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Local and global gradient estimates for Finsler p-harmonic functions

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In this paper, we give the local and global gradient estimates for positive Finsler p-eigenfunctions on a complete Finsler manifold M with the weighted Ricci curvature bounded from below by a negative constant. As applications, we obtain some Liouville and Harnack theorems, and the global gradient estimates for positive Finsler p-harmonic functions. As a by-product of the global estimate, we obtain an upper bound of the first p-eigenvalue $\lambda_{1,p}$ for Finsler p-Laplacian Δ_p. Further, we study the geometric structure at infinity of Finsler manifolds with $\lambda_{1,p}$ achieving its maximum value.

1. Introduction

Harmonic functions on Riemannian manifolds have been one of the important research objects in geometric analysis. It is well known that Cheng-Yau's local gradient estimate for positive harmonic functions is a standard

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result in Riemannian geometry ([CY]), which implies that a harmonic function with sublinear growth on a manifold with nonnegative Ricci curvature is a constant. Cheng-Yau's local gradient estimate has been generalized in different settings by many mathematicians. For example, Wang-Zhang obtained a local gradient estimate for positive $p(>1)$-harmonic functions in a geodesic ball $B_R(o)$ on a Riemannian manifold M ([WZ]) and Munteanu-Wang generalized the above local gradient estimate to the weighted Riemannian manifold with nonnegative weighted Ricci curvature ([MW]). Recently, this local gradient estimate has been extended to Alexandrov spaces by Zhang-Zhu ([ZZ]) and Finsler measure spaces by C. Xia ([Xc]). On the other hand, R. Moser recently established an interesting connection between the p-harmonic functions and the inverse mean curvature flow in [Mo] and proved the existence of the weak solution to the inverse mean curvature flow starting from the boundary of any smooth compact domain in the Euclidean space. After that, Kotschwar-Ni in [KN] had succeeded in carrying out the same scheme on general Riemannian manifolds with sectional curvature bounded from below. The existence of the weak solutions for these inverse mean curvature flows strongly relies on a uniform gradient estimate (independent of p) for the function $v = -(p-1)\log u$ with u being p-harmonic. Inspired by these, one of the objectives of this paper is to establish the local and global gradient estimates for Finsler $p(>1)$-harmonic functions and give some applications. The further applications of the global gradient estimates for Finsler p-harmonic functions will be studied elsewhere.

Recall that a *Finsler manifold* (M,F) means a smooth manifold M equipped with a Finsler metric (or Finsler structure) $F : TM \to [0,+\infty)$ such that $F_x = F|_{T_xM}$ is a Minkowski norm on T_xM at each point $x \in M$. Given a smooth measure m, the triple (M,F,m) is called a *Finsler measure space*. A Finsler measure space is not a metric space in usual sense because F may be nonreversible, i.e., $F(x,y) \neq F(x,-y)$ may happen. This non-reversibility causes the asymmetry of the associated distance function. We say that F satisfies *uniform convexity* (resp. *uniform smoothness or concavity*) if there exists a positive constant κ^* (resp. κ) such that for any $x \in M$, $V \in T_xM \setminus \{0\}$ and $y \in T_xM$, we have

$$\kappa^* F^2(x,y) \leq g_V(y,y), \qquad (\text{resp. } g_V(y,y) \leq \kappa F^2(x,y)), \tag{1.1}$$

where $g_V = (g_{ij}(V))$ is the Riemannian metric on M induced by F with reference vector V. (1.1) implies that $0 < \kappa^* \leq 1 \leq \kappa < \infty$ and F has finite reversibility Λ (see (2.2) below). The uniform smoothness and the uniform convexity were first introduced in Banach space theory by Ball, Carlen and

Lieb in [BCL]. Recently S. Ohta gave their geometric explanations in Finsler geometry ([Oh]). In fact there are many Finsler metrics satisfying (1.1). For example,

- any regular Randers metric $F = \alpha + \beta$ on a closed manifold M satisfies (1.1) with $\kappa = \left(\frac{1+b}{1-b}\right)^2$ and $\kappa^* = \left(\frac{1-b}{1+b}\right)^2$, where α is a Riemannian metric and β is a 1-form with $b = \max_{x\in M} \|\beta\|_\alpha < 1$ on M ([ZY], Corollary 6.3).
- any regular Randers metric $F = \alpha + \beta$ on a noncompact manifold satisfies uniform smoothness and convexity with κ and κ^* as above, where β is a closed and conformal, in particular, parallel 1-form with respect to α. In this case, $b = \|\beta\|_\alpha$ is a constant.

Note that F is not smooth at the zero section and F is nonlinear. This causes that the gradient ∇u of a smooth function u on M is only continuous on M and the Finsler Laplacian Δ_m or p-Laplacian $\Delta_{p,m}$ is not linear and not well defined on the whole manifold. It should be understood in a weak sense. The precise definitions of the gradient ∇ and the Finsler $p(>1)$-Laplacian Δ_p will be given in Section 2.

Moreover, it is well known that Ricci curvature plays a prominent role in Riemannian geometric analysis and geometric topology. In Finsler geometry, the global (analytic or topological) properties of manifolds are affected by not only Riemannian quantities, for example, flag curvature and Ricci curvature, but also non-Riemnanian quantities, such as, S-curvature, Douglas curvature and Landsberg curvature etc., which vanish in Riemannian geometry ([BCS], [Sh]). S.Ohta introduced the weighted Ricci curvature Ric_N for $N \in [n, \infty]$ in terms of the Ricci curvature and S-curvature, and proved that the condition that Ric_N has a lower bound is equivalent to the curvature-dimension condition, introduced by Lott-Villani and Sturm ([Oh1], [LV], [St1]-[St2]). Studies show the weighted Ricci curvature plays an important role in global Finsler geometry. Some progress of comparison geometry and geometric analysis on Finsler manifolds with weighted Ricci curvature bounded below has been made in recent years ([Oh1]-[Oh2], [OS1]-[OS2], [WX], [Xc], [Xia1]-[Xia3] and references therein). In this paper, we continue to pursue the study on this aspect. In particular, we give the local and global gradient estimates of eigenfunctions for the Finsler p-Laplacian on a complete Finsler manifold (M, F) equipped with a uniform smooth and uniform convex Finsler metric F if the weighted Ricci curvature is bounded from below by a negative constant. As applications, we obtain the Harnack or Liouville properties for Finsler p-harmonic functions. Further we obtain a global upper bound estimate for Finsler p-harmonic functions and a upper bound estimate of the first p-eigenvalue.

Theorem 1.1. *Let (M,F,m) be an $n(\geq 2)$-dimensional forward complete and noncompact Finsler measure space equipped with a uniformly convex and uniformly smooth Finsler metric F and a smooth measure m. Assume that $Ric_N \geq -K$ for some $N \in [n,\infty)$ and $K \geq 0$. Let u be a positive p-eigenfunction corresponding to the eigenvalue λ_p, i.e.,*

$$\Delta_p u = -\lambda_p |u|^{p-2} u \tag{1.2}$$

in a weak sense in a forward geodesic ball $B^+_{2R}(q) \subset M$ for any $q \in M$. Then there exists a positive constant $C = C(N,p,\kappa,\kappa^*)$ depending on N, the uniform constants κ and κ^*, such that

$$\sup_{x\in B^+_R(q)} \{F(x,\nabla \log u(x)), F(x,\nabla(-\log u(x)))\} \leq C\frac{1+\sqrt{K}R}{R}. \tag{1.3}$$

In particular, $F(x,\nabla \log u(x))$ and $F(x,\nabla \log(-u(x)))$ are bounded on M.

The precise definitions of the gradient ∇, the Finsler $p(>1)$-Laplacian Δ_p and the weighted Ricci curvature Ric_N etc. will be given in Section 2. When $p=2$ and $\lambda_p = 0$, that is, u is a harmonic function, Theorem 1.1 is reduced to Theorem 1.1 in [Xc]. If (M,F) is Riemnnian and $\lambda_p = 0$ (i.e., u is a p-harmonic function), then Theorem 1.1 is exactly Theorem 1.1 in [WZ]. It is worth mentioning that Theorem 1.1 does not coincide with the local gradient estimate for positive p-harmonic functions on weighted Riemannian manifold $(M, g_{\nabla u}, m)$ since Ric_N and the weighted Ricci curvature $\mathrm{Ric}_N^{\nabla u}$ of $(M, g_{\nabla u}, m)$ are different ([Oh], [Xc]). In fact, Ric_N depends only on the Finsler structure F and the measure m, while $\mathrm{Ric}_N^{\nabla u}$ depends on u. Moreover, in [CY], Cheng-Yau combined the Bochner technique and maximum principle to prove the local gradient estimates of eigenfunctions on Riemannian manifolds. Cheng-Yau's approach turned out to be very useful to estimate the first (Riemannian) eigenvalue, heat kernel and so on ([Li]). It also can be used to prove the local gradient estimate for harmonic functions on a compact Finsler manifold with the weighted Ricci curvature bounded below (Theorem 1.3, [ZX]), but does not work on a complete noncompact Finsler manifold because of the nonlinearity of Finsler Laplacian. We will use the method of Moser's iteration to prove Theorem 1.1, which is inspired from recent works by Wang-Zhang ([WZ]) in Riemannian case and C. Xia ([Xc]) in Finslerian case.

It is worth mentioning that the local gradient estimate for eigenfunctions has a uniform bound independent of λ_p. This is because the terms containing λ_p in (3.7) and (3.8) in §3 are just cancelled. Theorem 1.1 is also true in Riemannian case (see the proof of Theorem 2.2 in [SW]). In particular, when $\lambda_p = 0$, the weak solution of (1.2) is a *p-harmonic function* (cf. §2). Thus Theorem 1.1 also implies a local gradient estimate for p-harmonic functions on a Finsler manifold. As applications, one obtains the Harnack and Liouville properties by standard arguments (cf. §1.3 in [SY], [Xc], [ZZ]).

Corollary 1.1. *Let (M, F, m), Ric_N be as in Theorem 1.1 and u be a positive $p(>1)$-eigenfunction or p-harmonic function in a geodesic ball $B^+_{2R}(q) \subset M$. Then there exists a constant $C = C(N, p, \kappa, \kappa^*)$ such that*

$$\sup_{x\in B^+_R(q)} u(x) \leq e^{C(1+\sqrt{K}R)} \inf_{x\in B^+_R(q)} u(x).$$

If $K = 0$, then we have a uniform constant $c = c(N, p, \kappa, \kappa^*)$ independent of R such that

$$\sup_{x\in B^+_R(q)} u(x) \leq c \inf_{x\in B^+_R(q)} u(x).$$

Corollary 1.2. *Let (M, F, m) be as in Theorem 1.1 with $Ric_N \geq 0$. If u is a p-eigenfunction bounded from below on M or u is a p-eigenfunction of sublinear growth on M, then u is constant. In particular, any positive p-harmonic function on M must be a constant.*

Based on the local gradient estimate, we obtain a global gradient estimate for p-eigenfunctions as follows. In Riemannian case, $\Lambda = \kappa = \kappa^* = 1$ and Ric_N is the usual Ricci curvature on M. Such an estimate is due to Sung-Wang ([SW]).

Theorem 1.2. *Let (M, F, m) be an $n(\geq 2)$-dimensional forward complete and noncompact Finsler measure space equipped with a uniformly convex and uniformly smooth Finsler metric F and a smooth measure m. Assume that $Ric_N \geq -K$ for some $N \in [n, \infty)$ and $K > 0$. Let u be a positive $p(>1)$-eigenfunction on M corresponding to the first eigenvalue $\lambda_{1,p}$. Then*

$$\sup_{x\in M} \{F(x, \nabla \log u(x)), F(x, \nabla(-\log u(x)))\} \leq \chi, \tag{1.4}$$

where χ is the largest positive root of the equation

$$(p-1)\chi^p - \Lambda\sqrt{(N-1)K}\chi^{p-1} + \lambda_{1,p} = 0, \tag{1.5}$$

where Λ *is the reversibility of* F.

We should point out that (1.5) has two positive roots and the value χ is the bigger one, which is well defined by Theorem 1.3 below. A direct consequence of Theorem 1.2 gives a global gradient estimate for positive p-harmonic functions.

Corollary 1.3. *Let* (M, F, m) *and* Ric_N *be as in Theorem 1.2. If* u *is a positive* p*-harmonic function, then*

$$\sup_{x\in M}\{F(x, \nabla \log u(x)), F(x, \nabla(-\log u(x)))\} \leq \frac{\Lambda\sqrt{(N-1)K}}{p-1}. \tag{1.6}$$

As a by-product of the proof of Theorem 1.2, we obtain a upper bound estimate for the first eigenvalue of $p(>1)$-Laplacian on a forward complete and noncompact Finsler manifold as follows (also see Corollary 4.1). By the way, several (sharp) lower bound estimates for the first eigenvalue of p-Laplacian on a compact Finsler manifold without boundary or with smooth boundary were given by the author of the present paper, according to different arrange of p ([Xia1]-[Xia3]).

Theorem 1.3. *Let* (M, F, m) *be an* n*-dimensional forward complete noncompact Finsler manifold with finite reversibility* Λ *and* $\lambda_{1,p}$ *be the first eigenvalue for the Finsler* $p(>1)$*-Laplacian on* M*. Assume that* $Ric_N \geq -K$ *for some* $N \in [n, +\infty)$ *and* $K \geq 0$*. Then* $\lambda_{1,p} \leq \left(\frac{\Lambda\sqrt{(N-1)K}}{p}\right)^p$.

When (M, F) is Riemannian, Theorem 1.3 was given by S. Cheng in [Ch] for $p = 2$ and Sung-Wang for general $p > 1$ in [SW] in different ways. A natural question is to ask what geometric structure of Finsler manifolds with maximal eigenvalue $\lambda_{1,p}$ is, which is another objective of this paper. In Riemannian case, this question has been studied by P. Li and J. Wang etc. in different settings ([LW1]-[LW2], [SW]). Inspired by these works, one obtains the following result.

Theorem 1.4. *Let* (M, F, m) *be an* $n(\geq 2)$*-dimensional complete and noncompact Berwald space containing a straight line* $\tilde{\gamma} : \mathbb{R} \to M$ *and* $\mathbf{b}_{\tilde{\gamma}}$ *the Busemann function associated to* $\tilde{\gamma}$*. Assume that* F *satisfies*

$$\kappa^* F^2(x, y) \leq g_{\nabla \mathbf{b}_{\tilde{\gamma}}}(y, y) \leq F^2(x, y), \quad y \in T_xM \tag{1.7}$$

for some constant $0 \leq \kappa^* \leq 1$ *and* $Ric_N \geq -K$ *for some* $N \in [n, \infty)$ *and* $K > 0$. *If* $\lambda_{1,p} = \left(p^{-1}\Lambda\sqrt{(N-1)K}\right)^p$, *then either* (M, F, m) *has no finite volume ends containing* $\tilde{\gamma}$ *or* (M, F, m) *has a splitting as follows.*

(1) (M, m) *admits a diffeomorphic measure splitting* $(M, m) = (\mathbb{R} \times \breve{M}, e^{-t\sqrt{(N-1)K}} L^1 \times \breve{m})$, *where* L^1 *is the one-dimensional Lebesgue measure and* $\breve{m} := m|_{\breve{M}}$ *is the induced measure on* $\breve{M} = \mathbf{b}_{\tilde{\gamma}}^{-1}(0)$.

(2) (M, F) *is reversible and* $\Delta_p v = -\lambda_{1,p}|v|^{p-2}v$ *in the distribution sense, where* $\lambda_{1,p} = \left(p^{-1}\sqrt{(N-1)K}\right)^p$ *and* $v = e^{\frac{1}{p}\sqrt{(N-1)K}\ \mathbf{b}_{\tilde{\gamma}}}$. *Further, let* $M_t := \mathbf{b}_{\tilde{\gamma}}^{-1}(t)$ *and* $\{\varphi_t\}$ *be the one-parameter family of* C^∞*-transformations of* M *generated from* $\nabla \mathbf{b}_{\tilde{\gamma}}$. *Then* φ_t *is a homothetic transformation of* (M, F) *with a homothetic factor* $\frac{1}{2}\sqrt{K/(N-1)}$. *Moreover, it holds* $(M, F) = \cup_{t \in \mathbb{R}}(M_t, F_t)$, *the union of a family of reversible Finsler submanifolds* (M_t, F_t), *where* $M_t = \varphi_t(\breve{M})$ *with* $\varphi_t^*(F_t) = e^{t\sqrt{K/(N-1)}}\breve{F}$, $(\breve{M}, \breve{F})$ *is a compact and reversible Finsler manifold with nonnegative weighted Ricci curvature and* $\breve{F}$*(resp.*F_t*) is the induced Finsler metric on* $\breve{M}$*(resp.*M_t*) from* F.

It is worth mentioning that there are non-Riemannian reversible Berwald metrics (cf. [CS], Example 4.3.1). It is not known if there is a splitting as in Theorem 1.4 for a complete Finsler space (M, F, m) equipped with a uniformly smooth and convex Finsler metric under the same assumptions on Ric_N and $\lambda_{1,p}$ as in Theorem 1.4. Moreover, it is a natural question to study the structure of manifolds at infinity with maximal $\lambda_{1,p}$ if M admits a straight line contained in an end with infinite volume. If F is Riemannian, then the equalities in (1.7) hold identically, $N = n$ and Ric_N is the usual Ricci curvature in Riemannian geometry. Thus Theorem 1.4 (also Proposition 5.2) is exactly Theorem 3.1 in [SW]. It is very interesting that there exists a splitting phenomenon on a weighted Riemannian manifold with the weighted Ricci curvature Ric_N bounded from below for $N < 0$ ([Mai]).

2. Preliminaries

In this section, we briefly review some basic concepts in Finsler geometry, as well as some recent progress on the global analysis on Finsler manifolds. For more details, we refer to [BCS], [Sh], [Oh1], [OS2], [WX] and [Xia2].

2.1. Finsler metrics and geodesics

Let M be an n-dimensional smooth manifold and TM the tangent bundle of M. A *Finsler metric* F on M means a function $F : TM \to [0, \infty)$

with the following properties: (1) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (2) for each $x \in M$, $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM, i.e., F satisfies $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$ and the matrix $[g_{ij}(x, y)] := [\frac{1}{2}(F^2)_{y^i y^j}]$ is positive definite for any nonzero $y \in T_xM$. Such a pair (M, F) is called a *Finsler manifold* and $g := g_{ij}(x, y)dx^i dx^j$ is called the *fundamental tensor* of F. Given a smooth measure m, the triple (M, F, m) is called a *Finsler measure space*. For any Finsler metric F, its dual $F^*(x, \xi) = \sup_{F_x(y)=1} \xi(y)$ is a Finsler co-metric on M, where $\xi \in T_x^*(M)$.

We define the *reverse metric* $\overleftarrow{F}$ of F by $\overleftarrow{F}(x, y) := F(x, -y)$ for all $(x, y) \in TM$. It is easy to see that $\overleftarrow{F}$ is also a Finsler metric on M. A Finsler metric F on M is said to be *reversible* if $\overleftarrow{F}(x, y) = F(x, y)$ for all $y \in TM$. Otherwise, we say F is nonreversible. In this case, we define the *reversibility* $\Lambda = \Lambda(M, F)$ of F by

$$\Lambda := \sup_{(x,y)\in TM\setminus\{0\}} \frac{F(x, -y)}{F(x, y)}.$$

Obviously, $\Lambda \in [1, \infty]$ and $\Lambda = 1$ if and only if F is reversible.

Given a non-vanishing smooth vector field V, one introduces the weighted Riemannian metric g_V on M given by

$$g_V(y, w) = g_{ij}(x, V_x)y^i w^j, \qquad \text{for } y, w \in T_xM \text{ and } x \in M. \tag{2.1}$$

Thus, we have $F^2(V) = g_V(V, V)$. If F satisfies the uniform smoothness or convexity (see (1.1)), then Λ is finite with

$$1 \le \Lambda \le \min\{\sqrt{\kappa}, \sqrt{1/\kappa^*}\}. \tag{2.2}$$

F is Riemannian if and only if $\kappa = 1$ if and only if $\kappa^* = 1$ ([Oh]).

For $x_1, x_2 \in M$, the *distance* from x_1 to x_2 is defined by

$$d_F(x_1, x_2) := \inf_\gamma \int_0^1 F(\dot\gamma(t))dt,$$

where the infimum is taken over all C^1 curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Note that $d_F(x_1, x_2) \ne d_F(x_2, x_1)$ unless F is reversible. If $\Lambda < \infty$, then $d_F(x_1, x_2)$ and $d_F(x_2, x_1)$ are comparable. Now we define the *forward and backward geodesic balls* of radius R with center at $x \in M$

by

$$B_R^+(x) := \{z \in M \mid d_F(x,z) < R\}, \quad B_R^-(x) := \{z \in M \mid d_F(z,x) < R\}.$$

A C^1 curve $\eta : [0,\ell] \to M$ is called a *geodesic* if it has constant speed (i.e., $F(\eta,\dot{\eta})$ is constant) and if it is locally minimizing. Such a geodesic is in fact a C^∞ curve. Given $x \in M$ and $v \in T_xM$, we define the *exponential map* by $\exp_x v = \eta(1)$ if there exists a geodesic $\eta : [0,1] \to M$ with $\eta(0) = x$ and $\dot{\eta}(0) = v$. A Finsler manifold (M,F) is said to be *forward complete (resp. backward complete)* if each geodesic defined on $[0,\ell)$ (resp. $(-\ell,0]$) can be extended to a geodesic defined on $[0,\infty)$ (resp. $(-\infty,0]$). (M,F) is backward complete if and only if $(M,\overleftarrow{F})$ is forward complete. We say (M,F) is *complete* if it is both forward complete and backward complete. By Hopf-Rinow Theorem (Theorem 6.6.1, [BCS]), if (M,F) is forward complete, then any two points $x,z \in M$ can be connected by a minimal geodesic from x to z and the forward closed balls $\overline{B_R^+(x)}$ are compact. For $x \in M$ and a unit vector $v \in T_xM$, let $t_0 = \sup\{t \in (0,\infty] | d_F(x,\exp_x(tv)) = t\}$. If $t_0 < \infty$, we call $\exp_x(t_0v)$ a *cut point* of x. The set of all cut points of x is said to be the *cut locus* of x, denoted by $\mathrm{Cut}(x)$. $\mathrm{Cut}(x)$ has zero measure and $d_F(x,\cdot)$ is smooth on $M \setminus (Cut(x) \cup \{x\})$. Moreover, $F(\nabla d_F) = 1$ ([BCS], [Sh]).

2.2. Connection and curvatures

Let $\pi : TM_0 \to M$ be the projective map. The pull-back π^*TM on TM_0 admits a unique linear connection, which is called the *Chern connection*. Given a nonzero vector field $V = V^i \frac{\partial}{\partial x^i}$, the Chern connection D is determined by the following equations

$$D_X^V Y - D_Y^V X = [X,Y], \tag{2.3}$$

$$Zg_V(X,Y) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + C_V(D_Z^V V, X, Y), \tag{2.4}$$

where $X,Y,Z \in TM$, g_V is defined by (2.1) and

$$C_V(X,Y,Z) := C_{ijk}(x,V)X^iY^jZ^k = \frac{1}{4}\frac{\partial^3 F^2(x,V)}{\partial V^i \partial V^j \partial V^k} X^iY^jZ^k$$

is the *Cartan tensor* of F. $D_X^V Y$ is the *covariant derivative* with respect to the reference vector V. Note that $C_V(V,X,Y) = 0$ from the homogeneity of F ([BCS], [Sh]). In terms of the Chern connection, a geodesic γ satisfies $D_{\dot{\gamma}}^{\dot{\gamma}} \dot{\gamma} = 0$. If V is a geodesic field, i.e., all integral curves of a non-vanishing smooth vector field V are geodesics, then we have

$$D_V^V W = D_V^{g_V} W, \qquad D_W^V V = D_W^{g_V} V \tag{2.5}$$

for any differentiable vector field W, where D^{g_W} stands for the covariant derivative with respect to the Levi-Civita connection of the Riemannian metric g_V (Lemma 6.2.1, [Sh]). A Finsler space (M, F, m) is said to be a *Berwald space* if the connection coefficients Γ_{ij}^k of the Chern connection are constants on $T_xM \setminus \{0\}$ for every $x \in M$, which shows that the parallel translation along any geodesic with respect to the Chern connection preserves the Minkowski norms on T_xM for each $x \in M$ ([Sh]).

Given a non-vanishing vector field V on M, the *Riemannian curvature* R^V is defined by

$$R^V(X,Y)Z := D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X,Y]}^V Z,$$

for any vector fields X, Y, Z on M. R^V is independent of the choices of connections. For two linearly independent vectors $v, w \in T_xM \setminus \{0\}$, the *flag curvature* with the pole v is defined by

$$K(v,w) = \frac{g_V(R^V(v,w)w, v)}{g_V(v,v)g_V(w,w) - g_V(v,w)^2},$$

where $v \in T_xM$ is extended to a geodesic field V near x, i.e., $D_V^V V = 0$. The flag curvature $K(v,w)$ coincides with the sectional curvature of the 2-plane spanned by v, w with respect to the Riemannian metric g_V on a neighbourhood of x. The *Ricci curvature* is defined by

$$Ric(v) := \sum_{i=1}^{n-1} K(v, e_i),$$

where $e_1, \cdots, e_{n-1}, \frac{v}{F(v)}$ form the orthonomal basis of T_xM with respect to g_V.

For any $v \in T_xM \setminus \{0\}$, let $\eta(t)$ be a geodesic with $\eta(0) = x$, $\dot{\eta}(0) = v$ and $dm = \sigma(x)dx$. Then the *distortion* of F is defined by $\tau(x,v) := \log\left\{\frac{\sqrt{det(g_{ij}(x,v))}}{\sigma(x)}\right\}$, and *S-curvature* S is defined as a rate of change of the distortion τ along the geodesic $\gamma(t)$, i.e.,

$$S(x,v) = \frac{d}{dt}\tau(\eta(t), \dot{\eta}(t))|_{t=0}.$$

Next we recall the definition of the weighted Ricci curvature on a Finsler manifold, which was introduced by S.Ohta ([Oh1]).

Definition 2.1. *([Oh1]) Given a vector $v \in T_xM$, let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic with $\eta(0) = x$ and $\dot{\eta}(0) = v$. We set $dm = e^{-\Psi(\eta(t))} vol_{\dot{\eta}}$ along η, where $vol_{\dot{\eta}}$ is the volume form of $g_{\dot{\eta}}$. Define the weighted Ricci curvature involving a parameter $N \in (n, \infty)$ by*

$$Ric_N(v) := Ric(v) + (\Psi \circ \eta)''(0) - \frac{(\Psi \circ \eta)'(0)^2}{(N-n)}.$$

Also define $Ric_\infty(v) := Ric(v) + (\Psi \circ \eta)''(0)$ and $Ric_n(v) := \lim_{N \to n} Ric_N$ as limits. Finally, for any $\lambda \geq 0$ and $N \in [n, \infty]$, define $Ric_N(\lambda v) := \lambda^2 Ric(v)$.

We say that $\mathrm{Ric}_N \geq K$ for some $K \in \mathbb{R}$ if $\mathrm{Ric}_N(v) \geq KF^2(v)$ for all $v \in TM$. We remark that $(\Psi \circ \eta)'(0) = S(x, v)$ and $(\Psi \circ \eta)''(0)$ is exactly the change rate of the S-curvature along the geodesic $\eta(t)$. The following point-wise Bochner-Weitzenböck formula is very important to derive the gradient estimates and study the structure of manifolds.

Theorem 2.1. *([OS2]) Given $u \in C^\infty(M)$, we have*

$$\Delta^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) = D(\Delta u)(\nabla u) + Ric_\infty(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)} \tag{2.6}$$

as well as

$$\Delta^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) \geq D(\Delta u)(\nabla u) + Ric_N(\nabla u) + \frac{(\Delta u)^2}{N} \tag{2.7}$$

for $N \in [n, \infty]$, point-wise on $M_u = \{x \in M | du(x) \neq 0\}$, where $\|\nabla^2 u\|^2_{HS(\nabla u)}$ stands for the Hilbert-Schmidt norm with respect to $g_{\nabla u}$.

In fact, we have a more refined inequality than (2.7) (Theorem 2.2, [WX]), that is,

$$\begin{aligned}\Delta^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) &\geq D(\Delta u)(\nabla u) + \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N} \\ &\quad + \frac{N}{N-1}\left(\frac{\Delta u}{N} - \frac{D(F^2(x, \nabla u))(\nabla u)}{2F^2(x, \nabla u)}\right)^2.\end{aligned} \tag{2.8}$$

All the above inequalities also hold on M in the integral sense ([OS2], [WX]).

2.3. Gradient and p-Laplacian

Let $J : T^*M \to TM$ be the *Legendre transform* associated with F and its dual norm F^*, that is, a transformation J from T^*M to TM, defined locally by $J(x,\xi) = J^i(x,\xi)\frac{\partial}{\partial x^i}$, $J^i(x,\xi) := \frac{1}{2}[F^{*2}(x,\xi)]_{\xi_i}(x,\xi)$, is sending $\xi \in T^*M$ to a unique element $y \in T_xM$ such that $F(x,y) = F^*(x,\xi)$ and $\xi(y) = F^2(y)$ ([Sh]). Let

$$(g^*_\xi)_{ij} := g^*_{ij}(x,\xi) = \frac{1}{2}[F^{*2}(x,\xi)]_{\xi_i\xi_j}.$$

One can verify that $g^*_{ij}(x,\xi) = g^{ij}(x,y)$, where $(g^{ij}(x,y)) = (g_{ij}(x,y))^{-1}$. If F is uniformly smooth and convex, then, by (1.1), (g^{ij}) is uniformly elliptic in the sense that there exist two constants $\tilde{\kappa} = (\kappa^*)^{-1}$ and $\tilde{\kappa}^* = \kappa^{-1}$ such that for any $x \in M$, $\xi \in T^*_xM \setminus \{0\}$ and $\eta \in T^*_xM$,

$$\tilde{\kappa}^* F^{*2}(x,\eta) \le g^*_\xi(\eta,\eta) = g^{ij}(x,J(\xi))\eta_i\eta_j \le \tilde{\kappa}F^{*2}(x,\eta). \tag{2.9}$$

In particular, F is Riemannian if and only if $\tilde{\kappa}^* = 1$ if and only if $\tilde{\kappa} = 1$.

For a smooth function $u : M \to \mathbb{R}$, the *gradient vector* $\nabla u(x)$ of u is defined by $\nabla u := J(du) \in T_xM$. Obviously, $\nabla u = 0$ if $du = 0$. In a local coordinate system, we can reexpress ∇u as

$$\nabla u := \begin{cases} g^{ij}(\nabla u)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j} & x \in M_u, \\ 0 & x \in M \setminus M_u, \end{cases}$$

where $M_u = \{x \in M | du(x) \ne 0\}$. In general, ∇u is only continuous on M, but smooth on M_u.

Given a weakly differentiable vector field V on M and a smooth measure $dm = \sigma(x)dx$, the *divergence* of V is defined by

$$\int_M \varphi \mathrm{div}(V)dm = -\int_M D\varphi(V)dm,$$

where $\varphi \in C^\infty_0(M)$. The *Finsler Laplacian* Δ of a function u on M is formally defined by $\Delta u = \mathrm{div}(\nabla u)$. Note that ∇u is weakly differentiable, the Finsler Laplacian should be understood in a weak sense by

$$\int_M \varphi\Delta u dm = -\int_M D\varphi(\nabla u)dm$$

for $u \in W^{1,2}(M)$ and $\varphi \in C_0^\infty(M)$ ([Sh]). If we write the measure $dm = \sigma(x)dx$, then

$$\Delta u = \frac{1}{\sigma}\frac{\partial}{\partial x^i}\left(\sigma g^{ij}(\nabla u)\frac{\partial u}{\partial x^j}\right) \tag{2.10}$$

locally on M_u. The *Hessian* of u is defined by

$$\nabla^2 u(X,Y) = g_{\nabla u}(D_X^{\nabla u}\nabla u, Y) \tag{2.11}$$

for any $X, Y \in TM$, where $D^{\nabla u}$ is the covariant differentiation with respect to the Chern connection of F with the reference vector ∇u (see §2.2). It is easy to check that $\nabla^2 u(X,Y)$ is symmetric with respect to X and Y.

If the vector field V is smooth, then one can introduce the *weighted gradient vector* and the *weighted Laplacian* on a weighted Riemannian manifold (M, g_V) given by

$$\nabla^V u := \begin{cases} g^{ij}(x,V)\frac{\partial u}{\partial x^j}\frac{\partial}{\partial x^i} & \text{for} \quad V \in T_xM \setminus \{0\}, \\ 0 & \text{for} \quad V = 0, \end{cases} \tag{2.12}$$

and $\Delta^V u := \operatorname{div}(\nabla^V u)$ respectively. We remark that $\nabla^{\nabla u}u = \nabla u$ and $\Delta^{\nabla u}u = \Delta u$. Moreover, it is easy to see that $\Delta u = \operatorname{tr}_{\nabla u}\nabla^2 u - S(\nabla u)$ on M_u.

Likewise, the *Finsler p-Laplacian* is defined by

$$\int_M \varphi \Delta_p u dm = -\int_M F^{p-2}(\nabla u) D\varphi(\nabla u) dm.$$

It follows from the variation of the energy functional. It is easy to check that

$$\Delta_p u = \operatorname{div}\left[F^{p-2}(x,\nabla u)\nabla u\right] = F^{p-2}(\nabla u)[\Delta u + (p-2)H_u] \tag{2.13}$$

on M_u, where $H_u := \frac{\nabla^2 u(\nabla u, \nabla u)}{F^2(\nabla u)}$. Obviously, if $p = 2$, then Δ_p is the Finsler Laplacian Δ. We say that u is a *p-harmonic function* on M if u is a weak solution of $\Delta_p u = 0$. In particular, when $p = 2$, it is exactly a harmonic function on M ([ZX], [Xc]).

For any $\eta \in C^2(M)$, the linearization of Δ_p on M_u is given by

$$\begin{aligned} \mathcal{L}_u(\eta) &= \operatorname{div}\left\{F^{p-2}(\nabla u)\left[\nabla^{\nabla u}\eta + (p-2)F^{-2}(\nabla u)Du(\nabla^{\nabla u}\eta)\nabla u\right]\right\} \\ &= \operatorname{div}\left[F^{p-2}(\nabla u)h_u(\nabla^{\nabla u}\eta)\right], \end{aligned} \tag{2.14}$$

where $h_u = id + (p-2)\frac{Du \otimes \nabla u}{F^2(\nabla u)}$ ([Xia2]). Obviously, $\mathcal{L}_u(u) = (p-1)\Delta_p u$. If $p = 2$, then $\mathcal{L}_u$ is reduced to the weighted Laplacian $\Delta^{\nabla u}$.

Let $W^{1,p}(M)(p>1)$ be the space of functions $u \in L^p(M)$ with $\int_M [F(\nabla u)]^p dm < \infty$ and $W_0^{1,p}(M)(p>1)$ is the closure of $C_0^\infty(M)$ under the (absolutely homogeneous) norm

$$\|u\|_{1,p} := \|u\|_{L^p(M)} + \frac{1}{2}\|F(\nabla u)\|_{L^p(M)} + \frac{1}{2}\|\overleftarrow{F}(\overleftarrow{\nabla} u)\|_{L^p(M)}, \tag{2.15}$$

where $\overleftarrow{\nabla} u$ is the gradient of u with respect to the reverse metric $\overleftarrow{F}$. In fact, $\overleftarrow{F}(\overleftarrow{\nabla} u) = F(\nabla(-u))$. Then $W^{1,p}(M)$ is a Banach space with respect to the norm $\|\cdot\|_{1,p}$. We define the energy functional $E : W^{1,p}(M) \setminus \{0\} \to [0,+\infty)$ by

$$E(u) := \frac{\int_M [F^*(x,Du)]^p dm}{\int_M |u|^p dm}. \tag{2.16}$$

Note that E is C^1 on $W^{1,p}(M) \setminus \{0\}$. It is easy to check that $d_u E = 0$ if and only if u satisfies (1.2) in a weak sense, i.e.,

$$\int_M D\varphi[F^{p-2}(\nabla u)\nabla u]dm = \lambda_p \int_M \varphi |u|^{p-2} u dm \tag{2.17}$$

for any $\varphi \in W_0^{1,p}(M)$, where $\lambda_p = E(u)$. In this case, λ_p is called an *eigenvalue* of Δ_p or a *p-eigenvalue*, and u is called a *p-eigenfunction* of Δ_p corresponding to λ_p. If M is compact and ∂M (possible $\partial M = \emptyset$) is smooth, then there exists a p-eigenfunction u with $\|u\|_{L^p(M)} = 1$, which minimizes the energy functional $E(u)$. In this case, we call $\lambda_{1,p} := \inf E(u)$ the *first eigenvalue* of Δ_p or *first p-eigenvalue*, and a critical point u a *first p-eigenfunction* on M corresponding to $\lambda_{1,p}$. Further, $u \in C^{1,\alpha}(M) \cap W^{2,2}_{loc}(M) \cap W^{1,p}(M)$ if $p \geq 2$ and $u \in C^{1,\alpha}(M) \cap W^{2,p}_{loc}(M) \cap W^{1,p}(M)$ if $1 < p < 2$. In particular, $u \in L^\infty(M)$. Moreover, u is smooth on the set M_u for $p = 2$ or $M_u \setminus M_0$ for $p > 1$. If $x \in M_u \cap M_0$, then $u(x)$ is of $C^{3,\alpha}$ when $p > 2$ and of $C^{2,\alpha}$ when $1 < p < 2$, where $M_0 := \{x \in M | u(x) = 0\}$ (see [GS], [Xia2] for more details).

For a noncompact Finsler manifold (M, F, m) (it needs not be complete), we define the *first p-eigenvalue* on M by

$$\lambda_{1,p} := \inf_{\Omega} \lambda_{1,p}(\Omega) = \inf_{\Omega} \inf_{u|_{\partial\Omega}=0} \frac{\int_\Omega [F^*(x,Du)]^p dm}{\int_\Omega |u|^p dm},$$

where Ω runs through all the compact subdomains with C^1 boundary in M, and the associating eigenfunctions are said to be the *first p-eigenfunctions* on M. Obviously, for an exhaustion $\Omega_1, \Omega_2, \cdots$ of M such that $\bar{\Omega}_i \subset \Omega_{i+1}$ for all $i \geq 1$ and $M = \cup_{i=1}^{\infty} \Omega_i$, $\{\lambda_{1,p}(\Omega_i)\}$ is a decreasing sequence with respect to $\{\Omega_i\}$. Consequently, $\lambda_{1,p} = \lim_{i\to\infty} \lambda_{1,p}(\Omega_i)$, which is independent of the choice of $\{\Omega_i\}$. In this case, if F satisfies the uniform convexity and uniform smoothness, then the p-eigenfunctions have the same regularity as the compact case from the proof of Theorem 1.1 in [Xia2].

3. Local gradient estimates for Finsler p-eigenfunctions

Let u be a positive p-eigenfunction in the forward geodesic ball $B_{2R} := B^+_{2R}(x)$ for any $x \in M$, namely, (2.17) holds on B_{2R}. Then $u \in C^{1,\alpha}(B_{2R}) \cap W^{2,2}_{loc}(B_{2R})$ if $p \geq 2$ and $u \in C^{1,\alpha}(B_{2R}) \cap W^{2,p}_{loc}(B_{2R})$ if $1 < p \leq 2$. Moreover, $u \in L^{\infty}(B_{2R})$ and u is smooth on the set $M_u \cap B_{2R}$ (see §2.3).

Denote $v = (p-1)\log u$. Then $M_u = M_v$ and $\nabla v = \frac{p-1}{u}\nabla u$. For any $\varphi \in W^{1,p}_0(B_{2R}) \cap L^{\infty}(B_{2R})$, we have $\frac{\varphi}{u^{p-1}} \in W^{1,p}_0(B_{2R}) \cap L^{\infty}(B_{2R})$ from the regularity and boundness of u. Thus it follows from (2.17) that

$$\begin{aligned}\int_M D\varphi\left[F^{p-2}(\nabla v)\nabla v\right] dm &= \int_M \frac{(p-1)^{p-1}}{u^{p-1}} D\varphi\left(F^{p-2}(\nabla u)\nabla u\right) dm \\ &= \int_M (p-1)^{p-1} D\left(\frac{\varphi}{u^{p-1}}\right)\left(F^{p-2}(\nabla u)\nabla u\right) dm \\ &\quad + \int_M (p-1)^p \varphi \frac{F^p(\nabla u)}{u^p} dm \\ &= \int_M \varphi\left[F^p(\nabla v) + (p-1)^{p-1}\lambda_p\right] dm. \end{aligned} \tag{3.1}$$

Let $f(x) := F^2(x, \nabla v)$. Then $f \in W^{1,2}_{loc}(B_{2R}) \cap C^{\alpha}(B_{2R})$ if $p \geq 2$ and $f \in W^{1,p}_{loc}(B_{2R}) \cap C^{\alpha}(B_{2R})$ if $1 < p < 2$. Moreover, f is smooth on $M_v \cap B_{2R}$. Since the LHS of (3.1) is equal to

$$\begin{aligned}&\int_M D\varphi\left(f^{p/2-1}\nabla v\right) dm \\ &\qquad = -\int_M \varphi\left\{\frac{1}{2}(p-2)f^{p/2-2}Df(\nabla v) + f^{p/2-1}\Delta v\right\} dm,\end{aligned}$$

(3.1) is equivalent to

$$
\begin{aligned}
(3.2)\qquad & -\int_M \varphi f^{p/2-1}\Delta v dm \\
&= \int_M \varphi \left[\frac{1}{2}(p-2)f^{p/2-2}Df(\nabla v) + f^{p/2} + (p-1)^{p-1}\lambda_p\right] dm.
\end{aligned}
$$

Note that $df(\nabla v) = g_{\nabla v}(\nabla^{\nabla v} f, \nabla v)$. It follows from Lemma 3.5 and the proof of Theorem 3.6 in [OS2] that $\nabla^{\nabla v} f = 0$ and $\Delta v = 0$ a.e. on $f^{-1}(0) = M \setminus M_v$. Therefore the both sides of (3.2) are actually integrated over $M_v \cap B_{2R}$. Thus (3.2) implies that on $M_v \cap B_{2R}$,

$$
(3.3)\qquad \Delta v = -\frac{1}{2}(p-2)f^{-1}Df(\nabla v) - f - (p-1)^{p-1}\lambda_p f^{-p/2+1}.
$$

Lemma 3.1. *For any $N \in [n, \infty]$, we have*

$$
\begin{aligned}
\mathcal{L}_v(f) &= \frac{1}{2}(p-2)f^{p/2-2}\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)} + 2f^{p/2-1}\|\nabla^2 v\|^2_{HS(\nabla v)} \\
(3.4)\qquad &\quad + 2f^{p/2-1}Ric_\infty(\nabla v) - pf^{p/2-1}Df(\nabla v)
\end{aligned}
$$

as well as

$$
\begin{aligned}
\mathcal{L}_v(f) &\geq -\frac{1}{2}f^{p/2-2}\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)} + 2f^{p/2-1}Ric_N(\nabla v) + \frac{2}{N}f^{p/2+1} \\
(3.5)\qquad &\quad - \frac{p(N-2)+4}{N}f^{p/2-1}Df(\nabla v)
\end{aligned}
$$

point-wise on $M_v \cap B_{2R}$, where $\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)} = g_{\nabla v}(\nabla^{\nabla v} f, \nabla^{\nabla v} f)$.

Proof. Note that $f > 0$ and f, v are smooth on $M_v \cap B_{2R}$. By (2.14) and the definition of h_v, one obtains

$$
\begin{aligned}
\mathcal{L}_v(f) &= \operatorname{div}\left[F^{p-2}(\nabla v)h_v(\nabla^{\nabla v} f)\right] \\
&= f^{p/2-1}\Delta^{\nabla v} f + \frac{1}{2}(p-2)f^{p/2-2}Df(\nabla^{\nabla v} f) \\
&\quad + \frac{1}{2}(p-2)(p-4)f^{p/2-3}[Df(\nabla v)]^2 \\
&\quad + (p-2)f^{p/2-2}Dv(\nabla^{\nabla v} f)\Delta v \\
(3.6)\qquad &\quad + (p-2)f^{p/2-2}D(Dv(\nabla^{\nabla v} f))(\nabla v),
\end{aligned}
$$

where we used $Dv(\nabla^{\nabla v} f) = Df(\nabla v)$. By Theorem 2.1 and (3.3), we have

$$
\begin{aligned}
f^{p/2-1}\Delta^{\nabla v} f &= 2f^{p/2-1}\left(D(\Delta v)(\nabla v) + Ric_\infty(\nabla v) + \|\nabla^2 v\|^2_{HS(\nabla v)}\right) \\
&= -f^{p/2-1}\Big\{D\Big[(p-2)f^{-1}Df(\nabla v) + 2f \\
&\qquad + 2(p-1)^{p-1}\lambda_p f^{-p/2+1}\Big](\nabla v)\Big\} \\
&\quad + 2f^{p/2-1}\left(Ric_\infty(\nabla v) + \|\nabla^2 v\|^2_{HS(\nabla v)}\right) \\
&= (p-2)f^{p/2-3}(Df(\nabla v))^2 \\
&\quad - (p-2)f^{p/2-2}D(Df(\nabla v))(\nabla v) - 2f^{p/2-1}Df(\nabla v) \\
&\quad + (p-2)(p-1)^{p-1}\lambda_p f^{-1}Df(\nabla v) \\
&\quad + 2f^{p/2-1}\left(Ric_\infty(\nabla v) + \|\nabla^2 v\|^2_{HS(\nabla v)}\right),
\end{aligned}
\tag{3.7}
$$

and the fourth term in the RHS of (3.6) is equal to

$$
\begin{aligned}
&(p-2)f^{p/2-2}Df(\nabla v)\Big[-\frac{1}{2}(p-2)f^{-1}Df(\nabla v) - f \\
&\qquad - (p-1)^{p-1}\lambda_p f^{-p/2+1}\Big] \\
&= -\frac{1}{2}(p-2)Df(\nabla v)\Big[(p-2)f^{p/2-3}(Df(\nabla v)) \\
&\qquad + 2f^{p/2-1} + 2(p-1)^{p-1}\lambda_p f^{-1}\Big].
\end{aligned}
\tag{3.8}
$$

Plugging (3.7) and (3.8) into (3.6) and using $\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)} = Df(\nabla^{\nabla v} f)$ yield (3.4).

Next we prove the second assertion. It is clear if $N = \infty$. For $N \in (n, \infty)$, we need to estimate $\|\nabla^2 v\|^2_{HS(\nabla v)}$. Choose a local normal coordinate system $\{x^i\}$ with respect to $g_{\nabla v}$ at $x \in M_v \cap B_{2R}$ such that $\nabla v = F(\nabla v)\frac{\partial}{\partial x^1}$ and $\Gamma^i_{jk}(\nabla v(x)) = 0$ for all i, j, k. Thus, we have

$$
\begin{aligned}
\|\nabla^2 v\|^2_{HS(\nabla v)} = \sum_{i,j} v_{ij}^2 \geq \sum_i v_{ii}^2 \geq \frac{1}{n}\left(tr_{g_{\nabla v}}\nabla^2 v\right)^2 \\
= \frac{1}{n}\left[\Delta v + S(\nabla v)\right]^2.
\end{aligned}
\tag{3.9}
$$

Plugging (3.3) into the RHS of (3.9) and using the inequality $(a-b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$ with $\delta = (N-n)/n > 0$, one obtains

$$\|\nabla^2 v\|^2_{HS(\nabla v)} \geq \frac{1}{N}\left[f^2 + (p-2)Df(\nabla v)\right] - \frac{1}{N-n}S^2(\nabla v).$$

Combining this with (3.4) yields (3.5). The remaining case of $N = n$ is derived as the limit. □

Remark 3.1. *(3.4)-(3.5) can not be simply derived from the the formula of the weighted Riemannian manifold $(M, g_{\nabla v}, m)$. This is because $Ric_\infty(\nabla v)$ and $\|\nabla^2 v\|^2_{HS(\nabla v)}$ are different from those for $g_{\nabla v}$ unless all integral curves of ∇v are geodesics.*

From now on, we assume that $\mathrm{Ric}_N \geq -K$, in particular, $\mathrm{Ric}_N(\nabla v) \geq -Kf$, where $N \in [n,\infty)$ and $K > 0$ are real numbers. For any nonnegative smooth function φ with a compact support in $B_{2R} \cap M_v$, by integrating (3.5) by parts, one obtains

$$\begin{aligned}
&\int_{B_{2R}\cap M_v} D\varphi\left[f^{p/2-1}h_v(\nabla^{\nabla v} f)\right]dm \\
&\quad \leq \frac{1}{2}\int_{B_{2R}\cap M_v} \varphi f^{p/2-2}\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)}dm \\
&\qquad + 2K\int_{B_{2R}\cap M_v} \varphi f^{p/2}dm - \frac{2}{N}\int_{B_{2R}\cap M_v} \varphi f^{p/2+1}dm \\
&\qquad + c_1\int_{B_{2R}\cap M_v} \varphi f^{p/2-1}Df(\nabla v)dm. \qquad (3.10)
\end{aligned}$$

where $c_1 := \frac{(N-2)p+4}{N}$ is a positive constant since $N \geq n \geq 2$.

To extend the above integral inequality to B_{2R}, we consider the function $f_\varepsilon = (f-\varepsilon)^+$ for $\varepsilon > 0$ and the nonnegative function $\eta \in C_0^\infty(B_{2R})$ with $0 \leq \eta \leq 1$. Note that $f_\varepsilon = f - \varepsilon, Df_\varepsilon = Df$ a.e. in $\{f > \varepsilon\}$ and $f_\varepsilon = 0, Df_\varepsilon = 0$ a.e. in $\{f \leq \varepsilon\}$. Choose $\varphi = f_\varepsilon^t \eta^2$ as the test function in (3.10), where $t > 1$

is to be determined later. Then we have

$$\begin{aligned}
&\int_{B_{2R}\cap\{f>\varepsilon\}} t\eta^2 f_\varepsilon^{t-1} Df\left[f^{p/2-1}h_v(\nabla^{\nabla v}f)\right]dm\\
&\quad+2\int_{B_{2R}\cap\{f>\varepsilon\}}\eta f_\varepsilon^t D\eta\left[f^{p/2-1}h_v(\nabla^{\nabla v}f)\right]dm\\
&\qquad\le\frac{1}{2}\int_{B_{2R}\cap\{f>\varepsilon\}}\eta^2 f_\varepsilon^t f^{p/2-2}\|\nabla^{\nabla v}f\|^2_{HS(\nabla v)}dm\\
&\qquad\quad+2K\int_{B_{2R}\cap\{f>\varepsilon\}}\eta^2 f_\varepsilon^t f^{p/2}dm\\
&\qquad\quad-\frac{2}{N}\int_{B_{2R}\cap\{f>\varepsilon\}}\eta^2 f_\varepsilon^t f^{p/2+1}dm\\
&\qquad\quad+c_1\int_{B_{2R}\cap\{f>\varepsilon\}}\eta^2 f_\varepsilon^t f^{p/2-1}Df(\nabla v)dm.
\end{aligned}\tag{3.11}$$

Assume that the Finsler metric F satisfies the uniform convexity and uniform smoothness. Then, by (2.9), we have

$$\tilde\kappa^* F^2(x,\nabla f)\le\|\nabla^{\nabla v}f\|^2_{HS(\nabla v)}=g^{ij}(x,\nabla v)f_if_j\le\tilde\kappa F^2(x,\nabla f).\tag{3.12}$$

From this and the Cauchy-Schwarz inequality, we get

$$|Df(\nabla^{\nabla v}\eta)|=|D\eta(\nabla^{\nabla v}f)|\le\tilde\kappa F(x,\nabla\eta)F(x,\nabla f).\tag{3.13}$$

Consequently,

$$\begin{aligned}
&f_\varepsilon^{t-1}Df\left[f^{p/2-1}h_v(\nabla^{\nabla v}f)\right]\\
&\quad=f_\varepsilon^{t-1}f^{p/2-1}\left[g^{ij}(\nabla v)f_if_j+(p-2)f^{-1}(g^{ij}(\nabla v)f_iv_j)^2\right]\\
&\quad\ge\begin{cases} f_\varepsilon^{t-1}f^{p/2-1}g^{ij}(\nabla v)f_if_j & \text{if } p\ge 2\\ (p-1)f_\varepsilon^{t-1}f^{p/2-1}g^{ij}(\nabla v)f_if_j & \text{if } 1<p\le 2\end{cases}\\
&\quad\ge c_2\tilde\kappa^* f_\varepsilon^{t-1}f^{p/2-1}F^2(\nabla f),
\end{aligned}\tag{3.14}$$

where $c_2 = \min\{1, p-1\}$. In a similar way, we have

$$
\begin{aligned}
&2\eta f_\varepsilon^t D\eta \left(f^{p/2-1} h_v(\nabla^{\nabla v} f) \right) \\
&\quad = 2\eta f_\varepsilon^t f^{p/2-1} \big[g^{ij}(\nabla v) f_i \eta_j \\
&\qquad\qquad + (p-2) f^{-1} (g^{ij}(\nabla v) f_i v_j)(g^{ij}(\nabla v) v_i \eta_j) \big] \\
&\quad \geq - \begin{cases} 2(p-1)\tilde{\kappa}\eta f_\varepsilon^t f^{p/2-1} F(\nabla f) F(\nabla \eta) & \text{if } \ p \geq 2 \\ 2(3-p)\tilde{\kappa}\eta f_\varepsilon^t f^{p/2-1} F(\nabla f) F(\nabla \eta) & \text{if } \ 1 < p \leq 2 \end{cases} \\
&\quad = -c_3 \tilde{\kappa} \eta f_\varepsilon^t f^{p/2-1} F(\nabla f) F(\nabla \eta), \qquad (3.15)
\end{aligned}
$$

where $c_3 = \max\{2(p-1), 2(3-p)\}$. Plugging (3.12)-(3.15) into (3.11) and passing ε to 0 yield

$$
\begin{aligned}
&c_2 \tilde{\kappa}^* t \int_{B_{2R}} \eta^2 f^{p/2+t-2} F^2(\nabla f) dm \\
&\qquad - c_3 \tilde{\kappa} \int_{B_{2R}} \eta f^{p/2+t-1} F(\nabla f) F(\nabla \eta) dm \\
&\quad \leq \frac{\tilde{\kappa}}{2} \int_{B_{2R}} \eta^2 f^{p/2+t-2} F^2(\nabla f) dm + 2K \int_{B_{2R}} \eta^2 f^{p/2+t} dm \\
&\qquad - \frac{2}{N} \int_{B_{2R}} \eta^2 f^{p/2+t+1} dm + c_1 \int_{B_{2R}} \eta^2 f^{(p-1)/2+t} F(\nabla f) dm, \qquad (3.16)
\end{aligned}
$$

where we used Cauchy-Schwarz's inequality $Df(\nabla v) \leq f^{1/2} F(\nabla f)$ in the last term. Now we choose a sufficiently large $t > \tilde{\kappa}/(c_2 \tilde{\kappa}^*) \geq 1$, then $c_2 \tilde{\kappa}^* t - \frac{1}{2}\tilde{\kappa} \geq \frac{1}{2} c_2 \tilde{\kappa}^* t$ and hence (3.16) becomes

$$
\begin{aligned}
&\frac{1}{2} c_2 \tilde{\kappa}^* t \int_{B_{2R}} \eta^2 f^{p/2+t-2} F^2(\nabla f) dm \\
&\quad \leq c_3 \tilde{\kappa} \int_{B_{2R}} \eta f^{p/2+t-1} F(\nabla f) F(\nabla \eta) dm + 2K \int_{B_{2R}} \eta^2 f^{p/2+t} dm \\
&\qquad - \frac{2}{N} \int_{B_{2R}} \eta^2 f^{p/2+t+1} dm + c_1 \int_{B_{2R}} \eta^2 f^{(p-1)/2+t} F(\nabla f) dm. \qquad (3.17)
\end{aligned}
$$

Let $c_4 := 2(c_3 \tilde{\kappa})^2 / c_2$ and $c_5 := 2c_1^2 / c_2$. Obviously c_4, c_5 are positive constants depending on $p, \tilde{\kappa}, N$. Then the first term of the RHS in (3.17) is less than or equal to

$$
\frac{c_2 \tilde{\kappa}^* t}{8} \int_{B_{2R}} \eta^2 f^{p/2+t-2} F^2(\nabla f) dm + \frac{c_4}{\tilde{\kappa}^* t} \int_{B_{2R}} f^{p/2+t} F^2(\nabla \eta) dm \qquad (3.18)
$$

and the fourth term of the RHS in (3.17) is less than or equal to

$$\begin{aligned} &\frac{c_2\tilde{\kappa}^* t}{8}\int_{B_{2R}}\eta^2 f^{p/2+t-2}F^2(\nabla f)dm+\frac{c_5}{\tilde{\kappa}^* t}\int_{B_{2R}}\eta^2 f^{p/2+t+1}dm \\ &\quad\le\frac{c_2\tilde{\kappa}^* t}{8}\int_{B_{2R}}\eta^2 f^{p/2+t-2}F^2(\nabla f)dm+\frac{1}{N}\int_{B_{2R}}\eta^2 f^{p/2+t+1}dm \end{aligned} \tag{3.19}$$

if we take $t \ge \max\{\tilde{\kappa}/(c_2\tilde{\kappa}^*), c_5N/\tilde{\kappa}^*\}$ large enough. Hence, for

$$t > \max\{\tilde{\kappa}/(c_2\tilde{\kappa}^*), c_5N/\tilde{\kappa}^*\} \ge 1,$$

it follows from (3.17) and (3.18)-(3.19) that

$$\begin{aligned} &\frac{1}{4}c_2\tilde{\kappa}^* t\int_{B_{2R}}\eta^2 f^{p/2+t-2}F^2(\nabla f)dm \\ &\quad\le\frac{c_4}{\tilde{\kappa}^* t}\int_{B_{2R}}f^{p/2+t}F^2(\nabla\eta)dm \\ &\qquad+2K\int_{B_{2R}}\eta^2 f^{p/2+t}dm-\frac{1}{N}\int_{B_{2R}}\eta^2 f^{p/2+t+1}dm. \end{aligned} \tag{3.20}$$

Recall that $F(\nabla f) = F^*(Df)$ and $F^*(\xi+\eta) \le F^*(\xi)+F^*(\eta)$ (cf. Lemma 1.2.2, [Sh]). By (3.20), there exist positive constants $c_i = c_i(p, \tilde{\kappa}, \tilde{\kappa}^*, N)$ ($i = 6, 7, 8$) depending only on $p, \tilde{\kappa}, \tilde{\kappa}^*, N$ such that

$$\begin{aligned} &\int_{B_{2R}}F^{*2}\left(D(\eta f^{p/4+t/2})\right)dm \\ &\quad\le 2\int_{B_{2R}}f^{p/2+t}F^{*2}(D\eta)dm \\ &\qquad+2\left(\frac{p}{4}+\frac{t}{2}\right)^2\int_{B_{2R}}\eta^2 f^{p/2+t-2}F^{*2}(Df)dm \\ &\quad\le\frac{c_6}{4}\int_{B_{2R}}f^{p/2+t}F^{*2}(D\eta)dm+\frac{c_7}{4}Kt\int_{B_{2R}}\eta^2 f^{p/2+t}dm \\ &\qquad-\frac{c_8}{4}t\int_{B_{2R}}\eta^2 f^{p/2+t+1}dm \end{aligned} \tag{3.21}$$

for $t > \max\{\tilde{\kappa}/(c_2\tilde{\kappa}^*), c_5N/\tilde{\kappa}^*\}$ large enough.

Summing up, one obtains the following lemma.

Lemma 3.2. *Let (M, F, m) be an $n(\ge 2)$-dimensional forward complete Finsler measure space equipped with a uniformly convex and uniformly smooth Finsler structure F and a smooth measure m. Assume that $Ric_N \ge$*

$-K$ for some $N \in [n, \infty)$ and $K \geq 0$ and u is a positive $p(>1)$-eigenfunction corresponding to the eigenvalue λ_p in a forward geodesic ball B_{2R}. Let $v := (p-1)\log u$ and $f := F^2(\nabla v)$. Then there exist positive constants $c_i = c_i(p, \tilde{\kappa}, \tilde{\kappa}^*, N)(6 \leq i \leq 8)$ such that (3.21) holds for $t > \max\{\tilde{\kappa}/(c_2\tilde{\kappa}^*), c_5 N/\tilde{\kappa}^*\}$ large enough.

The following local Sobolev inequality, which was due to C.Xia, plays an important role in the subsequent arguments.

Lemma 3.3. *([Xc]) Let (M, F, m) be an n-dimensional forward complete Finsler space equipped with a uniformly convex and uniformly smooth Finsler structure F. Assume that $Ric_N \geq -K$ for some $K \geq 0$ and $N \in [n, \infty)$. Then there exist constants $\nu > 2$ and $c_0 = c_0(\kappa, \kappa^*, N)$ depending only on κ, κ^* and N, such that*

$$\begin{aligned}\left(\int_{B_R} |u|^{\frac{2\nu}{\nu-2}} dm\right)^{\frac{\nu-2}{\nu}} &\leq e^{c_0(1+\sqrt{K}R)} R^2 m(B_R)^{-\frac{2}{\nu}} \\ &\times \int_{B_R} \{F^{*2}(x, Du) + R^{-2}u^2\} dm. \qquad (3.22)\end{aligned}$$

for $B_R \subset M$ and $u \in W^{1,2}_{loc}(M)$.

Let $\tau := \frac{\nu}{\nu-2}$. Taking $u = \eta f^{p/4+t/2}$ in (3.22) and using (3.21), one obtains

$$\begin{aligned}&\left(\int_{B_{2R}} \eta^{2\tau} f^{\tau(p/2+t)} dm\right)^{\frac{1}{\tau}} \\ &\leq 4e^{c_0(1+2\sqrt{K}R)} R^2 m(B_{2R})^{-\frac{2}{\nu}} \\ &\quad \times \left\{\int_{B_{2R}} F^{*2}\left(D(\eta f^{p/4+t/2})\right) dm + \frac{1}{4}R^{-2} \int_{B_{2R}} \eta^2 f^{p/2+t} dm\right\} \\ &\leq e^{2c_0(1+\sqrt{K}R)} m(B_{2R})^{-\frac{2}{\nu}} \\ &\quad \times \left\{c_6 R^2 \int_{B_{2R}} f^{p/2+t} F^{*2}(D\eta) dm - c_8 t R^2 \int_{B_{2R}} \eta^2 f^{p/2+t+1} dm\right. \\ &\quad \left. + \max\{c_7, 1\} t (1+\sqrt{K}R)^2 \int_{B_{2R}} \eta^2 f^{p/2+t} dm\right\}. \qquad (3.23)\end{aligned}$$

Note that $\tilde{\kappa}^*, \tilde{\kappa}$ only depend on κ^* and κ. In the following, we always denote the positive constants c_i as $c_i = c_i(p, \kappa, \kappa^*, N)$ depending on p, κ, κ^*, N, where $i = 1, 2, \cdots$. Now we use (3.23) to prove the following lemma and Theorem 1.1.

Lemma 3.4. *There exits a positive constant* $c = c(p, \kappa, \kappa^*, N)$ *such that for* $t_0 = c_9(1 + \sqrt{K}R)$ *and* $t_1 = \tau(p/2 + t_0)$*, we have* $f \in L^{t_1}(B_{\frac{3}{2}R})$ *with*

$$\|f\|_{L^{t_1}(B_{\frac{3}{2}R})} \leq \frac{c(1+\sqrt{K}R)^2}{R^2} m(B_{2R})^{\frac{1}{t_1}}, \tag{3.24}$$

where $c_9 := \max\{\tilde{\kappa}/(c_2\tilde{\kappa}^*), c_5 N/\tilde{\kappa}^*\} \geq 1$.

Proof. Taking $t = t_0$ in (3.23), we have

$$\begin{aligned} &\left(\int_{B_{2R}} \eta^{2\tau} f^{\tau(p/2+t_0)} dm\right)^{\frac{1}{\tau}} \\ &\quad \leq e^{c_{10}t_0} m(B_{2R})^{-\frac{2}{\nu}} \left\{ c_6 R^2 \int_{B_{2R}} f^{p/2+t_0} F^{*2}(D\eta) dm \right. \\ &\qquad \left. - c_8 t_0 R^2 \int_{B_{2R}} \eta^2 f^{p/2+t_0+1} dm + c_{11} t_0^3 \int_{B_{2R}} \eta^2 f^{p/2+t_0} dm \right\}. \end{aligned} \tag{3.25}$$

Now we decompose the region B_{2R} into two subregions, one is $\{f \geq \frac{2c_{11}}{c_8}(\frac{t_0}{R})^2\}$, another is the complement of the first subregion in B_{2R}. Thus,

$$\begin{aligned} c_{11} t_0^3 \int_{B_{2R}} \eta^2 f^{p/2+t_0} dm &\leq \frac{1}{2} c_8 t_0 R^2 \int_{B_{2R}} \eta^2 f^{p/2+t_0+1} dm \\ &\quad + c_{12} t_0^3 \left(\frac{t_0}{R}\right)^{p+2t_0} m(B_{2R}). \end{aligned} \tag{3.26}$$

For the first term of the RHS in (3.25), we let $\eta = \psi^{p/2+t_0+1}$ with $\psi(z) = \tilde{\psi}(d_F(x,z)) \in C_0^\infty(B_{2R})$ satisfying

$$0 \leq \tilde{\psi} \leq 1, \quad \tilde{\psi} = 1 \ \text{ in } \ [0, 3R/2), \quad |\tilde{\psi}'| \leq \frac{c_{13}}{R}.$$

Note that $F^*(Dd_F(x,\cdot)) = 1$ a.e.in B_{2R}. Thus ψ satisfies $0 \leq \psi \leq 1$, $\psi = 1$ in $B_{3R/2}$ and $F^*(D\psi) \leq \frac{c_{13}\Lambda}{R}$. Since F is uniformly convex and smooth, we have $1 \leq \Lambda \leq \min\{\sqrt{\kappa}, \sqrt{1/\kappa^*}\}$. Hence, $F^*(D\psi) \leq \frac{c'_{13}}{R}$, where $c'_{13} = c'_{13}(p, \kappa, \kappa^*, N)$, and $c_6 R^2 F^{*2}(D\eta) \leq c_{14} t_0^2 \eta^{\frac{p+2t_0}{p/2+t_0+1}}$. By Hölder's and Young's inequalities, one obtains

$$\begin{aligned}
&c_6R^2\int_{B_{2R}} f^{p/2+t_0}F^{*2}(D\eta)dm\\
&\quad\le c_{14}t_0^2\int_{B_{2R}} f^{p/2+t_0}\eta^{\frac{p+2t_0}{p/2+t_0+1}}dm\\
&\quad\le c_{14}t_0^2\left(\int_{B_{2R}}\eta^2 f^{p/2+t_0+1}dm\right)^{\frac{p/2+t_0}{p/2+t_0+1}}\cdot m(B_{2R})^{\frac{1}{p/2+t_0+1}}\\
&\quad\le \frac{1}{2}c_8t_0R^2\int_{B_{2R}}\eta^2 f^{p/2+t_0+1}dm + c_{15}^{p/2+t_0}t_0^{p/2+t_0+2}R^{-(p+2t_0)}m(B_{2R}).
\end{aligned}$$

Plugging the above inequality and (3.26) into (3.25) yields

$$\begin{aligned}
&\left(\int_{B_R}\eta^{2\tau}f^{\tau(p/2+t_0)}dm\right)^{\frac{1}{\tau}}\\
&\quad\le e^{c_{10}t_0}m(B_{2R})^{1-\frac{2}{\nu}}\left\{c_{12}t_0^3\left(\frac{t_0}{R}\right)^{p+2t_0}+c_{15}^{p/2+t_0}t_0^{p/2+t_0+2}R^{-(p+2t_0)}\right\}\\
&\quad\le c_{16}^{p/2+t_0}e^{c_{10}t_0}t_0^3\left(\frac{t_0}{R}\right)^{p+2t_0}m(B_{2R})^{1-\frac{2}{\nu}}.
\end{aligned}$$

Note that $t_0^{\frac{3}{t_0+p/2}}\le e^3$. Taking the $\frac{1}{p/2+t_0}$-th power on both sides of the above inequality, we have

$$\|f\|_{L^{t_1}(B_{\frac{3}{2}R})}\le c_{17}\left(\frac{t_0}{R}\right)^2 m(B_{2R})^{\frac{1}{t_1}},$$

which implies (3.24). This finishes the proof. □

Proof of Theorem 1.1. By (3.23), we have

$$\begin{aligned}
&\left(\int_{B_{2R}}\eta^{2\tau}f^{\tau(p/2+t)}dm\right)^{\frac{1}{\tau}}\\
&\quad\le e^{c_{10}t_0}m(B_{2R})^{-\frac{2}{\nu}}\Big\{c_6R^2\int_{B_{2R}}f^{p/2+t}F^{*2}(D\eta)dm\\
&\qquad + c_{11}t_0^2t\int_{B_{2R}}\eta^2 f^{p/2+t}dm\Big\}.
\end{aligned}\tag{3.27}$$

Let t_0, t_1 be those given in Lemma 3.4 and $t_{k+1} = \tau t_k$. Moreover we choose $R_k = R + \frac{R}{2^k}$ and $\eta_k \in C_0^\infty(B_{R_k})$ satisfying

$$0 \leq \eta_k \leq 1, \qquad \eta_k = 1 \text{ in } B_{R_{k+1}}, \qquad F^*(x, D\eta_k) \leq \tilde{c}\frac{2^k}{R}, \quad k = 1, 2, \cdots,$$

where $\tilde{c}$ is a certain constant depending on the reversibility constant Λ on $B_{3R/2}$. Denote $c_{18} = \max\{c_6\tilde{c}^2, c_{11}\}$. Taking $p/2 + t = t_k, \eta = \eta_k$ in (3.27), one obtains

$$\begin{aligned}\|f\|_{L^{t_{k+1}}(B_{R_{k+1}})} &\leq \left(c_{18}e^{c_{10}t_0}\right)^{\frac{1}{t_k}} m(B_{2R})^{-\frac{2}{\nu t_k}} (4^k + t_0^2 t_k)^{\frac{1}{t_k}} \|f\|_{L^{t_k}(B_{R_k})} \\ &= \left(c_{18}e^{c_{10}t_0}\right)^{\frac{1}{t_k}} m(B_{2R})^{-\frac{2}{\nu t_k}} \left(4^k + t_0^2 \tau^{k-1} t_1\right)^{\frac{1}{t_k}} \|f\|_{L^{t_k}(B_{R_k})}.\end{aligned}$$

Note that $\sum_k \frac{1}{t_k} = \frac{\nu}{2t_1}$ and $\sum_k \frac{k}{t_k}$ converges. By the standard Moser iteration and using Lemma 3.4, we get

$$\begin{aligned}&\|F^2(x, \nabla v)\|_{L^\infty(B_R)} = \|f\|_{L^\infty(B_R)} \\ &\leq c_{19}\left(c_{18}e^{c_{10}t_0}\right)^{\sum_k \frac{1}{t_k}} m(B_{2R})^{-\frac{2}{\nu}\sum_k \frac{1}{t_k}} \left(t_0^3\right)^{\sum_k \frac{1}{t_k}} \|f\|_{L^{t_1}(B_{R_1})} \\ &\leq C\frac{(1 + \sqrt{K}R)^2}{R^2}.\end{aligned}$$

By the same arguments as above, the above inequality also holds for $\overleftarrow{F}(x, \overleftarrow{\nabla} v)$. Since $F(x, \nabla(-v)) = \overleftarrow{F}(x, \overleftarrow{\nabla} v)$, we get the conclusion. □

4. Global gradient estimates for Finsler p-eigenfunctions

In this section, we will focus on the global gradient estimate for p-eigenfunctions and prove Theorem 1.2 based on Theorem 1.1. For this, we need a upper bound estimate of the first eigenvalue $\lambda_{1,p}$.

Recall that $B_R := B_R^+(x)$ is the forward geodesic ball of radius R centered at any point $x \in M$ and $D_x = M \setminus (\{x\} \cup Cut(x))$, where $\mathrm{Cut}(x)$ is the cut locus of x, which has zero Hausdorff measure in M. Then, for any $z \in D_x$, we can choose the geodesic polar coordinates (r, θ) centered at x such that $r(z) = F(v)$ and $\theta^\alpha(z) = \theta^\alpha(\frac{v}{F(v)})$, where $r(z) = d_F(x, z)$ is the distance function on M from a fixed point $x \in M$ and $v = \exp^{-1}(z) \in T_x(M) \setminus \{0\}$. It is well known that the distance function r starting from $x \in M$ is smooth on D_x and $F(\nabla r) = 1$ ([BCS], [Sh]). By Gauss Lemma (Lemma 6.1.1, [BCS]), the unit radial coordinate vector $\frac{\partial}{\partial r}$ is orthogonal to coordinate vectors $\frac{\partial}{\partial \theta^\alpha}$ with respect to $g_{\nabla r}$ for $1 \leq \alpha \leq n-1$. So, we can write $dm|_{\exp_x(rv_0)} = \sigma(r,\theta)drd\theta$, where $v_0 = \frac{v}{F(v)} \in I_x = \{v \in T_xM | F(v) = 1\}$. Set $\mathcal{D}_x(r) = \{v_0 \in I_x | rv_0 \in \exp^{-1}(D_x \cap B_R)\}$ and $\tau(r) := \int_{\mathcal{D}_x(r)} \sigma(r,\theta)d\theta$. The volume of B_R with respect to dm is given by

$$\begin{aligned} m(B_R) &= \int_{B_R} dm = \int_{B_R \cap D_x} dm \\ &= \int_0^R dr \int_{\mathcal{D}_x(r)} \sigma(r,\theta)d\theta = \int_0^R \tau(r)dr. \end{aligned} \tag{4.1}$$

For any Finsler manifold (M,F,m) with $\mathrm{Ric}_N \geq -K$, $K \geq 0$, and any $0 < r_1 < r_2 < R$, by Laplacian comparison Theorem, we have the following inequality (Proposition 5.1, [Xia3])

$$\frac{m(B_{r_2})}{m(B_{r_1})} \leq \left(\frac{r_2}{r_1}\right)^N e^{r_2\sqrt{(N-1)K}}. \tag{4.2}$$

In particular, for any $r \geq 1$, there is a positive constant $c = m(B_1)$ such that

$$m(B_r) \leq cr^N e^{r\sqrt{(N-1)K}}. \tag{4.3}$$

Proposition 4.1. *Let (M,F,m) be a forward complete Finsler manifold with finite reversibility Λ and infinite volume, i.e., $m(M) = +\infty$. Assume that $m(B_r) \leq cr^k e^{ar}$ for any $r \geq r_0 > 0$, where $c > 0$, $k \geq 0$ and $a \geq 0$ are constants independent of r, and $\lambda_{1,p}$ is the first eigenvalue for the Finsler p-Laplacian. (1) If $a = 0$, then $\lambda_{1,p}(M) = 0$. (2) If $a > 0$, then $\lambda_{1,p}(M) \leq \left(\frac{a\Lambda}{p}\right)^p$.*

To prove Proposition 4.1, we need to study the oscillatory behavior of the following ODE on $\psi = \psi(t)$:

$$(\psi'^{(p-1)}\nu)' + \lambda\psi^{(p-1)}\nu = 0, \quad t \geq t_0, \tag{4.4}$$

where $\psi^{(p-1)}(t) = \psi(t)^{(p-1)} := |\psi(t)|^{p-2}\psi(t)$, $\nu := \nu(t)$ is a positive continuous function on $[t_0,\infty)$ and λ is a positive constant. Recall that the equation (4.4) is said to be *oscillatory* if all solutions of (4.4) have arbitrary large zeroes on $[T,\infty)$.

Lemma 4.1. *Let $\int_{t_0}^{+\infty} \nu(\zeta)d\zeta = +\infty$ and $v(t) := \int_{t_0}^{t} \nu(\zeta)d\zeta \leq ct^k e^{at}$ for some nonnegative constants k, a and a positive constant c. Then (4.4) is oscillatory provided either (i) $a = 0$ or (ii) $\lambda > \left(\frac{a}{p}\right)^p$ when $a > 0$.*

Proof. We argue this by a contradiction. Since (4.4) is invariant up to a sign, we may assume that there exist a solution ψ of (4.4) and a sufficiently large positive constant $T > t_0$ such that $\psi > 0$ on $[T, \infty)$. Set

$$\tilde{\psi} = -\frac{\psi'^{(p-1)}\nu}{\psi^{(p-1)}}, \qquad t \in [T, \infty).$$

Then, by (4.4) and Young's inequality, one obtains

$$\tilde{\psi}' = \lambda\nu + (p-1)\left(\frac{1}{\nu}|\tilde{\psi}|^p\right)^{\frac{1}{p-1}} \geq p\lambda^{1/p}|\tilde{\psi}|. \tag{4.5}$$

Obviously, $\tilde{\psi}(t)$ is increasing on $[T, \infty)$.

Case I. $\int_T^\infty \frac{1}{\nu(\zeta)^{1/(p-1)}} d\zeta < +\infty$.

Note that $\tilde{\psi}' \geq \lambda\nu$, which means $\tilde{\psi}(t) \geq \lambda(v(t) - v(T)) + \tilde{\psi}(T)$. We may assume $\tilde{\psi}(t) > 0$ for $t \geq T$ since $\lim\limits_{t\to+\infty} v(t) = +\infty$. Further, it follows from (4.5) that $\tilde{\psi}(t) \geq \tilde{\psi}(T)e^{p(t-T)\sqrt[p]{\lambda}}$. On the other hand, (4.5) implies that

$$\tilde{\psi}' \geq (p-1)\left(\frac{1}{\nu}\tilde{\psi}^p\right)^{\frac{1}{p-1}}. \tag{4.6}$$

Solving the above inequality yields

$$\left(\frac{1}{\tilde{\psi}(t)}\right)^{\frac{1}{p-1}} \leq \left(\frac{1}{\tilde{\psi}(t-1)}\right)^{\frac{1}{p-1}} - \int_{t-1}^t \frac{1}{\nu(\zeta)^{1/(p-1)}} d\zeta. \tag{4.7}$$

Note that the assumption that $\int_T^\infty \frac{1}{\nu(\zeta)^{1/(p-1)}} d\zeta < +\infty$ ensures that the second term on the right side of (4.7) is meaningful for any $t \geq T$. By Hölder's inequality, one obtains

$$1 = \int_{t-1}^t d\zeta \leq \left(\int_{t-1}^t \frac{1}{\nu^{1/(p-1)}} d\zeta\right)^{\frac{p-1}{p}} \cdot \left(\int_{t-1}^t \nu d\zeta\right)^{\frac{1}{p}},$$

which implies

$$\int_{t-1}^t \frac{1}{\nu^{1/(p-1)}} d\zeta \geq \left(\int_{t-1}^t \nu d\zeta\right)^{-\frac{1}{p-1}} \geq \left(\frac{1}{v(t)}\right)^{\frac{1}{p-1}}.$$

Plugging the above inequality into (4.7) yields

$$\begin{aligned} 0 < \left(\frac{1}{\tilde{\psi}(t)}\right)^{\frac{1}{p-1}} &\leq \left(\frac{1}{\tilde{\psi}(t-1)}\right)^{\frac{1}{p-1}} - \left(\frac{1}{v(t)}\right)^{\frac{1}{p-1}} \\ &\leq \left(\frac{1}{\tilde{\psi}(T)e^{p(t-1-T)\sqrt[p]{\lambda}}}\right)^{\frac{1}{p-1}} - \left(\frac{1}{ct^k e^{at}}\right)^{\frac{1}{p-1}} \\ &= \left[\left(\frac{ct^k}{\tilde{\psi}(T)e^{[(p\lambda^{1/p}-a)t-p\lambda^{1/p}(1+T)]}}\right)^{\frac{1}{p-1}} - 1\right]\left(\frac{1}{ct^k e^{at}}\right)^{\frac{1}{p-1}}. \end{aligned}$$

Since $\lambda > 0$, the RHS of the last equality is less than or equal to zero when t is large enough if $a = 0$ or $\lambda > \left(\frac{a}{p}\right)^p$ when $a > 0$. We have a contradiction.

Case 2. $\int_T^\infty \frac{1}{\nu(\zeta)^{1/(p-1)}}d\zeta = +\infty$. Let

$$s(t) = \int_T^t \frac{1}{\nu(\zeta)^{1/(p-1)}}d\zeta$$

be a non-degenerate transformation of parameters. Then we can write $s = s(t)$ and $t = t(s)$, which is increasing, and

$$\frac{d}{dt}\left(\psi'(t)^{(p-1)}\nu(t)\right) = \frac{d}{ds}\left(\psi'(s)^{(p-1)}\right)\cdot\left(\frac{1}{\nu(s)}\right)^{\frac{1}{p-1}},$$

where we used that $\psi'(s) = \frac{d}{ds}(\psi(t(s))) = \psi'(t)\nu(t)^{\frac{1}{p-1}}$. Thus, (4.4) becomes

$$\frac{d}{ds}\left(\psi'(s)^{(p-1)}\right) + \lambda\psi(s)^{(p-1)}\nu(s)^{\frac{p}{p-1}} = 0. \tag{4.8}$$

Let $\bar{\psi}(s) := -\frac{\psi'(s)^{(p-1)}}{\psi(s)^{(p-1)}}$. Then, by (4.8) and Young's inequality,

$$\begin{aligned} \bar{\psi}'(s) &= \lambda\nu(s)^{\frac{p}{p-1}} + (p-1)\left|\frac{\psi'(s)}{\psi(s)}\right|^p \\ &= \lambda\nu(s)^{\frac{p}{p-1}} + (p-1)|\bar{\psi}(s)|^{\frac{p}{p-1}} \geq p\lambda^{\frac{1}{p}}\nu(s)^{\frac{1}{p-1}}|\bar{\psi}(s)|. \end{aligned} \tag{4.9}$$

Obviously, $\bar{\psi}'(s) \geq \lambda\nu(s)^{\frac{p}{p-1}}$, which means

$$\bar{\psi}(s) - \bar{\psi}(s(T)) \geq \lambda\int_{s(T)}^s \nu(s)^{\frac{p}{p-1}}ds = \lambda\int_T^t \nu(\zeta)d\zeta \to +\infty \quad \text{as } t \to +\infty.$$

So, we may assume $\bar{\psi}(s) > 0$. Further, the inequality (4.9) implies that

$$\begin{aligned}\bar{\psi}(s) &\geq \bar{\psi}(s(T)) \exp\left(p\lambda^{1/p} \int_{s(T)}^{s} \nu(s)^{\frac{1}{p-1}} ds\right) \\ &= \bar{\psi}(s(T)) \exp\left(p\lambda^{1/p}(t-T)\right), \qquad (4.10)\end{aligned}$$

where $s = s(t)$. Note that $s(t)$ and $t(s)$ are increasing and $s \to +\infty$ if and only if $t \to +\infty$ by the assumption. By (4.10), we have $\lim\limits_{s\to+\infty} \bar{\psi}(s) = +\infty$. On the other hand, (4.9) implies

$$\bar{\psi}'(s) \geq (p-1)\bar{\psi}^{\frac{p}{p-1}}.$$

By a similar argument to (4.7), we have

$$0 < \left(\frac{1}{\bar{\psi}(s)}\right)^{\frac{1}{p-1}} \leq \left(\frac{1}{\bar{\psi}(s-1)}\right)^{\frac{1}{p-1}} - 1 < 0$$

for a sufficient large s. Thus we get a contradiction. The proof is completed. □

Proof of Proposition 4.1. By (4.1), we have

$$m(B_r) = \int_0^r \tau(t)dt.$$

Since $m(M) = +\infty$, we have $\int_{r_0}^{+\infty} \tau(t)dt = +\infty$ for some $r_0 > 0$. For each $\lambda > 0$, by Lemma 4.1, if $a = 0$ or $\lambda > \left(\frac{a}{p}\right)^p$ when $a > 0$, there exists a nontrivial oscillatory solution ψ_λ of (4.4) on $[r_0, +\infty)$ in which $\nu(t)$ is replaced by $\tau(t)$. Consequently, there exist two numbers $r_1^\lambda, r_2^\lambda \in [r_0, +\infty)$ with $r_1^\lambda < r_2^\lambda$ such that $\psi_\lambda(r_1^\lambda) = \psi_\lambda(r_2^\lambda) = 0$ and $\psi_\lambda(r) \neq 0$ for any $r \in (r_1^\lambda, r_2^\lambda)$. This means either $\psi_\lambda(r) > 0$ or $\psi_\lambda(r) < 0$ on $(r_1^\lambda, r_2^\lambda)$. Let $r(z) := d_F(x, z)$, $\Omega_\lambda := B_{r_2^\lambda} \setminus B_{r_1^\lambda} \subset M$ and $u_\lambda(z) := \psi_\lambda(r(z))$. Then we have $F^*(Du_\lambda) = F^*(\psi_\lambda' Dr) \leq \Lambda|\psi_\lambda'|$ a.e. on Ω_λ. By the principle of the variation, we have

$$\begin{aligned}0 &\leq \lambda_{1,p}(M) \leq \lambda_{1,p}(\Omega_\lambda) \\ &= \inf_{u_\lambda} \frac{\int_{\Omega_\lambda} F^{*p}(Du_\lambda)dm}{\int_{\Omega_\lambda} |u_\lambda|^p dm} \leq \frac{\Lambda^p \int_{\Omega_\lambda} |\psi_\lambda'(r)|^p dm}{\int_{\Omega_\lambda} |\psi_\lambda(r)|^p dm} \\ &= \frac{\Lambda^p \int_{r_1^\lambda}^{r_2^\lambda} |\psi_\lambda'(r)|^p \tau(r)dr}{\int_{r_1^\lambda}^{r_2^\lambda} |\psi_\lambda(r)|^p \tau(r)dr} = -\frac{\Lambda^p \int_{r_1^\lambda}^{r_2^\lambda} \left(\psi_\lambda'^{(p-1)}\tau\right)' \psi_\lambda dr}{\int_{r_1^\lambda}^{r_2^\lambda} |\psi_\lambda|^p \tau dr} = \lambda\Lambda^p.\end{aligned}$$

In the case of $a = 0$, $\lambda_{1,p}(M) = 0$ since λ is an arbitrary positive constant. For the case when $a > 0$, $\lambda_{1,p}(M) \leq \left(\frac{a\Lambda}{p}\right)^p$ since λ is an arbitrary positive constant greater than $\left(\frac{a}{p}\right)^p$. This finishes the proof. □

Corollary 4.1. *Let (M, F, m) be an n-dimensional forward complete Finsler manifold with finite reversibility Λ.*

(i) If $m(M) < +\infty$, then $\lambda_{1,p} = 0$.

(ii) If $m(M) = +\infty$ and $Ric_N \geq -K$ for some $K \geq 0$ and $N \in [n, +\infty)$, then $\lambda_{1,p}(M) \leq \left(\frac{\Lambda\sqrt{(N-1)K}}{p}\right)^p$. In particular, $\lambda_{1,p} = 0$ when $K = 0$.

Proof. (i) Note that the variational principle of $\lambda_{1,p}(M)$ asserts that

$$\lambda_{1,p}(M) \int_M |u|^p dm \leq \int_M F^{*p}(Du) dm \tag{4.11}$$

for any $u \in W_0^{1,p}(M)$. In particular, we choose $u(z) = -\psi(z)$, where ψ is a cut-off function defined by

$$\psi(z) = \begin{cases} 1 & \text{on } B_R \\ \frac{2R - d_F(x,z)}{R} & \text{on } B_{2R} \setminus B_R \\ 0 & \text{on } M \setminus B_{2R}. \end{cases} \tag{4.12}$$

Then $F^*(Du) \leq \frac{1}{R}$ a.e. on B_{2R} and hence (4.11) implies that

$$\lambda_{1,p} m(B_R) \leq R^{-p} m(B_{2R}),$$

which implies that $\lambda_{1,p}(M) = 0$ by taking $R \to \infty$. The assertion (ii) follows from (4.3) and Proposition 4.1. □

Proof of Theorem 1.3. It directly follows from Corollary 4.1. □

Remark 4.1. *In Proposition 4.1, the assumption that $m(B_r) \leq cr^k e^{ar}$ is actually originated from (4.3). Obviously, $a = 0$ corresponds to the case of $Ric_N \geq 0$. If F is a Riemannian metric with the Ricci curvature $Ric \geq -(n-1)K (K \geq 0)$ and $p = 2$, then Proposition 4.1 implies that $\lambda_1 \leq \frac{(n-1)^2 K}{4}$, which was first obtained by S. Cheng in a different way (see Theorem 4.2, [Ch]).*

Moreover, by the Bochner-Weitzenböck formula, we can get a more refined growth estimate for the volume measure of forward geodesic ball B_r

than (4.3). In Riemannian case, the volume estimate for a geodesic ball directly follows from the volume comparison theorem.

Proposition 4.2. *Let (M, F, m) be a forward complete Finsler measure space satisfying $Ric_N \geq -K$ for some $N \in [n, +\infty)$ and $K > 0$. Then there exists a positive constant $C = C(K, N, m(B_1))$ depending on $N, K, m(B_1)$ such that*

$$m(B_r) \leq Ce^{r\sqrt{(N-1)K}}$$

for all $r \geq 1$.

Proof. Let $\gamma : [0, r(z)] \to M$ be the minimizing geodesic from x to z with $\gamma(0) = x$ and $\gamma(r(z)) = z$, where r is the distance function from x. In a geodesic polar coordinates (r, θ), it follows from (2.10) that

$$\Delta r = \frac{\partial}{\partial r} \log \sigma(x, r, \theta). \tag{4.13}$$

In the following, we will omit the dependence of the quantities on θ. On the other hand, applying (2.8) to the function $r(z) = r(x, z)$ for any $z \in M$ and using $F(\nabla r) = 1$ yield

$$D(\Delta r)(\nabla r) + Ric_N(\nabla r) + \frac{1}{N-1}(\Delta r)^2 \leq 0,$$

which implies that

$$\frac{\partial^2}{\partial r^2}(\log \sigma) + \frac{1}{N-1}\left(\frac{\partial}{\partial r}\log \sigma\right)^2 \leq K.$$

Integrating this inequality from 1 to r and letting $u(r) := \frac{\partial \log \sigma}{\partial r}$ give

$$u(r) + \frac{1}{N-1}\int_1^r u^2(t)dt \leq Kr + C_0$$

for some constant $C_0 > 0$. The Cauchy-Schwarz inequality implies that

$$u(r) + \frac{1}{(N-1)r}\left(\int_1^r u(t)dt\right)^2 \leq Kr + C_0. \tag{4.14}$$

Now we estimate $\int_1^r u(t)dt$. Consider the function

$$v(r) := -\int_1^r u(t)dt + r\sqrt{(N-1)K} + C_0\sqrt{(N-1)/K}. \tag{4.15}$$

Obviously, $v(1) > 0$. Assume that $R > 1$ is the first number such that $v(R) = 0$, namely,

$$\int_1^R u(t)dt = R\sqrt{(N-1)K} + C_0\sqrt{(N-1)/K}.$$

Then

$$\begin{aligned}&\frac{1}{(N-1)R}\left(\int_1^R u(t)dt\right)^2\\ &\quad = \frac{1}{(N-1)R}\left[R\sqrt{(N-1)K} + C_0\sqrt{(N-1)/K}\right]^2 \geq KR + 2C_0.\end{aligned}$$

Consequently, $u(R) \leq -C_0 < 0$ by (4.14), which implies that $v'(R) = -u(R) + \sqrt{(N-1)K} > 0$. Thus, there is a number $\varepsilon > 0$ small enough such that $v(R-\varepsilon) < 0$. This contradicts the choice of R. Hence $v(r) > 0$ for all $r \geq 1$. From this and (4.15), we have

$$\log\sigma(r) - \log\sigma|_{r=1} \leq r\sqrt{(N-1)K} + C_0\sqrt{(N-1)/K}.$$

This implies that $m(B_r) \leq Ce^{r\sqrt{(N-1)K}}$ for some positive constant C depending on N, K and $m(B_1)$. □

With these preparations, we begin to prove Theorem 1.2.

Proof of Theorem 1.2. As in Section 3, let $v = (p-1)\log u$ and $f = F^2(x, \nabla v)$. It suffices to consider the case when $f > 0$. To obtain the global estimate, we need to estimate $\mathcal{L}_v(f)$ in a more refined way than (3.5)

Choose a local orthonormal basis $\{e_1, \cdots, e_n\}$ with respect to $g_{\nabla v}$ at $x \in M_v$ such that $\nabla v = F(\nabla v)e_1$ as in the proof of Lemma 3.1. Then $v_1 = F(\nabla v) = f^{1/2} > 0$ and $v_i = 0 (2 \leq i \leq n)$. Differentiating $f = v_1^2$ yields $f^{-1}Df(\nabla v) = 2v_{11}$ and $f^{-1/2}Df(e_j) = 2v_{1j}(\forall j)$, where v_{ij} stand for the covariant derivatives of v with respect to the Levi-Civita connection of $g_{\nabla v}$. Thus, (3.3) can be rewritten as

$$\sum_{i=2}^n v_{ii} + \frac{1}{2}(p-1)f^{-1}Df(\nabla v) - S(\nabla v) + f + (p-1)^{p-1}\lambda_p f^{1-p/2} = 0.$$

Hence,

$$\begin{aligned}
\|\nabla^2 v\|^2_{HS(\nabla v)} &= \sum_{i,j=1}^{n} v_{ij}^2 \geq \sum_{j=1}^{n} v_{1j}^2 + \frac{1}{n-1}(\sum_{i=2}^{n} v_{ii})^2 \\
&= \frac{1}{n-1}\left(f + \frac{1}{2}(p-1)f^{-1}Df(\nabla v) + (p-1)^{p-1}\lambda_p f^{1-p/2} - S(\nabla v)\right)^2 \\
&\quad + \frac{1}{4}f^{-1}\sum_{j=1}^{n} Df(e_j)^2 \\
&\geq \frac{1}{N-1}\left(f + \frac{1}{2}(p-1)f^{-1}Df(\nabla v) + (p-1)^{p-1}\lambda_p f^{1-p/2}\right)^2 \\
&\quad - \frac{1}{N-n}S^2(\nabla v) + \frac{1}{4}f^{-1}\|\nabla^{\nabla v} f\|^2_{HS(\nabla v)},
\end{aligned}$$

where we used $(a-b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$ with $\delta = (N-n)/(n-1) > 0$ in the last inequality. Plugging this into (3.4), and using $Ric_N \geq -K$ and (2.2) yield

$$\begin{aligned}
\mathcal{L}_v(f) \geq & \frac{2}{N-1}f^{1-p/2}\left(f^{p/2} + \Lambda^{-1}\sqrt{(N-1)K}f^{(p-1)/2} + (p-1)^{p-1}\lambda_p\right) \\
& \cdot \left(f^{p/2} - \Lambda\sqrt{(N-1)K}f^{(p-1)/2} + (p-1)^{p-1}\lambda_p\right) \\
& + \left(\frac{2(p-1)}{N-1} - p\right) f^{p/2-1}Df(\nabla v) \\
& + \frac{2(p-1)}{N-1}(p-1)^{p-1}\lambda_p f^{-1}Df(\nabla v). \qquad (4.16)
\end{aligned}$$

Note that the above inequality holds only on M_v.

Next, let w be the largest positive root of the equation

$$w^{\frac{p}{2}} - \Lambda\sqrt{(N-1)K}w^{\frac{p-1}{2}} + (p-1)^{p-1}\lambda_p = 0. \qquad (4.17)$$

In fact, $\sqrt{w}$ exactly corresponds to $(p-1)\chi$, where χ is the largest positive root of (1.5). For any $\delta > 0$, consider the nonnegative function $\hat{f} = (f - (w+\delta))^+$. We denote by $\Omega := \{f \geq w + \delta\} \subset M_v$. Then $0 < w + \delta \leq f \leq c(N, K, \kappa, \kappa^*, p)$ on Ω by Theorem 1.1. Since $\mathcal{L}_v(f) = \mathcal{L}_v(\hat{f})$ and

$$|Df(\nabla v)| \leq \sqrt{f}\sqrt{g^{ij}(\nabla v)f_i f_j} \leq \sqrt{\tilde{\kappa}}\sqrt{f}F(\nabla f),$$

by (4.16), there exist positive constants $c_i = c_i(N, K, \kappa, \kappa^*, p, \delta)(i = 1, 2)$ such that

$$\mathcal{L}_v(\hat{f}) \geq c_1 \left(f^{p/2} - \Lambda\sqrt{(N-1)K} f^{(p-1)/2} + (p-1)^{p-1}\lambda_p \right) \\ - c_2 F^*(D\hat{f}) \quad \text{on } \Omega. \tag{4.18}$$

Observe that

$$f^{p/2} - \Lambda\sqrt{(N-1)K} f^{(p-1)/2} + (p-1)^{p-1}\lambda_p \geq c_3 \hat{f} \tag{4.19}$$

for some positive constant $c_3 = c_3(N, K, \Lambda, p, \delta)$. In fact, (4.19) clearly holds when $f = w$ for any choices of c_3. On the other hand, we view both sides of (4.19) as a function of f, and the derivative of the left side is given by

$$\frac{1}{2} f^{\frac{p-3}{2}} \left[p f^{1/2} - (p-1)\Lambda\sqrt{(N-1)K} \right].$$

Now we take $\lambda_p = \lambda_{1,p}$. Then $\lambda_{1,p} \leq \left(\frac{\Lambda\sqrt{(N-1)K}}{p} \right)^p$ by Corollary 4.1. Hence $w \geq \left(\frac{p-1}{p} \right)^2 \Lambda^2 (N-1)K$ by (4.17). From this and $f > w + \delta$, we get $p f^{1/2} - (p-1)\Lambda\sqrt{(N-1)K} \geq c(N, K, \Lambda, p, \delta) > 0$. Thus, (4.19) is true on Ω by choosing $0 < c_3 < c$. Consequently,

$$\mathcal{L}_v(\hat{f}) \geq c_4 \hat{f} - c_2 F^*(D\hat{f}) \quad \text{on } \Omega, \tag{4.20}$$

where $c_4 = c_4(N, K, \kappa, \kappa^*, p, \delta)$ is a positive constant. Further, we claim (4.20) holds on M in a weak sense. In fact, we have $Z(\hat{f}) = 0$ for any nonzero vector field Z on $\partial\Omega$, i.e., $g_{\nabla\hat{f}}(\nabla\hat{f}, Z) = 0$, which means that $\tilde{\nu} = \frac{\nabla\hat{f}}{F(\nabla\hat{f})} = \frac{\nabla f}{F(\nabla f)}$ is a normal vector field on $\partial\Omega$ with respect to $g_{\nabla f}$. However, $\tilde{\nu}$ points inward. To apply the stokes theorem, we need a normal vector field pointing outward. Actually, $\nu = \frac{\nabla(-f)}{F(\nabla(-f))}$ is a normal vector field pointing outward with respect to $g_{\nabla(-f)}$ on $\partial\Omega$. Its dual is $J^{-1}(\nu) = -\frac{Df}{F(\nabla(-f))}$. Note that $F(\nabla(-f)) \neq F(\nabla f)$ in general. Hence, for any nonnegative function $\varphi \in W_0^{1,p}(M)$, one obtains from the stokes theorem (Theorem 2.4.2, [Sh]) that

$$\int_M \hat{f}\mathcal{L}_v(\varphi)dm = \int_\Omega \hat{f}\mathcal{L}_v(\varphi)dm = \int_\Omega \varphi\mathcal{L}_v(\hat{f})dm + \int_{\partial\Omega} \hat{f} g_\nu(\nu, f^{p/2-1}h_v(\nabla^{\nabla v}\varphi))dm_\nu - \int_{\partial\Omega} \varphi g_\nu(\nu, f^{p/2-1}h_v(\nabla^{\nabla v}\hat{f}))dm_\nu, \tag{4.21}$$

where we used $D\hat{f}(\nabla^{\nabla v}\varphi) = D\varphi(\nabla^{\nabla v}\hat{f})$ and dm_ν is the volume measure on $\partial\Omega$ induced from $dm|_\Omega$ by ν. Note that $\hat{f} = 0$ on $\partial\Omega$ and

$$\begin{aligned} &- g_\nu(\nu, f^{p/2-1}h_v(\nabla^{\nabla v}\hat{f})) \\ &\quad = F^{-1}(\nabla(-f))f^{p/2-1}Df(h_v(\nabla^{\nabla v}f)) \\ &\quad = F^{-1}(\nabla(-f))f^{p/2-1}\left[Df(\nabla^{\nabla v}f) + (p-2)f^{-1}(Dv(\nabla^{\nabla v}f))^2\right] \\ &\quad \geq \min\{1, p-1\}F^{-1}(\nabla(-f))f^{p/2-1}\|\nabla^{\nabla v}f\|^2_{HS(\nabla v)} \geq 0. \end{aligned} \tag{4.22}$$

Consequently, combining(4.20)-(4.22) together yields

$$\begin{aligned} \int_M \hat{f}\mathcal{L}_v(\varphi)dm &\geq \int_\Omega \varphi\mathcal{L}_v(\hat{f})dm \\ &\geq \int_\Omega \varphi\left(c_4\hat{f} - c_2F^*(D\hat{f})\right)dm \\ &= \int_M \varphi\left(c_4\hat{f} - c_2F^*(D\hat{f})\right)dm. \end{aligned} \tag{4.23}$$

This proves the claim.

Finally, we prove that $\hat{f} \equiv 0$ on M based on (4.23). In other words, $f \leq w$ since $0 < \delta$ is arbitrary. Equivalently, $F(\nabla \log u) \leq \chi$.

Let $\varphi = \eta^2\hat{f}^t$ be a cut-off function on M as in the proof of Lemma 3.2 for some constant $t \geq 1$. Plugging this into (4.23) yields

$$\begin{aligned} &-\int_\Omega f^{p/2-1}D\hat{f}(h_v(\nabla^{\nabla v}(\eta^2\hat{f}^t)))dm \\ &\qquad \geq \int_M \eta^2\hat{f}^t\left(c_4\hat{f} - c_2F^*(D\hat{f})\right)dm. \end{aligned} \tag{4.24}$$

By similar arguments to (3.12)-(3.15) and the boundedness of f, we get

$$c_4\int_M \eta^2\hat{f}^{t+1}dm \le c_2\int_\Omega \eta^2\hat{f}^tF^*(D\hat{f})dm + c_5\int_\Omega \eta\hat{f}^tF^*(D\hat{f})F(\nabla\eta)dm$$
$$- c_6t\int_\Omega \eta^2\hat{f}^{t-1}F^{*2}(D\hat{f})dm \tag{4.25}$$

for some positive constants $c_i = c_i(N, K, p, \kappa, \kappa^*, \delta)(i = 5, 6)$. Thus, for any $0 < \epsilon < c_4$, we have

$$c_4\int_M \eta^2\hat{f}^{t+1}dm \le \epsilon\int_\Omega \eta^2\hat{f}^{t+1}dm + \frac{c_2^2}{4\epsilon}\int_\Omega \eta^2\hat{f}^{t-1}F^{*2}(D\hat{f})dm$$
$$+ \epsilon\int_\Omega \hat{f}^{t+1}F^2(\nabla\eta)dm$$
$$+ \frac{c_5^2}{4\epsilon}\int_\Omega \eta^2\hat{f}^{t-1}F^{*2}(D\hat{f})dm - c_6t\int_\Omega \eta^2\hat{f}^{t-1}F^{*2}(D\hat{f})dm. \tag{4.26}$$

Choose t such that $c_2^2 + c_5^2 = 4\epsilon c_6 t$. Therefore,

$$(c_4-\epsilon)\int_M \eta^2\hat{f}^{t+1}dm \le \epsilon\int_M \hat{f}^{t+1}F^2(\nabla\eta)dm. \tag{4.27}$$

We choose the test functions $\eta_k = -1$ on B_k and 0 outside B_{k+1} as in the proof of Corollary 4.1. Then we have $F(\nabla\eta_k) \le 1$, where k is a positive integer. Thus, we have

$$\int_{B_{k+1}} \hat{f}^{t+1}dm \ge \left(\frac{c_4-\epsilon}{\epsilon}\right)\int_{B_k} \hat{f}^{t+1}dm \ge \left(\frac{c_4-\epsilon}{\epsilon}\right)^k\int_{B_1} \hat{f}^{t+1}dm.$$

Consequently, either $\hat{f} \equiv 0$ or for all $R \ge 1$,

$$\int_{B_R} \hat{f}^{t+1}dm \ge ce^{R\log\frac{c_4-\epsilon}{\epsilon}}$$

for some positive constant c independent of ϵ. However $\hat{f}$ is bounded and $m(B_R) \le Ce^{R\sqrt{(N-1)K}}$ for $R \ge 1$ by Proposition 4.2. This leads to a contradiction if ϵ is sufficiently small. So, $\hat{f} \equiv 0$. Since $F(x, \nabla(-v)) = \overleftarrow{F}(x, \overleftarrow{\nabla}v)$, by the same arguments as above for $\overleftarrow{F}$, we have $\overleftarrow{F}(x, \overleftarrow{\nabla}v) \le \chi$. The proof is finished. □

5. Splitting of Finsler manifolds

Let (M, F, m) be a complete noncompact Finsler manifold. In this section, we study geometric structure at infinity of manifolds M with maximal eigenvalue $\lambda_{1,p}$ and prove Theorem 1.4.

Recall that a geodesic $\gamma : \mathbb{R} \to M$ is called a *straight line* if it is globally minimizing and has the unit speed. In particular, a straight line defined on $[0, \infty)$ is called a *ray*. Since M is complete, for any two points $x, z \in M$, there is a minimal geodesic from x to z and the forward (resp. backward) closed balls $\overline{B^+_{R_0}(x)}$ (resp. $\overline{B^-_{R_0}(x)}$) are compact. An *end* E of M, with respect to a compact subset $D \subset M$, means a unbounded connected component of $M \setminus D$. In general, we say that E is an end we mean that it is an end with respect to some compact subset D. The number of ends with respect to D is the number of unbounded connected components of $M \setminus D$. For all practical purposes, we may assume that $D = \overline{B^+_{R_0}(x)}$ (resp. $D = \overline{B^-_{R_0}(x)}$) for some $x \in M$ and $R_0 > 0$, and denote by E^+ (resp. E^-) the corresponding end.

Definition 5.1. *([BK]) Let (M, F, m) be a Finsler measure space. For any $1 \leq q < \infty$, the end E is said to be q-parabolic if, for each $U \Subset M$ and $\varepsilon > 0$, there exists a Lipschitz function ϕ with a compact support, $\phi \geq 1$ on U, such that $\int_E g^q_\phi dm < \varepsilon$. Otherwise E is said to be q-nonparabolic, here $g_\phi(x)$ is defined as*

$$g_\phi(x) = \lim \inf_{r \to 0^+} \sup_{y \in B^+_r(x)} \frac{\phi(y) - \phi(x)}{d_F(x, y)}.$$

Note that $g_u(x) = F(\nabla u)$ if $\phi = u \in C^1(M)$ (see P1393, [OS1]).

Assume that (M, F) admits a straight line $\tilde{\gamma} : \mathbb{R} \to M$. Then $\gamma = \tilde{\gamma}|_{[0,\infty)} \subset M$ is a ray and the associated *Busemann function* $\mathbf{b}_\gamma : M \to \mathbb{R}$ is defined by

$$\mathbf{b}_\gamma(z) := \lim_{t \to \infty} \{t - d_F(z, \gamma(t))\}.$$

It is well defined and differentiable almost everywhere on M (cf. [Oh2]). Further, $F(\nabla \mathbf{b}_\gamma) = 1$, and every integral curve of $\nabla \mathbf{b}_\gamma$ defined on $[0, \infty)$ is a geodesic by Lemma 3.1(ii) in [Oh2]. Similarly, for the ray $\bar{\gamma}(t) = \tilde{\gamma}(-t), t \in [0, \infty)$, we also have the Busemann function

$$\mathbf{b}_{\bar{\gamma}}(z) := \lim_{t \to \infty} \{t - d_{\overleftarrow{F}}(z, \bar{\gamma}(t))\} = \lim_{t \to \infty} \{t - d_F(\bar{\gamma}(t), z)\}$$

with respect to $\overleftarrow{F}$. It follows from the triangle inequality

$$\mathbf{b}_\gamma(z) \leq d_F(\gamma(0), z), \quad \mathbf{b}_\gamma + \mathbf{b}_{\bar{\gamma}} \leq 0. \tag{5.1}$$

Lemma 5.1. *Assume that (M, F) is an n-dimensional Finsler manifold containing a ray γ and $Ric_N \geq -K$ for some $N \in [n, \infty)$ and $K \geq 0$. Then $\Delta \mathbf{b}_\gamma \geq -\sqrt{(N-1)K}$ in the distributional sense.*

Proof. The conclusion follows directly from Proposition 3.2 in [Oh2] when $K = 0$. It suffices to prove this in the case of $K > 0$. For any $x \in M$, let $r(z) = d_F(x, z)$ be the distance function from x to $z \in M$. The Laplacian comparison theorem (Theorem 5.2 in [OS1]) gives

$$\begin{aligned}\Delta r(z) &\leq \sqrt{(N-1)K} \coth\left(r\sqrt{K/(N-1)}\right) \\ &\leq \frac{N-1}{r}\left(1 + r\sqrt{\frac{K}{N-1}}\right)\end{aligned} \tag{5.2}$$

pointwise on D_x and in the distributional sense on $M \setminus \{x\}$. Fix an arbitrary bounded open set $\Omega \subset M$ and a nonnegative function $\varphi \in W_0^{1,2}(\Omega)$, put $r_i(x) := -d_F(x, \gamma(i))$ for $i \in \mathbb{N}$. Then r_i are differentiable almost everywhere and $\nabla r_i(x)$ coincides with the initial vector of the unique unit speed minimal geodesic from x to $\gamma(i)$. By Lemma 3.1 (iii) in [Oh2], we have $\lim\limits_{i\to\infty} \nabla r_i(x) = \nabla \mathbf{b}_\gamma(x)$ for x at where $\mathbf{b}_\gamma$ is differentiable. Thus, by the dominated convergence theorem, we have

$$\lim_{i\to\infty} \int_\Omega D\varphi(\nabla r_i) dm = \int_\Omega D\varphi(\nabla \mathbf{b}_\gamma) dm. \tag{5.3}$$

Observe that $\nabla r_i = -\overleftarrow{\nabla}(-r_i) = -\overleftarrow{\nabla}\left(\overleftarrow{d}(\gamma(i), \cdot)\right)$. By (5.3) and applying (5.2) to the Laplacian $\overleftarrow{\Delta}$ with respect to $\overleftarrow{F}$, we get

$$\begin{aligned}\int_\Omega D\varphi(\nabla \mathbf{b}_\gamma) dm &= \lim_{i\to\infty} \int_\Omega \varphi \overleftarrow{\Delta}(-r_i) dm \\ &\leq \lim_{i\to\infty} \int_\Omega \varphi \cdot \frac{N-1}{-r_i}\left(1 - r_i\sqrt{\frac{K}{N-1}}\right) dm \\ &= \int_\Omega \varphi \cdot \sqrt{(N-1)K} dm,\end{aligned}$$

which implies that $\Delta \mathbf{b}_\gamma \geq -\sqrt{(N-1)K}$. This finishes the proof. □

Observe that $1 \le \Lambda \le \frac{1}{\sqrt{\kappa^*}}$ if F satisfies (1.7). Based on Lemma 5.1, we further have

Proposition 5.1. *Suppose that (M, F, m) contains a ray $\gamma \subset E^+$ with $m(E^+) < \infty$ and F satisfies (1.7). If $Ric_N \ge -K$ for some $N \in [n, \infty)$ and $K > 0$ and $\lambda_{1,p} = \left(p^{-1}\Lambda\sqrt{(N-1)K}\right)^p$, then F is reversible and $\Delta \mathbf{b}_\gamma = -\sqrt{(N-1)K}$. Moreover, $\Delta_p v = -\lambda_{1,p}|v|^{p-2}v$ in the distribution sense, where $v = e^{\frac{1}{p}\sqrt{(N-1)K}\ \mathbf{b}_\gamma}$.*

Proof. Consider the function $v(z) := e^{k\mathbf{b}_\gamma(z)}$ for any $z \in M$, where $k = \frac{1}{p}\sqrt{(N-1)K}$. For any nonnegative function $\varphi \in C_0^\infty(M)$, it follows from $F(\nabla \mathbf{b}_\gamma) = 1$ and Lemma 5.1 that

$$\begin{aligned}
&\int_M D\varphi\left[F^{p-2}(\nabla v)\nabla v\right] dm \\
&\quad = k^{p-1}\int_M e^{k(p-1)\mathbf{b}_\gamma} D\varphi(\nabla \mathbf{b}_\gamma) dm \\
&\quad = -(p-1)k^{p-1}\int_M k\varphi e^{k(p-1)\mathbf{b}_\gamma} dm - k^{p-1}\int_M \varphi e^{k(p-1)\mathbf{b}_\gamma}\Delta \mathbf{b}_\gamma dm \\
&\quad \le k^{p-1}\left[\sqrt{(N-1)K} - (p-1)k\right]\int_M \varphi v^{p-1} dm \\
&\quad = k^p \int_M \varphi v^{p-1} dm \le \lambda_{1,p}\int_M \varphi v^{p-1} dm.
\end{aligned} \tag{5.4}$$

Replacing φ with $\varphi^p v$ in (5.4) gives

$$\int_M D(\varphi^p v)\left[F^{p-2}(\nabla v)(\nabla v)\right] dm \le \lambda_{1,p}\int_M (\varphi v)^p dm. \tag{5.5}$$

On the other hand, it follows that $F^{*2}(x,\xi) \le g^*_{D\mathbf{b}_\gamma}(\xi,\xi) \le \kappa F^{*2}(x,\xi)$ from (1.7), where $D\mathbf{b}_\gamma = J^{-1}(\nabla \mathbf{b}_\gamma)$ and $\kappa = 1/\kappa^*$. The variation principle implies that

$$\begin{aligned}
\lambda_{1,p}\int_M (\varphi v)^p dm &\le \int_M F^{*p}(D(\varphi v)) dm \\
&\le \int_M \left[g^*_{D\mathbf{b}\gamma}(D(\varphi v), D(\varphi v))\right]^{p/2} dm.
\end{aligned} \tag{5.6}$$

Since $g^*_{ij}(x, D\mathbf{b}_\gamma) = g^{ij}(x, \nabla \mathbf{b}_\gamma) = g^{ij}(x, \nabla v)$, we have

$$\begin{aligned}
&\left[g^*_{D\mathbf{b}_\gamma}\left(D(\varphi v), D(\varphi v)\right)\right]^{p/2} \\
&\quad \le \left[\varphi^2 F^2(\nabla v) + 2v\varphi D\varphi(\nabla v) + \kappa v^2 F^{*2}(D\varphi)\right]^{p/2} \\
&\quad \le \varphi^p F^p(\nabla v) + pv\varphi^{p-1} D\varphi(\nabla v) F^{p-2}(\nabla v) + cv^p F^{*2}(D\varphi) \\
&\quad = D(\varphi^p v)\left[F^{p-2}(\nabla v)\nabla v\right] + cv^p F^{*2}(D\varphi)
\end{aligned}$$

for some constant $c = c(p, N, K, \kappa)$ depending on p, N, K and κ, where we used $F(\nabla v) = kv$ and the boundedness of $F(D\varphi)$ (as $R \to \infty$, see the choice of φ below) in the second inequality. Substituting this in (5.6) yields

$$\begin{aligned}
\lambda_{1,p} \int_M (\varphi v)^p dm \le & \int_M D(\varphi^p v)\left[F^{p-2}(\nabla v)\nabla v\right] dm \\
& + c\int_M v^p F^{*2}(D\varphi) dm.
\end{aligned} \tag{5.7}$$

Let $B_R = B_R^+(x)$ as before, here $x = \gamma(0)$. We choose a cut off function $\varphi \in C_0^\infty(M)$ as in (4.12) such that φ is 1 on B_R and 0 on $M \setminus B_{2R}$, and $F(\nabla\varphi) \le \frac{\Lambda}{R}$. Then

$$\begin{aligned}
\int_M v^p F^{*2}(D\varphi) dm \le & \frac{\Lambda^2}{R^2}\int_{E^+\cap(B_{2R}\setminus B_R)} e^{\sqrt{(N-1)K}\ \mathbf{b}_\gamma} dm \\
& + \frac{\Lambda^2}{R^2}\int_{(M\setminus E^+)\cap(B_{2R}\setminus B_R)} e^{\sqrt{(N-1)K}\ \mathbf{b}_\gamma} dm.
\end{aligned} \tag{5.8}$$

Note that the assumption on completeness implies that (M, F, m) is proper, i.e., closed forward (backward) geodesic balls are compact, and $\lambda_{1,p} = \left(p^{-1}\Lambda\sqrt{(N-1)K}\right)^p > 0$ is equivalent to the statement that E^+ supports a (p, p, λ)-Sobolev inequality by the variation principle. For the p-nonparabolic end, we have $m(E^+) = \infty$ from the proof of Theorem 0.1(2) in [BK]. Thus, by the assumption that $m(E^+) < \infty$ and Theorem 0.1 in [BK] again, E^+ must be p-parabolic and

$$m(E^+ \setminus B_r) \le Ce^{-r\sqrt{(N-1)K}}.$$

From this and (5.1), one obtains

$$\begin{aligned}\int_{E^+\cap(B_{2R}\setminus B_R)} & e^{\sqrt{(N-1)K}\ \mathbf{b}_\gamma} dm \\ & \leq \sum_{i=1}^{[R]+1} \int_{E^+\cap(B_{R+i}\setminus B_{R+i-1})} e^{\sqrt{(N-1)K}\ \mathbf{b}_\gamma} dm \\ & \leq C \sum_{i=1}^{[R]+1} e^{(R+i)\sqrt{(N-1)K}} \cdot e^{-(R+i-1)\sqrt{(N-1)K}} \leq CR, \end{aligned} \tag{5.9}$$

where $[R]$ means the integer part of R. Thus the first term on the right side of (5.8) goes to zero as R goes to infinity. Moreover, Lemma 4.2 in [Oh2] implies that

$$\mathbf{b}_\gamma(z) \leq -r(z) + C$$

for some constant C on $M \setminus E^+$. From this and Proposition 4.2, we have

$$\int_{(M\setminus E^+)\cap(B_{2R}\setminus B_R)} e^{\sqrt{(N-1)K}\ \mathbf{b}_\gamma} dm \leq CR,$$

which means that the second term on the right side of (5.8) also goes to zero as $R \to \infty$. Hence the equalities in (5.4) and (5.5) hold as limits of $R \to \infty$, which imply that $\Lambda = 1$, i.e., F is reversible, $\Delta \mathbf{b}_\gamma = -\sqrt{(N-1)K}$ and $\Delta_p v = -\left(p^{-1}\sqrt{(N-1)K}\right)^p |v|^{p-2} v$ in a distribution sense. □

In fact, Proposition 5.1 implies that $\Delta \mathbf{b}_\gamma = -\sqrt{(N-1)K}$ in the pointwise sense since F is reversible and hence $\mathbf{b}_\gamma$ is C^∞ (cf. Proposition 4.1, [Oh2]). Note that $\nabla \mathbf{b}_\gamma$ is a geodesic field with $F(\nabla \mathbf{b}_\gamma) = 1$, namely, the integral curve $\tilde{\eta} := \tilde{\eta}(t)$ of $\nabla \mathbf{b}_\gamma$ is a unit speed geodesic. Therefore, $\mathrm{Ric}_N(\nabla \mathbf{b}_\gamma)=$ $\mathrm{Ric}_N^{g_{\nabla \mathbf{b}_\gamma}}(\nabla \mathbf{b}_\gamma)$. Let $dm = e^{-\Psi(\tilde{\eta})}\mathrm{Vol}_{\dot{\tilde{\eta}}}$ along $\tilde{\eta}$. We have the isometric splitting of the weighted Riemannian manifold $(M, g_{\nabla \mathbf{b}_\gamma})$ as follows.

Proposition 5.2. *Assume that (M, F) admits a straight line $\tilde{\gamma} : \mathbb{R} \to M$ and F satisfies (1.7). If $Ric_N \geq -K$ for some $N \in [n, \infty)$ and $K > 0$ and $\lambda_{1,p} = \left(p^{-1}\Lambda\sqrt{(N-1)K}\right)^p$, then either (M, F, m) has no finite volume ends containing $\tilde{\gamma}$ or $(M, g_{\nabla \mathbf{b}_{\tilde{\gamma}}})$ splits isometrically as $M = \mathbb{R} \times \breve{M}$ with $\breve{M} = \mathbf{b}_{\tilde{\gamma}}^{-1}(0)$ and $g_{\nabla \mathbf{b}_{\tilde{\gamma}}} = dt^2 + e^{2ct}\breve{g}_{\nabla \mathbf{b}_{\tilde{\gamma}}}$ for some compact weighted Riemannian manifold $(\breve{M}, \breve{g}_{\nabla \mathbf{b}_{\tilde{\gamma}}}, m)$, where $c = \sqrt{K/(N-1)}$. In the latter case, $\Psi(\tilde{\eta}(t)) = (N-n)ct + \Psi(\tilde{\eta}(0))$, which is a linear function of t along the integral curve $\tilde{\eta}$ of $g_{\nabla \mathbf{b}_\gamma}$.*

Proof. First of all, we remark that F is reversible (i.e, $\Lambda = 1$) by the assumption and Proposition 5.1. Thus the (forward) end E^+ coincides with the (backward) end E^-. We write $\tilde{\gamma} = \gamma \cup \bar{\gamma}$, where $\gamma = \tilde{\gamma}|_{[0,\infty)}$ is a ray on M and $\bar{\gamma}(t) = \tilde{\gamma}(-t)$ for $t \in [0,\infty)$. Note that $m(M) = \infty$ since $\lambda_{1,p} > 0$ by Corollary 4.1, which implies that M must have an infinite volume end E'. If $\tilde{\gamma}$ is contained in an end with infinite volume, then the first case occurs. Otherwise, (M, F) has a finite volume end E containing $\tilde{\gamma}$. Note that it will not happen that one of $\{\tilde{\gamma}(+\infty), \tilde{\gamma}(-\infty)\}$ is contained in an end with finite volume and the other is contained in an end with infinite volume. In fact, if this is the case, then these two ends are connected into an end with infinite volume since $\tilde{\gamma}$ is connected. Let $x = \tilde{\gamma}(0) \in M$ be a fixed point such that the compact set $D = \overline{B_{R_0}(x)}$ separates the ends E and E' for some $R_0 > 0$, i.e., E and E' are two disjoint connected components of $M \setminus D$. Note that the Busemann function associated to a ray is invariant after a linear parameter transformation preserving the orientation of the ray. Thus, we always may assume that $E = E^+$ with $m(E^+) < \infty$ such that $\gamma \subset E^+$ (maybe after a reparameterization preserving the orientation of γ) because of the reversibility of F.

Applying (2.6) to $\mathbf{b}_\gamma$ and using Proposition 5.1, one obtains

$$\mathrm{Ric}_\infty(\nabla \mathbf{b}_\gamma) + \|\nabla^2 \mathbf{b}_\gamma\|^2_{HS(\nabla \mathbf{b}_\gamma)} = 0. \tag{5.10}$$

Then, by Definition 2.1,

$$\mathrm{Ric}_\infty(\nabla \mathbf{b}_\gamma) = \mathrm{Ric}_N(\nabla \mathbf{b}_\gamma) + \frac{S^2(\nabla \mathbf{b}_\gamma)}{N-n} \geq -K + \frac{S^2(\nabla \mathbf{b}_\gamma)}{N-n}, \tag{5.11}$$

where $S(\nabla \mathbf{b}_\gamma) = D\Psi(\nabla \mathbf{b}_\gamma)$ is the S-curvature in the direction $\nabla \mathbf{b}_\gamma$ and $S(\nabla \mathbf{b}_\gamma) = 0$ if $N = n$. Plugging (5.11) into (5.10) yields

$$\|\nabla^2 \mathbf{b}_\gamma\|^2_{HS(\nabla \mathbf{b}_\gamma)} \leq K - \frac{S^2(\nabla \mathbf{b}_\gamma)}{N-n}. \tag{5.12}$$

On the other hand, choose a local orthonormal frame $\{e_i\}_{i=1}^n$ with respect to $g_{\nabla \mathbf{b}_\gamma}$ such that $e_1 = \nabla \mathbf{b}_\gamma$. Thus $\nabla \mathbf{b}_\gamma = b_i e_i$, where $b_1 = 1$ and $b_i = 0 (2 \leq i \leq n)$. Denote by b_{ij} the components of $\nabla^2 \mathbf{b}_\gamma$ with respect to the Levi-Civita connection of $g_{\nabla\mathbf{b}_\gamma}$. We have $b_{11}=0$ and

$$\|\nabla^2\mathbf{b}_\gamma\|^2_{HS(\nabla\mathbf{b}_\gamma)}=\sum_{i,j}b_{ij}^2\geq\frac{1}{n-1}\left(\sum_{i=1}^n b_{ii}\right)^2=\frac{1}{n-1}\left[\Delta\mathbf{b}_\gamma+S(\nabla\mathbf{b}_\gamma)\right]^2$$

$$=\frac{1}{n-1}\left[-\sqrt{(N-1)K}+S(\nabla\mathbf{b}_\gamma)\right]^2\geq K-\frac{S^2(\nabla\mathbf{b}_\gamma)}{N-n}, \tag{5.13}$$

where we used the inequality $(a-b)^2\geq\frac{a^2}{1+\delta}-\frac{b^2}{\delta}$ with $\delta=\frac{N-n}{n-1}$ when $N>n$ and $S(\nabla\mathbf{b}_\gamma)=0$ when $N=n$. Combining (5.12) and (5.13) together, one obtains

$$\|\nabla^2\mathbf{b}_\gamma\|^2_{HS(\nabla\mathbf{b}_\gamma)}=K-\frac{S^2(\nabla\mathbf{b}_\gamma)}{N-n}$$

and all the above inequalities become equalities, which imply

$$\mathrm{Ric}_N(\nabla\mathbf{b}_\gamma)=-K,\quad S(\nabla\mathbf{b}_\gamma)=(N-n)c, \tag{5.14}$$

$$b_{11}=b_{ij}=0(i\neq j),\quad b_{22}=\cdots=b_{nn}=-c, \tag{5.15}$$

$$\|\nabla^2\mathbf{b}_\gamma\|^2_{HS(\nabla\mathbf{b}_\gamma)}=(n-1)c^2,$$

where we used $\Delta\mathbf{b}_\gamma=-\sqrt{(N-1)K}$ and $c=\sqrt{K/(N-1)}$. Consequently,

$$D\Psi(\dot{\tilde{\eta}})=S(\nabla\mathbf{b}_\gamma)=(N-n)c,$$

which means that $\Psi(\tilde{\eta}(t))=(N-n)ct+\Psi(\tilde{\eta}(0))$ is a linear function of t along $\tilde{\eta}$.

Let $\{\varphi_t\}$ be a local one-parameter transformation group generated by $\nabla\mathbf{b}_\gamma$ and

$$M_t:=\{x\in M|\mathbf{b}_\gamma(x)=t\}$$

be the level set of $\mathbf{b}_\gamma$. Obviously, $\breve{M}=M_0$ and $g_{\nabla\mathbf{b}_\gamma}(\nabla\mathbf{b}_\gamma,X)=0$ for any $X\in T_{\breve{x}}(\breve{M})$. Since $F(\nabla\mathbf{b}_\gamma)=\|\nabla\mathbf{b}_\gamma\|_{HS(\nabla\mathbf{b}_\gamma)}=1$, we have $\mathbf{b}_\gamma\circ\varphi_t=t$ for any fixed $t\in\mathbb{R}^+$, which means $\varphi_t(\breve{x})\in M_t$ for any $\breve{x}\in\breve{M}$. Similarly, for fixed $t\in\mathbb{R}^-$, we have $\varphi_t(\breve{x})\in M_t$ for any $\breve{x}\in\breve{M}$ if we use $\gamma(-t)$ instead of $\gamma(t)$. This defines a map $\Phi:\mathbb{R}\times\breve{M}\to M$, $\Phi(t,\breve{x})=\varphi_t(\breve{x})$. Obviously, it is injective. For any $q\in M$, letting $\breve{x}\in\breve{M}$ be the nearest point to q with respect to F and $\tau:[0,\ell]\to M$ be the unit speed minimal geodesic of F from $\breve{x}$ to q. By the first variation formula of τ, one obtains that $g_{\dot\tau(0)}(\dot\tau(0),X)=0$ for any $X\in T_{\breve{x}}\breve{M}$. Consequently, $g_{\nabla\mathbf{b}_\gamma}(\nabla\mathbf{b}_\gamma,X)=g_{\dot\tau(0)}(\dot\tau(0),X)=0$ at $\breve{x}$ for all $X\in T_{\breve{x}}\breve{M}$, which implies $\dot\tau(0)=\nabla\mathbf{b}_\gamma(\breve{x})$ by Lemma 1.2.4 in [Sh].

Hence $\tau(t) = \varphi_t(\breve{x})$ by the uniqueness of the minimal geodesic. This means $q \in \tau \subset Im(\Phi)$ and hence Φ is surjective. Thus, $M_t = \varphi_t(\breve{M})$ and Φ is a diffeomorphism.

Further, for any vector field X on $\breve{M}$,

$$g_{\nabla \mathbf{b}_\gamma}(d\Phi(\partial/\partial t), d\Phi(X)) = g_{\nabla \mathbf{b}_\gamma}(\nabla \mathbf{b}_\gamma, \varphi_{t*}(X)) = X(\mathbf{b}_\gamma \circ \varphi_t) = 0.$$

This shows that $\nabla \mathbf{b}_\gamma$ is a normal vector field on M_t with respect to $g_{\nabla \mathbf{b}_\gamma}$. We choose a local orthonormal frame $\{e_i\}_{i=1}^n$ as above. With respect to the induced metric from $g_{\nabla \mathbf{b}_\gamma}$, the second fundamental form $II = (h_{\alpha\beta})$ of M_t is given by $h_{\alpha\beta} = b_{\alpha\beta} = -c\delta_{\alpha\beta}$ $(2 \leq \alpha, \beta \leq n)$ by (5.15). This implies that $g_{\nabla \mathbf{b}_\gamma} = dt^2 + e^{2ct}\breve{g}_{\nabla \mathbf{b}_\gamma}$, where $\breve{g}_{\nabla \mathbf{b}_\gamma}$ is a Riemannian metric on $\breve{M}$ induced by $g_{\nabla \mathbf{b}_\gamma}$. Since M has (at least) two disconnected ends E^+ and E', $\breve{M}$ must be compact. Otherwise E^+ and E' are connected, which is a contracdiction. This finishes the proof. $\square$

Assume that (M, F, m) admits a straight line $\tilde{\gamma}$ contained in the end E with $m(E) < \infty$. Then Proposition 5.2 implies that

$$dm = e^{-(N-n)ct - \Psi(\tilde{\eta}(0))} \mathrm{Vol}_{\dot{\tilde{\eta}}} = e^{-t\sqrt{(N-1)K}} dt \cdot e^{-\Psi(\tilde{\eta}(0))} \mathrm{Vol}_{\dot{\tilde{\eta}}|_{\breve{M}}} \tag{5.16}$$

along $\tilde{\eta}$ with $\dot{\tilde{\eta}} = \nabla \mathbf{b}_{\tilde{\gamma}}$, where $\Psi(\tilde{\eta}(0)) = \Psi(0, \breve{x})$ for $\breve{x} \in \breve{M}$. Thus, we have the following diffeomorphic splitting of (M, m).

Corollary 5.1. *Let (M, F, m), Ric_N and $\lambda_{1,p}$ be as in Proposition 5.2. If (M, F, m) admits a straight line $\tilde{\gamma}$ contained in the end E with $m(E) < \infty$, then (M, m) admits a diffeomorphic measure splitting $(M, m) = (\mathbb{R} \times \breve{M},\ e^{-t\sqrt{(N-1)K}} L^1 \times \breve{m})$, where L^1 is the one-dimensional Lebesgue measure and $\breve{m} := m|_{\breve{M}}$ is the induced measure on $\breve{M}$.*

Although we have an isometric splitting of $(M, g_{\nabla b_{\tilde{\gamma}}})$ and a diffeomorphic splitting of (M, m), it is not known whether the splitting in Proposition 5.2 and Corollary 5.1 hold for Finsler mesure spaces (M, F, m) with the same assumptions as in Proposition 5.2. If (M, F, m) is a Berwald space, then one can obtain a splitting of (M, F, m) stated in Theorem 1.4. We remark that, different from the Riemannian case, one can not simply write $F^2(x, y)$ as a form of warped product since $F(x, y)$ is nonlinear in y.

Proof of Theorem 1.4. Assume that (M, F, m) has no finite volume ends containing $\tilde{\gamma}$. Then it has a finite volume end containing $\tilde{\gamma}$ as in the proof of Proposition 5.2. Then F is reversible by Proposition 5.1 and (M, m) admits a diffeomorphic measure splitting $(\mathbb{R}\times\breve{M},\ e^{-t\sqrt{(N-1)K}}L^1\times\breve{m})$ by Corollary 5.1. Next we further consider the geometric structure of (M,F).

Let $\{\varphi_t\}$ be the family of one-parameter transformation group generated by $\nabla\mathbf{b}_\gamma$ as before. Then every integral curve φ_t is a geodesic and $M_t=\varphi_t(\breve{M})$ for any $t\in\mathbb{R}$ from the proof of Proposition 5.2, where $\breve{M}$ is compact. For any $\breve{x}\in\breve{M}$, choose a local coordinate system $(\mathbb{R}\times U,x)$ on M with $x=(t,\breve{x})=(t,x^2,\cdots,x^n)$ such that $t=\mathbf{b}_\gamma(x)$, and a local orthonormal frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ with respect to $g_{\nabla\mathbf{b}_\gamma}$ such that $\frac{\partial}{\partial x^1}=\frac{\partial}{\partial t}=\nabla\mathbf{b}_\gamma$ and $\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=2}^n$ is a local orthonormal frame on $(\breve{M},\breve{g}_{\nabla\mathbf{b}_\gamma})$. Thus, the connection coefficients $\gamma^\alpha_{1\beta}(x)=\gamma^\alpha_{1\beta}(\nabla\mathbf{b}_\gamma(x))=c\delta^\alpha_\beta$ of $g_{\nabla\mathbf{b}_\gamma}$. Given a vector $X\in T_{\breve{x}}\breve{M}$ for any $\breve{x}\in\breve{M}$, let $V(t)=\varphi_{t*}(X)$ be a vector field along φ_t. Note that the isometric splitting of $(M,g_{\nabla\mathbf{b}_\gamma})$ in Proposition 5.2 shows that $\varphi_{t*}(\frac{\partial}{\partial x^\alpha}|_{\breve{M}})=e^{ct}\frac{\partial}{\partial x^\alpha}$. Then $D^{g_{\nabla\mathbf{b}_\gamma}}_{\dot\varphi}V=cV$. Since (M,F) is Berwaldian and $V(0)=X$, we have by (2.5)

$$\frac{d}{dt}[F^2(V)]=\frac{d}{dt}[g_V(V,V)]=2g_V(D^V_{\dot\varphi}V,V)=2g_V(D^{g_{\nabla\mathbf{b}_\gamma}}_{\dot\varphi}V,V)=2cF^2(V),$$

which implies that $F(V(t))=e^{ct}F(X)$, equivalently, $\varphi_t^*(F_t)=e^{ct}\breve{F}$, where $F_t=F|_{M_t}$ and $\breve{F}=F|_{\breve{M}}$, which are also reversible. In fact, for any vector $X\in T_xM$, let $V(t)=\varphi_{t*}(X)$. We have $F(V(t))=e^{ct}F(X)$ as above, i.e., $\varphi_t^*F=e^{ct}F$, which means that $\nabla\mathbf{b}_\gamma$ is a homothetic vector field of F and φ_t is a homothetic transformation of M with a homothetic factor $\frac{c}{2}$ ([SX]). Thus, $(M,F)=\cup_{t\in\mathbb{R}}(M_t,F_t)$, where $M_t=\varphi_t(\breve{M})$.

To see that $Ric^{\breve{M}}_{N-1}\geq 0$, we first claim that $Ric^{\breve{g}_{\nabla\mathbf{b}_\gamma}}\geq 0$. We simply denote $\hat{g}:=g_{\nabla\mathbf{b}_\gamma}$ and $\breve{g}:=\breve{g}_{\nabla\mathbf{b}_\gamma}$. In a local coordinate system taken as above, we can write $\hat{g}^2(x,y)=(y^1)^2+e^{2ct}\breve{g}^2(\breve{x},\breve{y})$, where $y=(y^1,\breve{y})=(y^1,y^2,\cdots,y^n)\in\mathbb{R}\times T_xU$. For the Riemannian metric $\hat{g}$, its geodesic coefficients (see (5.2) in [Sh]) are given by

$$\begin{aligned}\hat{G}^1(x,y)&=-\frac{c}{2}e^{2ct}\breve{g}(\breve{x},\breve{y})^2,\\ \hat{G}^\alpha(x,y)&=\breve{G}^\alpha(\breve{x},\breve{y})+cy^1y^\alpha\ (2\leq\alpha\leq n).\end{aligned}\tag{5.17}$$

By (6.4) in [Sh], the Riemannian curvature tensors of $\hat{g}$ are given by

$$\begin{aligned}\hat{R}^1{}_1(x,y)&=-c^2e^{2ct}\breve{g}(\breve{x},\breve{y})^2,\\ \hat{R}^\alpha{}_\alpha(x,y)&=\breve{R}^\alpha{}_\alpha(\breve{x},\breve{y})-(n-1)c^2\hat{g}(x,y)^2+c^2e^{2ct}\breve{g}(\breve{x},\breve{y})^2,\end{aligned}$$

where $\breve{R}^{\alpha}{}_{\beta}$ are Riemannian curvature tensors on $(\breve{M}, \breve{g})$. Thus, the Ricci curvature of $\hat{g}$ is

$$\hat{Ric}(x,y) = \hat{R}^1{}_1(x,y) + \hat{R}^{\alpha}{}_{\alpha}(x,y) = \breve{Ric}(\breve{x},\breve{y}) - (n-1)c^2\hat{g}(x,y)^2, \tag{5.18}$$

where $\breve{Ric}$ is the Ricci curvature of $\breve{g}$. Note that $\nabla \mathbf{b}_\gamma$ is a nonzero geodesic vector field with $F^2(\nabla \mathbf{b}_\gamma) = \hat{g}(\nabla \mathbf{b}_\gamma, \nabla \mathbf{b}_\gamma) = 1$. Then the weighted Ricci curvature of $(M, \hat{g}, m)$ satisfies $Ric_N^{\hat{g}}(\nabla \mathbf{b}_\gamma) = Ric_N(\nabla \mathbf{b}_\gamma) \geq -K$. On the other hand, along the integral curve $\tilde{\eta}$ of $\nabla \mathbf{b}_\gamma$, we have

$$Ric_N^{\hat{g}}(x, \nabla \mathbf{b}_\gamma) = \hat{Ric}(x, \nabla \mathbf{b}_\gamma) + (\Psi \circ \tilde{\eta})''(0) - \frac{(\Psi \circ \tilde{\eta})'(0)^2}{N-n} = \breve{Ric}(\breve{x}) - K,$$

where we used $\Psi(\tilde{\eta}(t)) = (N-n)ct + \Psi(\tilde{\eta}(0))$ (see Proposition 5.2). Hence, $Ric^{\breve{g}_{\nabla \mathbf{b}_\gamma}} = \breve{Ric} \geq 0$.

Next we prove that $Ric_{N-1}^{\breve{M}} \geq 0$. Fix a unit vector $\breve{y} \in U_{\breve{x}}\breve{M}$ (unit sphere bundle), and extend it to a vector field $\breve{Y}$ on a neighbourhood $U \subset \breve{M}$ of $\breve{x}$ such that all integral curves of $\breve{Y}$ are geodesic. We further extend $\breve{Y}$ to Y on $(-\varepsilon, \varepsilon) \times U \subset M$ by $Y(t, \breve{x}) = \breve{Y}(0, \breve{x}) \in T_{(t,\breve{x})}M$. Then all integral curves of Y are geodesic. Thus, for any $\breve{y} \in T_{\breve{x}}\breve{M}$, the Ricci curvature $Ric^{\breve{M}}(\breve{y}) = Ric^M((0,\breve{y}))$ with respect to $\breve{F} = F|_{\breve{M}}$ coincides with the Ricci curvature $Ric^{g_{\breve{Y}}}(\breve{y}) = Ric^{g_Y}((0,\breve{y}))$ with respect to $g_{\breve{Y}} = g_Y|_{\breve{M}}$ (in particular, it is independent of the choice of Y). On the other hand, since (M, F) is Berwaldian, the Ricci curvature $Ric^{g_Y}(y)$ with respect to g_Y coincides with the Ricci curvature $Ric^{g_{\nabla \mathbf{b}_\gamma}}(y)$ with respect to $g_{\nabla \mathbf{b}_\gamma}$ for any $y \in T_xM$. In particular, $Ric^{g_{\breve{Y}}}(\breve{y}) = Ric^{g_Y}((0,\breve{y}))$ with respect to $g_{\breve{Y}}$ coincides with $Ric^{\breve{g}_{\nabla \mathbf{b}_\gamma}}(\breve{y}) = Ric^{g_{\nabla \mathbf{b}_\gamma}}((0,\breve{y}))$ with respect to $\breve{g}_{\nabla \mathbf{b}_\gamma}$, which is nonnegative by the above claim. Hence $Ric^{\breve{M}}(\breve{y}) \geq 0$. From Definition 2.1 and Corollary 5.1, one obtains $Ric_{N-1}^{\breve{M}}(\breve{y}) = Ric^{\breve{M}}(\breve{y}) \geq 0$, where we used $(N-1)-(n-1) = N-n$. This finishes the proof. □

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