r']} \to C^{k+1}_*([r,r'],G).$$

Proof. The claim is clear from Lemma 10.2). □

The product integral is given by[7]

$$\textstyle\int_a^b \phi := \mathrm{Evol}^k_{[a,b]}(\phi|_{[a,b]})(b), \qquad \int \phi := \int_r^{r'} \phi, \qquad \int_c^c \phi := e,$$
$$\textstyle\int_r^\bullet \phi\colon [r,r'] \ni t \mapsto \int_r^t \phi$$

for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, $r \leq a < b \leq r'$, and $c \in [r,r']$. Then,

a) We conclude from (39) that

$$\textstyle\int_r^t \phi \cdot \int_r^t \psi = \int_r^t \phi + \mathrm{Ad}_{\int_r^\bullet \phi}(\psi) \qquad \forall\, \phi,\psi \in \mathfrak{D}_{[r,r']},\ \ t \in [r,r'].$$

b) We conclude from (42) that

$$\textstyle\big[\int_r^t \phi\big]^{-1}\big[\int_r^t \psi\big] = \int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \forall\, \phi,\psi \in \mathfrak{D}_{[r,r']},\ \ t \in [r,r'].$$

c) We conclude from (40) that

$$\textstyle\big[\int_r^t \phi\big]^{-1} = \int_r^t -\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\phi) \qquad \forall\, \phi \in \mathfrak{D}_{[r,r']},\ \ t \in [r,r'].$$

d) For $r = t_0 < \ldots < t_n = r'$ and $\phi \in \mathfrak{D}_{[r,r']}$, we conclude from the first two identities in (38) that

$$\textstyle\int_r^t \phi = \int_{t_p}^t \phi \cdot \int_{t_{p-1}}^{t_p} \phi \cdot \ldots \cdot \int_{t_0}^{t_1} \phi \qquad \forall\, t \in (t_p, t_{p+1}],\ \ p = 0, \ldots, n-1.$$

e) For $\varrho\colon [\ell,\ell'] \to [r,r']$ of class C^1, we conclude from the last identity in (38) that

$$\textstyle\int_r^\varrho \phi = \big[\int_\ell^\bullet \dot\varrho \cdot (\phi \circ \varrho)\big] \cdot \big[\int_r^{\varrho(\ell)} \phi\big] \qquad \forall\, \phi \in \mathfrak{D}_{[r,r']}.$$

f) We conclude from (43) that for each C^1-Lie group homomorphism $\Psi\colon G \to H$, we have

$$\textstyle\Psi \circ \int_r^\bullet \phi = \int_r^\bullet \mathrm{d}_e\Psi \circ \phi \qquad \forall\, \phi \in \mathfrak{D}_{[r,r']}.$$

---

[7]Observe that the first expression is defined by the second equality in (38) as well as Lemma 10.

Example 1. *For $[r,r']\in\mathfrak{K}$ fixed, we let $\varrho\colon [r,r']\to[r,r'],\ t\mapsto r+r'-t$; and define*

$$\mathfrak{D}_{[r,r']}\ni \mathfrak{inv}(\phi):=\dot{\varrho}\cdot(\phi\circ\varrho)\colon [r,r']\ni t\mapsto -\phi(r+r'-t)\qquad \forall\,\phi\in\mathfrak{D}_{[r,r']}.$$

We let $[\ell,\ell']\equiv[r,r']$; and obtain from e) that

$$e=\textstyle\int_r^{\varrho(r')}\phi\overset{e)}{=}\big[\int_r^{r'}\mathfrak{inv}(\phi)\big]\cdot\big[\int_r^{r'}\phi\big]\qquad \textit{holds, thus}\qquad [\int\phi]^{-1}=\int\mathfrak{inv}(\phi),$$

which will be useful for our argumentation in Sect. 7.2.3. ‡

Lemma 11. *Let $[r,r']\in\mathfrak{K}$, and $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ be fixed; and suppose that we are given $\phi\in C^k([r,r'],\mathfrak{g})$ and $r=t_0<\ldots<t_n=r'$, such that $\phi|_{[t_p,t_{p+1}]}\in\mathfrak{D}^k_{[t_p,t_{p+1}]}$ holds for $p=0,\ldots,n-1$. Then, we have $\phi\in\mathfrak{D}^k_{[r,r']}$ with*

$$\textstyle\int_r^t\phi=\int_{t_p}^t\phi\cdot\int_{t_{p-1}}^{t_p}\phi\cdot\ldots\cdot\int_{t_0}^{t_1}\phi\qquad \forall\, t\in(t_p,t_{p+1}],\ \ p=0,\ldots,n-1.$$

Proof. The proof is elementary, and can be found in Appendix A.7. □

3.5.3. Semiregularity. We say that G is ***C^k*-semiregular** for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$ *iff* $\mathfrak{D}^k_{[0,1]}=C^k([0,1],\mathfrak{g})$ holds. Then,

Lemma 12. *G is C^k-semiregular iff*

$$\mathfrak{D}^k_{[r,r']}=C^k([r,r'],\mathfrak{g})\qquad \textit{holds for each}\qquad [r,r']\in\mathfrak{K}.$$

Proof. The one direction is evident. For the other direction, we fix $[r,r']\in\mathfrak{K}$, and let

$$\varrho\colon [r,r']\to[0,1],\qquad t\mapsto |t-r|/|r'-r|.$$

Then, for $\phi\in C^k([r,r'],\mathfrak{g})$ given, we define $\psi:=|r'-r|\cdot\phi\circ\varrho^{-1}\in C^k([0,1],\mathfrak{g})$, and choose $\nu\in C^{k+1}([0,1],\mathfrak{g})$ with $\delta^r(\nu)=\psi$. Then, the last identity in (38) gives

$$\delta^r(\nu\circ\varrho)=|r'-r|^{-1}\cdot(\psi\circ\varrho)=\phi,$$

which proves the claim. □

We say that G **admits an exponential map** *iff* $\phi_X|_{[0,1]}\in\mathfrak{D}_{[0,1]}$ holds for each constant curve $\phi_X\colon\mathbb{R}\ni t\mapsto X\in\mathfrak{g}$; i.e., *iff*

$$\exp\colon \mathfrak{g}\ni X\mapsto \textstyle\int_0^1\phi_X\in G$$

is defined.

3.5.4. Continuity. We say that $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous for $p \leq k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ (we let $0 \leq \mathrm{lip} \leq \mathrm{lip} \leq 1$) and $[r,r'] \in \mathfrak{K}$ *iff* it is continuous w.r.t. the seminorms $\{\boldsymbol{.}\mathfrak{p}^{\mathrm{s}}_\infty\}_{\mathfrak{p}\in\mathfrak{P},\, \mathrm{s}\preceq p}$. We say that G is

- **p.k-continuous** for $p \preceq k$ *iff* $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous for each $[r,r'] \in \mathfrak{K}$,
- **k-continuous** *iff* G is k.k-continuous.

Then,

Lemma 13. *We have* $\mathrm{Ad}_\mu(\phi) \in C^k([r,r'],\mathfrak{g})$ *for each* $\mu \in C^{k+1}([r,r'],G)$, $\phi \in C^k([r,r'],\mathfrak{g})$, *and* $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$.

Proof. Since $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ is smooth, the claim is clear for $k \in \mathbb{N} \sqcup \{\infty\}$. The case where $k = \mathrm{lip}$ holds is proven in Appendix A.8. □

Lemma 14. *Let* $[r,r'] \in \mathfrak{K}$, $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, *and* $\phi \in \mathfrak{D}^k_{[r,r']}$ *be fixed. Then, for each* $\mathfrak{p} \in \mathfrak{P}$ *and* $\mathrm{s} \preceq k$, *there exists some* $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ *with*

$$\boldsymbol{.}\mathfrak{p}^{\mathrm{p}}\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)\big) \leq \boldsymbol{.}\mathfrak{q}^{\mathrm{p}}(\psi) \qquad \forall\, \psi \in C^k([r,r'],\mathfrak{g}),\;\; 0 \leq \mathrm{p} \leq \mathrm{s}.$$

Proof. Decomposing $[r,r']$ if necessary, we can assume that $\mathrm{im}[\int_r^\bullet \phi]$ is contained in the domain of a fixed chart $\widetilde{\Xi}$. The claim then follows from Lemma 5.2), applied to $\Omega \equiv \mathrm{Ad}(\mathrm{inv} \circ \widetilde{\Xi}^{-1}(\cdot), \cdot)$, $\mathrm{u} \equiv \mathrm{s}$, and the C^{s}-curve $\gamma \equiv \widetilde{\Xi} \circ \int_r^\bullet \phi$. □

We obtain that

Lemma 15. G *is* p.k*-continuous iff* $\mathrm{evol}^k_{[0,1]}$ *is* C^p*-continuous at zero.*

Proof. The one direction is evident; and the other direction follows from Lemma 13, Lemma 14, and *b)* once we have shown that $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous at zero if $\mathrm{evol}^k_{[r,r']}$ is C^p-continuous at zero. For this, we apply *e)* to the map

$$\varrho\colon [0,1] \to [r,r'], \qquad t \mapsto r + t \cdot |r' - r|;$$

and conclude that $\mathrm{evol}^k_{[r,r']} = \mathrm{evol}^k_{[0,1]} \circ \eta$ holds, for the C^p-continuous map (use **d)**)

$$\eta\colon C^k([r,r'],\mathfrak{g}) \to C^k([0,1],\mathfrak{g}), \qquad \phi \mapsto \dot{\varrho} \cdot (\phi \circ \varrho) \equiv |r' - r| \cdot (\phi \circ \varrho).$$

From this, the claim is clear. □

3.6. Supplementary material

In this subsection, we provide the proofs of the supplementary statements made but not verified in Sect. 2. First,

Lemma 16. *Suppose that G is abelian; and let $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ be fixed. Then, G is* k*-continuous iff G is* $0.\mathrm{k}$*-continuous.*

Proof. The one directions is evident. For the other direction, we suppose that G is k-continuous. Then, $\mathfrak{p} \in \mathfrak{P}$ given, there exist $\mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that

$$(44) \qquad {}_{\bullet}\mathfrak{q}^{\mathrm{s}}_{\infty}(\psi) \leq 1 \quad \text{for} \quad \psi \in \mathfrak{D}^k_{[0,1]} \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)(\textstyle\int \psi) \leq 1.$$

Then, for $\phi \in \mathfrak{D}^k_{[0,1]}$ with ${}_{\bullet}\mathfrak{q}_{\infty}(\phi) \leq 1$, we choose $n \geq 1$ such large that ${}_{\bullet}\mathfrak{q}^{\mathrm{s}}_{\infty}(\phi) \leq n$ holds; and define

$$\psi_p := \phi \circ \varrho_p \quad \text{for} \quad \varrho_p \colon [0, 1/n] \ni t \mapsto p/n + t \in [p/n, (p+1)/n]$$
$$\forall\, p = 0, \dots, n-1.$$

By *e)*, we have $\int \phi|_{[p/n,(p+1)/n]} = \int \psi_p$ for $p = 0, \dots n-1$; and obtain from *a)*, *d)*, and *e)* that[8]

$$(45) \qquad \begin{aligned} \textstyle\int \phi = \int \psi_{n-1} \cdot \ldots \cdot \int \psi_0 &= \textstyle\int_0^{1/n} \psi_{n-1} + \ldots + \psi_0 \\ &= \textstyle\int_0^1 \underbrace{1/n \cdot (\psi_0 + \ldots + \psi_{n-1}) \circ \varrho}_{\psi \in \mathfrak{D}^k_{[0,1]}} \end{aligned}$$

holds, for $\varrho\colon [0,1] \ni t \mapsto t/n \in [0, 1/n]$. Then, **d**) gives $\mathfrak{q}^{\mathrm{s}}_{\infty}(\psi) \leq 1$; so that (44) provides us with

$$(\mathfrak{p} \circ \Xi)\big(\textstyle\int \phi\big) \overset{(45)}{=} (\mathfrak{p} \circ \Xi)\big(\textstyle\int \psi\big) \leq 1.$$

The rest is clear from Lemma 15. □

Second, let us say that G is L^1-continuous *iff* $\mathrm{evol}^0_{[r,r']}$ is continuous w.r.t. the seminorms (12) for each $[r, r'] \in \mathfrak{K}$. Then,

Lemma 17. *G is 0-continuous iff G is L^1-continuous.*

[8] It is obvious from the definitions that $\mathrm{Ad}_g = \mathrm{id}_{\mathfrak{g}}$ holds for each $g \in G$ if G is abelian.

Proof. The one direction is evident. Let thus G be 0-continuous, fix $\mathfrak{p} \in \mathfrak{P}$; and choose $\mathfrak{q} \in \mathfrak{P}$ with

$$(46) \qquad \mathfrak{q}_\infty(\psi) \leq 1 \quad \text{for} \quad \psi \in \mathfrak{D}^0_{[0,2]} \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)(\textstyle\int \psi) \leq 1.$$

Then, for $\phi \in \mathfrak{D}^0_{[r,r']}$ with $\mathfrak{q}_\int(\phi) \leq 1$, we define

$$\lambda\colon [r,r'] \to [0,2], \qquad t \mapsto \tfrac{t-r}{r'-r} \cdot (2 - \mathfrak{q}_\int(\phi)) + \textstyle\int_0^t \mathfrak{q}(\phi(s))\,\mathrm{d}s;$$

and consider the C^1-diffeomorphism $\varrho := \lambda^{-1}\colon [0,2] \to [r,r']$. Then, $\int \phi = \int \psi$ holds for $\psi := \dot\varrho \cdot (\phi \circ \varrho) \in \mathfrak{D}^0_{[0,2]}$ by *e)*, with

$$\dot\varrho = (\dot\lambda \circ \varrho)^{-1} = (2 - \mathfrak{q}_\int(\phi))/|r' - r| + \mathfrak{q}(\phi \circ \varrho))^{-1} \leq \mathfrak{q}(\phi \circ \varrho)^{-1}.$$

We thus have $\mathfrak{q}_\infty(\psi) \leq 1$; so that the claim is clear from (46). □

Finally, let us collect some properties of the exponential map.

Remark 2.

1) *Suppose we have* $\phi_X|_{[0,1]} \in \mathfrak{D}_{[0,1]}$ *for some* $X \in \mathfrak{g}$*. Then, Lemma 11 (and* e)*) shows that* $\phi_X|_{[0,n]} \in \mathfrak{D}_{[0,n]}$ *holds for each* $n \geq 1$*; and,* e) *applied to* $\varrho\colon [0,1] \to [0, s \cdot n], \quad t \mapsto s \cdot n \cdot t$ *for* $0 < s \leq 1$*, gives*

$$\textstyle\int_0^{s\cdot n} \phi_X \overset{\text{e)}}{=} \int_0^1 \phi_{s\cdot n\cdot X} \equiv \exp(s \cdot n \cdot X) \qquad \forall\, 0 < s \leq 1.$$

We thus have $\mathbb{R}_{\geq 0} \cdot X \subseteq \mathrm{dom}[\exp]$ *with*

$$(47) \qquad \exp(t \cdot X) = \textstyle\int_0^t \phi_X \qquad \forall\, t \geq 0.$$

It follows that $\mathbb{R} \ni t \mapsto \exp(t \cdot X)$ *is a smooth Lie group homomorphism, cf. Appendix A.9.*

2) *Suppose that* G *is* C^∞*-semiregular; and that* $\mathrm{evol}^\infty_{[0,1]}$ *is of class* C^p *w.r.t. the* C^∞*-topology, for some* $p \in \mathbb{N} \sqcup \{\infty\}$*. Then,* $\exp$ *is of class* C^p*, because*

$$\mathfrak{p}^{\mathrm{s}}_\infty(\phi_X) = \mathfrak{p}(X) \qquad \forall\, \mathfrak{p} \in \mathfrak{P},\ \mathrm{s} \in \mathbb{N},\ X \in \mathfrak{g}$$

shows that $\mathfrak{g} \ni X \mapsto \phi_X \in C^\infty([0,1],\mathfrak{g})$ *is smooth.*

3) If G is abelian with $\exp\colon \mathfrak{g} \to G$ *of class C^1, then we have (cf. Appendix A.10)*

$$\textstyle\int \phi = \exp(\int \phi(s)\,\mathrm{d}s) \qquad \textit{for each} \qquad \phi \in C^0([0,1],\mathfrak{g})$$
$$\textit{with} \qquad \textstyle\int_0^t \phi(s)\,\mathrm{d}s \in \mathfrak{g} \qquad \textit{for each} \qquad t \in [0,1];$$

which is obviously continuous w.r.t. the seminorms $\mathbf{.}\mathfrak{p}_\infty, \mathbf{.}\mathfrak{p}_\int$ *for* $\mathfrak{p} \in \mathfrak{P}$. ‡

4. Auxiliary results

In this section, we prove further continuity statements for the evolution map; and discuss piecewise integrable curves.

4.1. Continuity of the evolution map

Lemma 18. *Suppose that G is* p.k-continuous; and let $[r, r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exist $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq p$, such that

$$\mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) \leq 1 \quad \text{for} \quad \phi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)(\textstyle\int_r^\bullet \phi) \leq 1.$$

Proof. By continuity, there exist $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq p$, such that

$$\mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\psi) \leq 1 \quad \text{for} \quad \psi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)(\textstyle\int_r^\bullet \psi) \leq 1. \tag{48}$$

Let now $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) \leq 1$, and $r < \tau \leq r'$ be fixed. We define $\psi := \phi|_{[r,\tau]}$ as well as

$$\varrho\colon [r,r'] \to [r,\tau], \qquad t \mapsto r + |t-r| \cdot c \qquad \text{for} \qquad c := \tfrac{\tau - r}{r' - r} \leq 1.$$

Then, $\int_r^\tau \phi \equiv \int \psi = \int \dot{\varrho} \cdot (\psi \circ \varrho)$ holds by *e)*, with $\dot{\varrho} \cdot (\psi \circ \varrho) \in \mathfrak{D}^k_{[r,r']}$ as well as

$$\mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\dot{\varrho} \cdot (\psi \circ \varrho)) = \mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(c \cdot (\psi \circ \varrho)) \overset{\mathbf{d)}}{\leq} \mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) \leq 1.$$

We thus obtain from (48) that

$$(\mathfrak{p} \circ \Xi)(\textstyle\int_r^\tau \phi) = (\mathfrak{p} \circ \Xi)(\int \psi) = (\mathfrak{p} \circ \Xi)(\int \dot{\varrho} \cdot \psi \circ \varrho) \leq 1$$

holds, from which the claim is clear. □

We inductively obtain

Lemma 19. *Suppose that G is* k*-continuous; and let $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{u} \preceq k$, there exist $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that*

$$\boldsymbol{.}\mathfrak{q}^{\mathrm{s}}_{\infty}(\phi) \leq 1 \quad \textit{for} \quad \phi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad \mathfrak{p}^{\mathrm{u}}_{\infty}\big(\Xi \circ \textstyle\int_r^\bullet \phi\big) \leq 1.$$

Confer [3] for the case that G is C^k-semiregular .

Proof. By Lemma 18, we can assume that the claim is proven for some $0 \leq \mathrm{u} < k$. In particular, there exist $\mathfrak{m} \in \mathfrak{P}$ and $\mathrm{o} \preceq k$, such that

$$\begin{aligned} \boldsymbol{\gamma} := \Xi \circ \mathrm{Evol}^k_{[r,r']} \colon \{\phi \in \mathfrak{D}^k_{[r,r']} \,|\, \boldsymbol{.}\mathfrak{m}^{\mathrm{o}}_{\infty}(\phi) \leq 1\} &\to C^{k+1}_*([r,r'],\mathcal{V}) \\ \phi &\mapsto \Xi \circ \textstyle\int_r^\bullet \phi \end{aligned}$$

is defined. Let thus $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\boldsymbol{.}\mathfrak{m}^{\mathrm{o}}_{\infty}(\phi) \leq 1$ be given. Then,

- We have $\boldsymbol{\gamma}(\phi)^{(1)} = \upsilon(\boldsymbol{\gamma}(\phi),\phi)$, for υ defined by (30); so that Corollary 3 shows that $\boldsymbol{\gamma}(\phi)^{(\mathrm{u}+1)} = \sum_{i=1}^d \alpha_i(\phi)$ holds, with

  $$\alpha_i \colon \phi \mapsto ([\partial_1]^{m_i}\upsilon)\big(\boldsymbol{\gamma}(\phi), \boldsymbol{\gamma}(\phi)^{(z[i]_1)}, \dots, \boldsymbol{\gamma}(\phi)^{(z[i]_{m_i})}, \phi^{(q_i)}\big)$$

  for certain $0 \leq z[i]_1, \dots, z[i]_{m_i}, q_i \leq \mathrm{u}$ and $m_i \geq 1$, for $i = 1,\dots,d$.
- For $\mathfrak{p} \in \mathfrak{P}$ fixed, Lemma 1 provides us with an open neighbourhood $V \subseteq \mathcal{V}$ of 0, as well as $\mathfrak{w} \in \mathfrak{P}$, such that

  $$\begin{aligned} &(\mathfrak{p} \circ [\partial_1]^{m_i}\upsilon)\big(x, \boldsymbol{\gamma}(\phi)^{(z[i]_1)}, \dots, \boldsymbol{\gamma}(\phi)^{(z[i]_{m_i})}, \phi^{(q_i)}\big) \\ &\qquad \leq \mathfrak{w}\big(\boldsymbol{\gamma}(\phi)^{(z[i]_1)}\big) \cdot \ldots \cdot \mathfrak{w}\big(\boldsymbol{\gamma}(\phi)^{(z[i]_{m_i})}\big) \cdot \boldsymbol{.}\mathfrak{w}\big(\phi^{(q_i)}\big) \\ &\qquad \leq \big[\mathfrak{w}^{\mathrm{u}}_{\infty}(\boldsymbol{\gamma}(\phi))\big]^{m_i} \cdot \boldsymbol{.}\mathfrak{w}^{\mathrm{u}}_{\infty}(\phi) \end{aligned} \tag{49}$$

  holds, for each $x \in V$ and $i = 1,\dots,d$.

We choose $V \prec \mathfrak{v} \in \mathfrak{P}$ with $d \cdot \mathfrak{w}, \mathfrak{p}, \mathfrak{m} \leq \mathfrak{v}$; and apply the induction hypotheses in order to fix $\mathfrak{v} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{o} \preceq \mathrm{s} \preceq k$, such that

$$\boldsymbol{.}\mathfrak{q}^{\mathrm{s}}_{\infty}(\phi) \leq 1 \quad \text{for} \quad \phi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad \mathfrak{v}^{\mathrm{u}}_{\infty}(\boldsymbol{\gamma}(\phi)) \equiv \mathfrak{v}^{\mathrm{u}}_{\infty}\big(\Xi \circ \textstyle\int_r^\bullet \phi\big) \leq 1.$$

In particular, then for $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\boldsymbol{.}\mathfrak{q}^{\mathrm{s}}_{\infty}(\phi) \leq 1$, we have

- $\mathrm{im}[\boldsymbol{\gamma}(\phi)] \subseteq V$ and $\boldsymbol{.}\mathfrak{q}^{\mathrm{o}}_{\infty}(\phi) \leq 1$; so that (49) gives

  $$\mathfrak{p}\big(\boldsymbol{\gamma}(\phi)^{(\mathrm{u}+1)}\big) \leq d \cdot \boldsymbol{.}\mathfrak{w}^{\mathrm{u}}_{\infty}(\phi) \leq \boldsymbol{.}\mathfrak{v}^{\mathrm{u}}_{\infty}(\phi) \leq \boldsymbol{.}\mathfrak{q}^{\mathrm{u}}_{\infty}(\phi).$$

- $\mathfrak{p}^{\mathrm{u}}_{\infty}(\boldsymbol{\gamma}(\phi)) \leq \mathfrak{v}^{\mathrm{u}}_{\infty}(\boldsymbol{\gamma}(\phi)) \leq 1.$

For $\widetilde{\mathrm{s}} := \max(\mathrm{s},\mathrm{u})$ and $\phi \in \mathfrak{D}^k_{[r,r']}$ with ${\scriptscriptstyle\blacksquare}\mathfrak{q}^{\widetilde{\mathrm{s}}}_\infty(\phi) \leq 1$, we thus have

$$\mathfrak{p}\big(\boldsymbol{\gamma}(\phi)^{(\mathrm{u}+1)}\big) \leq {\scriptscriptstyle\blacksquare}\mathfrak{q}^{\mathrm{u}}_\infty(\phi) \leq {\scriptscriptstyle\blacksquare}\mathfrak{q}^{\widetilde{\mathrm{s}}}_\infty(\phi) \leq 1 \qquad \text{and} \qquad \mathfrak{p}^{\mathrm{u}}_\infty(\boldsymbol{\gamma}(\phi)) \leq 1;$$

thus, $\mathfrak{p}^{\mathrm{u}+1}_\infty(\boldsymbol{\gamma}(\phi)) \leq 1$. The claim thus follows inductively. □

We furthermore obtain that

Lemma 20. *Suppose that G is* p.k*-continuous; and let $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exist $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq p$, such that*

$${\scriptscriptstyle\blacksquare}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) \leq 1 \quad \textit{for} \quad \phi \in \mathfrak{D}^k_{[r,r']} \qquad \Longrightarrow \qquad (\mathfrak{p} \circ \Xi)\big(\textstyle\int_r^\bullet \phi\big) \leq \int_r^\bullet {\scriptscriptstyle\blacksquare}\mathfrak{q}(\phi(s))\,\mathrm{d}s.$$

Proof. We choose $\mathfrak{w}$ as in (33) for $\mathfrak{v} \equiv \mathfrak{p}$ there; and let $\mathfrak{q}$, s be as in Lemma 18 for $\mathfrak{p} \equiv \mathfrak{w}$ there (i.e., we have $\mathfrak{p} \leq \mathfrak{w} \leq \mathfrak{q}$). Then, for $\phi \in \mathfrak{D}^k([r,r'],\mathfrak{g})$ with ${\scriptscriptstyle\blacksquare}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) \leq 1$, we have $(\mathfrak{w} \circ \Xi)\big(\int_r^\bullet \phi\big) \leq 1$ by Lemma 18; and obtain from (33) that for $\gamma := \Xi \circ \mu$ with $\mu := \int_r^\bullet \phi$ we have

$$\mathfrak{p}(\dot{\gamma}) = \mathfrak{p}\big(\upsilon(\gamma, \delta^r(\mu))\big) \leq {\scriptscriptstyle\blacksquare}\mathfrak{w}(\phi) \leq {\scriptscriptstyle\blacksquare}\mathfrak{q}(\phi).$$

The claim thus follows from Lemma 6. □

4.2. Estimates in charts

In this subsection, we prove certain statements that we will need for our differentiability discussions in Sect. 8. We start with a variation of Lemma 14.

Lemma 21. *Suppose that G is* k*-continuous; and let $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{u} \preceq k$, there exist $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that*

$${\scriptscriptstyle\blacksquare}\mathfrak{p}^{\mathrm{p}}\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)\big) \leq {\scriptscriptstyle\blacksquare}\mathfrak{m}^{\mathrm{p}}(\psi) \qquad \forall\, \psi \in C^k([r,r'],\mathfrak{g}),\ \ 0 \leq \mathrm{p} \leq \mathrm{u}$$

holds for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with ${\scriptscriptstyle\blacksquare}\mathfrak{m}^{\mathrm{s}}_\infty(\phi) \leq 1$.

Proof. Since $\mathfrak{P}$ is filtrating, Lemma 5.1) applied to

$$\Omega\colon \mathcal{V} \times \mathfrak{g} \to \mathfrak{g}, \qquad (x, X) \mapsto \mathrm{Ad}((\mathrm{inv} \circ \Xi^{-1})(x), X)$$

provides us with some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$, such that for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with ${\scriptscriptstyle\blacksquare}\mathfrak{q}^{\mathrm{u}}_\infty\big(\Xi \circ \int_r^\bullet \phi\big) \leq 1$, we have

$${\scriptscriptstyle\blacksquare}\mathfrak{p}^{\mathrm{p}}\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi)\big) \leq {\scriptscriptstyle\blacksquare}\mathfrak{q}^{\mathrm{p}}(\psi) \qquad \forall\, \psi \in C^k([r,r'],\mathfrak{g}),\ \ 0 \leq \mathrm{p} \leq \mathrm{u}.$$

By Lemma 19, there exist $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that $\text{.}\mathfrak{q}^{\mathrm{u}}_\infty\big(\Xi \circ \int_r^\bullet \phi\big) \leq 1$ holds for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\text{.}\mathfrak{m}^{\mathrm{s}}_\infty(\phi) \leq 1$; from which the claim is clear. □

Together with Lemma 19, this shows

Lemma 22. *Suppose that G is* k*-continuous; and let $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exist $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ and $\mathrm{u} \preceq k$, such that*

$$(\mathfrak{p} \circ \Xi)\big([\textstyle\int_r^\bullet \phi]^{-1}[\int_r^\bullet \psi]\big) \leq \int_r^\bullet \text{.}\mathfrak{m}(\psi(s) - \phi(s))\, \mathrm{d}s$$

holds for all $\phi, \psi \in \mathfrak{D}^k_{[r,r']}$ with $\text{.}\mathfrak{m}^{\mathrm{u}}_\infty(\phi)$, $\text{.}\mathfrak{m}^{\mathrm{u}}_\infty(\psi - \phi) \leq 1$.

Proof. We choose $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$ as in Lemma 20. Then, Lemma 21 provides us with some $\mathfrak{q} \leq \mathfrak{m} \in \mathfrak{P}$ and $\mathrm{o} \preceq k$, such that for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\text{.}\mathfrak{m}^{\mathrm{o}}_\infty(\phi) \leq 1$, we have

$$\text{.}\mathfrak{q}^{\mathrm{p}}\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\chi)\big) \leq \text{.}\mathfrak{m}^{\mathrm{p}}(\chi) \qquad \forall\, \chi \in C^k([r,r'],\mathfrak{g}),\ 0 \leq \mathrm{p} \leq \mathrm{s}. \tag{50}$$

We let $\mathrm{u} := \max(\mathrm{o},\mathrm{s})$; and recall that, cf. *b)*

$$[\textstyle\int_r^t \phi]^{-1}[\int_r^t \psi] = \int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \forall\, t \in [r,r'],\ \phi,\psi \in \mathfrak{D}^k_{[r,r']} \tag{51}$$

holds. For $\phi, \psi \in \mathfrak{D}^k_{[r,r']}$ with $\text{.}\mathfrak{m}^{\mathrm{u}}_\infty(\phi)$, $\text{.}\mathfrak{m}^{\mathrm{u}}_\infty(\psi - \phi) \leq 1$, we thus have

$$\text{.}\mathfrak{q}^{\mathrm{s}}_\infty\big(\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi)\big) \overset{(50)}{\leq} \mathfrak{m}^{\mathrm{s}}_\infty(\psi - \phi) \leq \mathfrak{m}^{\mathrm{u}}_\infty(\psi - \phi) \leq 1;$$

so that the claim is clear from Lemma 20, (51), and (50) for $\mathrm{p} \equiv 0$ there. □

We conclude that

Proposition 1. *Suppose that G is* k*-continuous; and let $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exist $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that*

$$\mathfrak{p}\big(\Xi\big(\textstyle\int_r^\bullet \phi\big) - \Xi\big(\int_r^\bullet \psi\big)\big) \leq \int_r^\bullet \text{.}\mathfrak{m}(\psi(s) - \phi(s))\, \mathrm{d}s$$

holds for all $\phi, \psi \in \mathfrak{D}^k_{[r,r']}$ with $\text{.}\mathfrak{m}^{\mathrm{s}}_\infty(\phi)$, $\text{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi)$, $\text{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi - \phi) \leq 1$.

Proof. We let $\mathfrak{p} \leq \mathfrak{u} \in \mathfrak{P}$ and V be as in Lemma 8 for $\mathrm{C} \equiv \{e\}$ there, i.e., we have $\overline{\mathrm{B}}_{\mathfrak{u},1} \subseteq \Xi(V)$. By Lemma 18, there exist $\mathfrak{u} \leq \mathfrak{q} \in \mathfrak{P}$ and $\mathrm{o} \preceq k$, such that

$$\mathbf{.}\mathfrak{q}^{\mathrm{o}}_\infty(\chi)\leq 1 \quad\text{ for }\quad \chi\in\mathfrak{D}^k_{[r,r']} \qquad\Longrightarrow\qquad (\mathfrak{u}\circ\Xi)\big(\textstyle\int_r^\bullet \chi\big)\leq 1$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Longrightarrow\qquad \textstyle\int_r^\bullet \chi\in V.$$

Then, for ϕ,ψ with $\mathbf{.}\mathfrak{q}^{\mathrm{o}}_\infty(\phi)\leq 1$, $\mathbf{.}\mathfrak{q}^{\mathrm{o}}_\infty(\psi)\leq 1$, Lemma 8 applied to $q\equiv \int_r^\bullet\phi$, $q'\equiv\int_r^\bullet\psi$, $h\equiv\int_r^\bullet\phi\in V$, and $g\equiv e$ gives

$$\begin{aligned}\mathfrak{p}\big(\Xi\big(\textstyle\int_r^\bullet\phi\big)-\Xi\big(\int_r^\bullet\psi\big)\big) &\leq \mathfrak{u}\big(\Xi_{\int_r^\bullet\phi}\big(\textstyle\int_r^\bullet\phi\big)-\Xi_{\int_r^\bullet\phi}\big(\int_r^\bullet\psi\big)\big)\\ &= (\mathfrak{u}\circ\Xi)\big([\textstyle\int_r^\bullet\phi]^{-1}[\int_r^\bullet\psi]\big).\end{aligned}$$

We choose $\mathfrak{u}\leq\mathfrak{m}\in\mathfrak{P}$ and $\mathrm{u}\preceq k$ as in Lemma 22 for $\mathfrak{p}\equiv\mathfrak{u}$ there, define $\mathrm{s}:=\max(\mathrm{o},\mathrm{u})$; and can additionally assume that $\mathfrak{p}\leq\mathfrak{q}\leq\mathfrak{m}$ holds. Then, for $\phi,\psi\in\mathfrak{D}^k_{[r,r']}$ with $\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\phi)\leq 1$, $\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi)$, $\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi-\phi)\leq 1$, we have

$$\mathfrak{p}\big(\Xi\big(\textstyle\int_r^\bullet\phi\big)-\Xi\big(\int_r^\bullet\psi\big)\big)\leq(\mathfrak{u}\circ\Xi)\big([\int_r^\bullet\phi]^{-1}[\int_r^\bullet\psi]\big)\leq\int_r^\bullet\mathbf{.}\mathfrak{m}(\psi(s)-\phi(s))\,\mathrm{d}s$$

by Lemma 22. □

We finally observe that

Lemma 23. *Suppose that G is* k-continuous*; and let $\phi\in\mathfrak{D}_{[r,r']}$ be fixed. Then, for each open neighbourhood $V\subseteq G$ of e, there exist $\mathfrak{m}\in\mathfrak{P}$ and $\mathrm{s}\preceq k$, such that*

$$\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi-\phi)\leq 1 \quad\textit{for}\quad \psi\in\mathfrak{D}^k_{[r,r']} \qquad\Longrightarrow\qquad \textstyle\int_r^\bullet\psi\in\int_r^\bullet\phi\cdot V.$$

Proof. We fix $V\prec\mathfrak{p}\in\mathfrak{P}$; and choose $\mathfrak{q}\in\mathfrak{P}$, $\mathrm{s}\preceq k$ as in Lemma 18. Then, Lemma (14) provides us with some $\mathfrak{m}\in\mathfrak{P}$, such that

$$\mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty\big(\mathrm{Ad}_{[\int_r^\bullet\phi]^{-1}}(\chi)\big)\leq\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\chi)\qquad\forall\,\chi\in\mathfrak{D}^k_{[r,r']}$$

holds. Then, for each $\psi\in\mathfrak{D}^k_{[r,r']}$ with $\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\psi-\phi)\leq 1$, we obtain from *b)* and Lemma 18 that

$$(\mathfrak{p}\circ\Xi)\big(\big[\textstyle\int_r^t\phi\big]^{-1}\big[\int_r^t\psi\big]\big)=(\mathfrak{p}\circ\Xi)\big(\int_r^t\mathrm{Ad}_{[\int_r^\bullet\phi]^{-1}}(\psi-\phi)\big)\leq 1\qquad\forall\,t\in[r,r']$$

holds; thus, $[\int_r^\bullet\phi]^{-1}[\int_r^\bullet\psi]\in V$, implying $\int_r^\bullet\psi\in\int_r^\bullet\phi\cdot V$. □

4.3. Piecewise integrable curves

For $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ and $[r,r'] \in \mathfrak{K}$, we let $\mathfrak{D}\mathrm{P}^k([r,r'],\mathfrak{g})$ denote the set of all $\phi\colon [r,r'] \to \mathfrak{g}$, such that there exist $r = t_0 < \ldots < t_n = r'$ and $\phi[p] \in \mathfrak{D}^k_{[t_p,t_{p+1}]}$ with

$$\phi|_{(t_p,t_{p+1})} = \phi[p]|_{(t_p,t_{p+1})} \qquad \forall\, p = 0,\ldots,n-1. \tag{52}$$

In this situation, we define $\int_r^r \phi := e$, as well as

$$\textstyle\int_r^t \phi := \int_{t_p}^t \phi[p] \cdot \int_{t_{p-1}}^{t_p} \phi[p-1] \cdot \ldots \cdot \int_{t_0}^{t_1} \phi[0] \qquad \forall\, t \in (t_p, t_{p+1}]. \tag{53}$$

A standard refinement argument in combination with *d)* then shows that this is well defined; i.e., independent of any choices we have made. It is furthermore not hard to see that (cf. Appendix B.1) for $\phi,\psi \in \mathfrak{D}\mathrm{P}^k([r,r'],\mathfrak{g})$, we have $\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \in \mathfrak{D}\mathrm{P}^k([r,r'],\mathfrak{g})$ with

$$\textstyle\big[\int_r^t \phi\big]^{-1}\big[\int_r^t \psi\big] = \int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\psi - \phi) \qquad \forall\, t \in [r,r']. \tag{54}$$

We now finally will extend Lemma 20 for the 0.k-continuous case to the piecewise integrable setting. For this, we fix (a bump function) $\boldsymbol{\rho}\colon [0,1] \to [0,2]$ smooth with

$$\begin{gathered} \boldsymbol{\rho}|_{(0,1)} > 0 \qquad \text{and} \qquad \textstyle\int_0^1 \boldsymbol{\rho}(s)\,\mathrm{d}s = 1 \\ \text{as well as} \\ \boldsymbol{\rho}^{(k)}(0) = 0 = \boldsymbol{\rho}^{(k)}(1) \qquad \forall\, k \in \mathbb{N}. \end{gathered} \tag{55}$$

Then, $[r,r'] \in \mathfrak{K}$ and $r = t_0 < \ldots < t_n = r'$ given, we set

$$\rho_p\colon [t_p,t_{p+1}] \to [0,2], \qquad t \mapsto \boldsymbol{\rho}(|t - t_p|/|t_{p+1} - t_p|) \qquad \forall\, p = 0,\ldots,n-1$$

and define $\rho\colon [r,r'] \to [0,2]$ by

$$\rho|_{[t_p,t_{p+1}]} := \rho_p \qquad \forall\, p = 0,\ldots,n-1.$$

Then, ρ is smooth with $\rho^{(k)}(t_p) = 0$ for each $k \in \mathbb{N}$, $p = 0,\ldots,n$; and (25) shows that

$$\varrho\colon [r,r'] \to [r,r'], \qquad t \mapsto r + \textstyle\int_r^t \rho(s)\,\mathrm{d}s$$

holds, with $\varrho(t_p) = t_p$ for $p = 0,\ldots,n-1$. We are ready for

Lemma 24. *Suppose that G is* $0.\mathrm{k}$*-continuous for* $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$*; and let* $[r, r'] \in \mathfrak{K}$ *be fixed. Then, for each* $\mathfrak{p} \in \mathfrak{P}$*, there exists some* $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$*, such that*

$$\begin{aligned} &\centerdot\mathfrak{m}_\infty(\phi) \leq 1 \quad \textit{for} \quad \phi \in \mathfrak{D}\mathrm{P}^k([r,r'],\mathfrak{g}) \\ &\implies \qquad (\mathfrak{p} \circ \Xi)(\textstyle\int_r^\bullet \phi) \leq \int_r^\bullet \centerdot\mathfrak{m}(\phi(s))\, \mathrm{d}s. \end{aligned}$$

Proof. We let $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$ be as in Lemma 20 ($s \equiv 0$); and define $\mathfrak{m} := 2 \cdot \mathfrak{q}$. Then, for $\phi \in \mathfrak{D}\mathrm{P}^k([r,r'],\mathfrak{g})$ with $\centerdot\mathfrak{m}_\infty(\phi) \leq 1$ given, we choose $r = t_0 < \ldots < t_n = r'$ and $\phi[0], \ldots, \phi[n-1]$ as in (52), and fix $\mu[p]\colon I_p \to G$ of class C^{k+1} ($I_p \subseteq \mathbb{R}$ an open interval containing $[t_p, t_{p+1}]$) with

$$\phi[p] = \widetilde{\phi}[p]|_{[t_p,t_{p+1}]} \qquad \text{for} \qquad \widetilde{\phi}[p] \equiv \delta^r(\mu[p]) \qquad \forall\, 0 \leq p \leq n-1.$$

We construct ρ and ϱ as described above; and define $\varrho[p] \in C^\infty(I_p, \mathbb{R})$ by

$$\begin{gathered} \varrho[p]|_{(-\infty,t_p)\cap I_p} := \varrho(t_p) \qquad \varrho[p]|_{[t_p,t_{p+1}]} := \varrho|_{[t_p,t_{p+1}]} \\ \varrho[p]|_{(t_{p+1},\infty)\cap I_p} := \varrho(t_{p+1}) \end{gathered}$$

for $p = 0, \ldots, n-1$. It follows that (cf. Appendix B.2) $\psi := \rho \cdot (\phi \circ \varrho) \in C^k([r,r'],\mathfrak{g})$ holds, with

$$\psi|_{[t_p,t_{p+1}]} = \delta^r(\mu[p] \circ \varrho[p]|_{[t_p,t_{p+1}]}) \in \mathfrak{D}^k_{[t_p,t_{p+1}]} \qquad \forall\, p = 0, \ldots, n-1. \tag{56}$$

Then, Lemma 11 shows that $\psi \in \mathfrak{D}^k_{[r,r']}$ holds, with

$$\begin{aligned} \textstyle\int_r^t \psi &= \textstyle\int_{t_p}^t \rho \cdot (\phi[p] \circ \varrho) \cdot \int_{t_{p-1}}^{t_p} \rho \cdot (\phi[p-1] \circ \varrho) \cdot \ldots \cdot \int_{t_0}^{t_1} \rho \cdot (\phi[0] \circ \varrho) \\ &\overset{e)}{=} \textstyle\int_{t_p}^{\varrho(t)} \phi[p] \cdot \int_{t_{p-1}}^{t_p} \phi[p-1] \cdot \ldots \cdot \int_{t_0}^{t_1} \phi[0] \\ &= \textstyle\int_r^{\varrho(t)} \phi \end{aligned} \tag{57}$$

for each $t \in (t_p, t_{p+1}]$ and $0 \leq p \leq n-1$. Since

$$\centerdot\mathfrak{q}_\infty(\psi) = 1/2 \cdot \centerdot\mathfrak{m}_\infty(\psi) \leq \centerdot\mathfrak{m}_\infty(\phi) \leq 1$$

holds by construction, Lemma 20 provides us with

$$\begin{aligned} (\mathfrak{p} \circ \Xi)(\textstyle\int_r^{\varrho(t)} \phi) \overset{(57)}{=} (\mathfrak{p} \circ \Xi)(\int_r^t \psi) &\leq \textstyle\int_r^t \centerdot\mathfrak{q}(\psi(s))\, \mathrm{d}s \\ &\leq \textstyle\int_r^t \centerdot\mathfrak{m}(\psi(s))\, \mathrm{d}s = \int_r^{\varrho(t)} \centerdot\mathfrak{m}(\phi(s))\, \mathrm{d}s; \end{aligned}$$

whereby the last step is due to (25) and (26). □

5. Local μ-convexity

In this section, we show that 0-continuity can be encoded in a property of the Lie group multiplication. More specifically, we will show that

Theorem 1. *G is 0-continuous iff G is locally μ-convex iff G is $0.\infty$-continuous.*

Here, G is said to be **locally μ-convex** *iff* for each $\mathfrak{u} \in \mathfrak{P}$, there exists some $\mathfrak{u} \leq \mathfrak{o} \in \mathfrak{P}$, such that

$$(\mathfrak{u} \circ \Xi)(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n)) \leq \mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \tag{58}$$

holds for each $X_1, \ldots, X_n \in E$ with $\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \leq 1$. According to Theorem 1, this definition does not depend on the explicit choice of Ξ.

For instance,

Example 2.

1) *Let $\Gamma \subseteq E$ be a discrete subgroup. Then, E/Γ is locally μ-convex, cf. Appendix C.1.*
2) *Banach-Lie groups are locally μ-convex, cf. Proposition 14.6 in [3] or Appendix C.2.*
3) *The unit group[9] $\mathcal{A}^\times$ of a continuous inverse algebra $\mathcal{A}$ fulfilling the condition $(*)$ from [5] is locally μ-convex, cf. Appendix C.3. We recall that the condition $(*)$ imposed on the algebra multiplication in [5] states that for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$ with*

$$\mathfrak{v}(a_1 \cdot \ldots \cdot a_n) \leq \mathfrak{w}(a_1) \cdot \ldots \cdot \mathfrak{w}(a_n) \qquad \forall\, a_1, \ldots, a_n \in \mathcal{A} \tag{59}$$

for each $n \geq 1$.[10] ‡

We break up the proof of Theorem 1 into the two directions.

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[9] Confer [4] for a proof of the fact that $\mathcal{A}^\times$ is a Lie group.

[10] In the Theorem proven in [5], $(\mathcal{A}, +)$ is additionally assumed to be Mackey complete. We will discuss this condition in Example 3.3) in Sect. 6.

5.1. The triangle inequality

We first show that (58) holds if G is $0.\infty$-continuous. For this, we recall that

$$\delta^r(\Xi^{-1}\circ\gamma)=\omega(\gamma,\dot{\gamma})\qquad \forall\,\gamma\in C^1([r,r'],\mathcal{V}),\ \ [r,r']\in\mathfrak{K} \tag{60}$$

holds, for ω defined by (29); and conclude from (32) that

Lemma 25. *For each $\mathfrak{m}\in\mathfrak{P}$, there exists some $\mathcal{V}\prec\mathfrak{o}\in\mathfrak{P}$ with $\mathfrak{m}\leq\mathfrak{o}$, such that*

$$\mathbf{.}\mathfrak{m}(\delta^r(\Xi^{-1}\circ\gamma))\leq\mathfrak{o}(\dot{\gamma})\qquad \textit{holds for each}\qquad \gamma\in\textstyle\bigsqcup_{[r,r']\in\mathfrak{K}}C^1([r,r'],E)$$
$$\textit{with}\qquad \mathrm{im}[\gamma]\subseteq\overline{\mathrm{B}}_{\mathfrak{o},1}.$$

Proof. Up to renaming seminorms, this is clear from (60) and (32). □

We obtain that

Lemma 26. *G is locally μ-convex if G is $0.\infty$-continuous.*

Proof. For $\mathfrak{u}\equiv\mathfrak{p}\in\mathfrak{P}$ fixed, we let $\mathfrak{p}\leq\mathfrak{m}$ be as in Lemma 24 for $[r,r']\equiv[0,2]$ there; and choose $\mathfrak{m}\leq\mathfrak{o}\in\mathfrak{P}$ as in Lemma 25. Then, for $X_1,\dots,X_n\in E$ with $\mathfrak{o}(X_1)+\ldots+\mathfrak{o}(X_n)=:\varepsilon\leq 1$ fixed, we define $Y_p:=X_{n-p}$ for $p=0,\dots,n-1$ as well as $Y_n:=0$. We let $\emptyset\neq\mathrm{J}\subseteq\{0,\dots,n\}$ denote the set of all indices $0\leq p\leq n$ with $\mathfrak{o}(Y_p)=0$; and denote its cardinality by $\mathrm{d}:=|\mathrm{J}|\geq 1$. We define

$$\delta_p:=\begin{cases}1/\mathrm{d} & \text{for each}\quad p\in\mathrm{J}\\ \mathfrak{o}(Y_p) & \text{for each}\quad p\in\{0,\dots,n\}-\mathrm{J};\end{cases}$$

and let $t_0:=0$, as well as $t_p:=\delta_0+\ldots+\delta_{p-1}$ for $p=1,\dots,n+1$. We furthermore consider

$$\phi[p]:=\delta^r(\Xi^{-1}\circ\gamma[p])\qquad\text{with}\qquad \gamma[p]\colon[t_p,t_{p+1}]\colon t\mapsto(t-t_p)\cdot\delta_p^{-1}\cdot Y_p$$

for $p=0,\dots,n$; and define $\phi\in\mathfrak{DP}^\infty([0,2],\mathfrak{g})$ by $\phi|_{[1+\varepsilon,2]}:=0$, as well as

$$\phi|_{[t_p,t_{p+1})}:=\phi[p]|_{[t_p,t_{p+1})}\qquad\forall\,p=0,\dots,n.$$

Then, $\mathfrak{o}_\infty(\gamma[p])\leq 1$ holds by construction for each $0\leq p\leq n$; so that Lemma 25 shows

$$\mathbf{.}\mathfrak{m}_\infty(\phi|_{[t_p,t_{p+1})})\leq\delta_p^{-1}\cdot\mathfrak{o}(Y_p)=\begin{cases}0 & \text{for}\quad p\in\mathrm{J}\\ 1 & \text{for}\quad p\in\{0,\dots,n\}-\mathrm{J}.\end{cases}$$

We thus have $\boldsymbol{.}\mathfrak{m}_\infty(\phi) \leq 1$, as well as

$$\begin{aligned}\int \boldsymbol{.}\mathfrak{m}(\phi(s))\, \mathrm{d}s &= \textstyle\sum_{p\in\{0,\dots,n\}-\mathrm{J}} \int \boldsymbol{.}\mathfrak{m}(\phi[p](s))\, \mathrm{d}s \\ &\leq \textstyle\sum_{p\in\{0,\dots,n\}-\mathrm{J}} \delta_p = \mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) = \varepsilon;\end{aligned}$$

so that Lemma 24 shows

$$\begin{aligned}\varepsilon &\geq (\mathfrak{u}\circ\Xi)(\textstyle\int \phi) \\ &= (\mathfrak{u}\circ\Xi)\big(\textstyle\int_{1+\varepsilon}^{2} \phi \cdot \int_{t_n}^{t_{n+1}} \phi[n] \cdot \ldots \cdot \int_{t_0}^{t_1} \phi[0]\,\big) \\ &= (\mathfrak{u}\circ\Xi)\big(\Xi^{-1}(Y_n)\cdot \ldots \cdot \Xi^{-1}(Y_0)\big) \\ &= (\mathfrak{u}\circ\Xi)\big(\Xi^{-1}(X_1)\cdot \ldots \cdot \Xi^{-1}(X_n)\big),\end{aligned}$$

from which the claim is clear. □

5.2. Continuity of the integral

Let us next show that G is 0-continuous if it is locally μ-convex. For this, we recall that

$$\dot\gamma = \upsilon(\gamma,\phi) \qquad \text{holds for} \qquad \gamma := \Xi\circ \textstyle\int_r^\bullet \phi, \tag{61}$$

for each $\phi \in \mathfrak{D}^k_{[r,r']}$ with $\int_r^\bullet \phi \in \mathcal{U}$; and conclude from (33) that

Lemma 27. *For each $\mathfrak{o}\in\mathfrak{P}$, there exists some $\mathcal{V} \prec \mathfrak{w} \in \mathfrak{P}$ with $\mathfrak{o} \leq \mathfrak{w}$, such that*

$$(\mathfrak{o}\circ\Xi)\big(\textstyle\int_r^\bullet \phi\big) \leq \int_r^\bullet \boldsymbol{.}\mathfrak{w}(\phi(s))\,\mathrm{d}s \qquad \textit{holds for each} \qquad \phi\in\mathfrak{D}_{[r,r']} \quad \textit{with} \quad \textstyle\int_r^\bullet \phi \in \Xi^{-1}(\overline{\mathrm{B}}_{\mathfrak{w},1}).$$

Proof. We choose $\mathfrak{o} \leq \mathfrak{w} \in \mathfrak{P}$ as in (33) for $\mathfrak{v} \equiv \mathfrak{o}$ there. Then, the rest is clear from (61) and Lemma 6. □

In addition to that, we observe that

Lemma 28. *For each $\mathfrak{w}\in\mathfrak{P}$ and $\phi\in\mathfrak{D}_{[r,r']}$, there exist $r = t_0 < \ldots < t_n = r'$ with*

$$(\mathfrak{w}\circ\Xi)\big(\textstyle\int_{t_p}^\bullet \phi|_{[t_p,t_{p+1}]}\big) \leq 1 \qquad \forall\, p = 0,\ldots,n-1.$$

Proof. We fix $\mu\colon I \to G$ ($I \subseteq \mathbb{R}$ open with $[r,r'] \subseteq I$) of class C^1 with $\delta^r(\mu)|_{[r,r']} = \phi$, choose $d > 0$ such small that $K_d \equiv [r-d, r'+d] \subseteq I$ holds, and define

$$\alpha\colon I \times I \ni (t,s) \mapsto \mu(t)\cdot\mu(s)^{-1} \in G.$$

Since $[r,r']$ is compact, and since α is continuous with $\alpha(t,t) = e$ for each $t \in [r,r']$, to each open neighbourhood U of e, there exists some $0 < \delta_U \leq d$, such that

$$U \ni \alpha(t+s,t) = \textstyle\int_t^{t+s} \delta^r(\mu) \qquad \forall\, t \in [r,r'],\ \ 0 \leq s \leq \delta_U$$

holds; from which the claim is clear. □

We conclude from Lemma 27 and Lemma 28 that

Proposition 2. *Suppose that G is locally μ-convex. Then, for each $\mathfrak{p} \in \mathfrak{P}$, there exists some $\mathfrak{p} \leq \mathfrak{q} \in \mathfrak{P}$, such that*

$$\begin{aligned}&\textstyle\int \boldsymbol{\cdot}\mathfrak{q}(\phi(s))\,\mathrm{d}s \leq 1 \quad \textit{for} \quad \phi \in \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})\\ &\Longrightarrow \qquad (\mathfrak{p}\circ\Xi)\big(\textstyle\int_r^\bullet \phi\big) \leq \int_r^\bullet \boldsymbol{\cdot}\mathfrak{q}(\phi(s))\,\mathrm{d}s,\end{aligned}$$

for each $[r,r'] \in \mathfrak{K}$.

Proof. For $\mathfrak{u} \equiv \mathfrak{p}$ fixed, we let $\mathfrak{u} \leq \mathfrak{o} \in \mathfrak{P}$ be as in (58); and choose $\mathfrak{o} \leq \mathfrak{w} \equiv \mathfrak{q} \in \mathfrak{P}$ as in Lemma 27. Then, since $\phi|_{[\ell,\ell']} \in \mathfrak{D}\mathrm{P}^0([\ell,\ell'],\mathfrak{g})$ holds for each $\phi \in \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})$ and $\mathfrak{K} \ni [\ell,\ell'] \subseteq [r,r'] \in \mathfrak{K}$, the claim follows if we show that

$$\begin{aligned}&\textstyle\int \boldsymbol{\cdot}\mathfrak{w}(\phi(s))\,\mathrm{d}s \leq 1 \quad \text{for} \quad \phi \in \bigsqcup_{[r,r']\in\mathfrak{K}} \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})\\ &\Longrightarrow \qquad (\mathfrak{u}\circ\Xi)\big(\textstyle\int \phi\big) \leq \int \boldsymbol{\cdot}\mathfrak{w}(\phi(s))\,\mathrm{d}s.\end{aligned}$$

To verify this, we fix $\phi \in \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})$ with $\int \boldsymbol{\cdot}\mathfrak{w}(\phi(s))\,\mathrm{d}s \leq 1$; and let $r = t_0 < \ldots < t_n = r'$ as well as $\phi[0],\ldots,\phi[n-1]$ be as in (52). By Lemma 28, we can refine this decomposition in such a way that

$$\mu[p]\colon [t_p, t_{p+1}] \ni t \mapsto \textstyle\int_{t_p}^t \phi[p] \in \Xi^{-1}(\overline{\mathrm{B}}_{\mathfrak{w},1}) \qquad \forall\, p = 0,\ldots,n-1$$

holds; so that Lemma 27 shows

$$(\mathfrak{o}\circ\Xi)(\mu[p](t_{p+1})) \leq \textstyle\int_{t_p}^{t_{p+1}} \boldsymbol{\cdot}\mathfrak{w}(\phi(s))\,\mathrm{d}s \qquad \forall\, p = 0,\ldots,n-1.$$

We define $X_{n-p} := (\Xi \circ \mu[p])(t_{p+1})$ for each $0 \leq p \leq n-1$; and obtain

$$\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \leq \int \mathbf{.}\mathfrak{w}(\phi(s))\, \mathrm{d}s \leq 1.$$

Then, (58) provides us with

$$\begin{aligned}(\mathfrak{u} \circ \Xi)(\textstyle\int \phi) &= (\mathfrak{u} \circ \Xi)(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n)) \\ &\leq \mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) \leq \textstyle\int \mathbf{.}\mathfrak{w}(\phi(s))\, \mathrm{d}s;\end{aligned}$$

which proves the claim. □

We are ready for the

Proof of Theorem 1. Clearly, G is $0\mathbf{.}\infty$-continuous if G is 0-continuous. Moreover, if G is $0\mathbf{.}\infty$-continuous, then G is locally μ-convex by Lemma 26. Finally, if G is locally μ-convex, then $\mathrm{evol}^0_{[0,1]}$ is C^0-continuous at zero by Proposition 2; so that G is 0-continuous by Lemma 15. □

6. Completeness and approximation

In this section, we discuss completeness properties of Lie groups; and prove certain approximation statements for continuous-, and Lipschitz curves. Both will be relevant for our investigation of semiregularity in Sect. 7.

6.1. Completeness conditions

A sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ is said to be a

- **Cauchy sequence** *iff* for each $\mathfrak{p} \in \mathfrak{P}$ and $\varepsilon > 0$, there exists some $p \in \mathbb{N}$ with

(62) $$(\mathfrak{p} \circ \Xi)(g_m^{-1} \cdot g_n) \leq \varepsilon \qquad \forall\, m, n \geq p.$$

  We then clearly can assume that $g_m^{-1} \cdot g_n \in \mathfrak{U}$ holds for all $m, n \in \mathbb{N}$ in the following.
- **Mackey-Cauchy sequence** *iff*

(63) $$(\mathfrak{p} \circ \Xi)(g_m^{-1} \cdot g_n) \leq \mathfrak{c}_{\mathfrak{p}} \cdot \lambda_{m,n} \qquad \forall\, m, n \geq \mathfrak{l}_{\mathfrak{p}},\ \ \mathfrak{p} \in \mathfrak{P}$$

  holds for certain $\{\mathfrak{c}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.

  Clearly, each Mackey-Cauchy sequence is a Cauchy sequence.

Remark 3. *It is straightforward from Lemma 1 and Lemma 6 (applied to coordinate changes) that these definitions are independent of the explicit choice of* Ξ*, cf. Appendix D.1.*

We say that G is

- **sequentially complete** *iff* each Cauchy sequence in G converges in G.
- **Mackey complete** *iff* each Mackey-Cauchy sequence in G converges in G.

We say that a locally convex vector space F is sequentially/Mackey complete *iff* F is sequentially/Mackey complete when considered as the Lie group $(F, +)$. Obviously, these definitions coincide with the standard definitions given in the literature.

Remark 4.

1) *Since each Mackey-Cauchy sequence is a Cauchy sequence, sequentially completeness of* G *implies Mackey completeness of* G.
2) *It is straightforward from the definitions that a Cauchy/Mackey-Cauchy sequence converges iff one of its subsequences converges.*
3) *If* $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ *is a Cauchy/Mackey-Cauchy sequence, then* $\{h \cdot g_n\}_{n\in\mathbb{N}} \subseteq G$ *is a Cauchy/Mackey-Cauchy sequence for each* $h \in G$*; and (evidently)* $\{g_n\}_{n\in\mathbb{N}}$ *converges iff* $\{h \cdot g_n\}_{n\in\mathbb{N}}$ *converges for each* $h \in G$.
4) *If* $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ *is a Cauchy/Mackey-Cauchy sequence, and* $U \subseteq G$ *an open neighbourhood of* e*, then there exists some* $q \in \mathbb{N}$ *with* $\{g_q^{-1} \cdot g_n\}_{n\geq q} \subseteq U$.

   Thus, in order to show that G *is sequentially/Mackey-Cauchy, by the previous two points, it suffices to verify convergence of each Cauchy/ Mackey-Cauchy sequence that is contained in a fixed open neighbourhood* U *of* e. ‡

Example 3.

1) *Let* Γ *be a discrete subgroup of* $(E, +)$*. Then,* E *is sequentially/Mackey complete iff* E/Γ *is sequentially/Mackey complete, cf. Appendix D.2.*
2) *Banach Lie groups are sequentially complete, cf. Appendix D.3.*
3) *The unit group* $\mathcal{A}^\times$ *of a continuous inverse algebra* $\mathcal{A}$ *fulfilling the condition* $(*)$ *from [5] (i.e., condition* (59)*) is sequentially/Mackey complete if* $(\mathcal{A}, +)$ *is sequentially/Mackey complete, cf. Appendix D.4.* ‡

We now are going to show that

Theorem 2. *G is Mackey complete if G is C^∞-semiregular.*

Remark 5.

1) *The idea of the proof of Theorem 2 is to construct some* $\phi \in C^\infty([0,1],\mathfrak{g})$ *whose integral* $\int \phi$ *is the limit of a (subsequence of a) given Mackey-Cauchy sequence* $\{g_n\}_{n\in\mathbb{N}} \subseteq G$*. Roughly speaking, we will use the substitution formula* e) *in order to glue together smooth curves whose integrals equal* $g_n^{-1}\cdot g_{n-1}$ *via suitable bump functions. Here, we will use that, passing to a subsequence if necessary, we can achieve that* $(\mathfrak{p}\circ\Xi)(g_n^{-1}\cdot g_{n-1})$ *decreases suitably fast; namely, (up to a factor* $\mathfrak{c}_\mathfrak{p}$*) in the same way for all seminorms – This ensures that the so-constructed* ϕ *is defined and smooth at* 1 *(where all of its derivatives must necessarily be zero).*
2) *An analogous result cannot hold for the* C^0*-semiregular case; i.e., we cannot have that* C^0*-semiregularity implies sequentially completeness. This can be seen immediately by considering the special situation where* G *equals a Hausdorff locally convex vector space* $(E,+)$ *as then integrability of all continuous curves – which is* integral completeness *in the sense of [3] – is equivalent to the "metric convex compactness property" [15] that, in general, is strictly weaker than sequentially completeness [14], cf. proof of Theorem C.(d) in [3].*

   Indeed, the strategy, sketched in 1) for the C^∞*-semiregular case does not work out in the* C^0*-semiregular situation; i.e., given a Cauchy sequence* $\{g_n\}_{n\in\mathbb{N}} \subseteq G$*, we cannot apply the same procedure to construct some* $\phi \in C^0([0,1],\mathfrak{g})$ *whose integral is the limit of (a subsequence of)* $\{g_n\}_{n\in\mathbb{N}}$*. The problem is that by passing to a subsequence, we can only assure that* $\lim_{t\to 1}(\mathfrak{p}\circ\phi)(t) = 0$ *holds for finitely many seminorms* $\mathfrak{p}\in\mathfrak{P}$*, but not for all of them.* ‡

Now, before we can prove Theorem 2, we first need some preparation:

- For $\gamma\colon [r,r'] \ni t \to |t-r|\cdot Y$ with $[r,r']\cdot Y \subseteq \mathcal{V}$, we have

$$\phi := \delta^r(\Xi^{-1}\circ\gamma) = \omega(\gamma, Y) = \omega[0](\gamma, Y) \in \mathfrak{D}^\infty_{[r,r']}.$$

- For $\varrho\colon [r,r'] \to [r,r']$ smooth, $p\in\mathbb{N}$, and $\rho \equiv \dot\varrho$, we define

$$C[\rho,p] := \max_{0\leq m,n\leq p}(\sup\{|\rho^{(m)}(t)|^{n+1} \mid t\in[r,r']\}) \qquad \forall\, p\in\mathbb{N};$$

  and obtain from **d**), **e**) that

$$\begin{aligned}(\rho\cdot(\phi\circ\varrho))^{(p)} &= \textstyle\sum_{q,m,n=0}^p h_p(q,m,n)\cdot\left(\rho^{(m)}\right)^{n+1}\cdot\omega[q](\gamma\circ\varrho, Y,\dots,Y)\\ &\leq (p+1)^3\cdot C[\rho,p]\cdot\omega[q](\gamma\circ\varrho,Y,\dots,Y)\end{aligned}$$

  holds, for a map $h_p\colon (0,\dots,p)^3 \to \{0,1\}$ that is independent of ϱ,ρ,Y.

- For $\mathfrak{v} \in \mathfrak{P}$, we choose $\mathcal{V} \prec \mathfrak{w} \in \mathfrak{P}$ with $\mathfrak{v} \leq \mathfrak{w}$ as in (35); and conclude that

$$\begin{aligned} &\mathfrak{w}(Y),\ \mathfrak{w}(\gamma \circ \varrho) \equiv \mathfrak{w}(|\varrho - r| \cdot Y) \leq 1 \qquad \text{implies} \\ &{\scriptstyle\bullet}\mathfrak{v}\big((\rho \cdot (\phi \circ \varrho))^{(q)}\big) \leq (p+1)^3 \cdot C[\rho, p] \cdot \mathfrak{w}(Y) \end{aligned}$$

  for $0 \leq q \leq p$, for each fixed $p \in \mathbb{N}$.

Let now $\boldsymbol{\rho}\colon [0,1] \to [0,2]$ be as in Sect. 4.3, cf. (55); and suppose that we are given $\{Y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{V}$, as well as $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ strictly increasing with $t_0 = 0$.

Then, for each $n \in \mathbb{N}$,

- we let $\delta_n := t_{n+1} - t_n$, and define

$$\begin{aligned} &\kappa_n\colon [t_n, t_{n+1}] \ni t \mapsto \delta_n^{-1} \cdot |t - t_n| \in [0,1] \\ &\text{as well as} \qquad \gamma_n\colon [t_n, t_{n+1}] \ni t \mapsto |t - t_n| \cdot Y_n. \end{aligned}$$

- we let $\phi_n := \delta^r(\Xi^{-1} \circ \gamma_n)$, and define

$$\begin{aligned} &\rho_n := \boldsymbol{\rho} \circ \kappa_n \qquad \text{as well as} \\ &\varrho_n\colon [t_n, t_{n+1}] \ni t \mapsto t_n + \textstyle\int_{t_n}^t \rho_n(s)\,\mathrm{d}s \in [t_n, t_{n+1}]. \end{aligned}$$

- we define $\phi\colon [0,1] \to \mathfrak{g}$ by $\phi(1) := 0$, and $\phi|_{[t_n, t_{n+1}]} := \rho_n \cdot (\phi_n \circ \varrho_n)$ for each $n \in \mathbb{N}$.

Then, the same arguments as in the proof of Lemma 24 show that $\phi|_{[0,t_n]} \in \mathfrak{D}^\infty_{[0,t_n]}$ holds, with

$$(64) \qquad \textstyle\int_0^{t_n} \phi = \Xi^{-1}(\delta_{n-1} \cdot Y_{n-1}) \cdot \ldots \cdot \Xi^{-1}(\delta_0 \cdot Y_0) \qquad \forall\, n \geq 1.$$

Moreover, for $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$ as above, $p \in \mathbb{N}$, and $n \in \mathbb{N}$ with $\mathfrak{w}(Y_n) \leq 1$ (thus, $\mathfrak{w}(|\varrho_n - t_n| \cdot Y_n) \leq 1$), we have

$$(65) \qquad \begin{aligned} {\scriptstyle\bullet}\mathfrak{v}\big((\rho_n \cdot (\phi_n \circ \varrho_n))^{(q)}\big) &\leq (p+1)^3 \cdot C[\rho_n, p] \cdot \mathfrak{w}(Y_n) \\ &\leq (p+1)^3 \cdot \delta_n^{-(p+1)^2} \cdot C[\boldsymbol{\rho}, p] \cdot \mathfrak{w}(Y_n) \end{aligned}$$

for $q = 0, \ldots, p$.

We are ready for the

Proof of Theorem 2. Let $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ be a Mackey-Cauchy sequence; and $U \subseteq G$ a symmetric open neighbourhood of e with $U \cdot U \subseteq \mathcal{U}$. By Remark 4.2), we can assume that $\{g_n\}_{n\in\mathbb{N}} \subseteq U$ holds; and, by Remark 4.2), it suffices to show that a subsequence of $\{g_n\}_{n\in\mathbb{N}}$ converges. Passing to a subsequence if necessary, we thus can assume that $\lambda_{n,n-1} \leq 2^{-n^2}$ holds for each $n \geq 1$. Then,

- We define $X_0 := 0$, as well as $\mathcal{V} \ni X_n := \Xi\big(g_n^{-1} \cdot g_{n-1}\big)$ for each $n \geq 1$.
- For each $\mathfrak{w} \in \mathfrak{P}$, we fix some $\mathfrak{j}_{\mathfrak{w}} \in \mathbb{N}$ with

$$\mathfrak{w}(X_n) \leq \mathfrak{c}_{\mathfrak{w}} \cdot 2^{-n^2} \qquad \forall\, n \geq \mathfrak{j}_{\mathfrak{w}}. \tag{66}$$

- We define $t_0 := 0$, as well as $t_n := \sum_{k=1}^{n} 2^{-k}$ for $n \geq 1$; and obtain

$$1/(1-h) \leq 2^{n+2} \qquad \forall\, h \in [t_n, t_{n+1}],\ \ n \in \mathbb{N}, \tag{67}$$

  from $1 - h \geq 1 - t_{n+1} = 1 - \sum_{k=1}^{n+1} 2^{-k} = \sum_{k=n+2}^{\infty} 2^{-k} \geq 2^{-(n+2)}$.
- We let $Y_n := 2^{n+1} \cdot X_n$ for each $n \in \mathbb{N}$; and define δ_n, γ_n, ϕ_n, as well as $\phi\colon [0,1] \to \mathfrak{g}$ as described above; i.e., we have $\delta_n \equiv |t_{n+1} - t_n| = 2^{-(n+1)}$ for each $n \in \mathbb{N}$.

The claim now follows once we have shown that ϕ is smooth, because then (64) provides us with

$$\begin{aligned}
\big(\textstyle\int \phi \cdot g_0^{-1}\big)^{-1} &= \lim_n \big(\big[\textstyle\int_0^{t_{n+1}} \phi\big] \cdot g_0^{-1}\big)^{-1} \\
&\overset{(64)}{=} \lim_n \big(\Xi^{-1}(\delta_n \cdot Y_n) \cdot \ldots \cdot \Xi^{-1}(\delta_0 \cdot Y_0) \cdot g_0^{-1}\big)^{-1} \\
&= \lim_n \big(\Xi^{-1}(X_n) \cdot \ldots \cdot \Xi^{-1}(X_1) \cdot g_0^{-1}\big)^{-1} \\
&= \lim_n g_n.
\end{aligned}$$

Since ϕ is smooth on $[0,1)$, here we only have to verify that

$$\lim_{[0,1)\ni h\to 1} \tfrac{1}{h} \cdot \phi^{(p)}(h) = 0 \qquad \forall\, p \in \mathbb{N}$$

holds.[11] To show this, we fix $p \in \mathbb{N}$ and $\mathfrak{v} \in \mathfrak{P}$, choose $\mathfrak{w}$ as in (65); and observe that

$$\mathfrak{w}(Y_n) = 2^{n+1} \cdot \mathfrak{w}(X_n) \leq \mathfrak{c}_{\mathfrak{w}} \cdot 2^{-n^2+n+1} \leq 1 \qquad \forall\, n \geq \mathfrak{j}'_{\mathfrak{w}}$$

holds, for some $\mathfrak{j}'_{\mathfrak{w}} \geq \max(2, \mathfrak{j}_{\mathfrak{w}})$ suitably large. We conclude from (65) that

$$\begin{aligned}
\mathfrak{v}\big((\rho_n \cdot (\phi_n \circ \varrho_n))^{(p)}\big) &\leq (p+1)^3 \cdot 2^{(n+1)\cdot(p+1)^2} \cdot C[\boldsymbol{\rho}, p] \cdot \mathfrak{c}_{\mathfrak{w}} \cdot 2^{-n^2+n+1} \\
&= (p+1)^3 \cdot C[\boldsymbol{\rho}, p] \cdot \mathfrak{c}_{\mathfrak{w}} \cdot 2^{-n^2+(n+1)\cdot((p+1)^2+1)}
\end{aligned}$$

[11] We then automatically have $\lim_{[0,1)\ni h\to 1} \phi^{(p)}(h) = 0$.

holds for each $n \geq \mathfrak{j}'_{\mathfrak{w}}$. Then, for $h \in [t_n, t_{n+1}]$ with $n \geq \mathfrak{j}'_{\mathfrak{w}}$, we obtain from (67) that

$$\begin{aligned} 1/(1-h) \cdot \mathbf{.}\mathfrak{v}\big(\phi^{(p)}(h)\big) &\leq 2^{n+2} \cdot \mathbf{.}\mathfrak{v}\big((\rho_n \cdot (\phi_n \circ \varrho_n))^{(p)}(h)\big) \\ &\leq (p+1)^3 \cdot C[\boldsymbol{\rho}, p] \cdot \mathfrak{c}_{\mathfrak{w}} \cdot 2^{1-n^2+(n+1)\cdot((p+1)^2+2)} \\ &= (p+1)^3 \cdot C[\boldsymbol{\rho}, p] \cdot \mathfrak{c}_{\mathfrak{w}} \cdot 2^{1-n\cdot[n-\frac{n+1}{n}\cdot((p+1)^2+2)]} \end{aligned}$$

holds; which clearly tends to zero for $n \to \infty$. □

6.2. Approximation

We now finally provide some approximation statements for curves that will be important for our discussions of particular situations in the context of Theorem 3 in Sect. 7.

In analogy to Sect. 6.1, we say that $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{CP}^0([r,r'],\mathfrak{g})$ is a

- Cauchy sequence *iff* for each $\mathfrak{p} \in \mathfrak{P}$ and $\varepsilon > 0$, there exists some $p \in \mathbb{N}$, such that
$$\mathbf{.}\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \varepsilon \qquad \forall\, m, n \geq p.$$
- Mackey-Cauchy sequence *iff*
$$\mathbf{.}\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \mathfrak{c}_{\mathfrak{p}} \cdot \lambda_{m,n} \qquad \forall\, m, n \geq \mathfrak{l}_{\mathfrak{p}},\ \ \mathfrak{p} \in \mathfrak{P} \tag{68}$$
holds for certain $\{\mathfrak{c}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.

We say that $\{\phi_n\}_{n\in\mathbb{N}} \to \phi$ (converges) uniformly for $\phi \in C^0([r,r'],\mathfrak{g})$ *iff*

$$\lim_{n\to\infty} \mathbf{.}\mathfrak{p}_\infty(\phi - \phi_n) = 0 \qquad \text{holds for each} \qquad \mathfrak{p} \in \mathfrak{P};$$

and obtain

Lemma 29. *Let $[r,r'] \in \mathfrak{K}$ be fixed.*

1) *For each $\phi \in C^0([r,r'],\mathfrak{g})$, there exists a* Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^\infty([r,r'],\mathfrak{g})$ *with $\{\phi_n\}_{n\in\mathbb{N}} \to \phi$ uniformly.*
2) *For each $\phi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$, there exists a* Mackey-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{DP}^\infty([r,r'],\mathfrak{g})$ *with $\{\phi_n\}_{n\in\mathbb{N}} \to \phi$ uniformly.*

Proof. We let $\phi \in C^0([r,r'],\mathfrak{g})$ be fixed; and, for the case that $\phi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ holds, we denote the Lipschitz constants of ϕ by $\{L_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$.

- We choose $\Delta > 0$ such small that $[0, \Delta] \cdot \mathrm{d}_e\Xi(\mathrm{im}[\phi]) \subseteq \mathcal{V}$ holds; and fix $m \geq 1$ with $|r' - r|/m \leq \Delta$.
- We define $\gamma[t']\colon [0, \Delta] \ni t \mapsto t \cdot \mathrm{d}_e\Xi(\phi(t'))$ for each $t' \in [r, r']$; and let

$$\Phi(t, t') := \omega(t \cdot \mathrm{d}_e\Xi(\phi(t')), \mathrm{d}_e\Xi(\phi(t'))) \equiv \delta^r(\Xi^{-1} \circ \gamma[t'])(t) \qquad \forall\, (t, t') \in [0, \Delta] \times [r, r'].$$

For each $n \geq m$, we construct $\phi_n \in \mathfrak{DP}^\infty([r, r'], \mathfrak{g})$ as follows:

- We define $\Delta_n := |r' - r|/n$; and let $t_{n,p} := r + p \cdot \Delta_n$ for $p = 0, \dots, n$.
- We define $\phi|_{[t_{n,n-1}, t_{n,n}]} := \Phi(\cdot - t_{n,n-1}, t_{n,n-1})$, as well as

$$\phi_n|_{[t_{n,p}, t_{n,p+1})} := \Phi(\cdot - t_{n,p}, t_{n,p}) \qquad \forall\, p = 0, \dots, n - 2.$$

By construction, we have

$$\phi_n(t_{n,p}) = \Phi(0, t_{n,p}) = \phi(t_{n,p}) \qquad \forall\, n \geq m,\ p = 0, \dots, n. \tag{69}$$

Let now $\mathfrak{v} \in \mathfrak{P}$ be fixed. We choose $\mathcal{V} \prec \mathfrak{w}$ as in (35) for $p \equiv 1$ there; and let $\mathfrak{l}_\mathfrak{v} \geq m$ be such large that $\Delta_{\mathfrak{l}_\mathfrak{v}} \cdot {\centerdot}\mathfrak{w}_\infty(\phi) \leq 1$ holds, i.e., we have $\mathfrak{w}_\infty(\gamma[t_{n,p}]|_{[0,\Delta_n]}) \leq 1$ for each $n \geq \mathfrak{l}_\mathfrak{v}$ and $p = 0, \dots, n - 1$.

Then, for each $n \geq \mathfrak{l}_\mathfrak{v}$,

- we obtain from (35) and Lemma 6 that

$$\begin{aligned} {\centerdot}\mathfrak{v}\big(\phi_n(t) - \phi_n(t_{n,p})\big) &= {\centerdot}\mathfrak{v}(\Phi(t - t_{n,p}, t_{n,p}) - \Phi(0, t_{n,p})) \\ &\leq \textstyle\int_0^{t - t_{n,p}} ({\centerdot}\mathfrak{v} \circ \omega[1])(\gamma[t_{n,p}](s), \mathrm{d}_e\Xi(\phi(t_{n,p})), \mathrm{d}_e\Xi(\phi(t_{n,p})))\, \mathrm{d}s \\ &\leq \mathfrak{w}(\mathrm{d}_e\Xi(\phi(t_{n,p})))^2 \cdot |t - t_{n,p}| \\ &\leq {\centerdot}\mathfrak{w}_\infty(\phi)^2 \cdot |t - t_{n,p}| \end{aligned} \tag{70}$$

holds, for each

$$t \in \begin{cases} [t_{n,p}, t_{n,p+1}) & \text{for} \quad 0 \leq p \leq n - 2, \\ [t_{n,n-1}, t_{n,n}] & \text{for} \quad p = n. \end{cases} \tag{71}$$

- we obtain from (69) and (70) that

$$\begin{aligned} {\centerdot}\mathfrak{v}(\phi(t) - \phi_n(t)) \leq{} & {\centerdot}\mathfrak{v}(\phi(t) - \phi(t_{n,p})) + {\centerdot}\mathfrak{v}(\phi(t_{n,p}) - \phi_n(t_{n,p})) \\ & + {\centerdot}\mathfrak{v}(\phi_n(t) - \phi_n(t_{n,p})) \\ \leq{} & {\centerdot}\mathfrak{v}(\phi(t) - \phi(t_{n,p})) + {\centerdot}\mathfrak{w}_\infty(\phi)^2 \cdot |t - t_{n,p}| \end{aligned} \tag{72}$$

holds, for t as in (71).

Clearly, (72) implies that $\{\phi_{n-m}\}_{n\in\mathbb{N}}$ is a Cauchy sequence with $\{\phi_{n-m}\}_{n\in\mathbb{N}} \rightarrow \phi$ uniformly. Moreover, for the case that $\phi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ holds, we define $\mathfrak{c}_{\mathfrak{v}} := L_{\mathfrak{v}} + {\scriptstyle\bullet}\mathfrak{w}_\infty(\phi)^2$, and obtain

$$
\begin{aligned}
{\scriptstyle\bullet}\mathfrak{v}(\phi(t) - \phi_n(t)) &\stackrel{(72)}{\leq} L_{\mathfrak{v}} \cdot |t - t_{n,p}| + {\scriptstyle\bullet}\mathfrak{w}_\infty(\phi)^2 \cdot |t - t_{n,p}| \\
&= \mathfrak{c}_{\mathfrak{v}} \cdot |t - t_{n,p}| \qquad \forall\, n \geq \mathfrak{l}_{\mathfrak{v}}
\end{aligned}
$$

for t as in (71); so that (68) is clear from the triangle inequality. □

Obviously, we also have

Lemma 30. *Let $[r,r'] \in \mathfrak{K}$ be fixed. Then,*

1) *For each $\phi \in C^0([r,r'],\mathfrak{g})$, there exists a* Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{CoP}([r,r'],\mathfrak{g})$ *with $\{\phi_n\}_{n\in\mathbb{N}} \rightarrow \phi$ uniformly.*
2) *For each $\phi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$, there exists a* Mackey-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{CoP}([r,r'],\mathfrak{g})$ *with $\{\phi_n\}_{n\in\mathbb{N}} \rightarrow \phi$ uniformly.*

This Lemma will be relevant for our discussion of the situation where G admits an exponential map, as then clearly $\mathrm{CoP}([r,r'],\mathfrak{g}) \subseteq \mathfrak{D}\mathrm{P}^\infty([r,r'],\mathfrak{g})$ holds for each $[r,r'] \in \mathfrak{K}$.

7. The confined condition

In this section, we clarify under which circumstance a locally μ-convex Lie group is C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ (partially for $k \equiv 0$). We first provide the basic definitions; and then prove the main result in Sect. 7.1. In the last part of this section, we will discuss several particular situations.

A sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})$ is said to be **tame** *iff* for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$ with

$$
(73) \qquad {\scriptstyle\bullet}\mathfrak{v} \circ \mathrm{Ad}_{[\int_r^\bullet \phi_n]^{-1}} \leq {\scriptstyle\bullet}\mathfrak{w} \qquad \forall\, n \in \mathbb{N}.
$$

We say that $\phi \in \bigsqcup_{[r,r']\in\mathfrak{K}} C^0([r,r'],\mathfrak{g})$ is

- $\mathfrak{s}$**-integrable** *iff* there exists a tame Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0(\mathrm{dom}[\phi],\mathfrak{g})$ with $\{\phi_n\}_{n\in\mathbb{N}} \rightarrow \phi$ uniformly.
  The set of all such ϕ will be denoted by $\mathfrak{Sequ}$ in the following.
- $\mathfrak{m}$**-integrable** *iff* there exists a tame Mackey-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0(\mathrm{dom}[\phi],\mathfrak{g})$ with $\{\phi_n\}_{n\in\mathbb{N}} \rightarrow \phi$ uniformly.
  The set of all such ϕ will be denoted by $\mathfrak{Mack}$ in the following.

Evidently,

Lemma 31. *We have* $\mathfrak{D} \subseteq \mathfrak{Mack} \subseteq \mathfrak{Sequ}$.

Proof. The second inclusion is evident. For the first inclusion, we fix $\phi \in \mathfrak{D}_{[r,r']}$ for $[r,r'] \in \mathfrak{K}$; and define $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})$ by $\phi_n := \phi$ for each $n \in \mathbb{N}$. Since $\mathrm{C} := \mathrm{inv}(\mathrm{im}[\int_r^\bullet \phi])$ is compact, the first inclusion is clear from (28). □

Conversely, we have, cf. Sect. 7.1

Proposition 3. *Suppose that G is locally μ-convex. Then,*

1) $\mathfrak{Sequ} \subseteq \mathfrak{D}$ holds if G is sequentially complete.
2) $\mathfrak{Mack} \subseteq \mathfrak{D}$ holds if G is Mackey complete.

We say that G is **k-confined**

- For $k \equiv 0$: *iff* $C^0([0,1],\mathfrak{g}) \subseteq \mathfrak{Sequ}$ holds.
- For $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip},\infty\}$: *iff* $C^k([0,1],\mathfrak{g}) \subseteq \mathfrak{Mack}$ holds.

We conclude from Lemma 31 and Proposition 3 that:

Theorem 3. *Suppose that G is locally μ-convex. Then,*

1) G is C^0-semiregular if G is sequentially complete and 0*-confined.*
2) G is C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip},\infty\}$ iff G is Mackey complete and k*-confined.*

Proof. If G is C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$, then G is Mackey complete by Theorem 2, as well as k-confined by Lemma 31. Moreover,

- If G is sequentially complete and 0-confined, then $C^0([0,1],\mathfrak{g}) \subseteq \mathfrak{Sequ} \subseteq \mathfrak{D}$ holds by Proposition 3.1); so that G is C^0-semiregular.
- If G is Mackey complete and k-confined for $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$, then $C^k([0,1],\mathfrak{g}) \subseteq \mathfrak{Mack} \subseteq \mathfrak{D}$ holds by Proposition 3.2); so that G is C^k-semiregular.

This proves the theorem. □

7.1. Semiregularity

We now provide the

Proof of Proposition 3. We fix $\phi \in \mathfrak{Sequ}/\mathfrak{Mack}$, and choose a tame Cauchy/ Mackey-Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}\mathrm{P}^0(\mathrm{dom}[\phi], \mathfrak{g})$ that converges uniformly to ϕ; i.e.,

- if $\phi \in \mathfrak{Sequ}$ holds, then for each $\mathfrak{p} \in \mathfrak{P}$ and $\varepsilon > 0$, there exists some $p \in \mathbb{N}$, such that
$$\cdot\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \varepsilon \qquad \forall\, m, n \geq p.$$
- if $\phi \in \mathfrak{Mack}$ holds, then we have
$$\cdot\mathfrak{p}_\infty(\phi_m - \phi_n) \leq \mathfrak{c}_\mathfrak{p} \cdot \lambda_{m,n} \qquad \forall\, m, n \geq \mathfrak{l}_\mathfrak{p},\ \ \mathfrak{p} \in \mathfrak{P}$$
  for sequences $\{\mathfrak{c}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$, $\{\mathfrak{l}_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{N}$, and $\mathbb{R}_{\geq 0} \supseteq \{\lambda_{m,n}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}} \to 0$.

We let $[r, r'] \equiv \mathrm{dom}[\phi]$, define $\mu_n := \int_r^\bullet \phi_n$ for each $n \in \mathbb{N}$; and fix an open neighbourhood $\mathcal{O} \subseteq G$ of e with $\overline{\mathcal{O}} \subseteq \mathcal{U}$.[12] Since

$$\mathrm{B} := \mathrm{im}[\phi] \cup \textstyle\bigcup_{n\in\mathbb{N}} \mathrm{im}[\phi_n]$$

is bounded, decomposing $[r, r']$ if necessary, we can assume that $\mathrm{im}[\mu_n] \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$ holds for each $n \in \mathbb{N}$: This is just clear from Lemma 11 and Lemma 24. We now will show in three steps that $\mu = \lim_n \mu_n$ exists, is of class C^1, and fulfills $\delta^r(\mu) = \phi$ with $\mu(0) = e$.

Existence of the Limit:

For $\mathfrak{p} \in \mathfrak{P}$, we choose $\mathfrak{q} \in \mathfrak{P}$ as in Proposition 2; and let $\mathfrak{q} \leq \mathfrak{w} \in \mathfrak{P}$ be as in (73) for $\mathfrak{v} \equiv \mathfrak{q}$ there. We choose $p \in \mathbb{N}$ such large that $|r' - r| \cdot \cdot\mathfrak{w}_\infty(\phi_m - \phi_n) \leq 1$ holds for each $m, n \geq p$, and obtain from (73) that

$$\int \cdot\mathfrak{q}\big(\mathrm{Ad}_{[\int_r^s \phi_m]^{-1}}(\phi_n(s) - \phi_m(s))\big)\, \mathrm{d}s \leq |r' - r| \cdot \cdot\mathfrak{w}_\infty(\phi_m - \phi_n) \leq 1 \qquad \forall\, m, n \geq p.$$

Then, *b)* in combination with Proposition 2 gives

$$\begin{aligned} (\mathfrak{p} \circ \Xi)\big(\mu_m^{-1}(t) \cdot \mu_n(t)\big) &= (\mathfrak{p} \circ \Xi)\big(\textstyle\int_r^t \mathrm{Ad}_{[\int_r^\bullet \phi_m]^{-1}}(\phi_n - \phi_m)\big) \\ &\leq \textstyle\int_r^t \cdot\mathfrak{q}\big(\mathrm{Ad}_{[\int_r^s \phi_m]^{-1}}(\phi_n(s) - \phi_m(s))\big)\, \mathrm{d}s \\ &\leq |r' - r| \cdot \cdot\mathfrak{w}_\infty(\phi_n - \phi_m) \end{aligned} \tag{74}$$

for each $m, n \geq p$, and each $t \in [r, r']$. Now,

[12] Recall that topological groups are T_3 spaces.

- This implies that $\{\mu_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for each $t\in[r,r']$; thus, converges to some $\mu(t)\in\overline{\mathcal{O}}\cap G\subseteq\mathcal{U}$ with $\mu(r)=e$, provided that G is sequentially complete.
- If $\{\phi_n\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence (i.e., we have $\phi\in\mathfrak{Mack}$), we replace $\mathfrak{l}_\mathfrak{p}$ by $\max(\mathfrak{l}_\mathfrak{p},p)$ as well as $\mathfrak{c}_\mathfrak{p}$ by $|r'-r|\cdot\max(\mathfrak{c}_\mathfrak{p},\mathfrak{c}_\mathfrak{w})$ for each $\mathfrak{p}\in\mathfrak{P}$. Then, $\{\mu_n(t)\}_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence for each $t\in[r,r']$; thus, converges to some $\mu(t)\in\overline{\mathcal{O}}\cap G\subseteq\mathcal{U}$ with $\mu(r)=e$, provided that G is Mackey complete.

The rest of the proof is the same for both situations, as we will only use the fact that $\{\phi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the following. We now first have to show that $\mu\colon[r,r']\ni t\mapsto\mu(t)\in G$ is continuous.

Continuity of the Limit:

We fix $\mathfrak{p}\in\mathfrak{P}$, $t\in[r,r']$, $1\geq\varepsilon>0$, and define $J_\delta:=[[r,r']-t]\cap(-\delta,\delta)$ for each $\delta>0$. We now have to show that for $\delta>0$ suitably small, we have

$$\mathfrak{p}\big(\Xi(\mu(t))-\Xi(\mu(t+\tau))\big)\leq\varepsilon\qquad\forall\,\tau\in J_\delta. \tag{75}$$

We choose $\mathfrak{p}\leq\mathfrak{u}\in\mathfrak{P}$ as in Lemma 8 for $\mathrm{C}\equiv\{\mu(t)\}$ there; and obtain

$$\begin{aligned}\mathfrak{p}\big(\Xi(\mu(t))-\Xi(\mu(t+\tau))\big)&\leq\mathfrak{u}\big(\Xi_{\mu(t)}(\mu(t))-\Xi_{\mu(t)}(\mu(t+\tau))\big)\\&=(\mathfrak{u}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu(t+\tau)\big)\end{aligned}$$

provided that $(\mathfrak{u}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu(t+\tau)\big)\leq 1$ holds. Thus, in order to prove (75), it suffices to show that there exist $p\in\mathbb{N}$ and $\delta>0$, such that

$$\begin{aligned}\varepsilon\geq(\mathfrak{u}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu(t+\tau)\big)=&(\mathfrak{u}\circ\Xi)\big((\Xi^{-1}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu_p(t)\big)\cdot\\&(\Xi^{-1}\circ\Xi)\big(\mu_p^{-1}(t)\cdot\mu_p(t+\tau)\big)\cdot\\&(\Xi^{-1}\circ\Xi)\big(\mu_p^{-1}(t+\tau)\cdot\mu(t+\tau)\big)\big)\end{aligned}\tag{76}$$

holds for each $\tau\in J_\delta$.

For this, we let $\mathfrak{u}\leq\mathfrak{o}\in\mathfrak{P}$ be as in (58); and will now show that there exist $p\in\mathbb{N}$, $\delta>0$, such that

$$\begin{aligned}(\mathfrak{o}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu_p(t)\big),\ (\mathfrak{o}\circ\Xi)&\big(\mu_p^{-1}(t)\cdot\mu_p(t+\tau)\big),\\&(\mathfrak{o}\circ\Xi)\big(\mu_p^{-1}(t+\tau)\cdot\mu(t+\tau)\big)\ \leq\ \varepsilon/3\end{aligned}$$

holds for all $\tau\in J_\delta$: Then, (76) is clear from (58).

Now,

- In order to estimate the second term,
  - We choose $\mathfrak{o} \leq \mathfrak{n}$ as in (36), for $\mathfrak{m} \equiv \mathfrak{o}$ there.
  - We choose $\mathfrak{n} \leq \mathfrak{q} \in \mathfrak{P}$ as in Proposition 2, for $\mathfrak{p} \equiv \mathfrak{n}$ there.
  - We choose $\mathfrak{q} \leq \mathfrak{w} \in \mathfrak{P}$ as in (73) for $\mathfrak{v} \equiv \mathfrak{q}$ there; and fix

$$1 \geq \delta := \varepsilon/3 \cdot \max(1, \sup\{X \in \mathrm{B} \mid \mathfrak{w}(X)\})^{-1}.$$

  We then have to discuss the cases $\tau \geq 0$ and $\tau < 0$ separately.

  ▷ Let $\tau \in J_\delta$ with $\tau \geq 0$. Then, for each $p \in \mathbb{N}$, we have

$$\begin{aligned}(\mathfrak{o} \circ \Xi)\big(\mu_p^{-1}(t) \cdot \mu_p(t+\tau)\big) &\leq (\mathfrak{n} \circ \Xi)\big(\mu_p^{-1}(t) \cdot \mu_p(t+\tau)\big) \\ &= (\mathfrak{n} \circ \Xi)\big(\mu_p^{-1}(t) \cdot \big[\textstyle\int_t^{t+\tau} \phi\big] \cdot \mu_p(t)\big) \\ &= (\mathfrak{n} \circ \Xi)\big(\textstyle\int_t^{t+\tau} \mathrm{Ad}_{\mu_p^{-1}(t)}(\phi_p)\big) \\ &\leq \textstyle\int_t^{t+\tau} \mathfrak{q}\big(\mathrm{Ad}_{\mu_p^{-1}(t)}(\phi_p(s))\big)\, \mathrm{d}s \\ &\leq \textstyle\int_t^{t+\tau} \mathfrak{w}(\phi_p(s))\, \mathrm{d}s \leq \varepsilon/3.\end{aligned}$$

  In the second step, we have used *d)*; and in the third step, we have applied *f)* to $\Psi \equiv \mathrm{Conj}_{\mu_p^{-1}(t)}$.

  ▷ Let $\tau \in J_\delta$ with $\tau < 0$. Then, for each $p \in \mathbb{N}$, we have

$$\begin{aligned}(\mathfrak{o} \circ \Xi)\big(\mu_p^{-1}(t) \cdot \mu_p(t-|\tau|)\big) &= (\mathfrak{o} \circ \Xi \circ \mathrm{inv})\big(\mu_p^{-1}(t-|\tau|) \cdot \mu_p(t)\big) \\ &\leq (\mathfrak{n} \circ \Xi)\big(\mu_p^{-1}(t-|\tau|) \cdot \mu_p(t)\big) \\ &= (\mathfrak{n} \circ \Xi)\big(\mu_p^{-1}(t-|\tau|) \cdot \big[\textstyle\int_{t-|\tau|}^{t} \phi_p\big] \cdot \mu_p(t-|\tau|)\big) \\ &= (\mathfrak{n} \circ \Xi)\big(\textstyle\int_{t-|\tau|}^{t} \mathrm{Ad}_{\mu_p^{-1}(t-|\tau|)}(\phi_p)\big) \\ &\leq \textstyle\int_{t-|\tau|}^{t} \mathfrak{q}\big(\mathrm{Ad}_{\mu_p^{-1}(t-|\tau|)}(\phi_p(s))\big)\, \mathrm{d}s \\ &\leq \textstyle\int_{t-|\tau|}^{t} \mathfrak{w}(\phi_p(s))\, \mathrm{d}s \leq \varepsilon/3.\end{aligned}$$

- In order to estimate the first-, and the third term, we let $\mathfrak{o} \leq \mathfrak{f} \in \mathfrak{P}$ be as in (58) for $\mathfrak{u} \equiv \mathfrak{o}$ and $\mathfrak{o} \equiv \mathfrak{f}$ there. We choose $\iota\colon \mathbb{N} \to \mathbb{N}$ strictly increasing with (use (74))

$$\begin{aligned}&\textstyle\sum_{n=0}^{\infty} (\mathfrak{f} \circ \Xi)\big(\mu_{\iota(n+1)}^{-1} \cdot \mu_{\iota(n)}\big) \leq \varepsilon/3 \\ \text{and} \quad &\textstyle\sum_{n=0}^{\infty} (\mathfrak{f} \circ \Xi)\big(\mu_{\iota(n)}^{-1} \cdot \mu_{\iota(n+1)}\big) \leq \varepsilon/3;\end{aligned}$$

  and observe that

$$\begin{aligned}\mu^{-1} \cdot \mu_{\iota(0)} &= \lim_n \big((\mu_{\iota(n)}^{-1} \cdot \mu_{\iota(n-1)}) \cdot (\mu_{\iota(n-1)}^{-1} \cdot \mu_{\iota(n-2)}) \cdot \ldots \cdot (\mu_{\iota(1)}^{-1} \cdot \mu_{\iota(0)})\big) \\ \mu_{\iota(0)}^{-1} \cdot \mu &= \lim_n \big((\mu_{\iota(0)}^{-1} \cdot \mu_{\iota(1)}) \cdot \ldots \cdot (\mu_{\iota(n-2)}^{-1} \cdot \mu_{\iota(n-1)}) \cdot (\mu_{\iota(n-1)}^{-1} \cdot \mu_{\iota(n)})\big)\end{aligned}$$

holds. It is thus clear from (58) that

$$(\mathfrak{o}\circ\Xi)\big(\mu^{-1}(t)\cdot\mu_p(t)\big)\leq\varepsilon/3 \qquad\text{and}\qquad (\mathfrak{o}\circ\Xi)\big(\mu_p^{-1}(t)\cdot\mu(t)\big)\leq\varepsilon/3$$

holds for each $t\in[r,r']$, for $p:=\iota(0)$. From this, the claim is clear.

Uniform Convergence:

We define $\gamma:=\Xi\circ\mu$, as well as $\gamma_n:=\Xi\circ\mu_n$ for each $n\in\mathbb{N}$; and now show that $\{\gamma_n\}_{n\in\mathbb{N}}$ converges uniformly to γ. For this, we let $\mathfrak{p}\in\mathfrak{P}$, and $1\geq\varepsilon>0$ be fixed; and observe that $\mathrm{C}\equiv\mathrm{im}[\mu]$ is compact, because μ is continuous. By Lemma 8, there thus exists some $\mathfrak{p}\leq\mathfrak{u}\in\mathfrak{P}$, such that (let $\mathrm{C}\equiv\mathrm{im}[\mu]$, $g\equiv g(t):=\mu(t)$, $q\equiv q(t):=\mu(t)$, $q'\equiv q'(t):=\mu_m(t)$, $h\equiv e$ there)

$$\begin{aligned}&(\mathfrak{u}\circ\Xi)\big(\mu^{-1}\cdot\mu_m\big)\leq 1 \quad\text{for}\quad m\in\mathbb{N}\\ \Longrightarrow\quad &\mathfrak{p}(\gamma-\gamma_m)\leq(\mathfrak{u}\circ\Xi)\big(\mu^{-1}\cdot\mu_m\big).\end{aligned}$$

We choose $\mathfrak{u}\leq\mathfrak{o}$ as in (58); and let $\iota\colon\mathbb{N}\to\mathbb{N}$ be strictly increasing with (use (74))

$$\begin{aligned}&(\mathfrak{o}\circ\Xi)\big(\mu_m^{-1}\cdot\mu_{\iota(0)}\big)\leq\varepsilon/2 \qquad \forall\, m\geq\iota(0)\\ \text{and}\qquad &\textstyle\sum_{n=0}^{\infty}(\mathfrak{o}\circ\Xi)\big(\mu_{\iota(n+1)}^{-1}\cdot\mu_{\iota(n)}\big)\leq\varepsilon/2.\end{aligned}$$

Then, (58) shows

$$\begin{aligned}(\mathfrak{u}\circ\Xi)\big(\mu^{-1}\cdot\mu_m\big)&=(\mathfrak{u}\circ\Xi)\big(\big(\mu^{-1}\cdot\mu_{\iota(0)}\big)\cdot\big(\mu_{\iota(0)}^{-1}\cdot\mu_m\big)\big)\\ &=\lim_n(\mathfrak{u}\circ\Xi)\big(\big(\mu_{\iota(n)}^{-1}\cdot\mu_{\iota(n-1)}\big)\cdot\big(\mu_{\iota(n-1)}^{-1}\cdot\mu_{\iota(n-2)}\big)\cdot\ldots\\ &\qquad\cdot\big(\mu_{\iota(1)}^{-1}\cdot\mu_{\iota(0)}\big)\cdot\big(\mu_{\iota(0)}^{-1}\cdot\mu_m\big)\big)\\ &\leq\varepsilon\end{aligned}$$

for each $m\geq\iota(0)$, which proves the claim.

We are ready to show

The solution property:

Let υ be as in (30). Then, it is straightforward from the definitions that

$$\gamma_n=\textstyle\int_r^{\bullet}\upsilon(\gamma_n(s),\phi_n(s))\,\mathrm{d}s \qquad \forall\, n\in\mathbb{N} \tag{77}$$

holds, cf. Appendix E.1. Moreover, since υ is continuous, since $\mathrm{im}[\gamma]\times\mathrm{im}[\phi]$ is compact, and since $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ converge uniformly to γ and ϕ, respectively, we additionally obtain

$$\lim_n \int_r^\bullet \upsilon(\gamma_n(s), \phi_n(s))\, \mathrm{d}s = \int_r^\bullet \upsilon(\gamma(s), \phi(s))\, \mathrm{d}s \in \overline{E}.$$

Together with (77), this shows

$$\gamma = \lim_n \gamma_n = \lim_n \int_r^\bullet \upsilon(\gamma_n(s), \phi_n(s))\, \mathrm{d}s = \int_r^\bullet \upsilon(\gamma(s), \phi(s))\, \mathrm{d}s;$$

i.e., that γ is of class C^1 with $\dot{\gamma} = \upsilon(\gamma, \phi) \in E$. We obtain

$$\begin{aligned}\delta^r(\mu) &= \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}\big(\mathrm{d}_\gamma \Xi^{-1}(\dot{\gamma})\big)\\ &= \mathrm{d}_\mu \mathrm{R}_{\mu^{-1}}\big(\mathrm{d}_\gamma \Xi^{-1}(\upsilon(\gamma, \phi))\big)\\ &= \big(\mathrm{d}_\mu \mathrm{R}_{\mu^{-1}} \circ \mathrm{d}_\gamma \Xi^{-1} \circ \mathrm{d}_{\Xi^{-1}(\gamma)} \Xi \circ \mathrm{d}_e \mathrm{R}_{\Xi^{-1}(\gamma)}\big)(\phi)\\ &= \big(\mathrm{d}_\mu \mathrm{R}_{\mu^{-1}} \circ \mathrm{d}_e \mathrm{R}_\mu\big)(\phi) = \phi,\end{aligned}$$

which proves the claim. □

7.2. Particular cases

In this subsection, we discuss several situations in which G is automatically k-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. We start with

7.2.1. Reliable Lie groups. We say that G is **reliable** *iff* for each $\mathfrak{v} \in \mathfrak{P}$, there exists a symmetric neighbourhood $V \subseteq G$ of e as well as a sequence $\{\mathfrak{w}_n\}_{n \in \mathbb{N}_{\geq 1}} \subseteq \mathfrak{P}$ with

$$(78) \qquad \mathfrak{v} \circ \mathrm{Ad}_{g_1} \circ \ldots \circ \mathrm{Ad}_{g_n} \leq \mathfrak{w}_n \qquad \forall\, g_1, \ldots, g_n \in V,\ n \geq 1.$$

For instance, G is reliable:

A) If G is abelian.
B) If for each $\mathfrak{v} \in \mathfrak{P}$, there exist $\mathfrak{w} \in \mathfrak{P}$, $C \geq 0$, and V open with $e \in V$, such that

$$\mathfrak{v} \circ \mathrm{Ad}_{g_1} \circ \ldots \circ \mathrm{Ad}_{g_n} \leq C^n \cdot \mathfrak{w} \qquad \forall\, g_1, \ldots, g_n \in V,\ n \geq 1.$$

In particular, this is the case for the unit group $\mathcal{A}^\times$ of a continuous inverse algebra $\mathcal{A}$ in the sense of [5], just by (59).

C) If for each $\mathfrak{v} \in \mathfrak{P}$, there exist $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, $C \geq 0$, and V open with $e \in V$, such that

$$\mathbf{.}\mathfrak{w} \circ \mathrm{Ad}_g \leq C \cdot \mathbf{.}\mathfrak{w} \qquad \forall\, g \in V.$$

In particular, this is the case if $(\mathfrak{g}, [\cdot,\cdot])$ is submultiplicative, cf. Proposition 6; so that Banach Lie groups are reliable (of course, this can also be directly seen from (27)).

Then,

Lemma 32. *Suppose that G is locally μ-convex and reliable, let $\mathrm{B} \subseteq \mathfrak{g}$ be bounded, and $[r,r'] \in \mathfrak{K}$ be fixed. Then, for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that*

$$\mathbf{.}\mathfrak{v} \circ \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}} \leq \mathbf{.}\mathfrak{w}$$

holds for each $\phi \in \mathfrak{D}\mathrm{P}^0([r,r'],\mathfrak{g})$ with $\mathrm{im}[\phi] \subseteq \mathrm{B}$.

Proof. We choose $\{\mathfrak{w}_n\}_{n\in\mathbb{N}_{\geq 1}} \subseteq \mathfrak{P}$ and V as in (78); and can assume that $\mathfrak{v} \leq \mathfrak{w}_1 \leq \mathfrak{w}_2 \leq \dots$ holds, just by replacing $\mathfrak{w}_n \to \mathfrak{w}_1 + \ldots + \mathfrak{w}_n$ for each $n \geq 1$ if necessary. Then,

- By Proposition 2, there exists some $\mathfrak{q} \in \mathfrak{P}$, such that $\int_\ell^\bullet \psi \in V$ holds for each $\psi \in \mathfrak{D}\mathrm{P}^0([\ell,\ell'],\mathfrak{g})$, $[\ell,\ell'] \in \mathfrak{K}$, with $\int \mathfrak{q}(\psi(s))\,\mathrm{d}s \leq 1$.
- We define $\lambda := \sup\{\mathfrak{q}(X) \mid X \in \mathrm{B}\}$; and choose $n \geq 1$ such large that $\lambda \cdot |r'-r|/n \leq 1$ holds.
- We define $t_p := r + p \cdot |r'-r|/n$ for $p = 0,\ldots,n$; and obtain $\big[\int_{t_p}^t \phi\big]^{-1} \in V$ for each $t \in [t_p, t_{p+1}]$, for $p = 0,\ldots,n-1$.

We define $\mathfrak{w} := \mathfrak{w}_n$, and obtain

$$\begin{aligned}\mathbf{.}\mathfrak{v} \circ \mathrm{Ad}_{[\int_r^t \phi]^{-1}} &\overset{d)}{=} \mathbf{.}\mathfrak{v} \circ \mathrm{Ad}_{[\int_{t_0}^{t_1} \phi]^{-1} \cdot \ldots \cdot [\int_{t_p}^{t} \phi]^{-1}} \\ &= \mathbf{.}\mathfrak{v} \circ \mathrm{Ad}_{[\int_{t_0}^{t_1} \phi]^{-1}} \circ \ldots \circ \mathrm{Ad}_{[\int_{t_p}^{t} \phi]^{-1}} \leq \mathbf{.}\mathfrak{w}_{p+1} \leq \mathbf{.}\mathfrak{w}\end{aligned}$$

for each $t \in [t_p, t_{p+1}]$, for $p = 0,\ldots,n-1$. □

We obtain

Lemma 33. *Suppose that G is locally μ-convex and reliable. Then, G is* k*-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty\}$.*

Proof. This is just clear from Lemma 29 and Lemma 32. □

We thus have

Proposition 4. *Suppose that G is locally μ-convex and reliable. Then,*

1) G is C^0-semiregular if G is sequentially complete.
2) G is C^{lip}-semiregular iff G is Mackey complete iff G is C^∞-semiregular.

Proof. By Lemma 33, G is k-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Thus,

- If G is sequentially complete, then G is C^0-semiregular by Theorem 3.1).
- If G is Mackey complete, then G is C^{lip}-semiregular by Theorem 3.2).

The converse direction in 2) is clear from Theorem 2. □

For instance,

Corollary 5. *Suppose that G is abelian and locally μ-convex. Then,*

1) G is C^0-semiregular if G is sequentially complete.
2) G is C^{lip}-semiregular iff G is Mackey complete iff G is C^∞-semiregular.

In particular, we recover the well known fact that[13]

Corollary 6. *E is Mackey complete iff the Riemann integral $\int \phi(s)\,\mathrm{d}s \in E$ exists for each $\phi \in C^\infty([0,1], E)$ iff the Riemann integral $\int \phi(s)\,\mathrm{d}s \in E$ exists for each $\phi \in \bigsqcup_{[r,r']\in\mathfrak{K}} C^{\mathrm{lip}}([r,r'], E)$.*

7.2.2. Constricted Lie groups. We say that G is **constricted** *iff* for each bounded subset $\mathrm{B} \subseteq \mathfrak{g}$, and each $\mathfrak{v} \in \mathfrak{P}$, there exist $C \geq 0$ and $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that

$$(79) \qquad \mathbf{.}\mathfrak{v} \circ \mathrm{ad}\,X_1 \circ \ldots \circ \mathrm{ad}\,X_n \leq C^n \cdot \mathbf{.}\mathfrak{w} \qquad \forall\, X_1, \ldots, X_n \in \mathrm{B},\ \ n \geq 1$$

holds, with $\mathrm{ad}\,X \colon \mathfrak{g} \ni Y \mapsto [X, Y] \in \mathfrak{g}$ for each $X \in \mathfrak{g}$. We define $\mathrm{ad}\,X^0 := \mathrm{id}_{\mathfrak{g}}$ as well as inductively

$$(\mathrm{ad}\,X)^n := \mathrm{ad}\,X \circ (\mathrm{ad}\,X)^{n-1} \qquad \forall\, n \geq 1.$$

Clearly, G is constricted:

- If $(\mathfrak{g}, [\cdot,\cdot])$ is asymptotic estimate in the sense of [1].

---

[13] Clearly, Corollary 5.1) proves the obvious fact that each $\phi \in \bigsqcup_{[r,r']\in\mathfrak{K}} C^0([r,r'], E)$ is Riemann integrable if E is sequentially complete.

- If $[\cdot,\cdot]$ is submultiplicative; i.e., *iff* for each $\mathfrak{v} \in \mathfrak{P}$, there exists some $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$, such that

$$\mathfrak{w}([X,Y]) \leq \mathfrak{w}(X) \cdot \mathfrak{w}(Y) \qquad \forall\, X,Y \in \mathfrak{g}. \tag{80}$$

- If $[\cdot,\cdot]$ is nilpotent in the sense that there exists some $n \geq 2$, such that

$$\mathrm{ad}\, X_1 \circ \ldots \circ \mathrm{ad}\, X_n = 0 \qquad \forall\, X_1,\ldots,X_n \in \mathfrak{g}.$$

We will now show step by step that

Proposition 5. *Suppose that G is constricted, and admits an exponential map; and that $\mathfrak{g}$ is sequentially complete. Then, G is* k*-confined for each $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$.*

Let us first recall that

Lemma 34. *Suppose that $\sum_{n=0}^\infty r^n \cdot a_n \in \overline{\mathfrak{g}}$ converges for some $r \in \mathbb{R}_{\neq 0}$, and $\{a_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathfrak{g}}$. Then, $\alpha\colon I \to \overline{\mathfrak{g}}, \quad t \mapsto \sum_{n=0}^\infty t^n \cdot a_n$ is of class C^1 (smooth) for each open interval $I \subseteq [-r,r]$, with*

$$\dot\alpha = \textstyle\sum_{n=1}^\infty n \cdot t^{n-1} \cdot a_n \qquad \textit{as well as} \qquad \int_0^t \alpha(s)\,\mathrm{d}s = \sum_{n=0}^\infty \frac{t^{n+1}}{n+1} \cdot a_n.$$

Proof. This just follows as in the case where $\mathfrak{g} = \overline{\mathfrak{g}} = \mathbb{C}$ holds. □

We obtain

Lemma 35. *Suppose that G is constricted, and that $\mathfrak{g}$ is sequentially complete. Then,*

$$\alpha_{X,Y}\colon \mathbb{R} \ni t \mapsto \textstyle\sum_{n=0}^\infty \frac{t^n}{n!} \cdot (\mathrm{ad}\, X)^n(Y) \in \mathfrak{g} \qquad \forall\, X,Y \in \mathfrak{g}$$

is of class C^1 with $\dot\alpha_{X,Y} = [X, \alpha_{X,Y}]$; thus, smooth by Corollary 2.

Proof. It is straightforward from the definitions that

$$\{\textstyle\sum_{k=0}^n \frac{t^k}{k!} \cdot (\mathrm{ad}\, X)^k(Y)\}_{n\in\mathbb{N}} \subseteq \mathfrak{g}$$

is a Cauchy sequence for each $t \in \mathbb{R}$, and $X, Y \in \mathfrak{g}$; thus, converges to some $\alpha_{[X,Y]}(t) \in \mathfrak{g}$. By Lemma 34, $\alpha_{X,Y}\colon \mathbb{R} \to \mathfrak{g} \subseteq \overline{\mathfrak{g}}$ is of class C^1 with $\dot\alpha_{X,Y} = [X, \alpha_{X,Y}]$; which implies $\mathrm{im}[\dot\alpha_{X,Y}] \subseteq \mathfrak{g}$. □

Let now $[r,r'] \in \mathfrak{K}$ be fixed; and recall that [12]

Lemma 36 (Omori). *Let $\phi \in \mathfrak{D}_{[r,r']}$, $Y \in \mathfrak{g}$, and $\alpha \in C^1([r,r'],\mathfrak{g})$ be fixed. Then, we have*

$$\begin{aligned} & \alpha = \mathrm{Ad}_\mu(Y) \quad \textit{for} \quad \mu := \textstyle\int_r^\bullet \phi \\ \Longleftrightarrow \quad & \dot{\alpha} = [\phi, \alpha] \quad \textit{holds with} \quad \alpha(r) = Y. \end{aligned}$$

Proof. The proof is elementary, and can be found in Appendix E.2. □

We conclude that

Corollary 7. *Suppose that G is constricted, and admits an exponential map; and that $\mathfrak{g}$ is sequentially complete. Then, we have $\mathrm{Ad}_{\exp(-t\cdot X)}(Y) = \alpha_{-X,Y}(t)$ for all $t \geq 0$, and $X, Y \in \mathfrak{g}$.*

Proof. By (47), we have $\alpha(t) := \mathrm{Ad}_{\exp(-t\cdot X)}(Y) = \mathrm{Ad}_{\int_0^t \phi_{-X}}(Y)$, i.e., $\dot{\alpha} = [\phi_{-X}, \alpha] \equiv [-X, \alpha]$ by Lemma 36. The claim is thus clear from Lemma 35 and Lemma 36. □

For the rest of this section, we let $\mathfrak{exp}$ denote the exponential function on $\mathbb{R}$.

We obtain

Lemma 37. *Suppose that G is constricted, and admits an exponential map; and that $\mathfrak{g}$ is sequentially complete. Then, for each $\mathfrak{p} \in \mathfrak{P}$, and each bounded subset $\mathrm{B} \subseteq \mathfrak{g}$, there exists some $\mathfrak{q} \in \mathfrak{P}$, such that $\boldsymbol{.}\mathfrak{p} \circ \mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}} \leq \boldsymbol{.}\mathfrak{q}$ holds for each $\phi \in \mathrm{CoP}([r,r'],\mathfrak{g})$ with $\mathrm{im}[\phi] \subseteq \mathrm{B}$.*

Proof. We let $\mathfrak{w}$ be as in (79), for $\mathfrak{v} \equiv \mathfrak{p}$ there; and choose $r = t_0 < \ldots < t_n = r'$ as well as $X_0, \ldots, X_{n-1} \in \mathfrak{g}$, with $\phi|_{(t_p,t_{p+1})} = X_p$ for all $p = 0, \ldots, n-1$. Then, for $0 \leq p \leq n-1$ and $t \in (t_p, t_{p+1}]$, we have

$$\mathrm{Ad}_{[\int_r^t \phi]^{-1}} = \mathrm{Ad}_{\exp(-|t_1-t_0|\cdot X_0)} \circ \ldots \circ \mathrm{Ad}_{\exp(-|t_p-t_{p-1}|\cdot X_{p-1})} \circ \mathrm{Ad}_{\exp(-|t-t_p|\cdot X_p)};$$

so that Corollary 7 together with (79) shows

$$\big(\boldsymbol{.}\mathfrak{p} \circ \mathrm{Ad}_{[\int_r^t \phi]^{-1}}\big)(Y) \leq \mathfrak{exp}(|t-r|\cdot C) \cdot \boldsymbol{.}\mathfrak{w}(Y) \qquad \forall\, Y \in \mathfrak{g},\ \ t \in [r,r'].$$

The claim thus holds for $\mathfrak{q} := \mathfrak{exp}(|r'-r|\cdot C)\cdot \mathfrak{w}$. □

We are ready for the

Proof of Proposition 5. The claim is clear from Lemma 30 and Lemma 37. □

7.2.3. Submultiplicative Lie algebras. We finally want to discuss the situation where $(\mathfrak{g},[\cdot,\cdot])$ is submultiplicative. Clearly, G is constricted in this case; but, as we are going to show now, there exists a sharper version of Proposition 5 neither presuming the existence of the exponential map nor sequentially completeness of $\mathfrak{g}$. More specifically, we will show that

Proposition 6. *If $(\mathfrak{g},[\cdot,\cdot])$ is submultiplicative, then G is* k*-confined for each $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$. Moreover, G is reliable as it fulfills the condition introduced in C).*

For this, let $[r,r']\in\mathfrak{K}$ be fixed; and recall that

Lemma 38 (Grönwall). *Let $\alpha,\beta\colon [r,r']\to\mathbb{R}_{\geq 0}$ be of class C^1, and $C\geq 0$. Then,*

$$\alpha\leq C+\int_r^\bullet(\alpha\cdot\beta)(s)\,\mathrm{d}s \qquad\Longrightarrow\qquad \alpha\leq C\cdot\mathfrak{exp}\big(\int_r^\bullet\beta(s)\,\mathrm{d}s\big).$$

We obtain that

Lemma 39. *Suppose that $(\mathfrak{g},[\cdot,\cdot])$ is submultiplicative, and let $\mathfrak{v}\leq\mathfrak{w}\in\mathfrak{P}$ be as in* (80)*. Then, for each $\phi\in C^0([r,r'],\mathfrak{g})$, $Y\in\mathfrak{g}$, and $\alpha\in C^1([r,r'],\mathfrak{g})$ with $\dot\alpha=[\phi,\alpha]$ and $\alpha(r)=Y$, we have*

$$\bullet\mathfrak{v}(\alpha)\leq\bullet\mathfrak{w}(Y)\cdot\mathfrak{exp}\big(\int_r^\bullet\bullet\mathfrak{w}(\phi(s))\,\mathrm{d}s\big).$$

Proof. We conclude from Lemma 6 that

$$\bullet\mathfrak{w}(\alpha)\leq\bullet\mathfrak{w}(Y)+\int_r^\bullet\bullet\mathfrak{w}(\dot\alpha(s))\,\mathrm{d}s\leq\bullet\mathfrak{w}(Y)+\int_r^\bullet\bullet\mathfrak{w}(\alpha(s))\cdot\bullet\mathfrak{w}(\phi(s))\,\mathrm{d}s$$

holds; so that the claim is clear from Lemma 38. □

Corollary 8. *Suppose that $(\mathfrak{g},[\cdot,\cdot])$ is submultiplicative, and let $\mathfrak{v}\leq\mathfrak{w}\in\mathfrak{P}$ be as in* (80)*. Then,*

$$\bullet\mathfrak{v}\big(\mathrm{Ad}_{\int_r^\bullet\phi}(Y)\big)\leq\mathfrak{exp}(\int_r^\bullet\bullet\mathfrak{w}(\phi(s))\,\mathrm{d}s)\cdot\bullet\mathfrak{w}(Y)$$

holds for each $Y\in\mathfrak{g}$ and $\phi\in\mathfrak{D}_{[r,r']}$; thus,

$$\bullet\mathfrak{w}\big(\mathrm{Ad}_{[\int_r^\bullet\phi]^\pm}(Y)\big)\leq\mathfrak{exp}(|r'-r|\cdot\bullet\mathfrak{w}_\infty(\phi))\cdot\bullet\mathfrak{w}(Y)\qquad\forall\,Y\in\mathfrak{g},\ \ \phi\in\mathfrak{D}_{[r,r']}.$$

Proof. The first statement is clear from Lemma 36, and Lemma 39. Then, the second statement is immediate from Example 1. □

We are ready for the

Proof of Proposition 6. The first statement is clear from Corollary 8 and Lemma 29. For the second statement, we let $\mathfrak{v} \leq \mathfrak{w} \in \mathfrak{P}$ be as in (80), choose $\mathfrak{w} \leq \mathfrak{o} \in \mathfrak{P}$ as in Lemma 25 for $\mathfrak{m} \equiv \mathfrak{w}$ there; and define $V := \Xi^{-1}(\mathrm{B}_{\mathfrak{o},1})$. We furthermore define

$$\gamma_x\colon [0,1] \ni t \mapsto t \cdot x \in V \quad \text{as well as} \quad \phi_x := \delta^r(\Xi^{-1} \circ \gamma_x) \quad \text{for each} \quad x \in V.$$

Then, Lemma 25 shows that $\mathfrak{w}_\infty(\phi_x) \leq \mathfrak{o}_\infty(\dot{\gamma}_x) \leq 1$ holds for each $x \in V$; so that Corollary 8 gives

$$\mathfrak{w}(\mathrm{Ad}_{\Xi^{-1}(x)}(Y)) = \mathfrak{w}(\mathrm{Ad}_{(\Xi^{-1}\circ\gamma_x)(1)}(Y)) = \mathfrak{w}\big(\mathrm{Ad}_{\int_0^1 \phi_x}(Y)\big) \leq \mathfrak{exp}(1) \cdot \mathfrak{w}(Y),$$

for each $Y \in \mathfrak{g}$; which shows the claim. □

8. Differentiation under the integral

In this section, we clarify under which circumstances $\mathrm{evol}^k_{[r,r']}$ for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ and $[r,r'] \in \mathfrak{K}$ is differentiable w.r.t. the (standard) the C^k-topology. In particular, we will show that[14]

Theorem 4.

1) *If G is 0-continuous and C^0-semiregular, then $\mathrm{evol}^0_{[r,r']}$ is smooth for each $[r,r'] \in \mathfrak{K}$ iff $\mathfrak{g}$ is integral complete iff $\mathrm{evol}^0_{[0,1]}$ is differentiable at zero.*
2) *If G is k-continuous and C^k-semiregular for $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$, then $\mathrm{evol}^k_{[r,r']}$ is smooth for each $[r,r'] \in \mathfrak{K}$ iff $\mathfrak{g}$ is Mackey complete iff $\mathrm{evol}^k_{[0,1]}$ is differentiable at zero.*

Here, for $k = 0$ in the first-, and $k \in \mathbb{N}_{\geq 1} \sqcup \{\infty\}$ in the second case, we have

$$\big(\mathrm{d}_\phi\, \mathrm{evol}^k_{[r,r']}\big)(\psi) = \mathrm{d}_e \mathrm{L}_{\int \phi}\big(\textstyle\int \mathrm{Ad}_{[\int_r^s \phi]^{-1}}(\psi(s))\, \mathrm{d}s\big)$$
$$\forall\, \phi, \psi \in C^k([r,r'], \mathfrak{g}), \;\; [r,r'] \in \mathfrak{K}.$$

Proof. Confer Sect. 8.4. □

[14] A C^1-version will also be proven for the Lipschitz case, cf. Corollary 13.

We recall [3] that a $\mathfrak{g}$ is said to be **integral complete** *iff* $\int \phi(s)\,\mathrm{d}s \in \mathfrak{g}$ exists for each $\phi \in C^0([0,1],\mathfrak{g})$.[15] Theorem 4 will be a consequence of the more general Theorem 5, being concerned with differentiation of parameter dependent integrals. The key point of the whole discussion is that if G is k-continuous and C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$, then the directional derivative of $\mathrm{evol}^k_{[r,r']}$ at zero along some $\phi \in C^k([r,r'],\mathfrak{g})$ always exists; namely, in the completion of $\overline{\mathfrak{g}}$ of $\mathfrak{g}$ – as explicitly given by

$$\tag{81} \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0} \int h\cdot\phi = \int \phi(s)\,\mathrm{d}s \in \overline{\mathfrak{g}}.$$

We thus have to clarify this elementary issues first.

8.1. Differentiation at zero

We fix $[r,r'] \in \mathfrak{K}$ in the following; and let $\overline{\mathfrak{g}}$ and $\overline{E}$ denote the completions of $\mathfrak{g}$ and E, respectively. By Lemma 2, then $\mathrm{d}_e\Xi\colon \mathfrak{g} \to E$ extends uniquely to a continuous isomorphism $\overline{\mathrm{d}_e\Xi}\colon \overline{\mathfrak{g}} \to \overline{E}$. In order to prove (81), we now first need to show that $\phi \in C^k([r,r'],\mathfrak{g})$, and $\mathrm{s} \preceq k \in \mathbb{N} \sqcup \{\infty\}$ given, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],\mathfrak{g})$ with

$$\lim_{n\to\infty} \mathfrak{p}^{\mathrm{s}}_\infty(\phi - \phi_n) = 0 \qquad \text{as well as} \qquad \int_r^\bullet \phi_n(s)\,\mathrm{d}s \in \mathfrak{g} \qquad \forall\, n \in \mathbb{N}.$$

Such a sequence can be obtained, e.g., by approximating $\phi^{(\mathrm{s})}$ by polygonal curves, smoothening them by convolution, and then integrating them s-times. Basically, then (81) follows from the triangle inequality and Proposition 1.

Polygons and Convolution:. We let F denote the set of all finite dimensional linear subspaces $F \subseteq \mathfrak{g}$ of $\mathfrak{g}$; and define

$$C^k(D,\mathrm{F}) := \bigsqcup_{F\in\mathrm{F}} C^k(D,F) \qquad \forall\, D \in \mathfrak{I},\;\; k \in \mathbb{N} \sqcup \{\infty\}.$$

Moreover, for each $n \geq 1$, we fix $\rho_n\colon (-1/n, 1/n) \to \mathbb{R}_{\geq 0}$ smooth and compactly supported with $\int \rho_n(s)\,\mathrm{d}s = 1$. Then, for $\chi \in C^0(I,\mathrm{F})$ with $[r,r'] \subseteq I$ ($I \subseteq \mathbb{R}$ an open interval) given, we choose $m \geq 1$ such large that $[r,r'] +$

[15] Clearly, this is equivalent to require that $\int \phi(s)\,\mathrm{d}s \in \mathfrak{g}$ exists for each $\phi \in C^0([r,r'],\mathfrak{g})$, for each $[r,r'] \in \mathfrak{K}$.

$(-1/m, 1/m) \subseteq I$ holds; and define (convolution)

$$C^\infty([r,r'],\mathrm{F}) \ni \chi * \rho_n\colon [r,r'] \ni t \mapsto \int_{t-1/n}^{t+1/n} \rho_n(t-s)\cdot\chi(s)\,\mathrm{d}s \qquad \forall\, n \geq m.$$

Clearly, $\{\chi * \rho_n\}_{n\geq m} \to \chi|_{[r,r']}$ converges uniformly (w.r.t. the seminorms $\{\boldsymbol{\cdot}\mathfrak{p}_\infty\}_{\mathfrak{p}\in\mathfrak{P}}$) with

$$(\chi * \rho_n)^{(p)} = \chi * \rho_n^{(p)} \qquad \forall\, p \in \mathbb{N},\ \ n \geq m.$$

Let now $\mathrm{Poly}([r,r'],\mathfrak{g}) \subseteq C^0([r,r'],\mathrm{F})$ denote the set of all maps $\chi\colon [r,r'] \to \mathfrak{g}$, such that there exist $r = t_0 < \ldots < t_n = r'$ and $X_0,\ldots,X_{n-1} \in \mathfrak{g}$ with

$$\chi(t_p+\tau) = \chi(t_p) + \tau\cdot X_p \qquad \forall\, p = 0,\ldots,n-1,\ \ \tau \leq t_{p+1}-t_p.$$

Clearly, for each $\psi \in C^0([r,r'],\mathfrak{g})$, there exists a sequence $\{\chi_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Poly}([r,r'],\mathfrak{g})$ with $\{\chi_n\}_{n\in\mathbb{N}} \to \psi$ uniformly; and, combining this with the statements made above, we easily obtain that

Lemma 40. *For each $\psi \in C^0([r,r'],\mathfrak{g})$, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],\mathrm{F})$ with $\{\psi_n\}_{n\in\mathbb{N}} \to \psi$ uniformly.*

Iterated Integration:. We define $\mathfrak{S}[p]\colon \mathfrak{g}^p \times C^0([r,r'],\overline{\mathfrak{g}}) \to C^p([r,r'],\overline{\mathfrak{g}})$ for $p \geq 1$, inductively by

$$\mathfrak{S}[1]\colon \mathfrak{g} \times C^0([r,r'],\overline{\mathfrak{g}}) \to C^1([r,r'],\overline{\mathfrak{g}}), \qquad (X,\phi) \mapsto X + \int_r^\bullet \phi(s)\,\mathrm{d}s$$

as well as

$$\mathfrak{S}[p](X_1,\ldots,X_p,\phi) := \mathfrak{S}[1](X_p, \mathfrak{S}[p-1](X_{p-1},\ldots,X_1,\phi))$$

for all $X_1,\ldots,X_p$ and $\phi \in C^0([r,r'],\overline{\mathfrak{g}})$, for $p \geq 2$. Evidently,

- for all $\phi \in C^0([r,r'],\mathrm{F})$ and $X_1,\ldots,X_p \in \mathfrak{g}$, we have $\mathfrak{S}[p](X_1,\ldots,X_p,\phi) \in C^p([r,r'],\mathrm{F})$.
- for all $\phi,\psi \in C^0([r,r'],\overline{\mathfrak{g}})$ and $X_1,\ldots,X_p \in \mathfrak{g}$, we have

$$\mathfrak{S}[p](X_1,\ldots,X_p,\phi) - \mathfrak{S}[p](X_1,\ldots,X_p,\psi) = \mathfrak{S}[p](0,\ldots,0,\phi-\psi).$$

- for all $\phi \in C^p([r,r'],\overline{\mathfrak{g}})$, we have $\phi = \mathfrak{S}[p]\big(\phi^{(p-1)}(r),\dots,\phi^{(0)}(r),\phi^{(p)}\big)$.
- for all $\phi \in C^0([r,r'],\overline{\mathfrak{g}})$, $p \geq 1$, and $X_1,\dots,X_p$, we have

$$\mathfrak{S}[p](X_1,\dots,X_p,\phi)^{(p)} = \phi$$

as well as

$$\mathfrak{S}[p](X_1,\dots,X_p,\phi)^{(\mathrm{s})} = \mathfrak{S}[p-\mathrm{s}](X_1,\dots,X_{p-\mathrm{s}},\phi) \qquad \forall\, 1 \leq \mathrm{s} \leq p-1.$$

- for all $\phi \in C^0([r,r'],\overline{\mathfrak{g}})$, $p \geq 1$, and $\mathfrak{q} \in \mathfrak{P}$, we have (apply Lemma 6 successively)

$$.\overline{\mathfrak{q}}_\infty(\mathfrak{S}[p](0,\dots,0,\phi)) \leq |r'-r|^p \cdot \overline{\mathfrak{q}}_\infty(\phi).$$

The previous (and the first) point thus shows that for $0 \leq \mathrm{u} \leq \mathrm{s} \in \mathbb{N}$, we have

$$\begin{aligned}&{}_{\bullet}\mathfrak{q}^{\mathrm{u}}_\infty(\mathfrak{S}[\mathrm{s}](0,\dots,0,\phi)) \leq \max(1,|r'-r|)^{\mathrm{s}} \cdot {}_{\bullet}\mathfrak{q}_\infty(\phi)\\ &\forall\, \mathfrak{q} \in \mathfrak{P},\ \ \phi \in C^0([r,r'],\mathrm{F}).\end{aligned}$$

Specific Estimates:. Let $\mathrm{s} \preceq k \in \mathbb{N} \sqcup \{\mathrm{lip},\infty\}$, $\mathfrak{m} \in \mathfrak{P}$, and $\phi \in C^k([r,r'],\mathfrak{g})$ be given.

- We choose $\{\psi_n\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],\mathrm{F})$ with $\{\psi_n\}_{n\in\mathbb{N}} \to \phi^{(\mathrm{s})}$ uniformly (Lemma 40); and define

$$\phi_n := \mathfrak{S}[\mathrm{s}]\big(\phi^{(\mathrm{s}-1)}(r),\dots,\phi^{(0)}(r),\psi_n\big) \in C^\infty([r,r'],\mathrm{F}) \qquad \forall\, n \in \mathbb{N}.$$

- We conclude from the third-, second-, and the last point in the previous part that

$$\begin{aligned}{}_{\bullet}\mathfrak{q}^{\mathrm{s}}_\infty(\phi-\phi_n) &= {}_{\bullet}\mathfrak{q}^{\mathrm{s}}_\infty\big(\mathfrak{S}[\mathrm{s}]\big(\phi^{(\mathrm{s}-1)}(r),\dots,\phi^{(0)}(r),\phi^{(\mathrm{s})}\big)\\ &\qquad - \mathfrak{S}[\mathrm{s}]\big(\phi^{(\mathrm{s}-1)}(r),\dots,\phi^{(0)}(r),\psi_n\big)\big)\\ &= {}_{\bullet}\mathfrak{q}^{\mathrm{s}}_\infty\big(\mathfrak{S}[\mathrm{s}]\big(0,\dots,0,\phi^{(\mathrm{s})}-\psi_n\big)\big)\\ &\leq \max(1,|r'-r|)^{\mathrm{s}} \cdot {}_{\bullet}\mathfrak{q}_\infty\big(\phi^{(\mathrm{s})}-\psi_n\big)\end{aligned} \tag{82}$$

holds for each $\mathfrak{q} \in \mathfrak{P}$; thus,

$$ {}_{\bullet}\mathfrak{q}^{\mathrm{s}}_\infty(\phi_n) \leq {}_{\bullet}\mathfrak{q}^{\mathrm{s}}_\infty(\phi) + \max(1,|r'-r|)^{\mathrm{s}} \cdot {}_{\bullet}\mathfrak{q}_\infty\big(\phi^{(\mathrm{s})}-\psi_n\big) \qquad \forall\, n \in \mathbb{N}. \tag{83}$$

- For each $h \in \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$\begin{aligned}\gamma_{h,n} &:= h \cdot \mathrm{d}_e\Xi(\textstyle\int_r^\bullet \phi_n(s)\,\mathrm{d}s)\\ &\equiv h \cdot \mathrm{d}_e\Xi \circ \mathfrak{S}[\mathrm{s}+1]\big(\phi^{(\mathrm{s}-1)}(r), \dots, \phi^{(0)}(r), 0, \psi_n\big) \in C^\infty([r,r'],E);\end{aligned}$$

and obtain from (83) that

$$\begin{aligned}\mathfrak{w}^{\mathrm{s}}_\infty(\gamma_{h,n}) &= |h| \cdot \mathbf{.}\mathfrak{w}^{\mathrm{s}}_\infty\big(\textstyle\int_r^\bullet \phi_n(s)\,\mathrm{d}s\big)\\ &\leq |h| \cdot |r'-r| \cdot \mathbf{.}\mathfrak{w}^{\mathrm{s}}_\infty(\phi_n)\\ &\leq |h| \cdot |r'-r| \cdot \big(\mathbf{.}\mathfrak{w}^{\mathrm{s}}_\infty(\phi)\\ &\qquad + \max(1,|r'-r|)^{\mathrm{s}} \cdot \mathbf{.}\mathfrak{w}_\infty\big(\phi^{(\mathrm{s})} - \psi_n\big)\big)\end{aligned} \tag{84}$$

holds for each $\mathfrak{w} \in \mathfrak{P}$.
- Since $\mathrm{B} := \mathrm{im}[\phi^{(\mathrm{s})}] \cup \bigcup_{n\in\mathbb{N}} \mathrm{im}[\psi_n]$ is bounded, there exists some $\delta > 0$, such that

$$\begin{aligned}&\mu_{h,n} := \Xi^{-1} \circ \gamma_{h,n} \qquad \text{and}\\ &\phi_{h,n} := \delta^r(\mu_{h,n}) = \omega(\gamma_{h,n}, \dot\gamma_{h,n}) = h \cdot \omega(\gamma_{h,n}, \mathrm{d}_e\Xi(\phi_n))\end{aligned}$$

are defined, for each $|h| \leq \delta$ and $n \in \mathbb{N}$.
- Then, for $\mathfrak{m} \in \mathfrak{P}$ fixed, Lemma 5.1) applied to $\Omega \equiv \omega(\cdot, \mathrm{d}_e\Xi(\cdot))$, $\gamma \equiv \gamma_{h,n}$, $\psi \equiv \phi_n$, $\mathfrak{p} \equiv \mathfrak{m}$, provides us with certain seminorms $\mathfrak{q}, \mathfrak{w} \in \mathfrak{P}$, such that

$$\mathbf{.}\mathfrak{w}^{\mathrm{s}}_\infty(\gamma_{h,n}) \leq 1 \qquad \Longrightarrow \qquad \begin{aligned}\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\phi_{h,n}) &= |h| \cdot \mathfrak{m}^{\mathrm{s}}_\infty(\Omega(\gamma_{h,n}, \phi_n))\\ &\leq |h| \cdot \mathbf{.}\mathfrak{q}^{\mathrm{s}}_\infty(\phi_n).\end{aligned}$$

Thus, shrinking δ if necessary, by (83) and boundedness of B, we can achieve that

$$\begin{aligned}&\mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(h\cdot\phi) \leq 1,\ \ \mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(\phi_{h,n}) \leq 1,\ \ \mathbf{.}\mathfrak{m}^{\mathrm{s}}_\infty(h\cdot\phi - \phi_{h,n}) \ \leq\ 1\\ &\forall\, |h| \leq \delta,\ \ n \in \mathbb{N}.\end{aligned} \tag{85}$$

We now have everything we need to prove

Proposition 7. *Suppose that G is* k-continuous *for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$; and that $(-\delta,\delta)\cdot\phi \subseteq \mathfrak{D}^k_{[r,r']}$ holds for some $\phi \in C^k([r,r'],\mathfrak{g})$ and $\delta > 0$. Then, we have*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0} \textstyle\int h\cdot\phi = \int \phi(s)\,\mathrm{d}s \in \overline{\mathfrak{g}}.$$

Proof. We fix $\mathfrak{p} \in \mathfrak{P}$, and have to show that[16]

$$\Delta_\phi(h) := \tfrac{1}{|h|} \cdot \overline{\mathfrak{p}}\big(\Xi(\textstyle\int h \cdot \phi) - h \cdot \overline{\mathrm{d}_e\Xi}(\textstyle\int \phi(s)\,\mathrm{d}s)\big)$$

tends to zero if h tends to zero. For this, we choose $\mathfrak{p} \leq \mathfrak{m}$ and $\mathrm{s} \preceq k$ as in Proposition 1; and let $\{\phi_{h,n}\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],\mathfrak{g})$, $\{\gamma_{h,n}\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],\mathfrak{g})$, $\{\mu_{h,n}\}_{n\in\mathbb{N}} \subseteq C^\infty([r,r'],G)$, $\delta > 0$ be as above. Then, Proposition 1 and (85) (fourth step) show that for $|h| \leq \delta$, we have

$$\begin{aligned}
\Delta_\phi(h) &\leq \tfrac{1}{|h|} \cdot \mathfrak{p}\big(\Xi(\textstyle\int h \cdot \phi) - h \cdot \mathrm{d}_e\Xi(\textstyle\int \phi_n(s)\,\mathrm{d}s)\big)\\
&\qquad + \overline{\mathfrak{p}}\big(\overline{\mathrm{d}_e\Xi}(\textstyle\int \phi(s)\,\mathrm{d}s) - \mathrm{d}_e\Xi(\textstyle\int \phi_n(s)\,\mathrm{d}s)\big)\\
&\leq \tfrac{1}{|h|} \cdot \mathfrak{p}\big(\Xi(\textstyle\int h \cdot \phi) - \gamma_{h,n}(r')\big)\\
&\qquad + \textstyle\int (\overline{\mathfrak{p}} \circ \overline{\mathrm{d}_e\Xi})(\phi(s) - \phi_n(s))\,\mathrm{d}s\\
&= \tfrac{1}{|h|} \cdot \mathfrak{p}\big(\Xi(\textstyle\int h \cdot \phi) - \Xi(\mu_{h,n}(r'))\big)\\
&\qquad + \textstyle\int {\scriptstyle\blacksquare}\mathfrak{p}(\phi(s) - \phi_n(s))\,\mathrm{d}s\\
&\leq \tfrac{1}{|h|} \cdot \textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(h \cdot \phi(s) - \phi_{h,n}(s)\big)\,\mathrm{d}s\\
&\qquad + \textstyle\int {\scriptstyle\blacksquare}\mathfrak{p}(\phi(s) - \phi_n(s))\,\mathrm{d}s\\
&= \textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(\phi(s) - \omega(\gamma_{h,n}(s), \mathrm{d}_e\Xi(\phi_n(s)))\big)\,\mathrm{d}s\\
&\qquad + \textstyle\int {\scriptstyle\blacksquare}\mathfrak{p}(\phi(s) - \phi_n(s))\,\mathrm{d}s.
\end{aligned}$$

Let now $\varepsilon > 0$ be fixed. By (82), there exists some $n_\varepsilon \in \mathbb{N}$, such that the second summand is bounded by $\varepsilon/3$ for each $n \geq n_\varepsilon$. Moreover, since $\phi = \omega(0, \mathrm{d}_e\Xi(\phi))$ holds, we can estimate the first summand by

$$\begin{aligned}
&\textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(\phi(s) - \omega(\gamma_{h,n}(s), \mathrm{d}_e\Xi(\phi_n(s)))\big)\,\mathrm{d}s\\
&\quad = \textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(\omega(0, \mathrm{d}_e\Xi(\phi(s)))) - \omega(\gamma_{h,n}(s), \mathrm{d}_e\Xi(\phi_n(s)))\big)\,\mathrm{d}s\\
&\quad \leq \textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(\omega(0, \mathrm{d}_e\Xi(\phi(s))) - \omega(\gamma_{h,n}(s), \mathrm{d}_e\Xi(\phi(s)))\big)\,\mathrm{d}s\\
&\qquad + \textstyle\int {\scriptstyle\blacksquare}\mathfrak{m}\big(\omega(\gamma_{h,n}(s), \mathrm{d}_e\Xi(\phi(s) - \phi_n(s)))\big)\,\mathrm{d}s.
\end{aligned} \tag{86}$$

Then,

- Since $\mathrm{im}[\phi]$ is compact, we can achieve that the second line in (86) is bounded by $\varepsilon/3$ for each $n \in \mathbb{N}$, just by shrinking δ if necessary.
- In order to estimate the third line in (86), we choose $\mathfrak{m} \leq \mathfrak{w}$ as in (32) for $\mathfrak{v} \equiv \mathfrak{m}$ there. Then, by (84), we can achieve that $\mathfrak{w}_\infty(\gamma_{h,n}) \leq \mathfrak{w}^{\mathrm{s}}_\infty(\gamma_{h,n}) \leq 1$ holds for each $|h| \leq \delta$, for $\delta > 0$ suitably small; and obtain

[16] For $|h| \leq \delta$ suitably small, this is defined by Lemma 20.

$$\boldsymbol{.}\mathfrak{m}_\infty\big(\omega(\gamma_{h,n}, \mathrm{d}_e\Xi(\phi - \phi_n))\big) \overset{(32)}{\leq} \boldsymbol{.}\mathfrak{w}_\infty(\phi - \phi_n) \qquad \forall\, |h| \leq \delta.$$

It is then clear from (82) that for $n'_\varepsilon \geq n_\varepsilon$ suitably large, the third line in (86) is bonded by $\varepsilon/3$ for each $n \geq n'_\varepsilon$ and $|h| \leq \delta$.

We thus have $\Delta_\phi(h) \leq \varepsilon$ for each $|h| \leq \delta$; and conclude that $\lim_{h\to 0} \Delta_\phi(h) = 0$ holds. □

We immediately obtain

Corollary 9.

1) *Suppose that G is 0-continuous and C^0-semiregular. Then,* $\mathrm{evol}^0_{[0,1]}$ *is differentiable at zero iff* $\mathfrak{g}$ *is* integral complete.
2) *Suppose that G is* k*-continuous and C^∞-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Then,* $\mathrm{evol}^k_{[0,1]}\big|_{C^\infty([0,1],\mathfrak{g})}$ *is differentiable at zero iff $\mathfrak{g}$ is Mackey complete.*

Proof. This is clear from Proposition 7 and Corollary 6. □

8.2. Integrals with parameters

Given an open interval $J \subseteq \mathbb{R}$ as well as $x \in J$, in the following, we denote

$$J[x] := \{h \in \mathbb{R}_{\neq 0} \mid x + h \in J\}.$$

We now will discuss the differentiation of parameter-dependent integrals. For this, we let $[r, r'] \in \mathfrak{K}$ be fixed; and observe that

Corollary 10. *Let G be C^k-semiregular for $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$. Then, for $\phi, \psi, \chi \in C^k([r, r'], \mathfrak{g})$, we have*

$$\begin{aligned} &\textstyle\int(\phi + \psi + \chi) = \alpha \cdot \beta \cdot \gamma \\ &\textit{for} \qquad \alpha := \textstyle\int \phi, \quad \beta := \textstyle\int \mathrm{Ad}_{\alpha^{-1}}(\psi), \quad \gamma := \textstyle\int \mathrm{Ad}_{(\alpha\cdot\beta)^{-1}}(\chi). \end{aligned}$$

Proof. Applying *b)* twice, we obtain

$$\beta^{-1} \cdot \alpha^{-1} \cdot [\textstyle\int \phi + \psi + \chi] = \beta^{-1} \cdot \textstyle\int_r^\bullet \mathrm{Ad}_{\alpha^{-1}}(\psi + \chi) = \textstyle\int_r^\bullet \mathrm{Ad}_{(\alpha\cdot\beta)^{-1}}(\chi),$$

because $\mathrm{Ad}_{\alpha^{-1}}(\psi + \chi),\ \mathrm{Ad}_{(\alpha\cdot\beta)^{-1}}(\chi) \in C^k([r, r'], \mathfrak{g})$ holds by Lemma 13. □

Moreover, let $\delta > 0$, and suppose that $\mu, \nu\colon [0,\delta] \to G$ are maps with

$$\begin{aligned} \lim_{h\to 0} \mu(h) = \mu(0) = e \qquad &\text{and} \qquad \lim_{h\to 0} \tfrac{1}{h}\cdot(\Xi\circ\mu)(h) = X \in \overline{E} \\ \lim_{h\to 0} \nu(h) = \nu(0) = e \qquad &\text{and} \qquad \lim_{h\to 0} \tfrac{1}{h}\cdot(\Xi\circ\nu)(h) = Y \in \overline{E}. \end{aligned}$$

Then, we obtain from Lemma 7 that, cf. Appendix F.1

$$\lim_{h\to 0} \tfrac{1}{h}\cdot \Xi(\mu(h)\cdot\nu(h)) = X + Y \in \overline{E} \tag{87}$$

holds; and are ready for

Theorem 5. *Suppose that G is* k*-continuous and C^k-semiregular for some $k \in \mathbb{N}\sqcup\{\mathrm{lip},\infty\}$; and let $\Phi\colon I\times[r,r'] \to \mathfrak{g}$ ($I \subseteq \mathbb{R}$ open) be fixed with $\Phi(z,\cdot) \in C^k([r,r'],\mathfrak{g})$ for each $z \in I$. Then,*

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\big([\textstyle\int\Phi(x,\cdot)]^{-1}[\int\Phi(x+h,\cdot)]\big) = \int \mathrm{Ad}_{[\int_r^s\Phi(x,\cdot)]^{-1}}(\partial_z\Phi(x,s))\,\mathrm{d}s \in \overline{\mathfrak{g}}$$

holds for $x \in I$, provided that

a) We have $(\partial_z\Phi)(x,\cdot) \in C^k([r,r'],\mathfrak{g})$.[17]
b) For each $\mathfrak{p} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, there exists $L_{\mathfrak{p},\mathrm{s}} \geq 0$, as well as $I_{\mathfrak{p},\mathrm{s}} \subseteq I$ open with $x \in I_{\mathfrak{p},\mathrm{s}}$, such that

$$1/|h|\cdot \mathfrak{p}^{\mathrm{s}}_\infty(\Phi(x+h,\cdot)-\Phi(x,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} \qquad \forall\, h \in I_{\mathfrak{p},\mathrm{s}}[x].$$

Proof. For $x + h \in I$, we have

$$\Phi(x+h,t) = \Phi(x,t) + h\cdot\partial_z\Phi(x,t) + h\cdot\varepsilon(x+h,t) \qquad \forall\, t\in[r,r'],$$

for some $\varepsilon\colon I\times[r,r'] \to \mathfrak{g}$ with

i) $\lim_{h\to 0}\varepsilon(x+h,t) = \varepsilon(x,t) = 0 \qquad \forall\, t \in [r,r'],$
ii) $\mathfrak{p}^{\mathrm{s}}_\infty(\varepsilon(x+h,\cdot)) \leq L_{\mathfrak{p},\mathrm{s}} + \mathfrak{p}^{\mathrm{s}}_\infty((\partial_z\Phi)(x,\cdot)) =: C_{\mathfrak{p},\mathrm{s}} < \infty \qquad \forall\, h \in I_{\mathfrak{p},\mathrm{s}}[x]$
for all $\mathfrak{p}\in\mathfrak{P}$ and $\mathrm{s}\preceq k$.

[17] More specifically, this means that for each $t \in [r,r']$ the map $I \ni z \mapsto \Phi(z,t)$ is differentiable at $z = x$ with derivative $(\partial_z\Phi)(x,t)$, such that $(\partial_z\Phi)(x,\cdot) \in C^k([r,r'],\mathfrak{g})$ holds. In particular, the latter condition ensures that $\mathfrak{p}^{\mathrm{s}}_\infty((\partial_z\Phi)(x,\cdot)) < \infty$ holds For each $\mathfrak{p}\in\mathfrak{P}$ and $\mathrm{s}\preceq k$, cf. **ii)**.

Then, *a)* together with Corollary 10 shows that $\int \Phi(x+h,\cdot) = \alpha(1)\cdot\beta(h,1)\cdot\gamma(h,1)$ holds, with

$$\begin{aligned}\alpha(t) &:= \textstyle\int_r^t \Phi(x,\cdot)\\ \beta(h,t) &:= \textstyle\int_r^t h\cdot \mathrm{Ad}_{\alpha^{-1}}(\partial_z\Phi(x,\cdot))\\ \gamma(h,t) &:= \textstyle\int_r^t h\cdot \mathrm{Ad}_{(\alpha\cdot\beta(h,\cdot))^{-1}}(\varepsilon(x+h,\cdot))\end{aligned}$$

for each $t\in[r,r']$; thus,

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\,\Xi\big([\textstyle\int\Phi(x,\cdot)]^{-1}[\int\Phi(x+h,\cdot)]\big) = \tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\,\Xi(\beta(h,1)\cdot\gamma(h,1))$$

provided that the right side exists. Now, since $\mathrm{Ad}_{\alpha^{-1}}(\partial_z\Phi(x,\cdot))$ is of class C^k by Lemma 13, Proposition 7 shows that

$$\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\,\beta(h,1) = \textstyle\int \mathrm{Ad}_{\alpha^{-1}(s)}(\partial_z\Phi(x,s))\,\mathrm{d}s = \int \mathrm{Ad}_{[\int_r^s\Phi(x,\cdot)]^{-1}}(\partial_z\Phi(x,s))\,\mathrm{d}s$$

holds; so that the claim follows from (87) once we have verified that

$$\lim_{h\to 0} 1/|h|\cdot(\mathfrak{p}\circ\Xi)(\gamma(h,1)) = 0 \qquad \forall\,\mathfrak{p}\in\mathfrak{P}. \tag{88}$$

To show this, we fix $\mathfrak{p}\in\mathfrak{P}$, and let

- $\mathfrak{p}\leq\mathfrak{q}\in\mathfrak{P}$, $\mathrm{u}\preceq k$ be as in Lemma 20 for $\mathrm{s}\equiv\mathrm{u}$ (and $p\equiv k$) there; i.e.,

$$\begin{aligned}&{\scriptstyle\blacksquare}\mathfrak{q}^{\mathrm{u}}_\infty(\phi)\leq 1 \quad\text{for}\quad \phi\in\mathfrak{D}^k_{[r,r']}\\ &\Longrightarrow\qquad (\mathfrak{p}\circ\Xi)\big(\textstyle\int_r^\bullet\phi\big)\leq\int_r^\bullet {\scriptstyle\blacksquare}\mathfrak{q}(\phi(s))\,\mathrm{d}s.\end{aligned}\tag{89}$$

- $\mathfrak{q}\leq\mathfrak{m}\in\mathfrak{P}$, $\mathrm{s}\preceq k$ be as in Lemma 21 for $\mathfrak{p}\equiv\mathfrak{q}$ there; i.e., we have

$${\scriptstyle\blacksquare}\mathfrak{q}^{\mathrm{p}}(\mathrm{Ad}_{\beta^{-1}(h,\cdot)}(\psi))\leq{\scriptstyle\blacksquare}\mathfrak{m}^{\mathrm{p}}(\psi) \qquad \forall\,\psi\in C^k([r,r'],\mathfrak{g}),\ \ 0\leq\mathrm{p}\leq\mathrm{u}, \tag{90}$$

  provided that ${\scriptstyle\blacksquare}\mathfrak{m}^{\mathrm{s}}_\infty(h\cdot\mathrm{Ad}_{\alpha^{-1}}(\partial_z\Phi(x,\cdot))\leq 1$ holds.
- $\mathfrak{m}\leq\mathfrak{n}\in\mathfrak{P}$ be as in Lemma 14 for $\mathfrak{p}\equiv\mathfrak{m}$, $\mathfrak{q}\equiv\mathfrak{n}$, $\mathrm{s}\equiv\mathrm{o}:=\max(\mathrm{s},\mathrm{u})$, and $\phi\equiv\Phi(x,\cdot)$ there; i.e., we have

$${\scriptstyle\blacksquare}\mathfrak{m}^{\mathrm{s}}(h\cdot\mathrm{Ad}_{\alpha^{-1}}(\partial_z\Phi(x,\cdot))\leq|h|\cdot{\scriptstyle\blacksquare}\mathfrak{n}^{\mathrm{s}}(\partial_z\Phi(x,\cdot)) \tag{91}$$

$${\scriptstyle\blacksquare}\mathfrak{m}^{\mathrm{p}}(\mathrm{Ad}_{\alpha^{-1}}(\varepsilon(x+h,\cdot)))\ \leq\ {\scriptstyle\blacksquare}\mathfrak{n}^{\mathrm{p}}(\varepsilon(x+h,\cdot)). \tag{92}$$

  for each $0\leq\mathrm{p}\leq\mathrm{o}$ and $h\in I[x]$.

We choose $\delta > 0$ such small that $(-\delta, 0) \cup (0, \delta) \subseteq I_{\mathfrak{n},\mathrm{o}}[x]$ holds, with $|h| \cdot \mathfrak{n}^{\mathrm{s}}_\infty(\partial_z\Phi(x,\cdot)) \leq 1$ for each $0 < |h| \leq \delta$. Then, (91), (90), (92), and **ii**) show that

$$
\begin{aligned}
&\mathfrak{q}^{\mathrm{p}}\big(h\cdot\mathrm{Ad}_{(\alpha\cdot\beta(h,\cdot))^{-1}}(\varepsilon(x+h,\cdot))\big)\\
&\qquad = |h|\cdot\mathfrak{q}^{\mathrm{p}}\big(\mathrm{Ad}_{\beta^{-1}(h,\cdot)}\circ\mathrm{Ad}_{\alpha^{-1}}(\varepsilon(x+h,\cdot))\big)\\
&\qquad \leq |h|\cdot\mathfrak{m}^{\mathrm{p}}(\mathrm{Ad}_{\alpha^{-1}}(\varepsilon(x+h,\cdot)))\\
&\qquad \leq |h|\cdot\mathfrak{n}^{\mathrm{p}}(\varepsilon(x+h,\cdot))\\
&\qquad \leq |h|\cdot\mathfrak{n}^{\mathrm{o}}_\infty(\varepsilon(x+h,\cdot))\\
&\qquad \leq |h|\cdot C_{\mathfrak{n},\mathrm{o}}
\end{aligned}
\tag{93}
$$

holds for each $0 < |h| \leq \delta$, and each $0 \leq \mathrm{p} \leq \mathrm{u}$. Thus, shrinking δ if necessary, we obtain from (89), as well as (93) for $\mathrm{p} \equiv \mathrm{u}$ there, that

$$
\begin{aligned}
&1/|h|\cdot(\mathfrak{p}\circ\Xi)(\gamma(h,t)) \leq \textstyle\int_r^t \mathfrak{q}(\mathrm{Ad}_{(\alpha(s)\cdot\beta(h,s))^{-1}}(\varepsilon(x+h,s)))\,\mathrm{d}s\\
&\forall\, 0 < |h| \leq \delta
\end{aligned}
$$

holds. Then, (93), for $\mathrm{p} \equiv 0$ there, gives

$$
1/|h|\cdot(\mathfrak{p}\circ\Xi)(\gamma(h,t)) \leq \textstyle\int \mathfrak{n}(\varepsilon(x+h,s))\,\mathrm{d}s \qquad \forall\, 0 < |h| \leq \delta.
$$

Now, the integrand (on the right side) is measurable, as well as bounded by **ii**). Then, (88) follows from the dominated convergence theorem, because

$$
\lim_{h\to 0}\mathfrak{n}(\varepsilon(x+h,\cdot)) = 0
$$

converges pointwise by **i**). □

Remark 6. *In the situation of Theorem 5, we obtain from Lemma 7 (and b)) that*

$$
\tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}\textstyle\int\Phi(x+h,\cdot) = \mathrm{d}_e\mathrm{L}_{\int\Phi(x,\cdot)}\big(\int\mathrm{Ad}_{[\int_r^s\Phi(x,\cdot)]^{-1}}(\partial_z\Phi(x,s))\,\mathrm{d}s\big) \in \mathfrak{g}
\tag{94}
$$

holds, provided that

- $\mathfrak{g}$ *is integral complete.*
- $\mathfrak{g}$ *is Mackey complete with* $\partial_z\Phi(x,\cdot) \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$.

Here, the first criterion is obvious, and the second criterion is clear from Lemma 13. ‡

8.3. Duhamel's formula

Suppose that G is ∞-continuous and C^∞-semiregular, and that $\mathfrak{g}$ is Mackey complete. We fix $\mathfrak{X}\colon I \to \mathfrak{g}$ of class C^1, and define

$$\Phi\colon I \times [0,1] \to \mathfrak{g}, \qquad (z,t) \mapsto \mathfrak{X}(z).$$

Then, Φ fulfills the presumptions of Theorem 5 for $[r,r'] \equiv [0,1]$ there, namely, for each $x \in I$; so that we have

Corollary 11. *Suppose that G is ∞-continuous and C^∞-semiregular, and that $\mathfrak{g}$ is Mackey complete. Then, for each $\mathfrak{X}\colon I \to \mathfrak{g}$ of class C^1, we have*

$$\partial_z \exp(\mathfrak{X}(x)) = \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big(\textstyle\int_0^1 \mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\partial_z\mathfrak{X}(x))\,\mathrm{d}s\big) \qquad \forall\, x \in I.$$

Proof. Clear. □

We want to provide a further version of this statement:

Referring to Lemma 35, we say that G is **quasi constricted** *iff*

$$\alpha_{X,Y}\colon \mathbb{R} \ni t \mapsto \textstyle\sum_{n=0}^\infty \frac{t^n}{n!}\cdot(\mathrm{ad}\,X)^n(Y) \in \mathfrak{g} \qquad \forall\, X,Y \in \mathfrak{g}$$

is defined and of class C^1 with $\dot\alpha_{X,Y} = [X,\alpha]$; thus, of class C^∞ by Corollary 2. Then, by Lemma 34, we have

$$\begin{aligned}\frac{\mathrm{id}_{\mathfrak{g}} - \exp(-\mathrm{ad}\,\mathfrak{X}(x))}{\mathrm{ad}\,\mathfrak{X}(x)}(Y) &:= \textstyle\int_0^1 \alpha_{-X,Y}(s)\,\mathrm{d}s \\ &= \textstyle\sum_{n=0}^\infty \frac{1}{(n+1)!}\cdot(\mathrm{ad}\,-\mathfrak{X}(x))^n(Y) \in \overline{\mathfrak{g}} \qquad \forall\, Y \in \mathfrak{g};\end{aligned} \tag{95}$$

and, in analogy to Corollary 7, we obtain

Corollary 12. *Suppose that G is quasi constricted, and admits an exponential map. Then,*

$$\mathrm{Ad}_{\exp(-t\cdot X)}(Y) = \alpha_{-X,Y}(t) \qquad \forall\, t \in \mathbb{R},\ X,Y \in \mathfrak{g}.$$

Proof. The proof is the same as for Corollary 7, whereby the statement in Lemma 35 now holds by definition. □

We obtain

Proposition 8 (Duhamel's formula). *Suppose that G is ∞-continuous, C^∞-semiregular, and quasi constricted; and that $\mathfrak{g}$ is Mackey complete. Then, for each $\mathfrak{X}\colon I \to \mathfrak{g}$ of class C^1, we have*

$$\partial_z \exp(\mathfrak{X}(x)) = \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\Big(\tfrac{\mathrm{id}_\mathfrak{g} - \exp(-\mathrm{ad}\,\mathfrak{X}(x))}{\mathrm{ad}\,\mathfrak{X}(x)}(\partial_z\mathfrak{X}(x))\Big) \qquad \forall\, x \in I.$$

Proof. By Corollary 11, we have

$$\partial_z \exp(\mathfrak{X}(x)) = \mathrm{d}_e\mathrm{L}_{\exp(\mathfrak{X}(x))}\big(\textstyle\int \mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\partial_z\mathfrak{X}(x))\,\mathrm{d}s\big) \qquad \forall\, x \in I.$$

We obtain from Corollary 12 and Lemma 34 that

$$\begin{aligned}\textstyle\int \mathrm{Ad}_{\exp(-s\cdot\mathfrak{X}(x))}(\partial_z\mathfrak{X}(x))\,\mathrm{d}s &= \textstyle\int \sum_{n=0}^\infty \frac{s^n}{n!}\cdot(\mathrm{ad}\,-\mathfrak{X}(x))^n(\partial_z\mathfrak{X}(x))\,\mathrm{d}s\\ &= \textstyle\sum_{n=0}^\infty \frac{1}{(n+1)!}\cdot\mathrm{ad}\,(-\mathfrak{X}(x))^n(\partial_z\mathfrak{X}(x))\end{aligned}$$

holds for each $x \in I$; which is necessarily in $\mathfrak{g}$. □

8.4. Smoothness of the integral

We now are going to prove Theorem 4. For this, we first observe that

Lemma 41. *Let $\Gamma\colon G \times \mathfrak{g} \to \mathfrak{g}$ be continuous, and $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ be fixed. Suppose furthermore that G is* k*-continuous and C^k-semiregular. Then,*

$$\widehat{\Gamma}\colon C^k([r,r'],\mathfrak{g}) \times C^k([r,r'],\mathfrak{g}) \to \overline{\mathfrak{g}}, \qquad (\phi,\psi) \mapsto \textstyle\int \Gamma\big(\int_r^s \phi, \psi(s)\big)\,\mathrm{d}s$$

is continuous for each $[r,r'] \in \mathfrak{K}$.

Proof. This follows by standard arguments from Lemma 23, cf. Appendix F.2. □

Let now $k \in \mathbb{N} \sqcup \{\mathrm{lip}, \infty\}$ be fixed; and suppose that G is k-continuous and C^k-semiregular, i.e, locally μ-convex and C^{lip}-semiregular for $k \equiv \mathrm{lip}$. Suppose furthermore that

- $\mathfrak{g}$ is integral complete if $k \equiv 0$ holds.
- $\mathfrak{g}$ is Mackey complete if $k \in \mathbb{N}_{\geq 1} \sqcup \{\mathrm{lip}, \infty\}$ holds.

Clearly,

$$\Phi[\phi,\psi]\colon (-1,1)\times[r,r']\to\mathfrak{g},\qquad (h,t)\mapsto\phi(t)+h\cdot\psi(t)$$

fulfills the presumptions of Theorem 5 for each $\phi,\psi\in C^k([r,r'],\mathfrak{g})$; i.e., we have, cf. Remark 6

$$\begin{aligned}&\big(\mathrm{d}_\phi\,\mathrm{evol}^k_{[r,r']}\big)(\psi)=\mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int\mathrm{Ad}_{[\int_r^s\phi]^{-1}}(\psi(s))\,\mathrm{d}s\big)\\&\forall\,\phi,\psi\in C^k([r,r'],\mathfrak{g}),\ \ [r,r']\in\mathfrak{K}.\end{aligned}\tag{96}$$

This can be written as, cf. (15)

$$\mathrm{d}_\phi\,\mathrm{evol}^k_{[r,r']}(\psi)=\mathrm{d}_{(\int\phi,e)}\mathrm{m}\big(0,\widehat{\Gamma}(\phi,\psi)\big)\qquad\text{for}\qquad\Gamma\equiv\mathrm{Ad}(\mathrm{inv}(\cdot),\cdot);$$

so that $\mathrm{evol}^k_{[r,r']}$ is of class C^1 by Lemma 41. We thus have

Corollary 13. *Let G be* k*-continuous and C^k-semiregular for $k\in\mathbb{N}\sqcup\{\mathrm{lip},\infty\}$. Suppose furthermore that $\mathfrak{g}$ is*

- *integral complete for $k\equiv 0$.*
- *Mackey complete for $k\in\mathbb{N}_{\geq 1}\sqcup\{\mathrm{lip},\infty\}$.*

Then, $\mathrm{evol}^k_{[r,r']}$ *is of class C^1 with*

$$\mathrm{d}_\phi\mathrm{evol}^k_{[r,r']}(\psi)=\mathrm{d}_e\mathrm{L}_{\int\phi}\big(\textstyle\int\mathrm{Ad}_{[\int_r^s\phi]^{-1}}(\psi(s))\,\mathrm{d}s\big)\qquad\forall\,\phi,\psi\in C^k([r,r'],\mathfrak{g})$$

for each $[r,r']\in\mathfrak{K}$.

Proof. Clear. □

We are ready for the

Proof of Theorem 4. By Corollary 9, it remains to show that $\mathrm{evol}^k_{[r,r']}$ is smooth for each $[r,r']\in\mathfrak{K}$

- if $\mathfrak{g}$ is integral complete for $k\equiv 0$.
- if $\mathfrak{g}$ is Mackey complete for $k\in\mathbb{N}_{\geq 1}\sqcup\{\infty\}$.

Now, since Corollary 13 shows that $\mathrm{evol}^k_{[0,1]}$ is of class C^1, Theorem E in [3] shows that $\int^k_{[0,1]}$ is smooth. Then, for $[r,r']\in\mathfrak{K}$ fixed, we define

$$\varrho\colon[0,1]\to[r,r'],\qquad t\mapsto r+t\cdot|r'-r|;$$

and recall that (cf. proof of Lemma 15) $\mathrm{evol}^k_{[r,r']} = \mathrm{evol}^k_{[0,1]} \circ \eta$ holds, for the k-continuous, linear map

$$\eta\colon C^k([r,r'],\mathfrak{g}) \to C^k([0,1],\mathfrak{g}), \qquad \phi \mapsto \dot{\varrho}\cdot(\phi\circ\varrho) \equiv |r'-r|\cdot(\phi\circ\varrho).$$

Since η is smooth by **b**), the claim follows. □

Remark 7. *It is to be expected that Theorem 4,2) also holds for* $k \equiv \mathrm{lip}$*; i.e., that we have:*

2') If G is lip*-continuous and C^{lip}-semiregular, then $\mathrm{evol}^{\mathrm{lip}}_{[r,r']}$ is smooth for each $[r,r'] \in \mathfrak{K}$ iff $\mathfrak{g}$ is Mackey complete iff $\mathrm{evol}^{\mathrm{lip}}_{[0,1]}$ is differentiable at zero.*

Indeed, by Corollary 9, it only remains to show that $\mathrm{evol}^{\mathrm{lip}}_{[r,r']}$ smooth; whereby (due to the explicit formula (96)*) Corollary 13 already shows that $\mathrm{evol}^{\mathrm{lip}}_{[r,r']}$ is of class C^1. Using similar arguments as in Lemma 41, it should follow inductively from* (96) *that $\mathrm{evol}^{\mathrm{lip}}_{[r,r']}$ is of class C^∞. The details, however, seem to be quite elaborate and technical; so that we leave this issue to a another paper.* ‡

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Appendix A. Appendix to Sect. 3

A.1.

Proof of Lemma 1. Since Φ is continuous with $\Phi(x,0,\dots,0)=0$, there exist $\mathfrak{q}_1 \in \mathfrak{Q}_1,\dots,\mathfrak{q}_n \in \mathfrak{Q}_n$ as well as $V \subseteq X$ open with $x \in V$, such that

$$(\mathfrak{p}\circ\Phi)(y,Y_1,\dots,Y_n) \leq 1 \qquad \forall\, y \in V \tag{A.1}$$

holds for all $Y_1 \in \overline{\mathrm{B}}_{\mathfrak{q}_1,1},\dots,Y_n \in \overline{\mathrm{B}}_{\mathfrak{q}_n,1}$. Let now $X_1 \in F_1,\dots,X_n \in F_n$ be fixed; and define

$$Y_k := \begin{cases} X_k & \text{for} \quad \mathfrak{q}_k(X_k) = 0, \\ 1/\mathfrak{q}_k(X_k)\cdot X_k & \text{for} \quad \mathfrak{q}_k(X_k) > 0, \end{cases}$$

for $k = 1, \ldots, n$. Then,

- if $\mathfrak{q}_1(X_1), \ldots, \mathfrak{q}_n(X_n) > 0$ holds, we obtain

$$(\mathfrak{p} \circ \Phi)(y, X_1, \ldots, X_n) \overset{(A.1)}{\leq} \mathfrak{q}_1(X_1) \cdot \ldots \cdot \mathfrak{q}_n(X_n) \qquad \forall\, y \in V.$$

- if $\mathfrak{q}_k(X_k) = 0$ holds for some $1 \leq k \leq n$, we have $\mathfrak{q}_k(n \cdot Y_k) = 0$ for each $n \geq 1$; thus,

$$\begin{aligned} (\mathfrak{p} \circ \Phi)(y, Y_1, \ldots, Y_n) &\overset{(A.1)}{\leq} 1/n \qquad \forall\, n \geq 1 \\ \implies \qquad (\mathfrak{p} \circ \Phi)(y, Y_1, \ldots, Y_n) &= 0 \\ \implies \qquad (\mathfrak{p} \circ \Phi)(y, X_1, \ldots, X_n) &= 0 \end{aligned}$$

  for each $y \in V$.

From this, the claim is clear. □

A.2.

Proof of Corollary 1. Since K is compact, by Lemma 1, there exist seminorms $\mathfrak{q}[p]_1 \in \mathfrak{Q}_1, \ldots, \mathfrak{q}[p]_n \in \mathfrak{Q}_n$ for $p = 1, \ldots, m$, as well as $V_1, \ldots, V_m \subseteq X$ open with $\mathrm{K} \subseteq V_1 \cup \ldots \cup V_m =: O$, such that

$$(\mathfrak{p} \circ \Phi)(y, X_1, \ldots, X_n) \leq \mathfrak{q}[p]_1(X_1) \cdot \ldots \cdot \mathfrak{q}[p]_n(X_n)$$
$$\forall\, y \in V_p, \ \ p = 1, \ldots, m$$

holds for all $X_1 \in F_1, \ldots, X_n \in F_n$. Evidently, then (17) holds for any $\mathfrak{q}_1 \in \mathfrak{Q}_1, \ldots, \mathfrak{q}_n \in \mathfrak{Q}_n$ with $\mathfrak{q}[1]_k, \ldots, \mathfrak{q}[m]_k \leq \mathfrak{q}_k$ for $k = 1, \ldots, n$. □

A.3.

Proof of Lemma 3. It is clear that $\gamma^{(k)}$ is of class C^1 if γ is of class C^{k+1}; and the other direction is clear if $D \equiv I$ is open. Thus, suppose that D is not open; and that γ is of class C^k with $\gamma^{(k)}$ of class C^1. We define $r := \inf\{D\}$ and $r' := \sup\{D\}$; and proceed as follows:

- If $r \notin D$ holds, we let $D' := D$ and $\gamma' := \gamma$.

- If $r \in D$ holds, we let $D' := (r - \varepsilon, r) \sqcup D$ for some $\varepsilon > 0$; and define $\gamma' \colon D' \to E$ by

$$\gamma'|_{(r-\varepsilon,r)} := (\cdot - r)^{k+1}/(k+1)! \cdot \big(\gamma^{(k)}\big)^{(1)}(r) + \textstyle\sum_{p=0}^{k} (\cdot - r)^p/p! \cdot \gamma^{(p)}(r)$$

and $\gamma'|_D := \gamma$.

Then,

- If $r' \notin D$ holds, we let $I := D'$ and $\gamma'' := \gamma'$.
- If $r' \in D$ holds, we let $I := D' \sqcup (r', r' + \varepsilon')$ for some $\varepsilon' > 0$; and define $\gamma'' \colon I \to E$ by

$$\gamma''|_{(r',r'+\varepsilon')} := (\cdot - r')^{k+1}/(k+1)! \cdot \big(\gamma^{(k)}\big)^{(1)}(r') + \textstyle\sum_{p=0}^{k} (\cdot - r')^p/p! \cdot \gamma^{(p)}(r')$$

and $\gamma''|_{D'} := \gamma'$.

By construction, I is open; and we have $\gamma = \gamma''|_D$, for γ'' of class C^{k+1}. □

A.4.

Proof of Lemma 4. Passing to C^k-extensions of γ_i for $i = 1, 2$, we can assume that $D \equiv I$ is open. Then, the first claim is clear from **c**), **d**). Moreover, for α as in (19) and $t \in I$, we have

$$\begin{aligned}\dot{\alpha}(t) &= \mathrm{d}_t(\Psi \circ \beta)(1) \overset{\mathbf{d)}}{=} \mathrm{d}\Psi(\beta(t), \mathrm{d}_t\beta(1)) \\ &\overset{\mathbf{c)}}{=} \mathrm{d}\Psi\big(\beta(t), \gamma_{i_1}^{(z_1+1)}(t) \times \ldots \times \gamma_{i_m}^{(z_m+1)}(t)\big) \\ &\overset{\mathbf{e)}}{=} \textstyle\sum_{u=1}^{m} \partial_u \Psi\big(\beta(t), \gamma_{i_u}^{(z_u+1)}(t)\big),\end{aligned}$$

for $\beta \equiv \gamma_{i_1}^{(z_1)} \times \ldots \times \gamma_{i_m}^{(z_m)}$; as well as

$$\partial_u \Psi = \mathrm{d}\Psi|_{V \times \{0\}^{u-1} \times F_{i_u} \times \{0\}^{m-u}} \qquad \forall\, u = 1, \ldots, m$$

smooth by **a**). The second claim thus follows inductively, as it clearly holds for $p = 0$. □

A.5.

Proof of Lemma 5. By Corollary 3, for $0 \leq \mathrm{p} \leq \mathrm{u}$, and each $\gamma \in C^{\mathrm{u}}([r,r'],W_1)$, $\psi \in C^{\mathrm{u}}([r,r'],F_2)$, we have $\Omega(\gamma,\psi)^{(\mathrm{p})} = \sum_{i=1}^{d_{\mathrm{p}}} \alpha_{\mathrm{p},i}(\gamma,\psi)$, with

$$\alpha_{\mathrm{p},i} \colon (\gamma,\psi) \mapsto ([\partial_1]^{m[\mathrm{p},i]}\Omega)\big(\gamma, \gamma^{(z[\mathrm{p},i]_1)}, \dots, \gamma^{(z[\mathrm{p},i]_{m[\mathrm{p},i]})}, \psi^{(q[\mathrm{p},i])}\big)$$

for certain $z[\mathrm{p},i]_1, \dots, z[\mathrm{p},i]_{m[\mathrm{p},i]}, q[\mathrm{p},i] \leq \mathrm{p}$ and $m[\mathrm{p},i] \geq 1$. Then,

1) For $\mathfrak{p} \in \mathfrak{P}$ fixed, Lemma 1 provides us with $\mathfrak{q}_1 \in \mathfrak{Q}_1$, $\mathfrak{q}_2 \in \mathfrak{Q}_2$, and an open neighbourhood $V \subseteq F_1$ of 0, such that

   $$\begin{aligned}&\mathfrak{p}(\alpha_{\mathrm{p},i}(\gamma,\psi)) \leq \mathfrak{q}_1\big(\gamma^{(z[\mathrm{p},i]_1)}\big) \cdot \ldots \cdot \mathfrak{q}_1\big(\gamma^{(z[\mathrm{p},i]_{m[\mathrm{p},i]})}\big) \cdot \mathfrak{q}_2\big(\psi^{(q[\mathrm{p},i])}\big)\\ &\forall\, i = 1,\dots,d_{\mathrm{p}},\ \ \mathrm{p} = 0,\dots,\mathrm{u}\end{aligned}$$

   holds, provided that we have $\mathrm{im}[\gamma] \subseteq V$. The claim thus holds for $\mathfrak{q} := \max(d_0,\dots,d_{\mathrm{p}}) \cdot \mathfrak{q}_2$, and each $V \prec \mathfrak{m} \in \mathfrak{Q}_1$ with $\mathfrak{q}_1 \leq \mathfrak{m}$.
2) For $\mathfrak{p} \in \mathfrak{P}$, $\gamma \in C^{\mathrm{u}}([r,r'],W_1)$ fixed, Corollary 1 provides us with $\mathfrak{q}_1 \in \mathfrak{Q}_1$, $\mathfrak{q}_2 \in \mathfrak{Q}_2$, such that

   $$\begin{aligned}&\mathfrak{p}(\alpha_{\mathrm{p},i}(\gamma,\psi)) \leq \mathfrak{q}_1\big(\gamma^{(z[\mathrm{p},i]_1)}\big) \cdot \ldots \cdot \mathfrak{q}_1\big(\gamma^{(z[\mathrm{p},i]_{m[\mathrm{p},i]})}\big) \cdot \mathfrak{q}_2\big(\psi^{(q[\mathrm{p},i])}\big)\\ &\forall\, i = 1,\dots,d_{\mathrm{p}},\ \ \mathrm{p} = 0,\dots,\mathrm{u}.\end{aligned}$$

   Since we have ${\mathfrak{q}_1}^{\mathrm{u}}_{\infty}(\gamma) < \infty$, the claim holds for $\mathfrak{q} = C \cdot \mathfrak{q}_2$, for $C \geq 0$ suitably large.

This proves the claim. □

A.6.

Proof of Lemma 7. Recall that $\overline{\mathrm{d}_{\gamma(t)} f}$ is defined, linear, and continuous by Lemma 2. We choose $\delta > 0$ such small that for each $h \in M := (D - t) \cap ((-\delta,0) \cup (0,\delta))$, we have

$$\gamma(t) + [0,1] \cdot \Delta_h \subseteq U \qquad \text{for} \qquad \Delta_h := \gamma(t+h) - \gamma(t).$$

We obtain from (18) that

$$\begin{aligned}\text{(A.2)}\qquad &\tfrac{1}{h} \cdot (f(\gamma(t+h)) - f(\gamma(t)))\\ &\quad= \tfrac{1}{h} \cdot \big(\mathrm{d}_{\gamma(t)} f(\Delta_h) + \textstyle\int_0^1 (1-s) \cdot \mathrm{d}^2_{\gamma(t)+s\cdot\Delta_h} f(\Delta_h,\Delta_h)\, \mathrm{d}s\big)\\ &\quad= \overline{\mathrm{d}_{\gamma(t)} f}\ (\tfrac{1}{h} \cdot \Delta_h)\ + \textstyle\int_0^1 (1-s) \cdot \mathrm{d}^2_{\gamma(t)+s\cdot\Delta_h} f(\tfrac{1}{h} \cdot \Delta_h, \Delta_h)\, \mathrm{d}s\end{aligned}$$

holds for each $h \in M$. Since $\overline{\mathrm{d}_{\gamma(t)} f}$ is continuous, we have

$$\lim_{h\to 0} \overline{\mathrm{d}_{\gamma(t)} f}\,(\tfrac{1}{h} \cdot \Delta_h) = \overline{\mathrm{d}_{\gamma(t)} f}\,(X).$$

The claim thus follows once we have shown that the second summand in (A.2) tends to zero if h tends to zero. For this, we fix $\mathfrak{p} \in \mathfrak{P}$; and choose $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathfrak{Q}$ as well as $V \subseteq U$ open with $\gamma(t) \in V$ as in Lemma 1, for $\Phi \equiv \mathrm{d}^2 f\colon U \times F \times F \to E$ and $x \equiv \gamma(t)$ there. Since $\lim_{h\to 0} \Delta_h = 0$ holds by continuity of γ at $t \in D$, we obtain

$$\begin{aligned}
\lim_{h\to 0} \mathfrak{p}\big(\textstyle\int_0^1 (1-s) \cdot \mathrm{d}^2_{\gamma(t)+s\cdot\Delta_h} f(\tfrac{1}{h} \cdot \Delta_h, \Delta_h)\, \mathrm{d}s\big) \\
&\overset{(22)}{\leq} \lim_{h\to 0} \mathfrak{p}\big(\mathrm{d}^2_{\gamma(t)+s\cdot\Delta_h} f(\tfrac{1}{h} \cdot \Delta_h, \Delta_h)\big)\\
&\leq \lim_{h\to 0} \mathfrak{q}_1(\tfrac{1}{h} \cdot \Delta_h) \cdot \mathfrak{q}_2(\Delta_h)\\
&= \lim_{h\to 0} \overline{\mathfrak{q}}_1(\tfrac{1}{h} \cdot \Delta_h) \cdot \mathfrak{q}_2(\Delta_h)\\
&= 0;
\end{aligned}$$

which shows the claim. □

A.7.

Proof of Lemma 11. By Lemma 10.2) and *d)*, it suffices to show that there exists some $\mu \in C^1(I, G)$, for $I \subseteq \mathbb{R}$ an open interval containing $[r, r']$, such that $\delta^r(\mu|_{[r,r']}) = \phi$ holds:

By assumption, for $p = 0, \dots, n-1$, we have

$$\text{(A.3)} \qquad \phi|_{[t_p,t_{p+1}]} = \delta^r(\mu[p]|_{[t_p,t_{p+1}]}) \qquad \text{for some} \qquad \mu[p] \in C^{k+1}(I_p, G)$$

with $I_p \subseteq \mathbb{R}$ an open interval containing $[t_p, t_{p+1}]$; and, due to the first identity in (38), we can assume that

$$\mu[p](t_{p+1}) = \mu[p+1](t_{p+1}) \qquad \forall\, p = 0, \dots, n-2$$

holds. We write $I_0 \equiv (\iota, \ell')$, $I_{n-1} \equiv (\ell, \iota')$, let $I \equiv (\iota, \iota')$, and define

- $\psi \in C^0(I, \mathfrak{g})$ by

$$\psi|_{(\iota,r)} := \delta^r(\mu[0]|_{(\iota,r)}), \quad \psi|_{[r,r']} := \phi, \quad \psi|_{(r',\iota')} := \delta^r(\mu[0]|_{(r',\iota')}).$$

- $\mu \in C^0(I, G)$ by

$$\begin{aligned} \mu|_{(\iota,r]} &:= \mu[0]|_{(\iota,r]}, \\ \mu|_{(t_p,t_{p+1}]} &:= \mu[p]|_{(t_p,t_{p+1}]} \qquad\qquad \forall\, p = 0,\dots,n-1, \\ \mu|_{(r',\iota')} &:= \mu[n-1]|_{(r',\iota')}. \end{aligned}$$

We obtain from (A.3) that

$$\lim_{h\to 0} \tfrac{1}{h} \cdot \Xi(\mu(t+h)\cdot \mu(t)^{-1}) = \mathrm{d}_e\Xi(\psi(t)) \qquad \forall\, t \in I \tag{A.4}$$

holds; and now will conclude from Lemma 7 that μ is of class C^1.

For this, we let $\tau \in I$ be fixed; and choose a chart $\Xi' \colon G \supseteq \mathcal{U}' \to \mathcal{V}' \subseteq E$ with $\mu(\tau) \in \mathcal{U}'$. Moreover, we choose $V \subseteq \mathcal{V}$ open with $0 \in V$, as well as $J \subseteq I$ open with $\tau \in J$, such that $\Xi^{-1}(V)\cdot \mu(J) \subseteq \mathcal{U}'$ holds. Then, shrinking J if necessary, we can assume that $(\Xi \circ \mathrm{m})(\mu(J), (\mathrm{inv} \circ \mu)(J)) \subseteq V$ holds. For each $s \in J$, we define $\gamma_s \colon J \ni t \mapsto (\Xi \circ \mathrm{m})(\mu(t), (\mathrm{inv}\circ\mu)(s)) \in V$, as well as

$$f_s \colon V \to \mathcal{V}', \qquad x \mapsto (\Xi' \circ \mathrm{m})(\Xi^{-1}(x), \mu(s)).$$

Then, Lemma 7 (second step) shows that

$$\begin{aligned} &\lim_{h\to 0} \tfrac{1}{h} \cdot ((\Xi' \circ \mu|_J)(s+h) - (\Xi' \circ \mu|_J)(s)) \\ &\qquad = \lim_{h\to 0} \tfrac{1}{h} \cdot (f_s(\gamma_s(s+h)) - f_s(\gamma_s(s))) \\ &\qquad = \mathrm{d}_{\gamma_s(s)} f_s(\lim_{h\to 0} \tfrac{1}{h}\cdot(\gamma_s(s+h) - \gamma_s(s))) \\ &\qquad \overset{\text{(A.4)}}{=} (\mathrm{d}_{\mu(s)}\Xi' \circ \mathrm{d}_e \mathrm{R}_{\mu(s)})(\psi(s)) \end{aligned}$$

holds, which shows that μ is of class C^1. □

A.8.

Proof of the Lipschitz Case in Lemma 13. We have to show that $\mathrm{Ad}_\mu(\phi) \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ holds for each $\mu \in C^1([r,r'],\mathfrak{g})$, and each $\phi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ with Lipschitz constants $\{L_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}} \subseteq \mathbb{R}_{\geq 0}$. For this, we fix $\mathfrak{p} \in \mathfrak{P}$; and observe that

$$\begin{aligned} &\mathfrak{p}\big(\mathrm{Ad}_{\mu(t)}(\phi(t)) - \mathrm{Ad}_{\mu(t')}(\phi(t'))\big) \\ &\qquad \leq \mathfrak{p}\big(\mathrm{Ad}_{\mu(t)}(\phi(t) - \phi(t'))\big) + \mathfrak{p}\big(\big(\mathrm{Ad}_{\mu(t)} - \mathrm{Ad}_{\mu(t')}\big)(\phi(t'))\big) \end{aligned}$$

holds. Then,

- We let C := im[μ], choose $\mathfrak{p} \leq \mathfrak{m} \in \mathfrak{P}$ as in (28) for $\mathfrak{n} \equiv \mathfrak{p}$ there; and obtain
$$\mathfrak{p}\big(\mathrm{Ad}_{\mu(t)}(\phi(t) - \phi(t'))\big) \leq \mathfrak{m}(\phi(t) - \phi(t')) \leq L_{\mathfrak{m}} \cdot |t' - t|$$
for $r \leq t < t' \leq r'$.
- Since $\alpha\colon [r, r'] \times \mathrm{im}[\phi] \ni (s, X) \to \partial_s \mathrm{Ad}_{\mu(s)}(X)$ is defined and continuous, Lemma 6 shows that
$$\mathfrak{p}\big(\big(\mathrm{Ad}_{\mu(t)} - \mathrm{Ad}_{\mu(t')}\big)(\phi(t'))\big) \leq \textstyle\int_t^{t'} \mathfrak{p}\big(\partial_s \mathrm{Ad}_{\mu(s)}(\phi(t'))\,\mathrm{d}s\big) \leq C \cdot |t' - t|$$
holds, for $C := \sup\{\mathfrak{p}(\alpha(s, X)) \mid (s, X) \in [r, r'] \times \mathrm{im}[\phi]\} < \infty$.

From this, the claim is clear. □

A.9.

Proof of the statement made in Remark 2.1). We obtain from (47), *d)*, *e)* that

$$\text{(A.5)} \qquad \begin{aligned} \exp(r \cdot X) \cdot \exp(s \cdot X) &= \exp((r + s) \cdot X) \\ &= \exp(s \cdot X) \cdot \exp(r \cdot X) \qquad \forall\, s, t \geq 0 \end{aligned}$$

holds. Then, (47) shows $\mathrm{Ad}_{\exp(t \cdot X)}(X) = X$ for each $t \geq 0$; thus,

$$\exp(t \cdot X)^{-1} \equiv [\textstyle\int_0^t \phi_X]^{-1} \overset{c)}{=} \textstyle\int_0^t -\phi_X \equiv \exp(-t \cdot X) \qquad \forall\, t \geq 0.$$

It follows that (A.5) even holds for all $s, r \in \mathbb{R}$; i.e., that $\beta\colon \mathbb{R} \ni t \mapsto \exp(t \cdot X) \in G$ is a group homomorphisms. Then, smoothness of β is clear from (38), (47), and Lemma 10.2). □

A.10.

Proof of the statement made in Remark 2.3). We define $\psi \in C^0([r - 2, r' + 2], \mathfrak{g})$ by

$$\psi|_{[r-2,r)} := \phi(r), \qquad \psi|_{[r,r']} := \phi, \qquad \psi|_{(r',r'+2]} := \phi(r'),$$

as well as $\beta \in C^1((r - 1, r' + 1), \mathfrak{g})$ by

$$\beta\colon (r - 1, r' + 1) \ni t \mapsto \textstyle\int_{r-1}^t \psi(s)\,\mathrm{d}s.$$

For $t \in (r-1, r'+1)$ fixed, and $0 < h \leq 1$, we let

$$Y := \int_{r-1}^{t+h} \psi(s)\,\mathrm{d}s, \qquad X := \int_{r-1}^{t} \psi(s)\,\mathrm{d}s, \qquad X_h := \int_{t}^{t+h} \psi(s)\,\mathrm{d}s;$$

and obtain

$$\begin{aligned}(\exp\circ\beta)(t+h) \equiv \textstyle\int_0^1 \phi_Y = \int_0^1 \phi_X + \phi_{X_h} &\overset{a)}{=} \textstyle\int_0^1 \phi_{X_h} \cdot \int_0^1 \phi_X \\ &= \exp\big(\textstyle\int_t^{t+h} \psi(s)\,\mathrm{d}s\big) \cdot \exp\big(\int_{r-1}^{t} \psi(s)\,\mathrm{d}s\big).\end{aligned}$$

Since exp is of class C^1, we obtain from (47) and **d**) that $\delta^r(\exp\circ\beta)|_{[r,r']} = \phi$ holds; which shows the claim. □

Appendix B. Appendix to Sect. 4

B.1.

Proof of Equation (54). Applying a standard refinement argument, we obtain $r = t_0 < \ldots < t_n = r'$ as well as $\phi[p], \psi[p] \in \mathfrak{D}^k_{[t_p,t_{p+1}]}$ for $p = 0, \ldots, n-1$ with

$$\phi|_{(t_p,t_{p+1})} = \phi[p]|_{(t_p,t_{p+1})},\ \psi|_{(t_p,t_{p+1})} = \psi[p]|_{(t_p,t_{p+1})} \qquad \forall\, p = 0, \ldots, n-1.$$

We let $\alpha := [\int_r^\bullet \phi]^{-1}[\int_r^\bullet \psi]$, $\mu := \int_r^\bullet \phi$, $\nu := \int_r^\bullet \psi$; and define

$$\text{(B.6)} \qquad \alpha_p := \alpha|_{[t_p,t_{p+1}]} \overset{(53)}{=} \mu(t_p)^{-1}\big[\textstyle\int_{t_p}^\bullet \phi[p]\big]^{-1}\big[\int_{t_p}^\bullet \psi[p]\big] \cdot \nu(t_p)$$

$$\text{(B.7)} \qquad \mu_p := \mu|_{[t_p,t_{p+1}]} \in C^{k+1}([t_p, t_{p+1}], \mathfrak{g})$$

for $p = 0, \ldots, n-1$. We obtain from *b)* that

$$\text{(B.8)} \quad \delta^r(\alpha_p)|_{(t_p,t_{p+1})} = \mathrm{Ad}_{\mu_p^{-1}}(\psi[p] - \phi[p])|_{(t_p,t_{p+1})} \qquad \forall\, p = 0, \ldots, n-1$$

holds; so that Lemma 13 and (B.7) show $\mathrm{Ad}_{\mu^{-1}}(\psi - \phi) \in \mathfrak{D}\mathrm{P}^k([r, r'], \mathfrak{g})$. Then, for $t \in (t_p, t_{p+1}]$ with $0 \leq p \leq n-1$, we have

$$\begin{aligned}\textstyle\int_r^t \mathrm{Ad}_{\mu^{-1}}(\psi - \phi) &\overset{(53),(B.8)}{=} \big[\alpha_p(t) \cdot \alpha_p(t_p)^{-1}\big] \cdot \big[\alpha_{p-1}(t_p) \cdot \alpha_{p-1}(t_{p-1})^{-1}\big] \\ &\qquad\qquad \cdot \ldots \cdot \big[\alpha_0(t_1) \cdot \alpha_0(t_0)^{-1}\big] \\ &\overset{(B.6)}{=} \alpha(t)\end{aligned}$$

which proves the claim. □

B.2.

Proof of Eq. (56) *and the C^k-statement made in the proof of Lemma 24.* It is straightforward from the triangle inequality, the properties of ρ, and smoothness of ϱ that $\psi \in C^{\mathrm{lip}}([r,r'],\mathfrak{g})$ holds for $k \equiv \mathrm{lip}$. Thus, in order to prove that ψ is of class C^k, and to verify Equation (56), we can assume that $k \in \mathbb{N} \sqcup \{\infty\}$ holds in the following.

Now, to prove the C^k-statement, we have to show that $\psi = \psi'|_{[r,r']}$ holds for some $\psi' \in C^k(I,\mathfrak{g})$ with $I \subseteq \mathbb{R}$ open containing $[r,r']$. For this, we define $\varrho' \in C^\infty(\mathbb{R},\mathbb{R})$ by

$$\varrho'|_{(-\infty,r)} := \varrho(r) \qquad \varrho'|_{[r,r']} := \varrho \qquad \varrho'|_{(r',\infty)} := \varrho(r');$$

and let $\psi' := \dot{\varrho}' \cdot (\phi \circ \varrho') \colon \mathbb{R} \to \mathfrak{g}$. Then,

- we have $\psi'^{(m)}|_{\mathbb{R}-[r,r']} = 0$ for $0 \leq m \leq k$, as well as

$$(\text{B.9}) \qquad \begin{aligned} &(\psi'|_{(t_p,t_{p+1})})^{(m)} = ((\dot{\varrho}[p] \cdot (\widetilde{\phi}[p] \circ \varrho[p]))|_{(t_p,t_{p+1})})^{(m)} \\ &\forall\, 0 \leq m \leq k,\ \ 0 \leq p \leq n-1. \end{aligned}$$

- we obtain from **d**) and **e**) that

$$(\text{B.10}) \qquad (\dot{\varrho}[p] \cdot (\widetilde{\phi}[p] \circ \varrho[p]))^{(m)}(t_p) = 0 = (\dot{\varrho}[p] \cdot (\widetilde{\phi}[p] \circ \varrho[p]))^{(m)}(t_{p+1})$$

  holds, for $0 \leq m \leq k$ and $p = 0, \dots, n-1$.

Now, since ψ' is of class C^0, we can assume that it is of class C^q for some $0 \leq q \leq k-1$. Then, (B.9) (for $m \equiv q$ there) shows that

$$(\text{B.11}) \quad \psi'^{(q)}|_{[t_p,t_{p+1}]} = (\dot{\varrho}[p] \cdot (\widetilde{\phi}[p] \circ \varrho[p]))^{(q)}|_{[t_p,t_{p+1}]} \qquad \forall\, p = 0, \dots, n-1$$

holds; with $\psi'^{(q)}|_{\mathbb{R}-[(t_0,t_1)\,\cup\,\dots\,\cup\,(t_{n-1},t_n)]} = 0$ by the first-, and by the second point (for $m \equiv q$ there). Together with (B.10) (for $m \equiv q$ there), this implies that $\psi'^{(q)}$ is differentiable with

$$\psi'^{(q+1)}|_{\mathbb{R}-[(t_0,t_1)\,\cup\,\dots\,\cup\,(t_{n-1},t_n)]} = 0;$$

so that (B.9) and (B.10) (for $m \equiv q+1$ there) show that $\psi'^{(q+1)}$ is continuous. It thus follows inductively that ψ' is of class C^k.

In particular, (56) is now clear from

$$\begin{aligned}\psi|_{[t_p,t_{p+1}]} = \psi'|_{[t_p,t_{p+1}]} &\stackrel{\text{(B.11)}}{=} (\dot{\varrho}[p]\cdot(\widetilde{\phi}[p]\circ\varrho[p]))|_{[t_p,t_{p+1}]}\\ &= \delta^r(\mu[p]\circ\varrho[p]|_{[t_p,t_{p+1}]})\end{aligned}$$

for $p = 0, \dots, n-1$. □

Appendix C. Appendix to Sect. 5

C.1.

Proof of the statement made in Example 2.1). We let $\pi\colon E \to G \equiv E/\Gamma$, $X \mapsto [X]$ denote the canonical projection, define $e := [0]$, and fix an open neighbourhood $\mathcal{O} \subseteq E$ of 0, such that $\mathcal{O} \cap [\mathcal{O} + [\Gamma - \{e\}]] = \emptyset$ holds.[18] Then, a chart of G that is centered at $e \equiv [0] \in G$, is given by

$$\Xi\colon \mathcal{U} \equiv \pi(\mathcal{O}) \to \mathcal{V} \equiv \mathcal{O}, \qquad [X] \mapsto \pi^{-1}(X) \cap \mathcal{O} \subseteq E.$$

Then, for $\mathcal{V} \prec \mathfrak{p} \in \mathfrak{P}$ and $X_1, \dots, X_n \in E$ with $\mathfrak{p}(X_1) + \ldots + \mathfrak{p}(X_n) \leq 1$, we have

$$\mathfrak{p}(X_1 + \ldots + X_n) \leq 1 \qquad \text{implying} \qquad [X_1 + \ldots + X_n] \in \mathcal{U};$$

and obtain

$$\begin{aligned}(\mathfrak{p}\circ\Xi)(\Xi^{-1}(X_1)\cdot\ldots\cdot\Xi^{-1}(X_n)) &= (\mathfrak{p}\circ\Xi)([X_1]\cdot\ldots\cdot[X_n])\\ &\equiv (\mathfrak{p}\circ\Xi)([X_1+\ldots+X_n])\\ &= \mathfrak{p}(X_1+\ldots+X_n)\\ &\leq \mathfrak{p}(X_1)+\ldots+\mathfrak{p}(X_n),\end{aligned}$$

which shows that G is locally μ-convex. □

C.2.

Proof of the statement made in Example 2.2). We let $\mathfrak{u} \equiv \|\cdot\|$ denote the Banach norm on E; and can assume that $\mathcal{V} \prec \mathfrak{u}$ holds, just by rescaling $\mathfrak{u}$ if

[18] Confer, e.g., Theorem 1.10 in [13] for the existence of such $\mathcal{O}$.

necessary. We fix $0 < \mathfrak{r} \leq 1$ with, cf. Remark 1

$$\big\|\mathrm{d}_{g\cdot\Xi^{-1}(x)\cdot q}\Xi\big(\mathrm{d}_{\Xi^{-1}(x)\cdot q}\mathrm{L}_g \circ \mathrm{d}_{\Xi^{-1}(x)}\mathrm{R}_q \circ \mathrm{d}_x\Xi^{-1}\big)\big\|_{\mathrm{op}} \leq \mathfrak{r}^{-1} \tag{C.12}$$

for all $g, q \in \mathcal{U}$ and $x \in \mathcal{V}$ with $(\mathfrak{u}\circ\Xi)(g), (\mathfrak{u}\circ\Xi)(q), \mathfrak{u}(x) \leq \mathfrak{r}$; and define $\mathfrak{o} := \mathfrak{r}^{-2}\cdot\mathfrak{u}$. Then, since we have $\mathfrak{r} \leq 1$, it suffices to show that

$$(\mathfrak{u}\circ\Xi)(\Xi^{-1}(X_1)\cdot\ldots\cdot\Xi^{-1}(X_n)) \leq \mathfrak{r}\cdot\varepsilon \tag{C.13}$$

holds for all $X_1, \ldots, X_n \in E$ with $\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_n) =: \varepsilon \leq 1$.

Now, (C.13) is clear for $n = 1$, as

$$\begin{aligned} &\mathfrak{o}(X) \leq \varepsilon \quad \text{for} \quad X \in E \\ &\Longrightarrow \qquad (\mathfrak{u}\circ\Xi)(\Xi^{-1}(X)) = \mathfrak{r}^2\cdot\mathfrak{o}(X) \leq \mathfrak{r}\cdot\varepsilon. \end{aligned}$$

We thus can assume that (C.13) holds for all $1 \leq q \leq n$ for some $n \geq 1$, fix $X_1, \ldots, X_{n+1} \in E$ with $\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_{n+1}) =: \varepsilon \leq 1$, and define

$$\rho\colon [0,1] \ni t \mapsto \Xi(\Xi^{-1}(t\cdot X_1)\cdot\ldots\cdot\Xi^{-1}(t\cdot X_{n+1})).$$

Then, applying the induction hypotheses, Lemma 6 together with (15) and (C.12) gives

$$\begin{aligned} \mathfrak{u}(\rho(1)) \leq \sup\nolimits_{t\in[0,1]}\mathfrak{u}(\dot\rho(t)) &\leq \mathfrak{r}^{-1}\cdot(\mathfrak{u}(X_1) + \ldots + \mathfrak{u}(X_{n+1})) \\ &= \mathfrak{r}\cdot(\mathfrak{o}(X_1) + \ldots + \mathfrak{o}(X_{n+1})) \leq \mathfrak{r}\cdot\varepsilon. \end{aligned}$$

Equation (C.13) thus follows inductively for each $n \geq 1$. □

C.3.

Proof of the statement made in Example 2.3). Let us first observe that

$$(1+\varepsilon_1)\cdot\ldots\cdot(1+\varepsilon_n) - 1 \leq 2\cdot\sum\nolimits_{k=1}^n \varepsilon_k \tag{C.14}$$

holds, for $\varepsilon_1, \ldots, \varepsilon_n > 0$ with $\sum_{k=1}^n \varepsilon_k \leq 1/2$. This is clear for $n = 1$; and follows inductively for each $n \geq 1$. In fact, suppose that (C.14) holds for $n \geq 1$, and let $\varepsilon_1, \ldots, \varepsilon_{n+1} > 0$ with $\sum_{k=1}^{n+1} \varepsilon_k \leq 1/2$. Then, we obtain from

(C.14) that

$$(1+\varepsilon_{n+1})\cdot(1+\varepsilon_1)\cdot\ldots\cdot(1+\varepsilon_n)-1 \\ \leq \big(2\cdot\textstyle\sum_{k=1}^n \varepsilon_k\big)+\big(\varepsilon_{n+1}\cdot(1+2\cdot\sum_{k=1}^n \varepsilon_k\big)\leq 2\cdot\sum_{k=1}^{n+1}\varepsilon_k\big).$$

Let now $\mathfrak{u}\in\mathfrak{P}$ be fixed. We choose $\mathfrak{u}\leq\mathfrak{w}\in\mathfrak{P}$ as in (59) for $\mathfrak{v}\equiv\mathfrak{u}$ there, let $\mathfrak{o}:=2\cdot\mathfrak{w}$; and consider the chart

$$\Xi\colon\mathcal{U}\equiv\mathcal{A}^\times\to\mathcal{V}\equiv\mathcal{A}^\times-\mathbf{1},\qquad a\mapsto a-\mathbf{1}$$

for $\mathbf{1}\equiv e$; and obtain from (59) that

$$\begin{aligned}(\mathfrak{u}\circ\Xi)(\Xi^{-1}(X_1)\cdot\ldots\cdot\Xi^{-1}(X_n))\\ =\mathfrak{u}((\mathbf{1}+X_1)\cdot\ldots\cdot(\mathbf{1}+X_n)-\mathbf{1})\\ \leq(1+\mathfrak{w}(X_1))\cdot\ldots\cdot(1+\mathfrak{w}(X_n))-1\end{aligned}\tag{C.15}$$

holds for all $X_1,\ldots,X_n\in\mathcal{V}$ with $n\geq 1$. Then,

$$\mathfrak{o}(X_1)+\ldots+\mathfrak{o}(X_n)=:\varepsilon\leq 1\qquad\Longrightarrow\qquad\textstyle\sum_{k=1}^n\mathfrak{w}(X_k)\leq 1/2;$$

and we conclude from (C.14) and (C.15) that

$$(\mathfrak{u}\circ\Xi)\big(\Xi^{-1}(X_1)\cdot\ldots\cdot\Xi^{-1}(X_n)\big)\leq 2\cdot\textstyle\sum_{k=1}^n\mathfrak{w}(X_k)=\sum_{k=1}^n\mathfrak{o}(X_k)=\varepsilon$$

holds, which shows the claim. □

Appendix D. Appendix to Sect. 6

D.1.

Proof of the statement made in Remark 3. Let $\Xi'\colon G\supseteq\mathcal{U}'\to\mathcal{V}'\subseteq E$ be a further chart of G with $e\in\mathcal{U}'$ and $\Xi'(e)=0$. Then, shrinking $\mathcal{V}$ if necessary, we can assume that

$$\Phi\equiv\mathrm{d}(\Xi'^{-1}\circ\Xi)\colon\mathcal{V}\times E\to E$$

is defined. Let now $\mathfrak{p}\in\mathfrak{P}$ be fixed. We choose $\mathfrak{q}\equiv\mathfrak{q}_1\in\mathfrak{P}$ and $V\subseteq\mathcal{V}$ as in Lemma 1, additionally convex; and define $\gamma_x\colon[0,1]\ni t\mapsto(\Xi'^{-1}\circ\Xi)(t\cdot x)\in$

V for each $x \in V$. Then, Lemma 6 shows

$$\begin{aligned}\mathfrak{p}(\Xi'^{-1} \circ \Xi)(x) &= \mathfrak{p}(\gamma_x(1) - \gamma_x(0)) \leq \int \mathfrak{p}(\dot{\gamma}_x(s))\, \mathrm{d}s \\ &= \int (\mathfrak{p} \circ \Phi)(\gamma_x(s), \dot{\gamma}_x(s))\, \mathrm{d}s \leq \int \mathfrak{q}(x)\, \mathrm{d}s = \mathfrak{q}(x)\end{aligned}$$

for each $x \in V$, from which the claim is clear. □

D.2.

Proof of the statement made in Example 3.1). We let $\Xi\colon \mathcal{U} \ni [X] \to X \in \mathcal{V}$ be defined as in Appendix C.1; and fix $V \subseteq \mathcal{V}$ symmetric open with $\overline{V} \subseteq \mathcal{V}$ and $\overline{V} + \overline{V} \subseteq \mathcal{V}$. Then, for $X, Y \in \overline{V}$ (or, alternatively, $[X], [Y] \in \Xi^{-1}(\overline{V})$), we have

$$\mathfrak{p}(-X + Y) = (\mathfrak{p} \circ \Xi)([-X + Y]) = (\mathfrak{p} \circ \Xi)([X]^{-1} \cdot [Y]) \qquad \forall\, \mathfrak{p} \in \mathfrak{P}.$$

The claim now follows easily from Remark 4.4), when applied to $U \equiv V$ as well as $U \equiv \Xi^{-1}(V)$ there. □

D.3.

Proof of the statement made in Example 3.2). Let $\|\cdot\|$ denote the Banach norm on E. Then, Lemma 8 applied to $\mathrm{C} \equiv \{e\}$ and $\mathfrak{p} \equiv \|\cdot\|$, provides us with an open neighbourhood V of e as well as some $C > 0$, such that

$$\|\Xi(q) - \Xi(q')\| \leq C \cdot \|\Xi_h(q) - \Xi_h(q')\| \qquad \forall\, q, q', h \in V \tag{D.16}$$

holds. We fix an open neighbourhood $U \subseteq G$ of e with $\overline{U} \subseteq V \cap \mathcal{U}$; and recall that in order to show that G is sequentially complete – by Remark 4.4) – it suffices to show that each Cauchy sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq U \subseteq G$ converges in G. Now, (D.16) applied to $h \equiv g_m$ gives

$$\|\Xi(g_m) - \Xi(g_n)\| \leq C \cdot \|\Xi(g_m^{-1} \cdot g_n)\| \qquad \forall\, m, n \in \mathbb{N},$$

which shows that $\{\Xi(g_n)\}_{n\in\mathbb{N}} \subseteq \Xi(U) \subseteq \mathcal{V}$ is a Cauchy sequence in E. By assumption, $\lim_n \Xi(g_n) = x \in \Xi(\overline{U}) \subseteq \mathcal{V}$ exists; so that $\{g_n\}_{n\in\mathbb{N}}$ converges to $\Xi^{-1}(x) \in G$. □

D.4.

Proof of the statement made in Example 3.3). Recall that $\mathcal{A}^\times$ is locally μ-convex by Example 2.3); and let $\Xi\colon \mathcal{U} \cong \mathcal{A}^\times \ni a \mapsto a - \mathbf{1} \in \mathcal{V} \equiv \mathcal{A}^\times - \mathbf{1}$ be as in Appendix C.3. Let furthermore $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^\times$ be a fixed sequence.

- We fix $\mathfrak{v} \in \mathfrak{P}$, choose $\mathfrak{v} \leq \mathfrak{m} \in \mathfrak{P}$ as in (59) for $\mathfrak{w} \equiv \mathfrak{m}$ there, and obtain

$$\text{(D.17)}\qquad \begin{aligned}\mathfrak{v}(a_n - a_{n-1}) &= \mathfrak{v}\big(a_{n-1} \cdot \big(a_{n-1}^{-1} \cdot a_n - \mathbf{1}\big)\big)\\ &\leq \mathfrak{m}(a_{n-1}) \cdot (\mathfrak{m} \circ \Xi)\big(a_{n-1}^{-1} \cdot a_n\big) \qquad \forall\, n \geq 1.\end{aligned}$$

- We choose $\mathfrak{m} \leq \mathfrak{u} \in \mathfrak{P}$ as in (59) for $\mathfrak{v} \equiv \mathfrak{m}$ and $\mathfrak{w} \equiv \mathfrak{u}$ there, and let $\mathfrak{u} \leq \mathfrak{o} \in \mathfrak{P}$ be as in (58). Then, passing to a subsequence if necessary, we can achieve that

$$\text{(D.18)}\quad \textstyle\sum_{n=1}^\infty \mathfrak{o}(X_n) \leq 1 \qquad \text{holds for} \qquad X_n := \Xi\big(a_{n-1}^{-1} \cdot a_n\big) \quad \forall\, n \geq 1.$$

  We obtain

$$\begin{aligned}\mathfrak{m}(a_n) \quad &= \quad \mathfrak{m}(a_0 \cdot \Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n))\\ &\overset{(59)}{\leq} \quad \mathfrak{u}(a_0) \cdot \mathfrak{u}(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n))\\ &\leq \quad \mathfrak{u}(a_0) \cdot \big(\mathfrak{u}(\mathbf{1}) + (\mathfrak{u} \circ \Xi)(\Xi^{-1}(X_1) \cdot \ldots \cdot \Xi^{-1}(X_n))\big)\\ &\overset{(58),(\text{D.18})}{\leq} \quad \mathfrak{u}(a_0) \cdot (\mathfrak{u}(\mathbf{1}) + 1),\end{aligned}$$

  implying $\sup\{\mathfrak{m}(a_n) \,|\, n \in \mathbb{N}\} < \infty$.

It is thus clear from (D.17) that:

- If $\mathcal{A}$ is sequentially complete, and $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^\times$ a Cauchy sequence, then $\lim_n a_n = a \in \mathcal{A}$ exists.
- If $\mathcal{A}$ is Mackey complete, and $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^\times$ a Mackey-Cauchy sequence, then $\lim_n a_n = a \in \mathcal{A}$ exists.

Now, since $\mathcal{A}^\times$ is open with $\mathbf{1} \in \mathcal{A}^\times$, there exists an open neighbourhood V of $\mathbf{1}$ with $\overline{V} \subseteq \mathcal{A}^\times$, as well as some $p \geq 0$ with $\{a_p^{-1} \cdot a_n\}_{n\geq p} \in V$. Then,

$$a_p^{-1} \cdot a = \lim_n (a_p^{-1} \cdot a_n) \in \overline{V} \subseteq \mathcal{A}^\times,$$

implies $a \in \mathcal{A}^\times$; which proves the claim. □

Appendix E. Appendix to Sect. 7

E.1.

Proof of Equation (77). We fix $q \in \mathbb{N}$; and choose $r = t_0 < \ldots < t_n = r'$, as well as $\phi_q[p]$ for $p = 0, \ldots, n-1$, as in (52) for $\phi \equiv \phi_q$ and $\phi[p] \equiv \phi_q[p]$ there. Then, it is clear from (53) that μ_q is of class C^1 on $J := \bigsqcup_{p=0}^{n-1}(t_p, t_{p+1})$ with $\phi_q = \mathrm{d}_{\mu_q}\mathrm{R}_{\mu_q^{-1}}(\dot{\mu}_q)$ thereon; so that we have

$$
\begin{aligned}
\upsilon(\gamma_q, \phi_q)|_J &= (\mathrm{d}_{\mu_q}\Xi \circ \mathrm{d}_e\mathrm{R}_{\mu_q})(\phi_q)|_J \\
&= (\mathrm{d}_{\mu_q}\Xi \circ \mathrm{d}_e\mathrm{R}_{\mu_q} \circ \mathrm{d}_{\mu_q}\mathrm{R}_{\mu_q^{-1}})(\dot{\mu}_q)|_J = \dot{\gamma}_q|_J.
\end{aligned} \tag{E.19}
$$

We define $\alpha_p := \upsilon(\gamma_q|_{[t_p,t_{p+1}]}, \phi_q[p])$ for $p = 0, \ldots, n-1$; and conclude from (24) and (E.19) that

$$
\gamma_q(\tau') - \gamma_q(\tau) = \int_\tau^{\tau'} \upsilon(\gamma_q(s), \phi_q(s))\, \mathrm{d}s = \int_\tau^{\tau'} \alpha_p(s)\, \mathrm{d}s \tag{E.20}
$$

holds, for each $\mathfrak{K} \ni [\tau, \tau'] \subseteq (t_p, t_{p+1})$. Since $\gamma_q, \alpha_0, \ldots, \alpha_{n-1}$ are continuous, we obtain

$$
\begin{aligned}
\gamma_q(\tau') - \gamma_q(\tau) &= \lim_{k\to\infty}(\gamma_q(\tau' - 1/k) - \gamma_q(\tau + 1/k)) \\
&\overset{\text{(E.20)}}{=} \lim_{k\to\infty} \int_{\tau+1/k}^{\tau'-1/k} \alpha_p(s)\, \mathrm{d}s = \int_\tau^{\tau'} \alpha_p(s)\, \mathrm{d}s \\
&= \int_\tau^{\tau'} \upsilon(\gamma_q(s), \phi_q[p](s))\, \mathrm{d}s
\end{aligned}
$$

for each $t_p \leq \tau < \tau' \leq t_{p+1}$, for $p = 0, \ldots, n-1$. The claim is thus clear from (26) and $\gamma_q(r) = 0$. □

E.2.

Proof of Lemma 36. Let $\frac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}^{>}$ denote the right derivative; and define $\mu := \int_r^\bullet \phi$. For the implication "$\Longrightarrow$",

- we observe that $\alpha := \mathrm{Ad}_\mu(Y)$ is of class C^1 with $\alpha(r) = Y$.
- we choose an extension $\psi \in \mathfrak{D}_{[r,r'+\delta]}$ of ϕ, for some $\delta > 0$; and define $\beta := \mathrm{Ad}_\nu(Y)$ for $\nu := \int_r^\bullet \psi$.
- we obtain from *d)* that

$$
\dot{\alpha}(t) = \dot{\beta}(t) = \frac{\mathrm{d}}{\mathrm{d}h}\Big|_{h=0}^{>} \mathrm{Ad}_{\int_t^{t+h}\psi}(\mathrm{Ad}_{\mu(t)}(Y)) = [\phi(t), \alpha(t)] \qquad \forall\, t \in [r, r'].
$$

For the implication "$\Longleftarrow$",

- we suppose that $\dot{\alpha} = [\phi, \alpha]$ holds for $\alpha \in C^1([r, r'], \mathfrak{g})$.
- we choose an extension $\psi \in \mathfrak{D}_{[r,r'+\delta]}$ of ϕ, and an extension $\beta \in C^1([r, r' + \delta], \mathfrak{g})$ of α, for some $\delta > 0$; and define $\gamma := \mathrm{Ad}_{[\int_r^\bullet \psi]^{-1}}(\beta)$.
- we recall that, cf. *c)*

$$\big[\textstyle\int_t^{t+h} \psi\big]^{-1} = \textstyle\int_t^{t+h} -\mathrm{Ad}_{[\int_t^\bullet \psi]^{-1}}(\psi) \qquad \forall\, 0 < h \leq \delta,\ \ t \in [r, r'];$$

  and conclude from *d)*, **b**), **e**) that

$$\begin{aligned}\dot{\gamma}(t) &= \tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}^{>} \mathrm{Ad}_{[\int_r^{t+h} \psi]^{-1}}(\beta(t+h)) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}h}\big|_{h=0}^{>} \mathrm{Ad}_{\mu^{-1}(t)}\big(\mathrm{Ad}_{[\int_t^{t+h} \psi]^{-1}}(\beta(t+h))\big) \\ &= \mathrm{Ad}_{\mu^{-1}(t)}\big([-\phi(t), \alpha(t)] + \dot{\alpha}(t)\big) = 0\end{aligned}$$

  holds for each $t \in [r, r']$.
- We thus conclude from (24) that $\mathrm{Ad}_{[\int_r^\bullet \phi]^{-1}}(\alpha) = \alpha(r) = Y$ holds; thus, $\alpha = \mathrm{Ad}_\mu(Y)$.

This proves the claim. □

Appendix F. Appendix to Sect. 8

F.1.

Proof of Equation (87)*.* We choose an open neighbourhood $V \subseteq E$ of 0, such that

$$f\colon V \times V \ni (x, y) \mapsto (\Xi \circ \mathrm{m})(\Xi^{-1}(x), \Xi^{-1}(y))$$

is defined. Then, shrinking δ if necessary, we can assume that

$$\gamma\colon [0, \delta] \ni t \mapsto ((\Xi \circ \mu)(t), (\Xi \circ \nu)(t)) \in V \times V$$

holds; and conclude from Lemma 7 (for $F \equiv E \times E$ and $U = V \times V$ there) that

$$\begin{aligned}\lim_{h\to 0} \tfrac{1}{h} \cdot \Xi(\mu(h) \cdot \nu(h)) &= \lim_{h\to 0} \tfrac{1}{h} \cdot (f(\gamma(h)) - f(\gamma(0))) \\ &= \overline{\mathrm{d}_{\gamma(0)} f}\,(X, Y) \\ &= X + Y\end{aligned} \tag{F.21}$$

holds. For the last step, observe that $\mathrm{d}_{(e,e)}\mathrm{m}(v,w) = v + w$ holds for all $v, w \in \mathfrak{g}$ by (15); thus,

$$\begin{aligned} \mathrm{d}_{\gamma(0)} f(Z, Z') &= (\mathrm{d}_e\Xi \circ \mathrm{d}_e\mathrm{m})(\mathrm{d}_0\Xi^{-1}(Z), \mathrm{d}_0\Xi^{-1}(Z')) \\ &= Z + Z' \qquad \forall\, Z, Z' \in E. \end{aligned}$$

The last step in (F.21) is thus clear from continuity of $\overline{\mathrm{d}_{\gamma(0)} f}$. □

F.2. Appendix

Proof of Lemma 41. By (22), it suffices to show that

$$\begin{aligned} \widetilde{\Gamma}\colon C^k([r,r'],\mathfrak{g}) \times C^k([r,r'],\mathfrak{g}) &\to C^0([r,r'],\mathfrak{g}), \\ (\phi,\psi) &\mapsto \big[t \mapsto \Gamma\big(\textstyle\int_r^t \phi, \psi(t)\big)\big] \end{aligned}$$

is continuous. For this, we let $\mathfrak{p} \in \mathfrak{P}$, $\varepsilon > 0$, and $(\phi,\psi) \in C^k([r,r'],\mathfrak{g}) \times C^k([r,r'],\mathfrak{g})$ be fixed; and have to show that there exist $\mathfrak{q} \in \mathfrak{P}$ and $\mathrm{s} \preceq k$, such that

$$\begin{aligned} &\mathbf{\cdot}\mathfrak{q}^{\mathrm{s}}_\infty(\phi' - \phi),\ \mathbf{\cdot}\mathfrak{q}^{\mathrm{s}}_\infty(\psi' - \psi) \leq 1 \\ \Longrightarrow \qquad &\mathbf{\cdot}\mathfrak{p}_\infty\big(\widetilde{\Gamma}(\phi',\psi') - \widetilde{\Gamma}(\phi,\psi)\big) \leq \varepsilon \end{aligned} \tag{F.22}$$

for $\phi', \psi' \in C^k([r,r'],\mathfrak{g})$. We let $\mu := \int_r^\bullet \phi$, and consider the continuous map

$$\alpha\colon G \times \mathfrak{g} \times G \times \mathfrak{g} \to \mathfrak{g}, \qquad ((g,X),(g',X')) \mapsto \mathbf{\cdot}\mathfrak{p}(\Gamma(g,X) - \Gamma(g',X')).$$

Then, for $t \in [r,r']$ fixed, there exists an open neighbourhood $W[t] \subseteq G$ of e, as well as $U[t] \subseteq \mathfrak{g}$ open with $0 \in U[t]$, such that

$$\begin{aligned} &\alpha((g,X),(g',X')) \leq \varepsilon \\ \forall\, (g,X),(g',X') \in &\big[\mu(t)\cdot W[t]\big] \times \big[\psi(t) + U[t]\big] \end{aligned} \tag{F.23}$$

holds. We choose

- $V[t] \subseteq G$ open with $e \in V[t]$ and $V[t]\cdot V[t] \subseteq W[t]$.
- $O[t] \subseteq \mathfrak{g}$ open with $0 \in O[t]$ and $O[t] + O[t] \subseteq U[t]$.
- $J[t] \subseteq \mathbb{R}$ open with $t \in J$, such that for $D[t] := J[t] \cap [r,r']$, we have

$$\begin{aligned} &\mu(D[t]) \subseteq \mu(t)\cdot V[t] \subseteq \mu(t)\cdot W[t] \\ \text{and} \qquad &\psi(D[t]) \subseteq \psi(t) + O[t] \subseteq \psi(t) + U[t]. \end{aligned} \tag{F.24}$$

Since $[r, r']$ is compact, there exist $t_0, \ldots, t_n \in [r, r']$, such that $[r, r'] \subseteq D_0 \cup \ldots \cup D_n$ holds.

- We define $V := V[t_0] \cap \ldots \cap V[t_n]$.

  Then, Lemma 23 provides us with some $\mathfrak{m}$ and $\mathrm{s} \preceq k$, such that

$$\int_r^\bullet \phi' \in \int_r^\bullet \phi \cdot V \qquad \text{holds for each} \qquad \phi' \in C^k([r, r'], \mathfrak{g}) \text{ with } {\scriptstyle\blacksquare}\mathfrak{m}^{\mathrm{s}}_\infty(\phi' - \phi) \leq 1. \tag{F.25}$$

- We define $O := O[t_0] \cap \ldots \cap O[t_n]$; and fix some $O \prec \mathfrak{q} \in \mathfrak{P}$ with $\mathfrak{m} \leq \mathfrak{q}$.

Let now $\phi', \psi' \in C^k([r, r'], \mathfrak{g})$ be given with ${\scriptstyle\blacksquare}\mathfrak{q}^{\mathrm{s}}_\infty(\phi' - \phi)$, ${\scriptstyle\blacksquare}\mathfrak{q}^{\mathrm{s}}_\infty(\psi' - \psi) \leq 1$. Then, for $\tau \in D_p$ wit $0 \leq p \leq n$, we obtain from (F.25), $O \prec \mathfrak{q}$, and (F.24) for $t \equiv t_p$ there that

- $\mu(t_p)^{-1} \cdot \int_r^\tau \phi' = \big(\mu(t_p)^{-1} \cdot \mu(\tau)\big) \cdot \big([\int_r^\tau \phi]^{-1}[\int_r^\tau \phi']\big) \in V \cdot V \subseteq W[t_p]$.
- $\psi'(\tau) - \psi(t_p) = (\psi'(\tau) - \psi(\tau)) + (\psi(\tau) - \psi(t_p)) \in O + O \subseteq U[t_p]$.

The claim is thus clear from (F.23) and (F.24). □

References

[1] H. Boseck, G. Czichowski, K.-P. Rudolph: *Analysis on topological groups-general Lie theory*, B.G. Teubner Verlagsgesellschaft, Leipzig, 1981.

[2] H. Glöckner: *Infinite-dimensional Lie groups without completeness restrictions*, pp. 43–59 in: Strasburger, A. et al. (Eds.), “Geometry and Analysis of Finite- and Infinite-Dimensional Lie Groups”, Banach Center Publications, Vol. **55**, Warsaw, 2002.

[3] H. Glöckner: *Regularity properties of infinite-dimensional Lie groups, and semiregularity* preprint, `arXiv:1208.0751v5 [math.FA]`, 2012.

[4] H. Glöckner, K.-H. Neeb: *Algebras whose groups of units are Lie groups*, Studia Math. **153** (2002), 147–177.

[5] H. Glöckner, K.-H. Neeb: *When unit groups of continuous inverse algebras are regular Lie groups*, Studia Math. **211** (2012), 95–109.

[6] R. Hamilton: *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[7] A. Kriegl, P.W. Michor: The Convenient Setting of Global Analysis, AMS, Providence, 1997.

[8] J. Milnor: *Remarks on infinite dimensional Lie groups*, pp. 1007–1057 in: B.S. DeWitt and R. Stora (eds.) “Relativité, groupes et topologie II”, North-Holland, 1984.

[9] K.-H. Neeb: *Representations of infinite dimensional groups*, in “Infinite Dimensional Kähler Manifolds”, Eds. A. Huckleberry, T. Wurzbacher, DMV-Seminar 31, Birkhäuser Verlag, 2001; 131–178.

[10] K.-H. Neeb: *Infinite-Dimensional Lie Groups.* 3rd cycle. Monastir (Tunisie), 2005, pp.76. cel-00391789

[11] K.-H. Neeb: *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006); 291–468.

[12] H. Omori: *Infinite-Dimensional Lie Groups,* Translations of Math. Monographs **158**, Amer. Math. Soc., 1997.

[13] W. Rudin: Functional Analysis, McGraw-Hill, 1991.

[14] J. Voigt: *On the convex compactness property of the strong operator topology*, Note di Mathematica Vol. XII, (1992) 259–269.

[15] H. von Weizsäcker: *In which spaces is every curve Lebesque-Pettis-integrable?*, preprint, `arXiv:1207.6034 [math.FA]`, 2012.

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