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Ribbon disks with the same exterior

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We construct an infinite family of inequivalent slice disks with the same exterior, which gives an affirmative answer to an old question asked by Hitt and Sumners in 1981. Furthermore, we prove that these slice disks are ribbon disks.

1. Introduction

One of the most outstanding problems in knot theory has been whether knots are determined by their complements or not. The celebrated theorem of Gordon and Luecke [12] states that if the complements of two classical knots in the 3-sphere S^3 are diffeomorphic, then these knots are equivalent. (For more recent results, see [8, 19, 25].) For higher-dimensional knots, there exist at most two inequivalent n-knots ($n \geq 2$) with diffeomorphic exteriors, see [5, 9, 20]. Examples of such n-knots are given in [6] and [11].

We consider an analogous problem for slice disks, that is, smoothly and properly embedded disks in the standard 4-ball B^4. The situation is quite different. Let X be the exterior of a slice disk, and define $\zeta(X)$ to be the number of inequivalent slice disks whose exteriors are diffeomorphic to X, where two slice disks D_1 and D_2 are equivalent if there exists a diffeomorphism $g\colon B^4 \to B^4$ such that $g(D_1) = D_2$. In 1981, Hitt and Sumners [15, Section 4] asked the following.

Question 1. Is there a slice disk exterior X with $\zeta(X) = +\infty$?

It seems that no essential progress has been made to Question 1 until recently. One of the reasons is that when we consider Question 1, we often encounter the smooth Poincaré conjecture in dimension four, which is one of the most challenging unsolved problems. In 2015, Larson and Meier [21] produced infinite families of inequivalent homotopy-ribbon disks with homotopy equivalent exteriors, which gives a partial answer to Question 1. In this paper, we prove the following.

Theorem 1.1. *There is a sequence of slice disks $D_n (n \geq 0)$ satisfying the following properties:*

1) *The exterior of each slice disk D_n is diffeomorphic to that of D_0.*
2) *The knots ∂D_n are mutually inequivalent, therefore D_n are mutually inequivalent.*
3) *D_n is a ribbon disk.*
4) *The knots ∂D_n are obtained from $4_1 \# 4_1$ by the n-fold annulus twist.*

As an immediate corollary of (1) and (2) in Theorem 1.1, we obtain the following.

Corollary 1.2. *There is a slice disk exterior X with $\zeta(X) = +\infty$.*

We do not know whether there exist inequivalent ribbon disks with the same exterior if their boundaries are equivalent. However, there is a related result. In 1991, Akbulut [2] found an interesting pair of ribbon disks using the Mazur cork (see also [3, Subsection 10.2]). Actually, he constructed a pair of ribbon disks E_1, E_2 satisfying the following.

(1) The two knots ∂E_1 and ∂E_2 coincide, that is, $\partial E_1 = \partial E_2$.
(2) The two ribbon disks E_1, E_2 are NOT isotopic rel the boundary.
(3) The two ribbon disks E_1, E_2 are equivalent, therefore, their exteriors are diffeomorphic.

We will give explicit pictures of the ribbon disks which clarify the symmetry which comes from the Mazur cork in the Appendix.

Here we give a historical remark on Question 1. In [15, Section 4], Hitt and Sumners also asked whether there exist infinitely many higher-dimensional slice disks with the same exterior or not. After pioneering work [15, 16, 23], Suciu [26] proved that there exist infinitely many inequivalent n-ribbon disks with the same exterior for $n \geq 3$ in 1985, and Question 1 remained open.

This paper is organized as follows: In Section 2, we recall some basic definitions. In Section 3, we prove the first half of Theorem 1.1. In Section 4, we prove the latter half of Theorem 1.1. In the Appendix, we will give explicit pictures of Akbulut's ribbon disks which clarify the symmetry.

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2. Basic definitions

In this short section, we recall some basic definitions and background.

We define a *slice disk* to be a smoothly and properly embedded disk $D \subset B^4$, and the boundary of D, $\partial D \subset S^3$, is called a *slice knot*. A knot $R \subset S^3$ is called *ribbon* if it bounds an immersed disk $\Delta \subset S^3$ with only ribbon singularities. For the definition of a ribbon singularity, see the left picture of Figure 1. By pushing the interior of Δ into the interior of B^4, we obtain a slice disk whose boundary is R, and the resulting slice disk is called a *ribbon disk*. It is well known that this ribbon disk is uniquely determined by $\Delta \subset S^3$ up to isotopy. The *Slice-Ribbon Conjecture*, also known as the Ribbon-Slice Problem, states that every slice knot is a ribbon knot, namely, it always bounds a ribbon disk. Our work is partially motivated by this conjecture[1]. Here we give an example of a ribbon knot.

Example 2.1. *The knot in the middle picture of Figure 1 is a ribbon knot since it bounds an immersed disk $\Delta \subset S^3$ as in the right picture of Figure 1.*

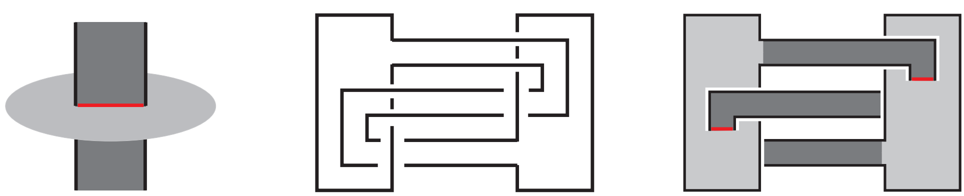

Figure 1: A ribbon singularity (colored red), a ribbon knot, and an immersed disk $\Delta \subset S^3$.

[1] Hass [13] proved that a slice disk is a ribbon disk if and only if it is isotopic to a minimal disk in B^4, see also [14, Appendix B], [4, p450]. Therefore the Slice-Ribbon Conjecture might be solved (affirmatively or negatively) using geometric analysis or geometric measure theory.

Two knots K_1, K_2 are *equivalent* if there exists a diffeomorphism $f\colon S^3 \to S^3$ such that $f(K_1) = K_2$. The *exterior* of a knot K is the 3-manifold $S^3 \setminus \nu(K)$, where $\nu(K)$ is an open tubular neighborhood of K in S^3. It is unique up to diffeomorphisms, and its interior is diffeomorphic to the complement of K. Similarly two slice disks D_1, D_2 are *equivalent* if there exists a diffeomorphism $g\colon B^4 \to B^4$ such that $g(D_1) = D_2$, and the *exterior* of a slice disk D is the 4-manifold $B^4 \setminus \nu(D)$, where $\nu(D)$ is an open tubular neighborhood of D in B^4. Note that two (ambient) isotopic slice disks D_1 and D_2 are equivalent.

3. Proof of the first half of Theorem 1.1

We prove the first half of Theorem 1.1, that is, the statements (1) and (2) in Theorem 1.1.

Throughout this paper, we only consider a 2-handlebody HD which consists of a 0-handle, 1-handles, and 2-handles. Also, note that the handle diagram of HD is drawn in the boundary of the 0-handle.

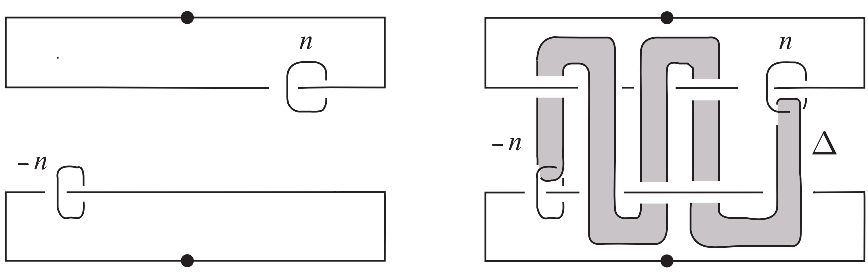

Figure 2: A handle diagram of B^4 and the definition of the slice disk D_n.

Proof of the first half of Theorem 1.1. We consider the handle decomposition of B^4 given by the handle diagram in the left picture in Figure 2. Let D_n be the slice disk obtained from the disk Δ in the right picture in Figure 2 by pushing the interior of Δ into the interior of the 0-handle, and X_n the exterior of D_n.

We will prove the statement (1) : X_n is diffeomorphic to X_0. By the definition of dotted circles, X_n is represented by the picture in Figure 3 (after an isotopy). For the dotted circle notation, see [2, 10]. Then X_n is diffeomorphic to X_0, which follows from the well-known fact that two handle diagrams that differ locally as in Figure 4 are related by a sequence of handle moves, see [2, 10].

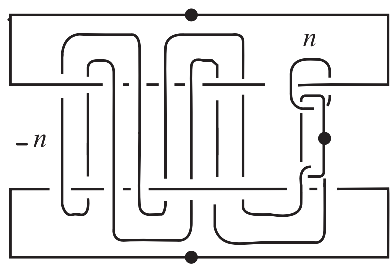

Figure 3: A handle diagram of X_n.

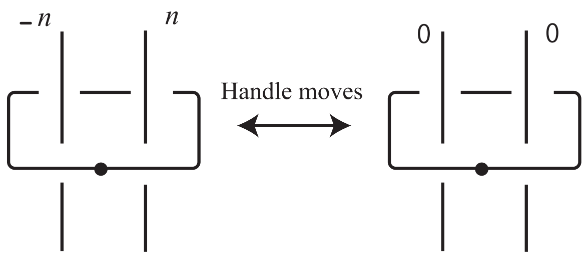

Figure 4: Two handle diagrams related by a sequence of handle moves.

Recently, Takioka [27] calculated the Γ-polynomial[2] of the knots ∂D_n. In particular, he showed that the span of the Γ-polynomial of ∂D_n is $2n + 4$ and proved that the knots ∂D_n are mutually inequivalent for $n \geq 0$, which implies the statement (2). □

Remark 3.1. *It is straightforward to see that the knot ∂D_0 is $4_1 \# 4_1$, that is, the connected sum of two figure-eight knots. We can also prove that ∂D_n is isotopic to ∂D_{-n} (by using the symmetry of $4_1 \# 4_1$).*

Remark 3.2. *It is not difficult to see that X_n is diffeomorphic to the exterior of the ribbon disk represented by the picture in Figure 5 (see [2, Subsection 1.4] or [10, Subsection 6.2]). However, this fact does not a priori imply that D_n is a ribbon disk. In our case, D_n is a ribbon disk as proven later.*

We conclude this section by asking the following question.

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[2]This is a polynomial invariant introduced by Akio Kawauchi in [18], which is a specialization of the Homflypt polynomial. This polynomial is independent of the Alexander polynomial and the Jones polynomial, and there is a polynomial time complexity algorithm for computing the Γ-polynomial, see [24].

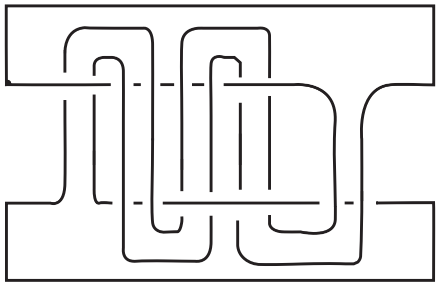

Figure 5: A ribbon knot, which determines an obvious ribbon disk.

Question 2. Let D be a slice disk whose exterior is diffeomorphic to a ribbon disk exterior. Then is D a ribbon disk ?

4. Proof of the latter half of Theorem 1.1

In this section, we observe Lemmas 4.1 and 4.2 which are important when we deal with ribbon disks in terms of handle diagrams of B^4, and prove the latter half of Theorem 1.1.

Lemma 4.1. *Let F and F' be the two smooth disks in the handle diagram of (a handle decomposition of) B^4 which locally differ as shown in Figure 6, and D and D' be the slice disks obtained by pushing the interiors of F and F' into the interior of the 0-handle. Then D and D' are (ambient) isotopic in B^4.*

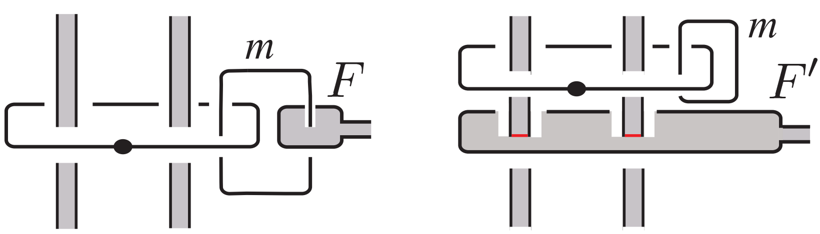

Figure 6: Two smooth disks F and F' in the handle diagram of B^4.

Proof. See Figure 7. □

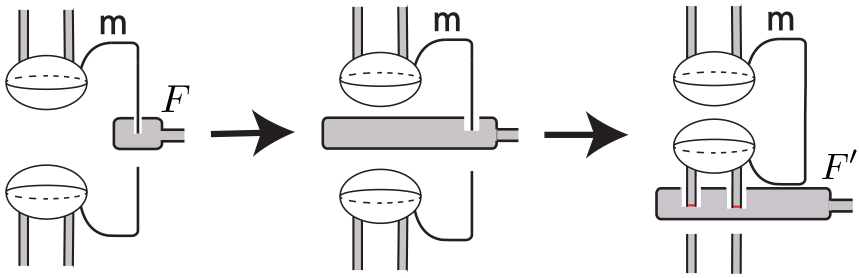

Figure 7: An isotopy between D and D' which are projected to F and F', respectively.

Lemma 4.2. *Let F and F' be two smooth disks in handle diagrams of B^4 which locally differ as shown in Figure 8, and D and D' be the slice disks obtained by pushing the interiors of F and F' into the interior of the 0-handle. Then D and D' are (ambient) isotopic in B^4.*

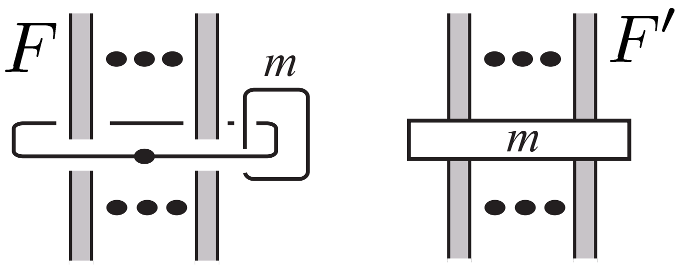

Figure 8: Two disks F and F'. Here the box named m means the m-full twists.

Proof. After sliding D over the 2-handle in the left picture of Figure 8, we cancel the 1/2-handle pair. Then we obtain the slice disk D' which are projected to F', see the right picture of Figure 8. □

We are ready to prove the latter half of Theorem 1.1.

Proof of the latter half of Theorem 1.1. Recall that the slice disk D_n is obtained by pushing the interior of the disk Δ in the left picture of Figure 9 into the interior of the 0-handle. By Lemmas 4.1 and 4.2, D_n is isotopic to the slice disk D'_n which is projected to the smooth disk Δ' in the right picture of Figure 9. This implies that D_n is a ribbon disk, and we have the statement (3). Here we note that the smooth disk Δ' has four ribbon singularities, however, we do not draw the whole picture of Δ' for simplicity.

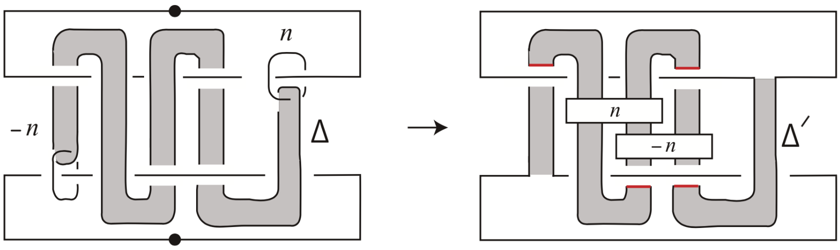

Figure 9: Δ' is an immersed disk with four ribbon singularities.

The right picture of Figure 9 tells us that the knot ∂D_n is isotopic to that in Figure 10. This implies that ∂D_n is obtained from the ribbon knot

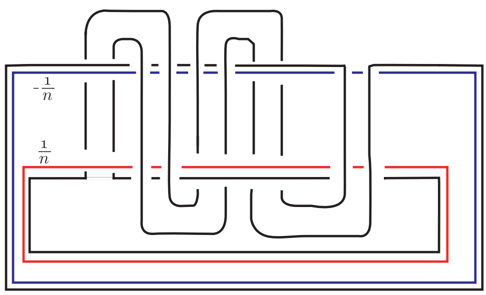

Figure 10.

$\partial D_0 = 4_1 \# 4_1$ by n-fold annulus twist, see [1]. We have now obtained the statement (4). □

Remark 4.3. *It turns out that the knots ∂D_n are the same as those in [22, Figure 7], however, we omit the proof since this fact is not used in this paper.*

Appendix

Akbulut [2] constructed a pair of ribbon disks E_1 and E_2 as in Figure 11, where ribbon disks are specified by dashed arcs. In [2], he gave explicit pictures of his ribbon disks; however, the symmetry which comes from the Mazur cork was not clear. Here we give explicit pictures of E_1 and E_2 which clarify the symmetry as in Figure 12. By the symmetry, E_1 and E_2 are equivalent, hence the ribbon disk exteriors are diffeomorphic. Figures 13 and 14 explain how to obtain explicit pictures of E_1 and E_2.

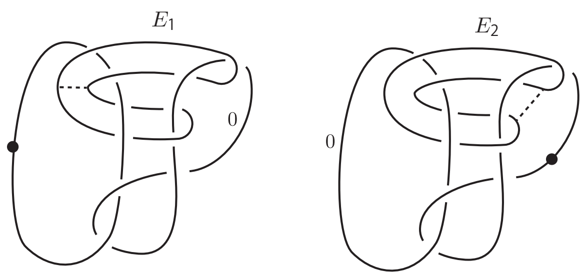

Figure 11: A pair of ribbon disks E_1 and E_2.

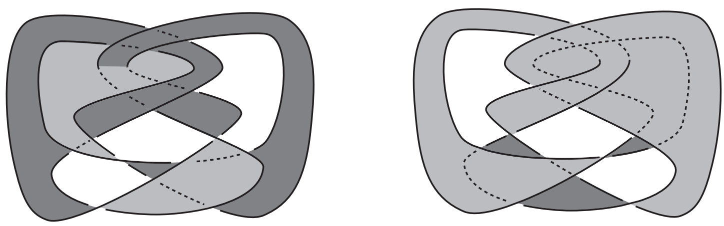

Figure 12: Explicit pictures of E_1 and E_2 which clarify the symmetry.

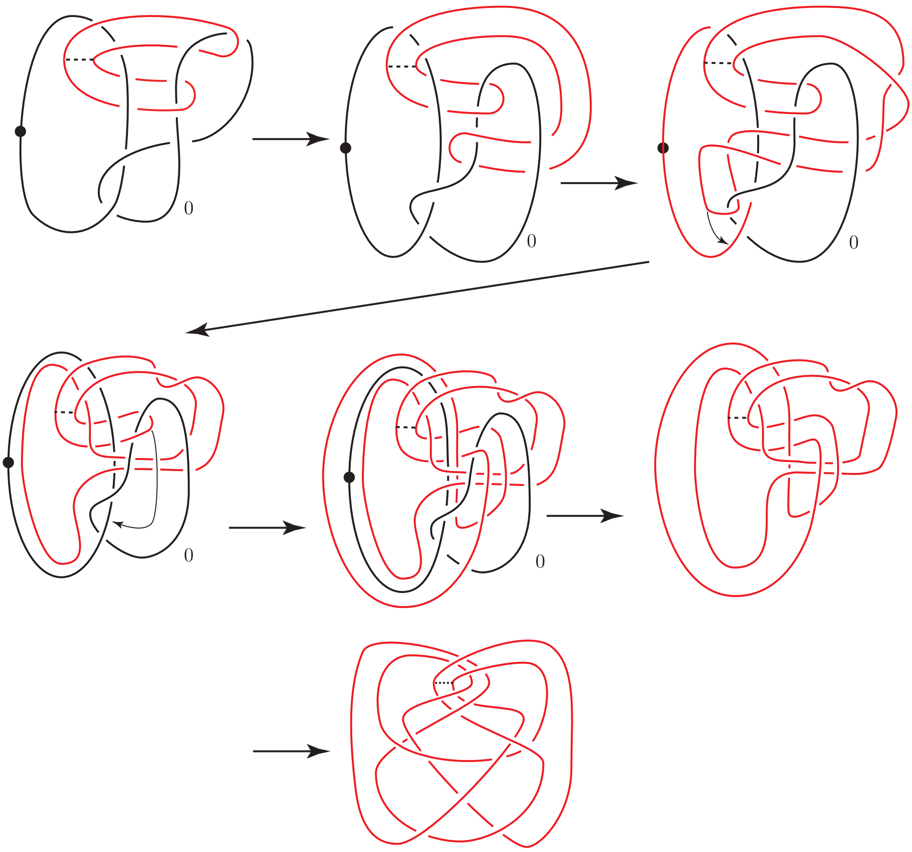

Figure 13: An isotopy.

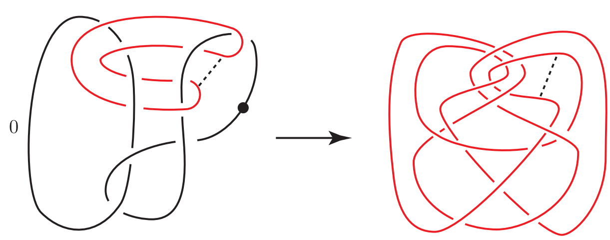

Figure 14: An isotopy.

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