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Relations in the tautological ring of the universal curve

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We bound the dimensions of the graded pieces of the tautological ring of the universal curve from below for genus up to 27 and from above for genus up to 9. As a consequence we obtain the precise structure of the tautological ring of the universal curve for genus up to 9. In particular, we see that it is Gorenstein for these genera.

1. Introduction

Chow rings of moduli spaces of curves are very large. However, Mumford [22] observed that one does not need the entire Chow ring in order to make interesting intersection theoretic computations and solve many enumerative questions - the tautological subring is enough. Very roughly speaking, this is the subring generated by the most geometrically interesting classes of the moduli space.

In this note we investigate the tautological ring of the universal curve $\mathcal{C}_g = \mathcal{M}_{g,1}$ by combining an extension of a method of Faber [5] with results of Liu and Xu [17]. In this way we are able to determine the structure of the tautological ring of $\mathcal{C}_g$ up to genus 9 and we determine its Gorenstein quotients up to genus 27. In particular, we show that the tautological ring of $\mathcal{C}_g$ is Gorenstein for $2 \leq g \leq 9$, see Theorem 3.6.

The research presented in this note was carried out at KTH in the spring of 2011 but has up to now only been presented in the somewhat obscure form [1]. Nevertheless, the results have gained some attention, see [20], [30], [31] and [32], and it therefore seems as though they should be presented in a way which is more accessible and easy to read. We also remark that the methods presented here are directly applicable to higher fiber powers of $\mathcal{C}_g$ over $\mathcal{M}_g$ and that similar ideas could plausibly be applied to other moduli spaces of interest.

The paper is structured as follows. In Section 2 we give the basic definitions and present some of the known results around tautological rings of moduli spaces of curves. In particular, we sketch a method for producing tautological relations due to Faber [5] in Section 2.3. In Section 3 we make an analogous construction for the tautological ring of the universal curve and we also present a result of Liu and Xu [17] in Section 3.2 which will be very important. By combining these results we are able to bound the dimensions of graded pieces of the tautological rings from below for $g \leq 27$ and from above for $g \leq 9$. The precise results are given in Section 3.3 and Section 3.4.

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2. Background

Let k be an algebraically closed field and let $g \geq 2$ be an integer. We let $\mathcal{M}_{g,n}$ denote the moduli space of curves of genus g with n points, i.e. of tuples $(C, p_1, \ldots, p_n)$, where C is a smooth curve of genus g over k and $p_1, \ldots, p_n$ are distinct points on C. The moduli space $\mathcal{M}_{g,1}$, is given the symbol $\mathcal{C}_g$ and we denote the morphism $\mathcal{C}_g \to \mathcal{M}_g$ forgetting the marked point by π. The space $\mathcal{C}_g$ is a universal curve over the dense open subset of $\mathcal{M}_g$ parameterizing curves without automorphisms. By abuse of terminology, $\mathcal{C}_g$ is therefore sometimes called the universal curve over $\mathcal{M}_g$.

We denote the n-fold fiber product of $\mathcal{C}_g$ over $\mathcal{M}_g$ by $\mathcal{C}_g^n$. The space $\mathcal{C}_g^n$ parametrizes smooth curves marked with n, not necessarily distinct, points. For notational convenience we shall sometimes write $\mathcal{C}_g^1$ to mean $\mathcal{C}_g$ and $\mathcal{C}_g^0$ to mean $\mathcal{M}_g$. For $m \geq n$ we have various morphisms $\mathcal{C}_g^m \to \mathcal{C}_g^n$ forgetting $m - n$ points. Especially important are the morphisms

$$\pi_{n,i} : \mathcal{C}_g^n \to \mathcal{C}_g^{n-1},$$

defined by forgetting the i'th point. For $n = 1$ we have $\pi = \pi_{1,1}$.

2.1. Tautological rings

The spaces $\mathcal{C}_g^n$ have Chow rings $A^\bullet(\mathcal{C}_g^n)$ (with rational coefficients). These rings are however believed to be very big so, following Mumford [22], one instead chooses to concentrate on a subring generated by the most important cycles, i.e. the tautological ring. Faber and Pandharipande [7] has given a very natural and general definition of this system of rings, here we choose to use Mumford's original definition.

Consider the morphism

$$\pi : \mathcal{C}_g \to \mathcal{M}_g.$$

Let ω_π denote the relative dualizing sheaf, i.e. the sheaf of rational sections of $\mathrm{Coker}(d\pi : \pi^*\Omega_{\mathcal{M}_g} \to \Omega_{\mathcal{C}_g})$. Define K to be the first Chern class of ω_π, i.e.

$$K = c_1(\omega_\pi) \in A^1(\mathcal{C}_g).$$

We use K to define the so-called κ-classes

$$\kappa_i = \pi_*(K^{i+1}) \in A^i(\mathcal{M}_g).$$

In particular we have $\kappa_{-1} = 0$ and $\kappa_0 = 2g - 2$. We may also consider the Hodge bundle

$$\mathbb{E} = \pi_*(\omega_\pi).$$

It is a vector bundle of rank g on $\mathcal{M}_g$ whose fiber at $[C] \in \mathcal{M}_g$ is the space of holomorphic differentials on C. We define the λ-classes as

$$\lambda_i := c_i(\mathbb{E}) \in A^i(\mathcal{M}_g).$$

In particular, $\lambda_0 = 1$ and $\lambda_i = 0$ if $i > g$. The κ- and λ-classes generate a $\mathbb{Q}$-subalgebra $R^\bullet(\mathcal{M}_g)$ of $A^\bullet(\mathcal{M}_g)$ called the tautological ring. By analogy, we introduce the relative dualizing sheaves $\omega_{\pi_{n,i}}$ of $\pi_{n,i} : \mathcal{C}_g^n \to \mathcal{C}_g^{n-1}$, the classes

$$K_i := c_1\left(\omega_{\pi_{n,i}}\right) \in A^1(\mathcal{C}_g^n)$$

and we also introduce the diagonal classes $D_{i,j}$ consisting of points

$$[(C, p_1, \ldots, p_n)] \in \mathcal{C}_g^n,$$

such that $p_i = p_j$, $i \neq j$.

By abuse of notation we shall also denote the pullback of κ_i and λ_i in $A^\bullet(\mathcal{C}_g^n)$ by κ_i and λ_i, respectively. We now define the tautological ring $R^\bullet(\mathcal{C}_g^n)$ of $\mathcal{C}_g^n$ as the subalgebra of $A^\bullet(\mathcal{C}_g^n)$ generated by the K_i-, $D_{i,j}$-, κ- and λ-classes.

2.2. Facts

An early result concerning the tautological ring is the following theorem of Mumford, [22].

Theorem 2.1 (Mumford). *The classes λ_i and κ_i are polynomials in the classes $\kappa_1, \dots, \kappa_{g-2}$.*

For instance, we have the following relation between the λ_i and the κ_j

$$\sum_{i=0}^{\infty} \lambda_i t^i = \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}\kappa_{2i-1}}{2i(2i-1)} t^{2i-1}\right),$$

where the B_{2i} are the signed Bernoulli numbers.

As conjectured by Faber [5] and proven by Ionel [13], Mumford's result can be improved quite a bit. In cohomology, the result was first obtained by Morita [21].

Theorem 2.2 (Ionel [13]). *The $\lfloor g/3 \rfloor$ classes $\kappa_1, \dots, \kappa_{\lfloor g/3 \rfloor}$ generate $R^\bullet(\mathcal{M}_g)$, where $\lfloor x \rfloor$ denotes the floor function of x.*

By combining the Madsen-Weiss theorem [19] and a stability result of Boldsen [2] (improving results of Harer [11]) one obtains the following.

Theorem 2.3 (Madsen-Weiss [19], Boldsen [2]). *There are no relations in $R^i(\mathcal{C}_g^n)$ for $i \leq g/3$.*

Remark 2.4. Even though Boldsen only claims the above result for $i < g/3$, the remaining case seems well known to experts, see e.g. Ionel [13].

We thus have a very good understanding of the tautological ring in low degrees. We now say something about what it known in high degrees. Since the dimension of $\mathcal{C}_g^n$ is $3g-3+n$ there could, a priori, be nonzero tautological classes in degrees up to $3g-3+n$. This is however far from the case as the following result of Looijenga shows.

Theorem 2.5 (Looijenga [18]). *$R^j(\mathcal{C}_g^n) = 0$ if $j > g+n-2$ and $R^{g+n-2}(\mathcal{C}_g^n)$ is at most one-dimensional.*

Looijenga's theorem was improved a bit by Faber, [4].

Theorem 2.6 (Faber [4]). *The class κ_{g-2} is non-zero in $R^{g-2}(\mathcal{M}_g)$.*

It follows that $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. The non-vanishing of $R^{g+n-2}(\mathcal{C}_g^n)$ extends easily to positive n.

Corollary 2.7. *$R^{g+n-2}(\mathcal{C}_g^n)$ is one-dimensional.*

In the case of $\mathcal{M}_g$, Faber [5] also conjectured explicit proportionalities in degree $g-2$. This conjecture was proven, first by Liu and Xu [16] and later by Buryak and Shadrin [3]. We also mention that a proof, conditional on the Virasoro conjecture for $\mathbb{P}^2$, had previously been given by Getzler and Pandharipande [8]. A proof of the Virasoro conjecture for $\mathbb{P}^n$ was in turn announced by Givental, see [9] and [10], although the details never seem to have appeared. By now, Teleman [29] has given a proof of the Virasoro conjecture for manifolds with semi-simple quantum cohomology.

To state the result we need some notation. Let $\overline{d} = (d_1, \ldots, d_k)$ be a partition of $g-2$ into positive integers. Let $\sigma \in S_k$ and let $\sigma = \alpha_1 \cdots \alpha_{\nu(\sigma)}$ be a decomposition of σ into disjoint cycles. For a cycle α we write $|\alpha(\overline{d})|$ to denote the sum

$$|\alpha(\overline{d})| = \sum_{i\in\alpha} d_i$$

and we write $\kappa_\sigma(\overline{d})$ to denote the product

$$\kappa_\sigma(\overline{d}) = \prod_{i=1}^{\nu(\sigma)} \kappa_{|\alpha_i(\overline{d})|}.$$

Theorem 2.8 (Liu and Xu [16]). *Let $\overline{d} = (d_1, \ldots, d_k)$ be a partition of $g-2$ into positive integers. Then*

$$\sum_{\sigma\in\mathfrak{S}_n} \kappa_\sigma(\overline{d}) = \frac{(2g-3+k)!(2g-1)!!}{(2g-1)!\prod_{j=1}^{k}(2d_j+1)!!}\kappa_{g-2}.$$

Together, the results 2.2, 2.7 and 2.8 prove two thirds of the Faber conjectures [5]. The remaining third, which asserts that the pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \tag{2.1}$$

is perfect, remains open but it is reasonable to view the results of Petersen and Tommasi [27] and of Petersen [26] as evidence against the conjecture.

The following easy identities, formulated in this form by Harris and Mumford [12], will be fundamental for our computations.

Lemma 2.9 (Harris and Mumford [12]). *The following identities hold in $R(\mathcal{C}_g^n)$:*

$$D_{i,n}D_{j,n} = D_{i,j}D_{i,n}, \qquad i<j<n, \tag{1}$$

$$D_{i,n}^2 = -K_i D_{i,n}, \qquad i<n, \tag{2}$$

$$K_n D_{i,n} = K_i D_{i,n}, \qquad i<n. \tag{3}$$

Using the above identities repeatedly, every monomial in the classes K_i and D_{ij} $(i<j<n)$ in $R(\mathcal{C}_g^n)$ can be rewritten as a monomial pulled back from $R(\mathcal{C}_g^{n-1})$ times either a single diagonal $D_{i,n}$ or a power of K_n.

If M is a monomial in $R(\mathcal{C}_g^n)$ which is pulled back from $R(\mathcal{C}_g^{n-1})$, then

$$\pi_{n*}(M \cdot D_{i,n}) = M, \tag{4}$$

$$\pi_{n*}(M \cdot K_n^k) = M \cdot \pi_n^*(\kappa_{k-1}) = M \cdot \kappa_{k-1}. \tag{5}$$

2.3. Faber's method

We shall now describe a method due to Faber [5] for producing relations in the tautological ring of $\mathcal{M}_g$.

Consider the morphism $\pi_{n+1} : \mathcal{C}_g^{n+1} \to \mathcal{C}_g^n$ that forgets the $(n+1)$'st point and let Δ_{n+1} denote the sum

$$\Delta_{n+1} = D_{1,n+1} + D_{2,n+1} + \cdots + D_{n,n+1}$$

Let ω_i be the line bundle on $\mathcal{C}_g^n$ obtained by pulling back ω_π along the projection $\pi_i : \mathcal{C}_g^n \to \mathcal{C}_g$ onto the i'th factor and define a coherent sheaf $\mathbb{F}_n$ on $\mathcal{C}_g^n$ by

$$\mathbb{F}_n = \pi_{n+1*}\left(\mathcal{O}_{\Delta_{n+1}} \otimes \omega_{n+1}\right).$$

The sheaf $\mathbb{F}_n$ is locally free of rank n.

Theorem 2.10 (Faber). *If $n \geq 2g-1$ and $j \geq n-g+1$, then $c_j(\mathbb{F}_n - \mathbb{E}) = 0$.*

Thus, if $P \in R^\bullet(\mathcal{C}_g^n)$ is any element, then $P \cdot c_j(\mathbb{F}_n - \mathbb{E}) = 0$ as long as $n \geq 2g-1$ and $j \geq n-g+1$. Pushing this down to $\mathcal{M}_g$ gives a relation in $R^\bullet(\mathcal{M}_g)$. This can be done by means of Lemma 2.9 as soon as we understand

$c_j(\mathbb{F}_n - \mathbb{E})$ in terms of tautological classes. Faber proves that

$$c(\mathbb{F}_d) = (1 + K_1)(1 + K_2 - \Delta_2)(1 + K_3 - \Delta_3) \cdots (1 + K_d - \Delta_d)$$

which together with Mumford's identity [22]

$$c(\mathbb{E})^{-1} = c(\mathbb{E}^\vee) = \sum_{i=0}^{g} (-1)^i \lambda_i = 1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g$$

gives an expression of the desired form. In this way one can compute a number of relations in $R^i(\mathcal{M}_g)$ and thus obtain an upper bound of its dimension.

We shall now discuss how to obtain a lower bound for the dimension. Recall that $\deg(\kappa_i) = i$ so if $\kappa_I = \kappa_{i_1}^{n_1} \cdot \kappa_{i_2}^{n_2} \cdots \kappa_{i_r}^{n_r}$, then

$$\deg(\kappa_I) = \sum_{j=1}^{r} n_j i_j.$$

Let κ_I be a monomial in the κ-classes of degree i and let κ_J be a monomial in the κ-classes of degree $j = g - 2 - i$. Then $\kappa_I \cdot \kappa_J$ is a monomial of degree $g - 2$. Since $R^{g-2}(\mathcal{M}_g)$ is one-dimensional and generated by κ_{g-2}, we may express $\kappa_I \cdot \kappa_J$ as a rational multiple of κ_{g-2}, $\kappa_I \cdot \kappa_J = r \cdot \kappa_{g-2}$. We therefore make the following definition.

Definition 2.11. Let κ_I be a monomial of degree $g - 2$ in the κ-classes. Define $r(\kappa_I)$ to be the rational number which satisfies

$$\kappa_I = r(\kappa_I) \cdot \kappa_{g-2}.$$

We remark that Theorem 2.8 may be used to calculate the numbers $r(\kappa_I)$.

From this point on we fix a monomial ordering $<_\kappa$ of the monomials in the κ-classes. Which one is of no importance so the reader may think of his or her favourite.

Recall that the partition function, p, is the function which for each non-negative integer gives the number of ways of writing it as an unordered sum of positive integers. For instance, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$ and $p(4) = 5$. Since it is not completely uncommon to define the partition function only for positive integers, we point out that $p(0) = 1$ (the empty partition).

Definition 2.12. Let $i \leq g - 2$ be a non-negative integer. We define the $p(i) \times p(g - 2 - i)$-matrix $P_{g,i}$ as follows. Let κ_k be the kth monic monomial of degree i and let κ_l be the lth monic monomial of degree $g - 2 - i$

(according to $<_\kappa$). Then the (k,l)th entry of $P_{g,i}$ is $r(\kappa_k \cdot \kappa_l)$. We shall refer to matrices of this type as pairing matrices.

The monomials κ_I, where I is a multi-index such that

$$\sum_{i_r \in I} r \cdot i_r = i,$$

generate $R^i(\mathcal{M}_g)$ by Theorem 2.1. Note that every $\mathbb{Q}$-linear relation among the monomials κ_I of degree i clearly gives a linear relation among the rows of $P_{g,i}$. Hence, if the rank of $P_{g,i}$ is n, then $R^i(\mathcal{M}_g)$ has dimension at least n.

Faber's two-step program to compute $R^\bullet(\mathcal{M}_g)$ for specific values of g is now the following. First, compute the rank of $P_{g,i}$ to obtain a lower bound n for the dimension of $R^i(\mathcal{M}_g)$. Then multiply the relation in Theorem 2.10 by monomials and use Lemma 2.9 to push these relations to $R^\bullet(\mathcal{M}_g)$. We then pick out the degree i part of the relation which must be a relation in $R^i(\mathcal{M}_g)$. By producing such relations one obtains an ideal $I \subset \mathbb{Q}(\kappa_1, \ldots, \kappa_{g-2})$ and consequently an upper bound m for the dimension of $R^i(\mathcal{M}_g)$ as the dimension of the degree i part of the quotient $\mathbb{Q}(\kappa_1, \ldots, \kappa_{g-2})/I$. If $m = n$ one may conclude that $R^i(\mathcal{M}_g) = \mathbb{Q}(\kappa_1, \ldots, \kappa_{g-2})/I$. Faber [5] used this idea to compute $R^\bullet(\mathcal{M}_g)$ for small values of g. It was this data that lead him to state his conjectures and it also lead to the formulation of the Faber-Zagier relations, later generalized by Pixton [28] and proven in this more general form by Pandharipande, Pixton and Zvonkine [25] in cohomology and by Janda, see [14] and [15], in Chow. Today, we know that the Faber-Zagier relations are all relations for $g \leq 23$. For $g = 24$ there is one "missing" relation in degree 11, i.e. there is a difference of 1 between the upper bound and the lower bound in this degree.

3. The universal curve

The aim of this project was to adapt the technique of Faber, described in the previous section, to $R^\bullet(\mathcal{C}_g)$. To do so we first note that we may stop pushing down at $R^\bullet(\mathcal{C}_g)$ instead of at $R^\bullet(\mathcal{M}_g)$. Thus, the method for generating relations extends to $R^\bullet(\mathcal{C}_g)$ without any trouble and we thus have a way to produce upper bounds for the dimension of $R^i(\mathcal{C}_g)$. We thus turn to the problem of finding lower bounds.

3.1. Pairing Matrices

Recall the matrices $P_{g,i}$, introduced in Definition 2.12. There are corresponding matrices related to the product structure of $R^\bullet(\mathcal{C}_g)$. To define these matrices we need a bit of preparation.

In $R^\bullet(\mathcal{C}_g)$ we only have one more generator than in $R^\bullet(\mathcal{M}_g)$, namely the class K. Hence, Theorem 2.1 gives that $R^\bullet(\mathcal{C}_g)$ is generated by the monomials in $\kappa_1, \ldots, \kappa_{g-2}$ and K. The class K has degree 1 so a monomial $M = K^j \kappa_1^{n_1} \cdots \kappa_{g-2}^{n_{g-2}}$ has degree

$$\deg(M) = j + \sum_{i=1}^{g-2} n_i \cdot i.$$

We extend the monomial ordering $<_\kappa$ on $R^\bullet(\mathcal{M}_g)$ to a monomial ordering $<_*$ on $R^\bullet(\mathcal{C}_g)$ as follows.

Definition 3.1. Let $M = K^r \kappa_I$ and $N = K^s \kappa_J$ be monomials in the κ-classes and K of the same degree. We define a monomial ordering $<_*$ by

(a) $M <_* N$ if $r < s$ or,
(b) $M <_* N$ if $r = s$ and $\kappa_I <_\kappa \kappa_J$.

By Theorem 2.5 and Corollary 2.7 any monomial M of degree $g-1$ is a rational multiple of K^{g-1}, i.e. $M = s(M) \cdot K^{g-1}$ for some rational number $s(M)$. By Lemma 2.9 we have that $\pi_*(M) = s(M) \cdot \kappa_{g-2}$.

Definition 3.2. Let M be the k'th monic monomial of degree i according to $<_*$ and let N be the lth monic monomial of degree $g-1-i$ according to $<_*$. Define $s^i_{k,l}$ as the rational number satisfying $\pi_*(M \cdot N) = s^i_{k,l}\kappa_{g-2}$ and let

$$Q_{g,i} = (s^i_{k,l}).$$

The dimensions of $Q_{g,i}$ are

$$\left(\sum_{r=0}^{i} p(r)\right) \times \left(\sum_{r=0}^{g-1-i} p(r)\right),$$

where p is the partition function. Just as for the matrices $P_{g,i}$, the rank of $Q_{g,i}$ determines a lower bound for the dimension of $R^i(\mathcal{C}_g)$.

To explain the relationship between the matrices $Q_{g,i}$ and $P_{g,i}$ in more detail it is convenient to introduce some notation.

Definition 3.3. Let $j \leq i$ be a positive integer. Define $P^j_{g,i}$ as the $p(i-j) \times p(g-2-i)$-submatrix of $P_{g,i}$ consisting of the rows of $P_{g,i}$ which are labeled by monomials κ_I containing at least one factor κ_j. We also define

$$P^0_{g,i} = (2g-2) \cdot P_{g,i}$$

and

$$P^{-1}_{g,i} = \text{ the zero matrix of size } p(i+1) \times p(g-2-i).$$

We are now ready to state the following Proposition.

Proposition 3.4. *(a) Let $Q_{g,i}$ and $P^r_{g,j}$ be defined as above and let $i \geq 1$. Then,*

$$Q_{g,i} = \begin{pmatrix} P^{-1}_{g,i-1} & P^0_{g,i} & P^1_{g,i+1} & P^2_{g,i+2} & \cdots & P^{g-2-i}_{g,g-2} \\ P^0_{g,i-1} & P^1_{g,i} & P^2_{g,i+1} & \cdots & \cdots & \vdots \\ P^1_{g,i-1} & P^2_{g,i} & \ddots & & & \vdots \\ P^2_{g,i-1} & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ P^{i-1}_{g,i-1} & \cdots & \cdots & \cdots & \cdots & P^{g-2}_{g,g-2} \end{pmatrix}.$$

(b) The rank of $Q_{g,0}$ is 1.

Proof. (a) Denote the monomial labeling the rth row of $Q_{g,i}$ by N_r and the monomial labeling the sth column of $Q_{g,i}$ by N_s. Consider first the submatrix of $Q_{g,i}$ corresponding to rows and columns labeled by monomials N_r and N_s not containing a factor K. Then $N_r \cdot N_s$ projects to 0 so this submatrix consists entirely of zeros. With the above notation, this submatrix is equal to $P^{-1}_{g,i-1}$.

Now consider a submatrix C of $Q_{g,i}$ corresponding to rows and columns labeled by monomials N_r and N_s such that ,

(i) $N_r = K^{n_r} N'_r$ and $N_s = K^{n_s} N'_s$ where K does not divide N'_r or N'_s and,

(ii) not both n_r and n_s are zero.

Then,

$$\pi_*(N_r \cdot N_s) = \pi_*(K^{n_r+n_s} \cdot N'_r \cdot N'_s) = \kappa_{n_r+n_s-1} \cdot N'_r \cdot N'_s.$$

Note that $\kappa_{n_r+n_s-1} \cdot N'_r$ is a monomial in the κ_i's of degree $i+n_s-1$ containing a factor $\kappa_{n_r+n_s-1}$ and that N'_s is a monomial of degree $g-1-i-n_s = g-2-(i+n_s-1)$ in the κ_i's. Further, every monomial in the κ_i's of degree $i+n_s-1$ containing a factor $\kappa_{n_r+n_s-1}$ is the image of some monomial $K^{n_r+n_s} \cdot N'_r$ and every polynomial of degree $g-2-(i+n_s-1)$ is represented by the N'_s's. By our choice of monomial order labeling the rows and columns of $Q_{g,i}$ we now see that $C = P^{n_r+n_s-1}_{g,i+n_s-1}$. This completes the proof of (a).

(b) The only row of $Q_{g,0}$ is labeled by 1. The last column of $Q_{g,0}$ is labeled by K^{g-1}. Hence, the last entry of $Q_{g,0}$ is 1 and we conclude that the rank is one. □

The merit of Proposition 3.4 is that it tells us how to compute the matrices $Q_{g,i}$ without having to project monomials of $R^\bullet(\mathcal{C}_g)$ down to $R^\bullet(\mathcal{M}_g)$. Hence, we have reduced the problem of computing the matrices $Q_{g,i}$ to computing the matrices $P_{g,i}$, which are smaller and easier to compute. We shall describe a rather efficient way of doing this shortly. However, we first make a few observations which reduce the calculations a bit.

Firstly, let M be the kth monic monomial of degree i and let N be the lth monic monomial of degree $g-1-i$. By definition we have $\pi_*(M \cdot N) = s^i_{k,l} \cdot \kappa_{g-2}$ and $\pi_*(N \cdot M) = s^{g-1-i}_{l,k} \cdot \kappa_{g-2}$. We thus see that $Q_{g,g-1-i} = Q^T_{g,i}$. Similarly, we have $P_{g,g-2-i} = P^T_{g,i}$. Hence, we only have to compute $P_{g,i}$ for $i \leq \lceil (g-2)/2 \rceil$ and we only have to compute the rank of $Q_{g,i}$ for $i \leq \lceil (g-1)/2 \rceil$.

Secondly, Theorem 2.3 states that there are no relations in degrees less than $g/3$. In other words, the matrices $Q_{g,i}$ have full rank for $i < \lfloor g/3 \rfloor$. Thus, what needs to be computed is the rank of $Q_{g,i}$ for $\lfloor g/3 \rfloor < i \leq \lceil g/2 \rceil$. This is done by means of Proposition 3.4 and the following algorithm of Liu and Xu [17].

3.2. Computing pairing matrices

In this section we describe an algorithm due to Liu and Xu [17] by means of which one may efficiently compute the matrices $P_{g,i}$.

Let $\mathbf{m} = (m_1, m_1, \cdots)$ be a sequence of non-negative integers with only finitely many of the m_i nonzero. The set of such sequences is a monoid under coordinatewise addition. Define

$$|\mathbf{m}| = \sum_{i=1}^{\infty} i \cdot m_i, \quad ||\mathbf{m}|| = \sum_{i=1}^{\infty} m_i, \quad \mathbf{m}! = \prod_{i=1}^{\infty} m_i!.$$

A sequence $\mathbf{m}$ determines a monomial $\kappa_{\mathbf{m}}$ as

$$\kappa_{\mathbf{m}} = \prod_{m_i \in \mathbf{m}} \kappa_i^{m_i}$$

Inductively define constants $\beta_{\mathbf{m}}$ by setting $\beta_{\mathbf{0}} = 1$ and requiring

$$\sum_{\mathbf{m}'+\mathbf{m}''=\mathbf{m}} \frac{(-1)^{||\mathbf{m}'||}\beta_{\mathbf{m}'}}{\mathbf{m}''!(2|\mathbf{m}''|+1)!!} = 0 \quad \text{when } \mathbf{m} \neq \mathbf{0}$$

and constants $\gamma_{\mathbf{m}}$ as

$$\gamma_{\mathbf{m}} = \frac{(-1)^{||\mathbf{m}||}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}.$$

The constants $\beta_{\mathbf{m}}$ and $\gamma_{\mathbf{m}}$ can be used to define new constants $C_{\mathbf{m}}$

$$C_{\mathbf{m}} = \sum_{\mathbf{m}'+\mathbf{m}''=\mathbf{m}} 2|\mathbf{m}'|\beta_{\mathbf{m}'}\gamma_{\mathbf{m}''}.$$

Now let $|\mathbf{m}| \leq g-2$ and define further constants $F_g(\mathbf{m})$ via

$$|\mathbf{m}| \cdot F_g(\mathbf{m}) = (g-1) \cdot \sum_{\substack{\mathbf{m}'+\mathbf{m}''=\mathbf{m} \\ \mathbf{m}'\neq 0}} C_{\mathbf{m}'} F_g(\mathbf{m}''),$$

and let $F_g(\mathbf{0}) = 1$. We can now state the following result of Liu and Xu [17].

Theorem 3.5 (Liu and Xu [17]). *Let $|\boldsymbol{m}| = g-2$ and let $r(\kappa_{\boldsymbol{m}})$ be as defined in Definition 2.11. Then $r(\kappa_{\boldsymbol{m}})$ is given by*

$$r(\kappa_{\boldsymbol{m}}) = \frac{(2g-3)!! \cdot \boldsymbol{m}!}{2g-2} \cdot F_g(\boldsymbol{m}).$$

Theorem 3.5 gives a very efficient method to compute $P_{g,i}$. The method is especially nice if one wants to compute many different $P_{g,i}$, since much of the work can be reused, so the theorem suits our purposes very well.

Since the definitions are somewhat involved it might be helpful to see an example in order to decipher them.

Example 3.1. *Let* $g=4$ *and consider* $r(\kappa_{(2,0,\cdots)})$. *First take* $\boldsymbol{m}=(1,0,0,\cdots)$. *Then*

$$\begin{aligned} 0 &= \frac{(-1)^{||\boldsymbol{0}||}\beta_{\boldsymbol{0}}}{(1,0,0,\cdots)!(2|(1,0,0,\cdots)|+1)!!} + \frac{(-1)^{||(1,0,0,\cdots)||}\beta_{(1,0,0,\cdots)}}{\boldsymbol{0}!(2|\boldsymbol{0}|+1)!!} = \\ &= \frac{1\cdot 1}{1\cdot(2\cdot 1+1)!!} - \frac{\beta_{(1,0,0,\cdots)}}{1\cdot(2\cdot 0+1)!!} = \\ &= \frac{1}{3} - \beta_{(1,0,0,\cdots)}. \end{aligned}$$

Hence, $\beta_{(1,0,0,\cdots)}=\frac{1}{3}$. *A similar computation for* $\boldsymbol{m}=(2,0,0,\cdots)$ *gives* $\beta_{(2,0,\cdots)}=\frac{7}{90}$. *We also compute* $\gamma_{\boldsymbol{0}}=1$, $\gamma_{(1,0,\cdots)}=\frac{1}{3}$ *and* $\gamma_{(2,0,\cdots)}=\frac{1}{30}$. *We continue by computing the* $C_{\boldsymbol{m}}$. *For instance we get*

$$C_{(1,0,\cdots)} = 2|(1,0,\cdots)|\beta_{(1,0,\cdots)}\gamma_{\boldsymbol{0}} + 2|\boldsymbol{0}|\beta_{\boldsymbol{0}}\gamma_{(1,0,\cdots)} = 2\cdot 1\cdot\frac{1}{3}\cdot 1 = \frac{2}{3}.$$

A similar computation gives $C_{(2,0,\cdots)}=\frac{4}{45}$.

Up to this point, the computations are valid for all $g\geq 2$. *However,* $F_g(\boldsymbol{m})$ *depends on* g, *which in our case is* 4. *We get*

$$|(1,0,\cdots)|F_4((1,0,\cdots)) = (4-1)\cdot\frac{2}{3}\cdot 1$$

so $F_4((1,0,\cdots))=2$. *A similar computation gives that* $F_4((2,0,\cdots))=\frac{32}{15}$. *Lemma 3.5 now gives that*

$$r(\kappa_{(2,0,\cdots)}) = \frac{(2\cdot 4-3)!!\cdot(2,0,\cdots)!}{2\cdot 4-2}\cdot\frac{32}{15} = \frac{15\cdot 2}{6}\cdot\frac{32}{15} = \frac{32}{3}.$$

Since $\kappa_{(2,0,\cdots)}=\kappa_1^2$, *this is another way of expressing that in* $R^2(\mathcal{M}_4)$, *the relation*

$$\kappa_1^2 = \frac{32}{3}\cdot\kappa_2$$

holds. This relation can also be found in [5].

3.3. Ranks of pairing matrices

Using Proposition 3.4 and Theorem 3.5 we have constructed a Maple[1] program for computing the rank of $Q_{g,i}$. The results for $g\leq 27$ are shown in Table 1 below.

[1]Maple©is a trademark of Waterloo Maple Inc.

Write $g = 3k - l - 1$ with k a positive integer and l a non-negative integer. In [5], Faber remarked that the computational evidence suggests that the dimension of the degree k part of the kernel of the homomorphism

$$\varphi : \mathbb{Q}[x_1, \ldots, x_{g-2}] \to R^\bullet(\mathcal{M}_g)$$

sending x_i to κ_i only depends on l as long as $2k \leq g - 2$. Under this assumption, $a(l)$ is defined to be $\dim(\ker(\varphi)_k)$. The numbers $a(l)$ have been computed in [5] for $0 \leq l \leq 9$. This has later been extended to $l \leq 14$ in [17]. We show the results for $0 \leq l \leq 11$ in Table 2.

Faber and Zagier have guessed that $a(l)$ equals the number of partitions of l without any parts other than 2 which are congruent to 2 modulo 3. The guess is supported by the following (see also [6]). Let $\mathbf{p} = \{p_1, p_3, p_4, p_6, p_7, p_8, p_9, \ldots\}$ be a collection of variables indexed by the positive integers not congruent to 2 modulo 3. Define

$$\Psi(t, \mathbf{p}) = \sum_{i=0}^{\infty} t^i p_{3i} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} t^j + \sum_{i=0}^{\infty} t^i p_{3i+1} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} \frac{6j+1}{6j-1} t^j,$$

where we take $p_0 = 1$. Let $\sigma = (\alpha_1, 0, \alpha_3, \alpha_4, 0, \alpha_6 \ldots)$ be a sequence of non-negative integers with all coordinates with indices congruent to 2 modulo 5 equal to zero. Define

$$\mathbf{p}^\sigma = p_1^{\alpha_1} p_3^{\alpha_3} p_4^{\alpha_4} \cdots .$$

Define constants $C_r(\sigma)$ via

$$\log\left(\Psi(t, \mathbf{p})\right) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r(\sigma) t^r \mathbf{p}^\sigma.$$

We use these constants to define

$$\gamma = \sum_{\sigma} \sum_{i=0}^{\infty} C_r(\sigma) \kappa_r t^r \mathbf{p}^\sigma.$$

It was shown by Faber and Zagier that the relation

$$\left[\exp\left(-\gamma\right)\right]_{t^r \mathbf{p}^\sigma} = 0,$$

holds in the Gorenstein quotient of $R^\bullet(\mathcal{M}_g)$ when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \mod 2$. These are the so-called FZ-relations. It has been shown by Pandharipande and Pixton, see [23] and [24], that these relations also hold in $R^\bullet(\mathcal{M}_g)$. These relations are sufficiently many for codimensions $\leq \lfloor (g-2)/2 \rfloor$, but it is not clear whether these relations are linearly independent or not. Note the central role of positive integers not congruent to 2 modulo 3 in the above - this has now been explained in terms of 3-spin structures, see [25].

With the above in mind, it might be interesting to investigate whether a similar behaviour can be observed in $R^\bullet(\mathcal{C}_g)$. We therefore introduce the homomorphism

$$\hat{\varphi} : \mathbb{Q}[x_1, \ldots, x_{g-2}, y] \to R^\bullet(\mathcal{C}_g)$$

sending x_i to κ_i and y to K and note that note that the expected dimension of the degree k part of $\dim(\ker(\hat{\varphi}))$ is given through the formula

$$n = \sum_{i=0}^{k} p(i) - \mathrm{rank}(Q_{g,k}).$$

Here $p(i)$ is the partition function extended with $p(0) = 1$. The computations for $l \leq 9$ suggested that the number n is a function of l only, as long as $2k \leq g-1$, but for $l \geq 10$ this pattern does not persist. Nevertheless, we shall momentarily pretend that n is a function of l. We show the computations of n for $0 \leq l \leq 11$ in Table 3.

Using Table 3, a formula $b(l)$ for n as a function of l was guessed by Faber

$$b(l) = \sum_{\substack{i=0 \\ i \not\equiv 2 \pmod 3}}^{l} a(l-i), \tag{3.1}$$

where a is the a-function discussed above. As is easily shown by induction, $b(l)$ satisfies the following recursive formula

$$b(l) = 2\sum_{i=0}^{l-1} a(i) + a(l) - b(l-1) - b(l-2), \quad l \geq 2,$$

with initial values $b(0) = a(0)$ and $b(1) = a(0) + a(1)$.

Our guess $b(l)$, gives the right number of relations n when $0 \leq l \leq 9$ but it gives the value $b(10) = 90$ instead of the value $n = 91$ which was obtained by computing the rank of $Q_{25,12}$. To investigate the matter further I computed the rank of $Q_{28,13}$ and $Q_{31,14}$. Both computations gave the predicted value $n = b(10) = 90$ which suggests that $Q_{25,12}$ is exceptional. Noteworthy is that the anomaly occurs in the middle degree, $(g-1)/2$.

$g \setminus i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
2	1	1																									
3	1	2	1																								
4	1	2	2	1																							
5	1	2	3	2	1																						
6	1	2	4	4	2	1																					
7	1	2	4	5	4	2	1																				
8	1	2	4	6	6	4	2	1																			
9	1	2	4	7	9	7	4	2	1																		
10	1	2	4	7	10	10	7	4	2	1																	
11	1	2	4	7	11	13	11	7	4	2	1																
12	1	2	4	7	12	16	16	12	7	4	2	1															
13	1	2	4	7	12	17	20	17	12	7	4	2	1														
14	1	2	4	7	12	18	24	24	18	12	7	4	2	1													
15	1	2	4	7	12	19	27	31	27	19	12	7	4	2	1												
16	1	2	4	7	12	19	28	35	35	28	19	12	7	4	2	1											
17	1	2	4	7	12	19	29	39	45	39	29	19	12	7	4	2	1										
18	1	2	4	7	12	19	30	42	53	53	42	30	19	12	7	4	2	1									
19	1	2	4	7	12	19	30	43	57	64	57	43	30	19	12	7	4	2	1								
20	1	2	4	7	12	19	30	44	61	75	75	61	44	30	19	12	7	4	2	1							
21	1	2	4	7	12	19	30	45	64	83	94	83	64	45	30	19	12	7	4	2	1						
22	1	2	4	7	12	19	30	45	65	87	106	106	87	65	45	30	19	12	7	4	2	1					
23	1	2	4	7	12	19	30	45	66	91	117	131	117	91	66	45	30	19	12	7	4	2	1				
24	1	2	4	7	12	19	30	45	67	94	125	150	150	125	94	67	45	30	19	12	7	4	2	1			
25	1	2	4	7	12	19	30	45	67	95	129	162	181	162	129	95	67	45	30	19	12	7	4	2	1		
26	1	2	4	7	12	19	30	45	67	96	133	173	208	208	173	133	96	67	45	30	19	12	7	4	2	1	
27	1	2	4	7	12	19	30	45	67	97	136	181	227	253	227	181	136	97	67	45	30	19	12	7	4	2	1

Table 1: The rank of $Q_{g,i}$ for $2 \leq g \leq 27$ and $0 \leq i \leq 26$.

l	0	1	2	3	4	5	6	7	8	9	10	11
$a(l)$	1	1	2	3	5	6	10	13	18	24	33	41

Table 2: The a-function for $0 \leq l \leq 11$. The values for $l \leq 9$ can be found in [5] while $a(10)$ and $a(11)$ are found in [17].

The above results suggest that n may exhibit a similar behaviour in the middle degree also for $g > 25$. If this is so, we expect an anomaly for $g = 27$, $k = 13$. The rank of $Q_{27,13}$ gives $n = 120$ while $b(11) = 119$. Computing the rank of $Q_{30,14}$ again yields the predicted value, $n = b(11) = 119$.

One way to avoid this anomaly would be to require $2k \leq g - 2$ instead of $2k \leq g - 1$, although this is not very appealing (and very ad hoc). It might be interesting to recall that the method of Faber has been unsuccessful in proving the Faber conjectures in $R^{11}(\mathcal{M}_{24})$. Note that also here the problem arises in the middle degree.

3.4. Generating Relations

We earlier described a method for generating relations. Even though the method is rather easy in principle, its computational complexity is quite an

l	0	1	2	3	4	5	6	7	8	9	10	11
n	1	2	3	6	10	14	22	33	45	64	90 (91)	119 (120)
#	8	7	6	6	5	4	4	3	2	2	2 (1)	1 (1)

Table 3: n for $0 \leq l \leq 11$. # is the number of g for which n has been computed. The numbers in parentheses are values for which the expected behaviour fails along with how many times that happened for each l.

obstacle. We shall therefore discuss a few tricks which have helped to make the computations more efficient.

The first step of the algorithm is to pick a monomial M in $R^\bullet(\mathcal{C}_g^{2g-1})$ in the K and $D_{i,j}$-classes. However, the set of all such polynomials is much too large already for low g. The computations so far suggest that the algorithm described below produces enough relations.

Suppose that we want to produce relations in $R^i(\mathcal{C}_g)$ by multiplying the relation $c_j(\mathbb{F}_{2g-1} - \mathbb{E}) \in R^j(\mathcal{C}_g^{2g-1})$ by a monomial M and then pushing down. Since the degree drops by $2g-2$ and since $c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ has degree j, the degree d of the monomial must be $d = i + 2g - 2 - j$. Choose $q = 2g + 2i - 2j + 1$ and define monomials in the following way.

(a) Define $M_0 = D_{1,2}D_{1,3} \cdots D_{1,q}D_{q+1,q+2}D_{q+3,q+4} \cdots D_{2g-2,2g-1}$,

(b) for $r = 0, 1, \ldots, q-3$, replace $D_{1,q-r}$ by $D_{q-r,q-r+1}$ in M_r to obtain M_{r+1}.

Each M_r is a monomial of degree $i + 2g - 2 - j$ and $M_r c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ will thus give a relation in $R^i(\mathcal{C}_g)$ when pushed down.

The second step is to calculate $M \cdot c_j(\mathbb{F}_{2g-1} - \mathbb{E})$ for suitable choices of j. As stated earlier, we have

$$c(\mathbb{F}_{2g-1}) = (1 + K_1)(1 + K_2 - \Delta_2)(1 + K_3 - \Delta_3) \cdots (1 + K_{2g-1} - \Delta_{2g-1}),$$

and

$$c(\mathbb{E})^{-1} = \sum_{i=0}^{g} (-1)^i \lambda_i.$$

Hence

$$c(\mathbb{F}_{2g-1} - \mathbb{E}) = c_0(\mathbb{F}_{2g-1}) + c_1(\mathbb{F}_{2g-1}) - \lambda_1 c_0(\mathbb{F}_{2g-1}) + c_2(\mathbb{F}_{2g-1}) - \\ - \lambda_1 c_1(\mathbb{F}_{2g-1}) + \lambda_2 c_0(\mathbb{F}_{2g-1}) + \cdots$$

If we identify the degree k part we obtain the formula

$$c_k(\mathbb{F}_{2g-1} - \mathbb{E}) = \sum_{i=0}^{k} (-1)^i \lambda_i c_{k-i}(\mathbb{F}_{2g-1}). \tag{1}$$

As pointed out in [5], we have

$$c_k(\mathbb{F}_n) = c_k(\mathbb{F}_{n-1}) + (K_n - \Delta_n) c_{k-1}(\mathbb{F}_{n-1}).$$

No term of $c_j(\mathbb{F}_{n-1})$ has a factor K_n or $D_{i,n}$. Hence, if P is a polynomial in K_i and $D_{i,j}$ then, $\pi_{n*}(P \cdot c_j(\mathbb{F}_{n-1})) = \pi_{n*}(P) \cdot c_j(\mathbb{F}_{n-1})$. By putting the pieces together we obtain

$$\pi_{n*}(M c_k(\mathbb{F}_n)) = \pi_{n*}(M) c_k(\mathbb{F}_{n-1}) + \pi_{n,n*}(M(K_n - \Delta_n)) c_{k-1}(\mathbb{F}_{n-1}). \tag{2}$$

Using formulas (1) and (2), the computations become more manageable.

Several Maple procedures has been written for performing these computations. These procedures has then been used to find the necessary number of relations for $2 \leq g \leq 9$. In other words, we have the following.

Theorem 3.6. *The tautological ring $R^\bullet(\mathcal{C}_g)$ is Gorenstein for $2 \leq g \leq 9$.*

No higher genera have been attempted since the computations are expected to take unfeasibly long time. However, shortly after our results first appeared, Yin [32] was able to prove that $R^\bullet(\mathcal{C}_g)$ is Gorenstein for g up to 19 using completely different methods. Below, we present the relations for $g = 2$, 3 and 4. The other relations, as well as the Maple code, are available from the author upon request.

The case $g = 2$. Since $\kappa_0 = 2g - 2 = 2$, there should be no relation in degree zero. In degree one there should be one relation. Multiplying $c_2(\mathbb{F}_3 - \mathbb{E})$ by $D_{2,3}$ and pushing down to $R^*(\mathcal{C}_2)$ yields the relation $\frac{5}{3}\kappa_1 = 0$. Hence, $K \neq 0$ and $\kappa_1 = 0$. This is no surprise, since κ_1 is the pullback of κ_1 in $R^*(\mathcal{M}_g)$, which is zero by [5]. The result also follows from Theorem 2.5 and Theorem 2.6.

The case $g = 3$. Since $g/3 = 1$ we should have no relations in degrees zero and one. In degree two we should have three relations (and will have, by Theorems 2.5 and 2.6). Multiplying $c_3(\mathbb{F}_5 - \mathbb{E})$ with $D_{1,2}D_{1,3}D_{4,5}$ respectively $D_{1,2}D_{3,4}D_{4,5}$ and pushing down to $R^*(\mathcal{C}_3)$ yields the relations

$$42K^2 - \frac{21}{2}K\kappa_1 + \frac{7}{48}\kappa_1^2 = 0, \quad 126K^2 - \frac{63}{2}K\kappa_1 + \frac{41}{48}\kappa_1^2 - 6\kappa_2 = 0.$$

Multiplying $c_4(\mathbb{F}_5 - \mathbb{E})$ with $D_{2,3}D_{4,5}$ and pushing down yields the relation

$$56K^2 - 14K\kappa_1 + \frac{47}{12}\kappa_1^2 - 20\kappa_2 = 0.$$

These three relations are linearly independent, so we are done. If we solve the equations we see that

$$\kappa_1^2 = \kappa_2 = 0, \quad \text{and} \quad K\kappa_1 = 4K^2.$$

The case $g = 4$. We expect to find two relations in degree 2 and six in degree 3. Multiplying $c_4(\mathbb{F}_7 - \mathbb{E})$ with $D_{1,2}D_{1,3}D_{4,5}D_{6,7}$ respectively $D_{1,2}D_{3,4}D_{4,5}D_{6,7}$ and pushing down yields the relations

$$420K^2 - 70K\kappa_1 + \frac{115}{6}\kappa_1^2 - 150\kappa_2 = 0,$$
$$120K^2 - 20K\kappa_1 + \frac{10}{3}\kappa_1^2 - 20\kappa_2 = 0.$$

These relations are linearly independent so we are done in degree 2. We solve the equations to obtain

$$\kappa_1^2 = \frac{32}{3}\kappa_2, \quad \text{and} \quad K\kappa_1 = 6K^2 + \frac{7}{9}\kappa_2.$$

Note that the first of these relations is the relation we obtained in $R(\mathcal{M}_4^2)$ in Example 3.1.

In degree 3 we have the six linearly independent relations which can be written as

$$\kappa_3 = \kappa_2\kappa_1 = \kappa_1^3 = 0, \quad K_1^2\kappa_1 = \frac{32}{3}K_1^3, \quad K_1\kappa_1^2 = 64K_1^3, \quad K_1\kappa_1 = 6K_1^3.$$

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