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Nearly-Kähler 6-manifolds of cohomogeneity two: principal locus

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We study nearly-Kähler 6-manifolds equipped with a cohomogeneity-two Lie group action for which the principal orbits are coisotropic. If the metric is complete, then we show that this last condition is automatically satisfied, and both the acting Lie group and the principal orbits are finite quotients of $\mathbb{S}^3 \times \mathbb{S}^1$.

We then partition the class of such nearly-Kähler structures into three types (called I, II, III) and prove a local existence and generality result for each type. Metrics of Types I and II are shown to be incomplete.

1. Introduction

Nearly-Kähler 6-manifolds are a class of Riemannian 6-manifolds (M^6, g) whose geometry is in some sense modeled on the round 6-sphere $\mathbb{S}^6 \simeq \mathrm{G}_2/\mathrm{SU}(3)$. Like the round $\mathbb{S}^6$, they carry a triple (J, Ω, Υ) consisting of a compatible almost-complex structure J, a non-degenerate 2-form Ω, and a $(3,0)$-form Υ, and these are asked to satisfy the defining differential equations

$$\begin{aligned} d\Omega &= 3\,\mathrm{Im}(\Upsilon) \\ d\,\mathrm{Re}(\Upsilon) &= 2\,\Omega \wedge \Omega. \end{aligned}$$

Here, the almost-complex structure J is *not* integrable, and the 2-form Ω is *not* closed.

Yet, in spite of these two shortcomings, nearly-Kähler 6-manifolds enjoy several remarkable characterizations that have led to increased attention as of late, especially in connection with exceptional holonomy metrics and real Killing spinors. Indeed:

Theorem (Bär [1], Grunewald [20]): Let (M^6, g) be a simply-connected, spin, complete Riemannian 6-manifold. Let $\mathrm{Cone}(M,g) = (\mathbb{R}^+ \times M, dt^2 + t^2 g)$ be its Riemannian cone. The following are equivalent:
(i) (M,g) admits a nearly-Kähler structure.
(ii) (M,g) has a real Killing spinor.
(iii) $\mathrm{Cone}(M,g)$ has a parallel spinor.
(iv) $\mathrm{Hol}(\mathrm{Cone}(M,g)) = \mathrm{G}_2$ or $(M^6,g) \cong (\mathbb{S}^6, g_{\mathrm{round}})$.

In fact, nearly-Kähler 6-manifolds are Einstein of positive scalar curvature. Thus, complete examples are compact with finite fundamental group (by Bonnet-Myers).

A central problem in the study of nearly-Kähler 6-manifolds is the present dearth of compact, simply-connected examples. Indeed, as of this writing, only six such examples are known. Four of these are the homogeneous spaces [28]

$$\begin{aligned} \mathbb{S}^6 &= \frac{\mathrm{G}_2}{\mathrm{SU}(3)}, & \mathbb{S}^3 \times \mathbb{S}^3 &= \frac{\mathrm{SU}(2)^3}{\Delta\mathrm{SU}(2)}, \\ \mathrm{CP}^3 &= \frac{\mathrm{Sp}(2)}{\mathrm{U}(1) \times \mathrm{Sp}(1)}, & \mathrm{Flag}(\mathbb{C}^3) &= \frac{\mathrm{SU}(3)}{T^2}, \end{aligned}$$

and it has been shown [9] that these are the only possible homogeneous examples. Here, we caution that the metric on $\mathbb{S}^3 \times \mathbb{S}^3$ is *not* the product metric, and the almost-complex structure on CP^3 is *not* the standard one.

Following work of Conti and Salamon [11], Fernández, Ivanov, Muñoz and Ugarte [14], and Podestà and Spiro [23] [24], recently Foscolo and

Haskins [16] succeeded in constructing *inhomogeneous* nearly-Kähler metrics on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$ that are cohomogeneity-one under an $(\mathrm{SU}(2) \times \mathrm{SU}(2))$-action. Their approach involves cohomogeneity-one techniques, drawing on methods of Eschenburg and Wang [13] and Böhm [4], guided by the idea that such examples might arise as desingularizations of the sine-cone over the Sasaki-Einstein $\mathbb{S}^2 \times \mathbb{S}^3$. In fact, Foscolo and Haskins conjectured that there are no further (compact, simply-connected) cohomogeneity-one examples to be found; their work [16] contains numerical evidence in support of this conjecture.

Thus, from the point of view of symmetries, the next natural question is the existence of compact simply-connected examples of cohomogeneity-two. This remains a difficult open problem, and is the primary motivation for this work.

1.1. Methods and main results

In this work, we study the geometry of cohomogeneity-two nearly-Kähler 6-manifolds. That is, we study nearly-Kähler 6-manifolds M equipped with a faithful G-action whose generic orbits have codimension two. We always suppose that G is a closed, connected subgroup of the isometry group of M, and that the G-action preserves (J, Ω, Υ).

We will restrict attention to the case where the principal G-orbits are *coisotropic*, meaning that the 4-form $\Omega \wedge \Omega$ vanishes on these (4-dimensional) orbits. Our first result shows that, in fact, this case is the one of most interest:

Theorem 1.1: Let M be a nearly-Kähler 6-manifold. Suppose that a connected Lie group G, closed in the isometry group of M, acts faithfully on M with cohomogeneity two, preserving (J, Ω, Υ).

(a) If M is complete, then the principal G-orbits in M are coisotropic.

(b) If the principal G-orbits are coisotropic, then G is 4-dimensional and non-abelian.

(c) If M is complete, then both G and the principal G-orbits in M are finite quotients of $\mathbb{S}^3 \times \mathbb{S}^1$.

Remark: The requirement that the G-action preserve (J, Ω, Υ) is not severe. Indeed, it is shown in §3 of [22] that on a complete nearly-Kähler 6-manifold M other than the round $\mathbb{S}^6$, any Killing vector field $X \in \Gamma(TM)$ will satisfy $\mathcal{L}_X J = \mathcal{L}_X \Omega = 0$. Thus, any isometric G-action on such an M will preserve (J, Ω, Υ). $\square$

Next, we turn to the question of local existence. That is, on sufficiently small open sets of $\mathbb{R}^6$ we ask whether cohomogeneity-two nearly-Kähler metrics can exist at all. If so, what is the initial data required to construct these metrics as solutions to a (sequence of) Cauchy problem(s)?

We approach this problem by an application of Cartan's Third Theorem [8]. This result generalizes Lie's Third Theorem on the "integration" of Lie algebras to local Lie groups. Its primary hypothesis is that "mixed partials commute," meaning the satisfaction of a set of integrability conditions (analogous to the Jacobi identity for Lie algebras).

In the case of cohomogeneity-two nearly-Kähler metrics with coisotropic principal orbits, these integrability conditions form a system of 80 quadratic equations for 55 unknown functions. Careful study of this system leads us to partition the class of metrics under consideration into three types, called Types I, II, and III.

We will show that metrics of each Type exist locally and in abundance: each Type is an infinite-dimensional family. More precisely:

Theorem 1.2: On sufficiently small open sets in $\mathbb{R}^6$:

(a) Nearly-Kähler structures of Type I exist, depending on 2 arbitrary functions of 1 variable. If M is of Type I, then G is a discrete quotient of $\mathrm{H}_3 \times \mathbb{R}$, where H_3 is the real Heisenberg group. In particular, metrics of Type I are incomplete.

(b) Nearly-Kähler structures of Type II exist, depending on 2 arbitrary functions of 1 variable. If M is of Type II, then G is solvable. In particular, metrics of Type II are incomplete.

(c) Nearly-Kähler structures of Type III with $G = (\mathbb{S}^3 \times \mathbb{S}^1)/(\text{Finite})$ exist, depending on 2 arbitrary functions of 1 variable.

The dependence on 2 arbitrary functions of 1 variable — the same initial data (or "local generality") required to construct holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ — suggests the possibility that cohomogeneity-two nearly-Kähler 6-manifolds may be recovered from holomorphic data. More precisely, one can ask:

1) Can the principal locus of cohomogeneity-two nearly-Kähler structures be reconstructed from solutions to an elliptic PDE system on a Riemann surface Σ?

2) Can solutions to this elliptic PDE system, in turn, be reinterpreted as pseudo-holomorphic curves in some almost-complex manifold?

3) Can this elliptic PDE system be recast as a single second-order elliptic PDE on Σ?

4) Can cohomogeneity-two nearly-Kähler structures be reconstructed from holomorphic data by means of a Weierstrass representation formula?

These questions provide interesting directions for further study.

1.2. Organization

This work is organized as follows. In §2, we review the fundamentals of H-structures and intrinsic torsion, the language in which this work is phrased. In particular, we state Cartan's Third Theorem (labeled Theorem 2.2), our main tool for proving local existence/generality results.

In §3.1, we compare and contrast various definitions of "nearly-Kähler 6-manifold" encountered in the literature, clarifying our own conventions. The material of §2 and §3.1 is standard, and experts may wish to skip these. In §3.2, we prove Theorem 1.1(a) (labeled Proposition 3.3).

Section 4 sets up the moving frame apparatus that we will use for calculations. In §4.2, we adapt frames to the (coisotropic) principal G-orbits. This frame adaptation defines an $\mathrm{O}(2)$-structure, and its study is central to this work. In §4.3, we describe the intrinsic torsion of our $\mathrm{O}(2)$-structure, concrete geometric interpretations of which are offered in §4.5. In §4.4, we prove Theorem 1.1(b) and 1.1(c) (labeled Proposition 4.5).

In §5, we describe our partition into Types I, II, and III. In §5.1, we examine Type I structures and prove Theorem 1.2(a) (labeled Theorem 5.4 and Proposition 5.5). Similarly, §5.2 pertains to Type II structures and contains a proof of Theorem 1.2(b) (labeled Theorem 5.8 and Proposition 5.9), and §5.3 contains a proof of Theorem 1.2(c) (labeled Theorem 5.14).

Notation: The following notation and terminology will be used throughout.

- Let $\pi\colon P \to M$ be a submersion. A k-form $\theta \in \Omega^k(P)$ is *π-semibasic* if $X \lrcorner\, \theta = 0$ for all vectors $X \in TP$ tangent to the π-fibers. We will simply say "semibasic" when π is clear from context.

- When $\omega = (\omega^1, \ldots, \omega^n)$ denotes the tautological 1-form on an H-structure $B \to M^n$, we will use the shorthand

$$\omega^{ij} := \omega^i \wedge \omega^j, \qquad \omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k, \qquad \text{etc.}$$

  to denote wedge products.
- For 1-forms $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$, we let $\langle \alpha_1, \ldots, \alpha_k \rangle$ denote the differential ideal in $\Omega^*(M)$ generated by these 1-forms. In particular, $\langle \alpha_1, \alpha_2 \rangle$ denotes an ideal (not an inner product).

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2. H-structures and Cartan's third theorem

Much of this work will be phrased in the language of H-structures, intrinsic torsion, and augmented coframings. As such, we use this section to recall this terminology, set notation, and describe our primary technical tool for proving local existence. The material in this section is standard; more information can be found in [8], [17], and [26].

2.1. H-structures and intrinsic torsion

Let M be a smooth n-manifold. A *coframe* at $x \in M$ is a vector space isomorphism $u \colon T_xM \to \mathbb{R}^n$. We let $\pi \colon FM \to M$ denote the *general coframe bundle*, which is the principal right $\mathrm{GL}_n(\mathbb{R})$-bundle over M whose fiber at $x \in M$ consists of the coframes at x. Here, the right $\mathrm{GL}_n(\mathbb{R})$-action on FM is by composition: for $g \in \mathrm{GL}_n(\mathbb{R})$ and $u \in FM$, we set

$$u \cdot g := g^{-1} \circ u.$$

A *coframing* on an open set $U \subset M$ is an n-tuple $\eta = (\eta^1, \ldots, \eta^n)$ of linearly independent 1-forms on U. We think of coframings as $\mathbb{R}^n$-valued 1-forms $\eta \in \Omega^1(U; \mathbb{R}^n)$ for which each $\eta_x \colon T_xU \to \mathbb{R}^n$ is a coframe. Alternatively, we regard coframings as local sections $\sigma_\eta \in \Gamma(U; FM)$ or as local trivializations $\psi_\eta \colon U \times \mathrm{GL}_n(\mathbb{R}) \to FM|_U$ via $\psi_\eta(x, g) = \eta_x \cdot g$.

To a local diffeomorphism $f \colon M_1 \to M_2$, we associate the bundle map $f^{(1)} \colon FM_1 \to FM_2$ defined by

$$f^{(1)}(u) = u \circ (f_*|_{\pi_1(u)})^{-1}.$$

One can check that $f \mapsto f^{(1)}$ is functorial.

For a subgroup $H \leq \mathrm{GL}_n(\mathbb{R})$, an *$H$-structure* B on an n-manifold M^n is an H-subbundle of the general coframe bundle $B \subset FM$. Note that, despite the terminology, an H-structure depends on the representation of H on $\mathbb{R}^n$, not just on the abstract group itself.

We say that H-structures $\pi_1 \colon B_1 \to M$ and $\pi_2 \colon B_2 \to M_2$ are *(locally) equivalent* if there is a (local) diffeomorphism $f \colon M_1 \to M_2$ for which $f^{(1)}(B_1) = B_2$.

The *tautological* 1*-form* on an H-structure B is the $\mathbb{R}^n$-valued 1-form $\omega = (\omega^1, \ldots, \omega^n) \in \Omega^1(B; \mathbb{R}^n)$ given by

$$\omega(v) = u(\pi_*(v)), \quad \text{for } v \in T_uB.$$

The tautological 1-form "reproduces" all of the local coframings of M, in that it satisfies the following property: For any coframing $\eta \in \Omega^1(U; \mathbb{R}^n)$, we have $\sigma_\eta^*(\omega^1, \ldots, \omega^n) = (\eta^1, \ldots, \eta^n)$, or equivalently, $\psi_\eta^*(\omega^1, \ldots, \omega^n)|_{(x,h)} = (\eta^1, \ldots, \eta^n)|_x \cdot h$.

One can show [17] that if H is connected, a smooth map $F \colon B_1 \to B_2$ between H-structures is a local equivalence if and only if $F^*(\omega_2) = \omega_1$.

A *connection* on an H-structure B is simply a connection on the principal H-bundle B. That is, it is an $\mathfrak{h}$-valued 1-form $\phi \in \Omega^1(B; \mathfrak{h})$ that sends H-action vector fields to their Lie algebra generators and is H-equivariant:

$$\begin{aligned} \phi(X^\#) &= X, \quad \text{for all } X \in \mathfrak{h} \\ R_h^*(\phi) &= \mathrm{Ad}_{h^{-1}}(\phi), \quad \text{for all } h \in H. \end{aligned}$$

Here, $X^\# \in \Gamma(TB)$ is the vector field given by $X^\#|_u = \frac{d}{dt}\big|_{t=0} u \cdot (\exp tX)$ at $u \in B$. Note that the first condition implies that ϕ restricts to each H-fiber to be the Maurer-Cartan form on H.

Given an H-structure $\pi\colon B \to M$ with connection $\phi \in \Omega^1(B;\mathfrak{h})$, one can differentiate the equation $\psi_\eta^*(\omega) = \eta \cdot h$ to derive *Cartan's first structure equation*

$$d\omega = -\phi \wedge \omega + \tfrac{1}{2}T_\phi(\omega \wedge \omega),$$

where $T_\phi\colon B \to \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^*$ is a function called the *torsion of the connection* ϕ. To emphasize the distinction between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$, let us write $V = \mathbb{R}^n$.

Let ϕ_1, ϕ_2 be two connections on B, with torsion functions T_{ϕ_1}, T_{ϕ_2}, respectively. The difference $\phi_1 - \phi_2$ is π-semibasic and so can be written $\phi_1 - \phi_2 = p(\omega)$ for some function $p\colon B \to \mathfrak{h} \otimes V^*$. A calculation shows [17], [26] that the difference in the torsion functions is

$$T_{\phi_1} - T_{\phi_2} = \delta(p),$$

where $\delta\colon \mathfrak{h} \otimes V^* \hookrightarrow V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$ is the H-equivariant linear map given by skew-symmetrization. Thus, the composite map

$$T\colon B \to V \otimes \Lambda^2 V^* \twoheadrightarrow \frac{V \otimes \Lambda^2 V^*}{\delta(\mathfrak{h} \otimes V^*)} =: H^{0,2}(\mathfrak{h})$$

is independent of the choice of connection ϕ. We refer to T as the *intrinsic torsion of the H-structure*, and the codomain $H^{0,2}(\mathfrak{h}) = (V \otimes \Lambda^2 V^*)/\mathrm{Im}(\delta)$ as the *intrinsic torsion space.*

Remark: The vector space $H^{0,2}(\mathfrak{h})$ can be regarded as a Spencer cohomology group, which explains the reason for the notation. $\square$

2.2. The case of $H \leq \mathrm{SO}(n)$

Suppose now that $H \leq \mathrm{SO}(n)$. We regard $B \subset F_{\mathrm{SO}(n)}$, where $F_{\mathrm{SO}(n)}$ is the orthonormal frame bundle corresponding to the underlying $\mathrm{SO}(n)$-structure. Let $\theta \in \Omega^1(F_{\mathrm{SO}(n)};\mathfrak{so}(n))$ denote the Levi-Civita connection. On $F_{\mathrm{SO}(n)}$, the Fundamental Lemma of Riemannian Geometry gives

$$d\omega = -\theta \wedge \omega.$$

Let us split $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ with respect to the Killing form of $\mathfrak{so}(n)$. Accordingly, we split

$$\theta|_B = \gamma_H + \tau_H, \tag{2.1}$$

where $\gamma_H \in \Omega^1(B; \mathfrak{h})$ and $\tau_H \in \Omega^1(B; \mathfrak{h}^\perp)$. One can check that γ_H is a connection on the H-structure B, while $\tau_H = t(\omega)$ for some $t\colon B \to \mathfrak{h}^\perp \otimes V^*$. Thus, on B,

$$d\omega = -\gamma_H \wedge \omega + \tfrac{1}{2}\delta(t)(\omega \wedge \omega),$$

and so the torsion of the connection γ_H takes values in $\delta(\mathfrak{h}^\perp \otimes V^*)$. In fact, since $\delta\colon \mathfrak{so}(n) \otimes V^* \to V \otimes \Lambda^2 V^*$ is injective, and since $\Lambda^2 V^* \cong \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, we have

$$V \otimes \Lambda^2 V^* \cong V \otimes (\mathfrak{h} \oplus \mathfrak{h}^\perp) = (V \otimes \mathfrak{h}) \oplus (V \otimes \mathfrak{h}^\perp) \cong \delta(\mathfrak{h} \otimes V^*) \oplus (\mathfrak{h}^\perp \otimes V^*),$$

whence

$$H^{0,2}(\mathfrak{h}) \cong \mathfrak{h}^\perp \otimes V^*.$$

We will return to this formula in §3.1 in the cases $H = \mathrm{U}(3) \leq \mathrm{SO}(6)$ and $H = \mathrm{SU}(3) \leq \mathrm{SO}(6)$.

2.3. Group actions on H-structures

We will be concerned with H-structures on manifolds M equipped with a G-action that preserves the H-structure. In this regard, we make a simple preliminary observation.

A G-action on M induces G-actions on both T^*M and FM. Explicitly, the G-action on FM is

$$g \cdot u = (g^{-1})^* u = u \circ (g^{-1})_*.$$

Note that if $g \in G$ stabilizes a coframe $u \in FM|_x$, then $gx = x$ and $(g^{-1})^* u = u$, so that g acts as the identity on $T^*_x M$. From this, we observe:

Lemma 2.1: Let $P \to M^n$ be an H-structure, where $H \leq \mathrm{SO}(n)$. Suppose M is equipped with a G-action that preserves the H-structure and acts by cohomogeneity-k on M. Then $n - k \leq \dim(G) \leq n + \dim(H)$.

Proof: Since G acts with cohomogeneity-k on M^n, so G acts transitively on the $(n-k)$-dimensional principal orbits in M, so $\dim(G) \geq n - k$.

On the other hand, if g stabilizes a coframe $u \in FM|_x$, then g acts as the identity on T^*_xM. Since g is an isometry (because $H \leq \mathrm{SO}(n)$), so $g = \mathrm{Id}$, so the G-action on P is free. Thus, $\dim(G) \leq \dim(P) = n + \dim(H)$. $\lozenge$

2.4. Cartan's third theorem

In order to prove the local existence of H-structures with desired properties, we encode the data of an H-structure in terms of an "augmented coframing."

Definition: An *augmented coframing* on an n-manifold P is a triple (η, a, b), where $\eta = (\eta^1, \ldots, \eta^n)$ is a coframing on P, and $a = (a^1, \ldots, a^s)\colon P \to \mathbb{R}^s$ and $b = (b^1, \ldots, b^r)\colon P \to \mathbb{R}^r$ are smooth functions.

The functions $a^1, \ldots, a^s\colon P \to \mathbb{R}$ are called the *primary invariants* of the augmented coframing, while the functions $b^1, \ldots, b^r\colon P \to \mathbb{R}$ are called *free derivatives.*

For the rest of this section, we fix index ranges $1 \leq i, j, k \leq n$ and $1 \leq \alpha, \beta \leq s$ and $1 \leq \rho \leq r$.

We will be interested in augmented coframings that satisfy a given set of *structure equations*, by which we mean a set of equations of the form

$$
\begin{aligned}
d\eta^i &= -\tfrac{1}{2} C^i_{jk}(a)\, \eta^j \wedge \eta^k \\
da^\alpha &= F^\alpha_i(a, b)\, \eta^i
\end{aligned}
\tag{2.2}
$$

for some given functions $C^i_{jk}(u) = -C^i_{kj}(u)$ on $\mathbb{R}^s$ and $F^\alpha_i(u, v)$ on $\mathbb{R}^s \times \mathbb{R}^r$.

Let $\pi\colon B \to M^n$ and $\theta \in \Omega^1(B; \mathfrak{h})$ be an H-structure-with-connection. Let $\omega \in \Omega^1(B; \mathbb{R}^n)$ denote the tautological 1-form on B. Then $\eta = (\omega, \theta) = (\omega^i, \theta^j_k)\colon TB \to \mathbb{R}^n \oplus \mathfrak{h}$ is a coframing of B whose exterior derivatives satisfy equations of the form

$$
d\omega^i = -\theta^i_j \wedge \omega^j + T^i_{jk}\, \omega^j \wedge \omega^k \tag{2.3a}
$$

$$
d\theta^i_j = -\theta^i_k \wedge \theta^k_j + R^i_{jk\ell}\, \omega^k \wedge \omega^\ell \tag{2.3b}
$$

$$
dT^i_{jk} = A^i_{jk\ell}(T)\, \theta^\ell + B^i_{jk\ell}\omega^\ell \tag{2.3c}
$$

$$
dR^i_{jk\ell} = C^i_{jk\ell m}(R)\, \theta^m + D^i_{jk\ell m}\omega^m. \tag{2.3d}
$$

for some functions $T = (T^i_{jk})\colon B \to V \otimes \Lambda^2 V^*$ and $R = (R^i_{jk\ell})\colon B \to \mathfrak{h} \otimes \Lambda^2 V^*$. Here, the $A^i_{jk\ell}(T)$ are functions of T alone, while $C^i_{jk\ell m}(R)$ are functions of R alone, and the $B^i_{jk\ell}, D^i_{jk\ell m} \in \Omega^0(B)$ are functions on B that need not depend on T or R.

Conversely, suppose P is a manifold with a coframing $\eta = (\omega, \theta) \colon TP \to \mathbb{R}^n \oplus \mathfrak{h}$ and functions $T = (T^i_{jk}) \colon P \to V \otimes \Lambda^2 V^*$ and $R = (R^i_{jk\ell}) \colon P \to \mathfrak{h} \otimes \Lambda^2 V^*$ satisfying (2.3a)-(2.3d). From (2.3a), there is a submersion $\pi \colon P \to M$ whose fibers are integral manifolds of the (Frobenius) ideal $\langle \omega^1, \ldots, \omega^n \rangle$. Further, one can construct a local diffeomorphism $\sigma \colon P \to FM$ whose image is an H-structure $B \subset FM$ such that σ sends π-fibers to H-orbits and has $\sigma^*(\omega_0) = \omega$, where ω_0 is the tautological form on B.

To prove the local existence of augmented coframings satisfying prescribed structure equations (2.2), we will appeal to a very general result. This theorem, due to Cartan, is a vast generalization of the converse to Lie's Third Theorem on the "integration" of a Lie algebra to a local Lie group. Roughly, the theorem says that the necessary first-order conditions for existence — namely, $d(d\eta^i) = 0$ and $d(da^\alpha) = 0$ — are very close to sufficient.

Let us be more explicit. The equations $d(d\eta^i) = 0$, meaning $d(C^i_{jk}(a)\, \eta^j \wedge \eta^k) = 0$, expand to

$$F^\alpha_j \frac{\partial C^i_{k\ell}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{\ell j}}{\partial u^\alpha} + F^\alpha_\ell \frac{\partial C^i_{jk}}{\partial u^\alpha} = C^i_{mj} C^m_{k\ell} + C^i_{mk} C^m_{\ell j} + C^i_{m\ell} C^m_{jk}. \tag{2.4}$$

Similarly, the equations $d(da^\alpha) = 0$, meaning $d(F^\alpha_i(a, b)\, \eta^i) = 0$, expand to

$$0 = \frac{\partial F^\alpha_i}{\partial v^\rho}\, db^\rho \wedge \eta^i + \frac{1}{2}\left(F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^\ell_{ij} F^\alpha_\ell \right) \eta^i \wedge \eta^j.$$

Since we lack formulas for db^ρ, it is not immediately clear how to satisfy this condition. However, if there exist functions G^ρ_j on $\mathbb{R}^s \times \mathbb{R}^r$ for which

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^\ell_{ij} F^\alpha_\ell = \frac{\partial F^\alpha_i}{\partial v^\rho} G^\rho_j - \frac{\partial F^\alpha_j}{\partial v^\rho} G^\rho_i, \tag{2.5}$$

then $d(da^\alpha) = 0$ reads simply

$$0 = \frac{\partial F^\alpha_i}{\partial v^\rho} \left(db^\rho - G^\rho_j\, \eta^j \right) \wedge \eta^i.$$

Thus, if functions G^ρ_j exist which satisfy (2.5), then there will exist an expression of the db^ρ in terms of η^i that will fulfill $d(da^\alpha) = 0$. We need one last piece of terminology before stating the theorem.

Definition: The *tableau of free derivatives* of the equations (2.2) at a point $(u, v) \in \mathbb{R}^s \times \mathbb{R}^r$ is the linear subspace $A(u, v) \subset \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^s)$ given by

$$A(u,v) = \mathrm{span}\left\{ \frac{\partial F_i^\alpha}{\partial v^\rho}(u,v)\, e_\alpha \otimes f^i \colon 1 \leq \rho \leq r \right\},$$

where here $\{e_1, \ldots, e_s\}$ is a basis of $\mathbb{R}^s$ and $\{f^1, \ldots, f^n\}$ is a basis of $(\mathbb{R}^n)^*$.

Theorem 2.2 (Cartan): Fix real-analytic functions $C^i_{jk} = -C^i_{kj}$ on U and F^α_i on $U \times V$, where $U \subset \mathbb{R}^s$ and $V \subset \mathbb{R}^r$ are open sets. Suppose that:

- The functions C^i_{jk} and F^α_i satisfy (2.4).
- There exist real-analytic functions G^ρ_i on $U \times V$ satisfying (2.5).
- The tableau of free derivatives $A(u, v)$ is involutive, has dimension r, and has Cartan characters $(\widetilde{s}_1, \ldots, \widetilde{s}_n)$ for all $(u, v) \in U \times V$.

Then for any $(a_0, b_0) \in U \times V$, there exists a real-analytic augmented coframing (η, a, b) on an open neighborhood of $0 \in \mathbb{R}^n$ that satisfies (2.2) and has $(a(0), b(0)) = (a_0, b_0)$.

Moreover, augmented coframings satisfying (2.2) depend (modulo diffeomorphism) on $\widetilde{s}_p$ functions of p variables (in the sense of exterior differential systems) where $\widetilde{s}_p$ is the last non-zero Cartan character of $A(u, v)$.

Remark: In outline, the proof of Theorem 2.2 is as follows: One constructs an exterior differential system on the manifold $\mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \times U \times V$ whose integral n-manifolds are in bijection with augmented coframings satisfying (2.2). An application of the Cartan-Kähler Theorem then yields the desired integral n-manifolds, and these depend on $\widetilde{s}_p$ functions of p variables. For details, see [8].

The Cartan-Kähler Theorem requires real-analyticity, which is why Theorem 2.2 does, too. However, since we will be using Theorem 2.2 to construct Einstein metrics — which are real-analytic in harmonic coordinates [12] — the real-analyticity hypothesis is not a significant limitation. $\square$

3. Nearly-Kähler 6-manifolds

3.1. Nearly-Kähler 6-manifolds

At present, several not-quite-equivalent definitions of "nearly-Kähler 6-manifold" exist in the literature, depending on whether one views a nearly-Kähler structure as an SU(3)-structure, as a U(3)-structure, or simply as a Riemannian metric. Moreover, in the context of SU(3)-structures, competing conventions exist for the nearly-Kähler equations. We take this opportunity to contrast the various notions, while also putting our work in its proper context.

In Gray's original formulation [18], a nearly-Kähler structure on a smooth 6-manifold M^6 referred to a certain kind of U(3)-structure on M^6. A U(3)-structure $B \subset FM$ is equivalent to specifying on M a triple (g, J, Ω) consisting of a Riemannian metric g, an almost-complex structure J, and a non-degenerate 2-form Ω satisfying the compatibility condition $g(u, v) = \Omega(u, Jv)$. A 6-manifold with U(3)-structure is called an *almost-Hermitian* 6-manifold.

In [19], the intrinsic torsion space of a U(3)-structure was calculated to be of the form

$$H^{0,2}(\mathfrak{u}(3)) = \mathfrak{u}(3)^{\perp} \otimes \mathbb{R}^6 = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where W_1, W_2, W_3, W_4 are certain irreducible U(3)-modules of real dimensions 2, 16, 12, 6, respectively.

A U(3)-structure was then defined to be *nearly-Kä* if its intrinsic torsion function $T\colon B \to H^{0,2}(\mathfrak{u}(3))$ takes values in W_1, the lowest-dimensional piece in the decomposition. This is equivalent (see [18], [25]) to requiring that ∇J satisfies $(\nabla_X J)(X) = 0$ for all vector fields $X \in \Gamma(TM)$, or equivalently that $\nabla\Omega = \frac{1}{3}d\Omega$, where ∇ is the Levi-Civita connection of the metric g.

Remark: Note that an almost-Hermitian 6-manifold is *Kähler* if its intrinsic torsion is identically zero. Equivalently, $\nabla J = 0$, or equivalently $\nabla\Omega = 0$. □

In this work, we will adopt a different definition of "nearly-Kähler" also encountered in the literature (see, e.g., [7] and [16]) which entails an additional bit of structure. For us, a "nearly-Kähler structure" refers to a certain kind of SU(3)-structure.

An SU(3)-structure $B \subset FM$ is equivalent to specifying on M a triple (g, J, Ω) as above together with a (3, 0)-form Υ such that $\Upsilon \wedge \overline{\Upsilon} = -\frac{4}{3}i\,\Omega^3$. In fact, the data (Ω, Υ), subject to appropriate algebraic conditions, is enough to reconstruct (g, J) (see [27]). Thus, an SU(3)-structure may be regarded as a pair $\Omega \in \Omega^2(M)$ and $\Upsilon \in \Omega^3(M; \mathbb{C})$ such that at each $x \in M$, there is an isomorphism $u\colon T_xM \to \mathbb{R}^6$ for which

$$\begin{aligned}
\Omega|_x &= u^*(dx^1 \wedge dx^4 + dx^2 \wedge dx^5 + dx^3 \wedge dx^6) \\
\Upsilon|_x &= u^*(dz^1 \wedge dz^2 \wedge dz^3)
\end{aligned}$$

where $(z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)$ are the standard coordinates on $\mathbb{C}^3 \cong \mathbb{R}^6$.

One can show [10] that the intrinsic torsion space of an SU(3)-structure is of the form

$$H^{0,2}(\mathfrak{su}(3)) = \mathfrak{su}(3)^\perp \otimes \mathbb{R}^6 = X_0^+ \oplus X_0^- \oplus X_2^+ \oplus X_2^- \oplus X_3 \oplus X_4 \oplus X_5,$$

where $X_0^\pm$, $X_2^\pm$, X_3, X_4, X_5 are certain irreducible SU(3)-modules of real dimensions 1, 8, 12, 6, 6, respectively. Following [2], we can give a more concrete description of $H^{0,2}(\mathfrak{su}(3))$ via exterior algebra. Indeed, the SU(3)-modules $\Lambda^2(\mathbb{R}^6)$ and $\Lambda^3(\mathbb{R}^6)$ decompose into irreducibles as ([2], [15])

$$\begin{aligned}
\Lambda^2(\mathbb{R}^6) &= \mathbb{R}\,\Omega \oplus \Lambda^2_6 \oplus \Lambda^2_8 \\
\Lambda^3(\mathbb{R}^6) &= \mathbb{R}\,\mathrm{Re}(\Upsilon) \oplus \mathbb{R}\,\mathrm{Im}(\Upsilon) \oplus \Lambda^3_6 \oplus \Lambda^3_{12},
\end{aligned}$$

where

$$\begin{aligned}
\Lambda^2_6 &= \{*(\alpha \wedge \mathrm{Re}(\Upsilon)) \colon \alpha \in \Lambda^1\} \\
\Lambda^2_8 &= \{\beta \in \Lambda^2 \colon \beta \wedge \mathrm{Re}(\Upsilon) = 0 \text{ and } *\beta = -\beta \wedge \Omega\} \\
\Lambda^3_6 &= \{\alpha \wedge \Omega \colon \alpha \in \Lambda^1\} \\
\Lambda^3_{12} &= \{\gamma \in \Lambda^3 \colon \gamma \wedge \Omega = 0 \text{ and } \gamma \wedge \mathrm{Re}(\Upsilon) = 0 \text{ and } \gamma \wedge \mathrm{Im}(\Upsilon) = 0\}.
\end{aligned}$$

This gives the description

$$H^{0,2}(\mathfrak{su}(3)) \cong \mathbb{R} \oplus \mathbb{R} \oplus \Lambda^2_8 \oplus \Lambda^2_8 \oplus \Lambda^3_{12} \oplus \Lambda^1 \oplus \Lambda^1.$$

It can be shown [10] that the intrinsic torsion of the SU(3)-structure can be completely encoded in the *exterior* derivatives of Ω and Υ. Moreover, borrowing the notation of [15], these exterior derivatives decompose as

$$\begin{aligned}
d\Omega &= 3\tau_0\,\mathrm{Re}(\Upsilon) + 3\hat{\tau}_0\,\mathrm{Im}(\Upsilon) + \tau_3 + \tau_4 \wedge \Omega \\
d\,\mathrm{Re}(\Upsilon) &= 2\hat{\tau}_0\,\Omega^2 + \tau_5 \wedge \mathrm{Re}(\Upsilon) + \tau_2 \wedge \Omega \\
d\,\mathrm{Im}(\Upsilon) &= -2\tau_0\,\Omega^2 - J\tau_5 \wedge \mathrm{Re}(\Upsilon) + \hat{\tau}_2 \wedge \Omega,
\end{aligned}$$

where $\tau_0, \hat{\tau}_0 \in \Omega^0$, $\tau_2, \hat{\tau}_2 \in \Omega^2_8$, $\tau_3 \in \Omega^3_{12}$, and $\tau_4, \tau_5 \in \Gamma(TM)$, and $\Omega^k_\ell = \Gamma(\Lambda^k_\ell(T^*M))$. This leads to:

Definition: Let M^6 be a real 6-manifold. A *nearly-Kähler structure* on M is an SU(3)-structure $B \subset FM$ whose intrinsic torsion function $T \colon B \to H^{0,2}(\mathfrak{su}(3))$ takes values in X_0^-.

In other words, a nearly-Kähler structure on M is an SU(3)-structure (Ω, Υ) having

$$\tau_0 = \tau_2 = \hat{\tau}_2 = \tau_3 = \tau_4 = \tau_5 = 0.$$

It is easy to check that this forces the remaining torsion form $\hat{\tau}_0 = c$ to be constant.

Thus, a nearly-Kähler structure is an SU(3)-structure (Ω, Υ) that satisfies

$$\begin{aligned} d\Omega &= 3c\,\mathrm{Im}(\Upsilon) \\ d\,\mathrm{Re}(\Upsilon) &= 2c\,\Omega^2 \\ d\,\mathrm{Im}(\Upsilon) &= 0. \end{aligned}$$

Of course, the third equation is a consequence of the first.

Remark: Note that other works (e.g. [16]) instead take the condition $\hat{\tau}_0 = \tau_2 = \hat{\tau}_2 = \tau_3 = \tau_4 = \tau_5 = 0$ as the definition of "nearly-Kähler." In that case, the torsion form $\tau_0 = c$ is constant. □

Note that a nearly-Kähler structure has $c = 0$ if and only if it is Calabi-Yau. Those with $c \neq 0$ are sometimes called *strict* nearly-Kähler structures. In this case, by rescaling the metric, we may take the constant $c = 1$. For simplicity, and following [15] and [16], we enact the following:

Convention: In this work, by a "nearly-Kähler structure" *we will always mean a "strict nearly-Kähler structure, scaled so that $c = 1$."*

3.2. The coisotropic orbit condition

We will need to understand how 4-planes in $\mathbb{R}^6$ behave under the usual SU(3)-action. This requires some linear algebraic preliminaries.

Consider $(\mathbb{R}^6, g_0, \Omega_0)$ with the standard metric g_0, symplectic form Ω_0, and orientation. Let $*$ denote the corresponding Hodge star operator. We let $(e_1, \ldots, e_6)$ be the standard basis of $\mathbb{R}^6$, and we identify $\mathbb{C}^3 \cong \mathbb{R}^6$ via $(z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)$. Explicitly,

$$\begin{aligned} g_0 &= (dx^1)^2 + \cdots + (dx^6)^2 \\ \Omega_0 &= dx^1 \wedge dx^4 + dx^2 \wedge dx^5 + dx^3 \wedge dx^6. \end{aligned}$$

In particular, we observe that

$$*\Omega_0 = \tfrac{1}{2}\,\Omega_0 \wedge \Omega_0. \tag{3.1}$$

Let $V_k(\mathbb{R}^6)$ denote the Stiefel manifold of ordered orthonormal k-frames in $\mathbb{R}^6$, and let $\mathrm{Gr}_k(\mathbb{R}^6)$ denote the Grassmannian of real k-planes in $\mathbb{R}^6$. Recall that the *symplectic complement* and *orthogonal complement* of a k-plane $E \in \mathrm{Gr}_k(\mathbb{R}^6)$ are the respective subspaces

$$\begin{aligned} E^\Omega &:= \{v \in \mathbb{R}^6 \colon \Omega_0(v,w) = 0,\ \forall w \in E\} \\ E^\perp &:= \{v \in \mathbb{R}^6 \colon g_0(v,w) = 0,\ \forall w \in E\}. \end{aligned}$$

We say that E is *isotropic* if $E \subset E^\Omega$, and that E is *coisotropic* if $E \supset E^\Omega$. Using (3.1), we see that for a 4-plane $E \in \mathrm{Gr}_4(\mathbb{R}^6)$:

$$\begin{aligned} (\Omega_0 \wedge \Omega_0)|_E = 0 &\iff E^\perp \text{ is an isotropic 2-plane} \\ &\iff E \text{ is a coisotropic 4-plane.} \end{aligned}$$

In particular, coisotropic 4-planes are in bijection with isotropic 2-planes.

We now seek to understand the $\mathrm{SU}(3)$-action on 2-planes (equivalently, 4-planes) in $\mathbb{R}^6$. For $\theta \in [0,\pi]$, let us set

$$\begin{aligned} V_2(\theta) &= \mathrm{SU}(3)\cdot(e_1, \cos(\theta)e_4 + \sin(\theta)e_2) \subset V_2(\mathbb{R}^6) \\ \mathrm{Gr}_2(\theta) &= \mathrm{SU}(3)\cdot \mathrm{span}(e_1, \cos(\theta)e_4 + \sin(\theta)e_2) \subset \mathrm{Gr}_2(\mathbb{R}^6). \end{aligned}$$

Of particular interest to us is the orbit

$$\mathrm{Gr}_2(\tfrac{\pi}{2}) = \mathrm{SU}(3)\cdot\mathrm{span}(e_1,e_2) = \{E \in \mathrm{Gr}_2(\mathbb{R}^6) \colon E \text{ isotropic}\}.$$

Lemma 3.1:

(a) Every $(v,w) \in V_2(\mathbb{R}^6)$ belongs to exactly one of the orbits $V_2(\theta)$, where $\theta \in [0,\pi]$. The transitive $\mathrm{SU}(3)$-actions on $V_2(0)$ and $V_2(\pi)$ have stabilizer $\mathrm{SU}(2)$. For $\theta \in (0,\pi)$, the transitive $\mathrm{SU}(3)$-action on $V_2(\theta)$ is free.

(b) Every $E \in \mathrm{Gr}_2(\mathbb{R}^6)$ belongs to exactly one of the orbits $\mathrm{Gr}_2(\theta)$, where $\theta \in [0,\pi)$. The transitive $\mathrm{SU}(3)$-action on $\mathrm{Gr}_2(0) \cong \mathbb{CP}^2$ has stabilizer $\mathrm{U}(2)$. For $\theta \in (0,\pi)$, the transitive $\mathrm{SU}(3)$-action on $\mathrm{Gr}_2(\theta)$ has stabilizer $\mathrm{O}(2)$.

(c) In particular, $\mathrm{SU}(3)$ acts transitively on

$$\mathrm{Gr}_2(\tfrac{\pi}{2}) \cong \{E \in \mathrm{Gr}_4(\mathbb{R}^6) \colon E \text{ coisotropic}\}$$

with stabilizer

$$\mathrm{O}(2) = \left\{ \begin{pmatrix} \cos\theta & \mp\sin\theta & 0 & & & \\ \sin\theta & \mp\cos\theta & 0 & & & \\ 0 & 0 & \pm 1 & & & \\ & & & \cos\theta & \pm\sin\theta & 0 \\ & & & \sin\theta & \mp\cos\theta & 0 \\ & & & 0 & 0 & \pm 1 \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$
$$\leq \mathrm{SU}(3) \leq \mathrm{SO}(6).$$

Proof: (a) We first show that every $(v, w) \in V_2(\mathbb{R}^6)$ belongs to some $V_2(\theta)$.

Let $(v, w) \in V_2(\mathbb{R}^6)$. Since $\mathrm{SU}(3)$ acts transitively on $V_1(\mathbb{R}^6) \cong \mathbb{S}^5$, there exists $A \in \mathrm{SU}(3)$ with $Av = e_1$, so $A \cdot (v, w) = (e_1, Aw)$. Since $Aw \perp e_1$, so $Aw \in \mathbb{R}e_4 \oplus \mathbb{C}^2$, where $\mathbb{C}^2 = \mathrm{span}_{\mathbb{R}}(e_2, e_5, e_3, e_6)$.

Now, the subgroup of $\mathrm{SU}(3)$ that fixes $e_1 \in \mathbb{R}^6$ is a copy of $\mathrm{SU}(2)$. This $\mathrm{SU}(2)$ acts on the orthogonal $\mathbb{R}e_4 \oplus \mathbb{C}^2$ in the usual way: it acts trivially $\mathbb{R}e_4$ and in the standard way on $\mathbb{C}^2$. In particular, every $x \in \mathbb{R}e_4 \oplus \mathbb{C}^2$ is $\mathrm{SU}(2)$-conjugate to an element of the form $c_4e_4 + c_2e_2$, where $c_4 \in \mathbb{R}$ and $c_2 \geq 0$.

Thus, there exists $B \in \mathrm{SU}(2) \leq \mathrm{SU}(3)$ with $B \cdot Aw = c_4e_4 + c_2e_2$ for some $c_4 \in \mathbb{R}$ and $c_2 \geq 0$, so $BA \cdot (v, w) = (e_1, c_4e_4 + c_2e_2)$. Since $1 = \|w\|^2 = \|BAw\|^2 = c_4^2 + c_2^2$, so we may write $(c_4, c_2) = (\cos\theta, \sin\theta)$ for some $\theta \in [0, \pi]$. Thus, $(v, w) \in V_2(\theta)$.

To see that the orbits are disjoint, note that the composition $\Omega_0 : V_2(\mathbb{R}^6) \hookrightarrow \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}$ is an $\mathrm{SU}(3)$-invariant function, so is constant on the $\mathrm{SU}(3)$-orbits $V_2(\theta)$. Indeed,

$$\Omega_0(e_1, \cos(\theta)e_4 + \sin(\theta)e_2) = \cos(\theta).$$

In particular, if $(v, w) \in V_2(\theta_1) \cap V_2(\theta_2)$, then $\cos(\theta_1) = \cos(\theta_2)$, so $\theta_1 = \theta_2$.

Note that $A \in \mathrm{SU}(3)$ stabilizes $(e_1, \cos(\theta)e_4 + \sin(\theta)e_2)$ if and only if $Ae_1 = e_1$ (so $Ae_4 = e_4$) and $\sin(\theta)Ae_2 = \sin(\theta)e_2$. For $\theta = 0$ and $\theta = \pi$, this describes $\mathrm{SU}(2)$. For $\theta \in (0, \pi)$, this describes the identity subgroup.

(b) This follows from part (a) and the fibration $\mathrm{O}(2) \to V_2(\mathbb{R}^6) \to \mathrm{Gr}_2(\mathbb{R}^6)$.

(c) Note that if $A \in \mathrm{SU}(3)$ stabilizes $\mathrm{span}(e_1, e_2)$, then A also stabilizes $\mathrm{span}(e_4, e_5)$, which forces A to lie in the $\mathrm{O}(2)$ subgroup described above. $\Diamond$

Thus, there are two geometrically natural first-order conditions that one could impose on the real 4-folds in a nearly-Kähler 6-manifold. In one direction, we could ask that the 4-fold be pseudo-holomorphic (normal planes lie in $\mathrm{Gr}_2(0)$). However, such submanifolds do not exist, even locally [6]. In the other direction, we could ask that the 4-fold be coisotropic (normal planes lie in $\mathrm{Gr}_2(\frac{\pi}{2})$).

There is, however, another reason to study coisotropic 4-folds: any *complete* nearly-Kähler 6-manifold of cohomogeneity-two must have coisotropic principal orbits, as we now show.

Lemma 3.2: Let N^n be a compact G-homogeneous Riemannian manifold. If $\chi \in \Omega^n(N)$ is a G-invariant exact n-form on N, then $\chi = 0$.

Proof: Let χ be such a G-invariant exact n-form. Write $\chi = f\,\mathrm{vol}_N$ for some function $f \in C^\infty(N)$. Since χ is G-invariant, so f is G-invariant. Since the G-action is transitive, so f is constant. Since N is compact and χ is exact, Stokes' Theorem gives

$$0 = \int_N \chi = \int_N f\,\mathrm{vol}_N = f \cdot \mathrm{vol}(N).$$

Thus, $f = 0$, whence $\chi = 0$. $\Diamond$

Proposition 3.3: Let M^6 be a nearly-Kähler 6-manifold equipped with a G-action of cohomogeneity-two that preserves the $\mathrm{SU}(3)$-structure, where $G \leq \mathrm{Isom}(M, g)$ is closed.

If M is complete, then M is compact, G is compact, the quotient space M/G is compact Hausdorff, and the principal G-orbits in M are coisotropic.

Proof: Suppose M is complete. Since M is Einstein of positive scalar curvature, by Bonnet-Myers, M is compact. By Myers-Steenrod [21], the isometry group $\mathrm{Isom}(M, g)$ is compact, so G is compact.

Let N^4 be any principal G-orbit in M. Note that N is a compact, G-homogeneous Riemannian manifold. Moreover, $\Omega^2 = \frac{1}{2}d(\mathrm{Im}(\Upsilon))$ is a G-invariant exact 4-form on N. Thus, by Lemma 3.2, we have $\Omega^2|_N = 0$, meaning that N is coisotropic. $\Diamond$

Remark: If, moreover, M is connected and simply-connected, and the Lie group G is connected, then the quotient space M/G is simply-connected. See, e.g., Chapter II: Corollary 6.3 (page 91) of [5]. □

Finally, although we will not need it here, we remark that the same argument establishes:

Proposition 3.4: Let M^6 be a nearly-Kähler 6-manifold equipped with a G-action of cohomogeneity-three that preserves the SU(3)-structure, where $G \leq \mathrm{Isom}(M, g)$ is closed.

If M is complete, then M is compact, G is compact, the quotient space M/G is compact Hausdorff, and the 3-form $\mathrm{Im}(\Upsilon)$ vanishes on the principal G-orbits.

4. Moving frame setup

4.1. The first structure equations of a nearly-Kähler 6-manifold

Let $\pi\colon B \to M$ be an SU(3)-structure on a 6-manifold M. Let $\omega = (\omega^1, \ldots, \omega^6) \in \Omega^1(B; \mathbb{R}^6)$ denote the tautological 1-form. We will identify $\mathbb{C}^3 \cong \mathbb{R}^6$ via

$$(z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)$$

and let $\zeta = (\zeta^1, \zeta^2, \zeta^3) \in \Omega^1(B; \mathbb{C}^3)$ denote the $\mathbb{C}$-valued tautological 1-form:

$$(\zeta^1, \zeta^2, \zeta^3) = (\omega^1 + i\omega^4, \omega^2 + i\omega^5, \omega^3 + i\omega^6).$$

Since B is an SU(3)-structure, the 6-manifold M is endowed with a metric g, a non-degenerate 2-form Ω, and a complex volume form Υ. Pulled up to B, these are exactly:

$$\begin{aligned}
\pi^*g &= \textstyle\sum(\zeta_j \circ \overline{\zeta}_j)^2 = (\omega^1)^2 + \cdots + (\omega^6)^2 \\
\pi^*\Omega &= \tfrac{i}{2}\textstyle\sum \zeta_j \wedge \overline{\zeta}_j = \omega^{14} + \omega^{25} + \omega^{36} \\
\pi^*\Upsilon &= \zeta_1 \wedge \zeta_2 \wedge \zeta_3 = (\omega^1 + i\omega^4) \wedge (\omega^2 + i\omega^5) \wedge (\omega^3 + i\omega^6).
\end{aligned}$$

In the special case where the SU(3)-structure B is nearly-Kähler, the exterior derivatives $d\zeta_i$ satisfy the *first structure equations* (see [7], [29]) given by

$$d\zeta_i = -\kappa_{i\bar{\ell}} \wedge \zeta_\ell + \overline{\zeta_j \wedge \zeta_k} \tag{4.1}$$

where $\kappa = (\kappa_{i\bar{\ell}}) \in \Omega^1(B; \mathfrak{su}(3))$ is a connection 1-form, and where (i, j, k) is an even permutation of $(1, 2, 3)$. In terms of the basis $(\omega^1, \ldots, \omega^6)$ for the π-semibasic 1-forms, the structure equations (4.1) read

$$\begin{aligned} d\begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} &= -\left(\begin{array}{ccc|ccc} 0 & \alpha_3 & -\alpha_2 & -\beta_{11} & -\beta_{12} & -\beta_{13} \\ -\alpha_3 & 0 & \alpha_1 & -\beta_{21} & -\beta_{22} & -\beta_{23} \\ \alpha_2 & -\alpha_1 & 0 & -\beta_{31} & -\beta_{32} & -\beta_{33} \\ \hline \beta_{11} & \beta_{12} & \beta_{13} & 0 & \alpha_3 & -\alpha_2 \\ \beta_{21} & \beta_{22} & \beta_{23} & -\alpha_3 & 0 & \alpha_1 \\ \beta_{31} & \beta_{32} & \beta_{33} & \alpha_2 & -\alpha_1 & 0 \end{array}\right) \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} \\ &\quad + \begin{pmatrix} \omega^{23} - \omega^{56} \\ -\omega^{13} + \omega^{46} \\ \omega^{12} - \omega^{45} \\ -\omega^{26} + \omega^{35} \\ \omega^{16} - \omega^{34} \\ -\omega^{15} + \omega^{24} \end{pmatrix} \end{aligned} \tag{4.2}$$

where $\alpha_i, \beta_{ij} \in \Omega^1(B; \mathbb{R})$ are connection 1-forms with $\beta_{ij} = \beta_{ji}$ and $\sum \beta_{ii} = 0$.

In this work, however, it will be convenient to express (4.1) in a different form. Indeed, in light of the $\mathrm{O}(2)$-representation on $\mathbb{R}^6$ described in Lemma 3.1(c), we will often prefer the basis of π-semibasic 1-forms given by $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6)$, where

$$\begin{aligned} \eta &= \omega^1 + i\omega^2 \\ \theta &= \omega^4 + i\omega^5. \end{aligned}$$

In terms of this basis, (4.1) is equivalent to

$$
d\begin{pmatrix}\eta\\ \omega^3\\ \theta\\ \omega^6\end{pmatrix} = -\left(\begin{array}{ccc|ccc} -i\alpha & 0 & 2i\xi_1 & -\xi_0 & -\xi_3 & -2\xi_2 \\ i\overline{\xi}_1 & -i\xi_1 & 0 & -\overline{\xi}_2 & -\xi_2 & 2\xi_0 \\ \hline \xi_0 & \xi_3 & 2\xi_2 & -i\alpha & 0 & 2i\xi_1 \\ \overline{\xi}_2 & \xi_2 & -2\xi_0 & i\overline{\xi}_1 & -i\xi_1 & 0 \end{array}\right) \wedge \begin{pmatrix}\eta\\ \overline{\eta}\\ \omega^3\\ \theta\\ \overline{\theta}\\ \omega^6\end{pmatrix}
$$

$$
+\, i\begin{pmatrix} -\eta\wedge\omega^3 + \theta\wedge\omega^6 \\ \frac{1}{2}(\eta\wedge\overline{\eta} - \theta\wedge\overline{\theta}) \\ -\omega^3\wedge\theta + \eta\wedge\omega^6 \\ \frac{1}{2}(\overline{\theta}\wedge\eta - \theta\wedge\overline{\eta}) \end{pmatrix} \tag{4.3}
$$

where $\alpha, \xi_0 \in \Omega^1(B;\mathbb{R})$ and $\xi_1, \xi_2, \xi_3 \in \Omega^1(B;\mathbb{C})$ are connection 1-forms. The structure equations (4.3) will be central to our calculations.

4.2. Frame adaptation: the O(2)-bundle *P*

Let M be a nearly-Kähler 6-manifold acted upon by a connected Lie group G with cohomogeneity-two. We suppose that this G-action preserves the SU(3)-structure and that the principal G-orbits are coisotropic. For simplicity, the following two conventions will be in force for the rest of this work.

Convention 4.1: Without loss of generality, we suppose that G acts faithfully on M, and that G is a closed subgroup of the isometry group of M. □

Convention 4.2: We restrict our attention entirely to the principal locus of M, by which we mean the union of principal G-orbits Gx in M. Henceforth, when we refer to the manifold M, *we shall always mean the principal locus of M*. □

We begin our study by adapting coframes to the foliation of M by coisotropic 4-folds. Define the subbundle $P \subset B$ of SU(3)-coframes $u = (u^1, \ldots, u^6)\colon T_xM \to \mathbb{R}^6$ for which $T_xGx = \mathrm{Ker}(u^1, u^2)$. In other words, letting $\{e_1, \ldots, e_6\}$ denote the standard basis of $\mathbb{R}^6$, we set

$$P = \{u \in B\colon u(T_xGx) = \mathrm{span}(e_3, e_4, e_5, e_6)\} \subset B.$$

Since SU(3) acts transitively on the Grassmannian of coisotropic 4-planes in $\mathbb{R}^6$ (Lemma 3.1), this adaptation is well-defined. Note that P is an

O(2)-subbundle, where the inclusion $\mathrm{O}(2) \leq \mathrm{SU}(3) \hookrightarrow \mathrm{GL}_6(\mathbb{R})$ is the one described in Lemma 3.1(c).

Remark: The Lie group G is contained in the group $\mathrm{Aut}_{\mathrm{O}(2)}$ of automorphisms that preserve the foliation of M by coisotropic 4-folds, which is itself contained in the full automorphism group $\mathrm{Aut}_{\mathrm{SU}(3)}$ of the SU(3)-structure:

$$G \leq \mathrm{Aut}_{\mathrm{O}(2)}(M) \leq \mathrm{Aut}_{\mathrm{SU}(3)}(M).$$

By Lemma 2.1, we see that:

$$4 \leq \dim(G) \leq 5, \quad 4 \leq \dim(\mathrm{Aut}_{\mathrm{O}(2)}) \leq 7, \quad 4 \leq \dim(\mathrm{Aut}_{\mathrm{SU}(3)}) \leq 14. \qquad \square$$

Henceforth, we work on the O(2)-subbundle $P \subset B$ and use the same letter $\pi \colon P \to M$ to denote the restricted projection map. Now, the connection 1-form $\kappa \in \Omega^1(B; \mathfrak{su}(3))$ does not remain a connection form when restricted to P. Indeed, for a choice of splitting $\mathfrak{su}(3) = \mathfrak{so}(2) \oplus W$, the 1-form $\kappa|_P \in \Omega^1(P; \mathfrak{su}(3))$ decomposes as

$$\kappa|_P = \gamma_{\mathrm{O}(2)} + \tau_{\mathrm{O}(2)},$$

where $\gamma_{\mathrm{O}(2)} \in \Omega^1(P; \mathfrak{so}(2))$ is a connection 1-form and $\tau_{\mathrm{O}(2)} \in \Omega^1(P; W)$ is π-semibasic. In terms of the basis $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6)$, this splitting reads

$$\begin{aligned}
\kappa|_P = &\left(\begin{array}{ccc|ccc} -i\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -i\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&+ \left(\begin{array}{ccc|ccc} 0 & 0 & 2i\xi_1 & -\xi_0 & -\xi_3 & -2\xi_2 \\ i\overline{\xi}_1 & -i\xi_1 & 0 & -\overline{\xi}_2 & -\xi_2 & 2\xi_0 \\ \hline \xi_0 & \xi_3 & 2\xi_2 & 0 & 0 & 2i\xi_1 \\ \overline{\xi}_2 & \xi_2 & -2\xi_0 & i\overline{\xi}_1 & -i\xi_1 & 0 \end{array}\right)
\end{aligned}$$

In particular, α is a connection 1-form for the O(2)-bundle $\pi \colon P \to M$, so $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \alpha)$ is a coframing on P.

On the other hand, the 1-forms $\xi_0 \in \Omega^1(P; \mathbb{R})$ and $\xi_1, \xi_2, \xi_3 \in \Omega^1(P; \mathbb{C})$

are π-semibasic, so that we may write

$$\begin{aligned} 2\xi_1 &= a_{11}\eta + a_{12}\overline{\eta} + a_{13}\omega^3 + a_{14}\theta + a_{15}\overline{\theta} + a_{16}\omega^6 \\ 2\xi_2 &= a_{21}\eta + a_{22}\overline{\eta} + a_{23}\omega^3 + a_{24}\theta + a_{25}\overline{\theta} + a_{26}\omega^6 \\ \xi_3 &= a_{31}\eta + a_{32}\overline{\eta} + a_{33}\omega^3 + a_{34}\theta + a_{35}\overline{\theta} + a_{36}\omega^6 \\ \xi_0 &= a_{01}\eta + a_{02}\overline{\eta} + a_{03}\omega^3 + a_{04}\theta + a_{05}\overline{\theta} + a_{06}\omega^6 \end{aligned} \tag{4.4}$$

for some 24 G-invariant functions $a_{ij} \colon P \to \mathbb{C}$. We will refer to these 24 functions as the *torsion functions* of the O(2)-structure. In the next section, we will see (Lemma 4.4) that they are not independent of one another.

4.3. The torsion of the O(2)-structure

We continue with the setup from §4.2. The purpose of this section is to derive relations (Lemma 4.4) on the 24 functions $a_{ij} \colon P \to \mathbb{C}$ of (4.4). In the next section, we will use this information to show (Proposition 4.5) that the acting Lie group G is 4-dimensional and non-abelian.

We begin with the following observation: Unlike a generic O(2)-structure, the O(2)-structures in our situation enjoy a special geometric feature. Namely, the (real) 4-plane field $\mathrm{Ker}(\omega^1, \omega^2) = \mathrm{Ker}(\eta, \overline{\eta})$ on M^6 is integrable, its leaf space is the orbit space $\Sigma = M/G$ (recall Convention 4.2), and the quadratic form $(\omega^1)^2 + (\omega^2)^2 = \eta \circ \overline{\eta}$ descends to a Riemannian metric on Σ. Consequently:

Lemma 4.3: There exist a 1-form $\phi \in \Omega^1(P;\mathbb{R})$ and a function $K \in \Omega^0(P;\mathbb{R})$ such that

$$d\eta = i\,\phi \wedge \eta \tag{4.5}$$

$$d\phi = \tfrac{i}{2}\,K\,\eta \wedge \overline{\eta}. \tag{4.6}$$

Proof: The quadratic form $\eta \circ \overline{\eta} \in \Gamma(\mathrm{Sym}^2(T^*P))$ is both O(2)-invariant and G-invariant, so descends to a Riemannian metric g_Σ on the surface Σ. Let $\varpi \colon F \to \Sigma$ denote the orthonormal frame bundle of g_Σ, and let $\widetilde{\eta} \in \Omega^1(F;\mathbb{C})$ denote the $\mathbb{C}$-valued tautological 1-form on F.

By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\widetilde{\phi} \in \Omega^1(F)$, the Levi-Civita connection of g_Σ, for which

$$d\widetilde{\eta} = i\,\widetilde{\phi} \wedge \widetilde{\eta}.$$

Now, the quotient map $\mathrm{pr}\colon M \to \Sigma$ induces a map $\widetilde{\mathrm{pr}}\colon P \to F$ via $\widetilde{\mathrm{pr}}(u)(v) := u(\widetilde{v})$, where $\widetilde{v} \in TM$ is the horizontal lift of $v \in T\Sigma$. Unwinding the definitions shows that $\widetilde{\mathrm{pr}}^*(\widetilde{\eta}) = \eta$, whence the equation $d\eta = i\,\widetilde{\mathrm{pr}}^*(\widetilde{\phi}) \wedge \eta$ holds on P. Setting $\phi := \widetilde{\mathrm{pr}}^*(\widetilde{\phi})$ establishes (4.5).

Equation (4.6) now follows by differentiating (4.5). That is, K is the Gauss curvature of g_Σ. $\Diamond$

Lemma 4.4: There exist seven $\mathbb{C}$-valued functions $p_1, p_2, p_3, p_4, q_1, q_2, r : P \to \mathbb{C}$ and four $\mathbb{R}$-valued functions $h_1, h_2, h_3, h_4\colon P \to \mathbb{R}$ for which

$$\begin{aligned} 2\xi_1 &= h_1\eta - ih_2\theta + q_1\overline{\theta} + p_1\omega^3 + p_2\omega^6 \\ 2\xi_2 &= ih_4\eta + (h_3 - i)\,\theta + q_2\overline{\theta} - ip_2\omega^3 + p_3\omega^6 \\ \xi_3 &= p_4\theta - ir\overline{\theta} - iq_1\omega^3 + q_2\omega^6 \\ \xi_0 &= \overline{p}_4\theta + p_4\overline{\theta} - h_2\omega^3 + h_3\omega^6. \end{aligned} \tag{4.7}$$

Their exterior derivatives modulo $\langle \omega^1, \omega^2\rangle = \langle \eta, \overline{\eta}\rangle$ satisfy

$$\begin{aligned} dp_1 &\equiv ip_1\phi & dq_1 &\equiv 2iq_1\phi & dh_1 &\equiv 0 \\ dp_2 &\equiv ip_2\phi & dq_2 &\equiv 2iq_2\phi & dh_2 &\equiv 0 \\ dp_3 &\equiv ip_3\phi & & & dh_3 &\equiv 0 \\ dp_4 &\equiv ip_4\phi & dr &\equiv 3ir\phi & dh_4 &\equiv 0. \end{aligned} \tag{4.8}$$

Moreover, we have the formula

$$\phi = \alpha + (h_1 + 1)\,\omega^3 - h_4\omega^6. \tag{4.9}$$

The upshot is that we have re-expressed the torsion of the O(2)-structure in terms of just seven $\mathbb{C}$-valued functions and four $\mathbb{R}$-valued functions p_i, q_i, r, h_i on P, all of which are G-invariant and O(2)-equivariant (for the O(2)-actions indicated by (4.8)). Accordingly, we will refer to

$$T = (p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4)\colon P \to \mathbb{C}^7 \oplus \mathbb{R}^4$$

as the (*intrinsic*) *torsion of the O*(2)*-structure*. Geometrically, the function T describes the 1-jet of the O(2)-structure (or the 2-jet of the underlying SU(3)-structure) up to diffeomorphism.

Equation (4.9) shows in particular that $(\omega^1, \ldots, \omega^6, \phi)\colon TP \to \mathbb{R}^7$ is a coframing on P. Going forward, we prefer to work with the coframings $(\omega^1, \ldots, \omega^6, \phi)$ and $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta},\ \omega^6, \phi)$ rather than with the original

$(\omega^1, \dots, \omega^6, \alpha)$ and $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \alpha)$.

Proof of Lemma 4.4: From (4.3) and (4.4), we have

$$\begin{aligned}(4.10)\quad d\eta = {} & i(\alpha + ia_{01}\theta + ia_{31}\overline{\theta} + (a_{11}+1)\,\omega^3 + ia_{21}\,\omega^6) \wedge \eta \\ & - ia_{12}\,\overline{\eta} \wedge \omega^3 + a_{22}\,\overline{\eta} \wedge \omega^6 + a_{32}\,\overline{\eta} \wedge \overline{\theta} + a_{02}\,\overline{\eta} \wedge \theta \\ & + (a_{05} - a_{34})\,\overline{\theta} \wedge \theta + (a_{03} + ia_{14})\,\omega^3 \wedge \theta + (a_{33} + ia_{15})\,\omega^3 \wedge \overline{\theta} \\ & + (a_{24} - a_{06} + i)\,\theta \wedge \omega^6 - (a_{25} - a_{36})\,\omega^6 \wedge \overline{\theta} \\ & + (ia_{16} + a_{23})\,\omega^3 \wedge \omega^6.\end{aligned}$$

Equating this with $d\eta = i\phi \wedge \eta$ yields the following relations:

$$\begin{array}{lllll} a_{12} = 0 & a_{32} = 0 & a_{15} = ia_{33} & a_{05} = a_{34} & a_{25} = a_{36} \\ a_{22} = 0 & a_{02} = 0 & a_{06} = a_{24} + i & a_{03} = -ia_{14} & a_{16} = ia_{23} \end{array}$$

From these relations, we may define

$$\begin{array}{lll} p_1 = a_{13} & q_1 = a_{15} = ia_{33} & h_1 = a_{11} \\ p_2 = a_{16} = ia_{23} & q_2 = a_{25} = a_{36} & h_2 = -a_{03} = ia_{14} \\ p_3 = a_{26} & & h_3 = a_{06} = a_{24} + i \\ p_4 = a_{05} = a_{34} & r = ia_{35} & h_4 = -ia_{21}. \end{array}$$

Moreover, since ξ_0 is real-valued, we see that a_{03} and a_{06} are real-valued, whence h_2 and h_3 are real-valued. The reality of ξ_0 also yields $a_{01} = \overline{a}_{02} = 0$ and $a_{04} = \overline{a}_{05} = \overline{p}_4$.

In this new notation, (4.10) and its complex conjugate now read as follows:

$$\begin{aligned} d\eta &= i(\alpha + ia_{31}\overline{\theta} + (h_1+1)\,\omega^3 - h_4\,\omega^6) \wedge \eta \\ d\overline{\eta} &= -i(\alpha - i\overline{a}_{31}\,\theta + (\overline{h}_1 + 1)\,\omega^3 - \overline{h}_4\,\omega^6) \wedge \overline{\eta}. \end{aligned}$$

Again equating with $d\eta = i\phi \wedge \eta$ and $d\overline{\eta} = -i\phi \wedge \overline{\eta}$, we see that $a_{31} = 0$, that h_1 and h_4 are real-valued, and that

$$\phi = \alpha + (h_1 + 1)\,\omega^3 - h_4\omega^6.$$

This proves (4.7) and (4.9). The proof of (4.8) is a direct calculation. $\Diamond$

4.4. The acting Lie group G

Proposition 4.5: The Lie group G is 4-dimensional and non-abelian. In particular, if M is complete, then both G and the principal G-orbits in M are finite quotients of $\mathrm{SU}(2) \times \mathrm{U}(1) \cong \mathbb{S}^3 \times \mathbb{S}^1$.

Proof: For $X \in \mathfrak{g}$, let $X^{\#} \in \Gamma(TP)$ be the corresponding G-action vector field on P, by which we we mean $X^{\#}|_p = \frac{d}{dt}\big|_{t=0} (\exp tX) \cdot p$.

Since $X^{\#}$ is tangent to the pre-images $\pi^{-1}(Gx) \subset P$, we have $\omega^1(X^{\#}) = \omega^2(X^{\#}) = 0$. From the real part (4.5), we have $d\omega^1 = -\phi \wedge \omega^2$, whence

$$0 = \mathcal{L}_{X^{\#}}\omega^1 = \iota_{X^{\#}}(d\omega^1) + d(\iota_{X^{\#}}\omega^1) = \iota_{X^{\#}}(-\phi \wedge \omega^2) = -\phi(X^{\#})\, \omega^2,$$

whence $\phi(X^{\#}) = 0$. Thus, at each $p \in P$, we have

$$\mathfrak{g} \cong \{X^{\#}|_p \in T_pP \colon X \in \mathfrak{g}\} \subset \mathrm{Ker}\,(\omega^1, \omega^2, \phi)\big|_p, \tag{4.11}$$

whence $\dim(G) \leq 4$. Since $\dim(G) \geq 4$, we have equality. In particular, the inclusion in (4.11) is an equality, so the G-orbits in P are the integral 4-folds of $\mathcal{I}_G := \langle \omega^1, \omega^2, \phi \rangle$.

Let us now identify G with an integral 4-fold of $\mathcal{I}_G$. Via this identification, $\{\omega^3, \omega^4, \omega^5, \omega^6\}$ is a basis of left-invariant 1-forms on G. Let $\{X_3, X_4, X_5, X_6\}$ be a basis of $\mathfrak{g} = \{\text{left-invariant vector fields on } G\}$ whose dual basis is $\{\omega^3, \omega^4, \omega^5, \omega^6\}$. From (4.3) and (4.7), one may calculate that

$$d\omega^3 \equiv -\tfrac{3}{2} i\, \theta \wedge \overline{\theta} = -3\, \omega^4 \wedge \omega^5 \pmod{\eta, \overline{\eta}, \omega^3, \omega^6, \phi}.$$

Thus, $[X_4, X_5] = 3X_3$, so G is non-abelian.

If M is complete, then (Proposition 3.3) G is a compact 4-dimensional non-abelian Lie group. Hence, by the classification of compact Lie groups (see, e.g., §0.6 of [5]), G must be a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$. In this case, since the principal G-orbits are 4-dimensional G-homogeneous spaces, they must also be finite quotients of $\mathrm{SU}(2) \times \mathrm{U}(1)$. $\Diamond$

4.5. Geometric interpretation of the torsion

We pause to interpret the torsion functions $p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4$ geometrically. This section is parenthetical to the rest of this work: the main results in §5 will not draw on these remarks.

4.5.1. Background: Riemannian submersions. Let $\mathrm{pr}\colon (M^n, g) \to (\Sigma^k, g_\Sigma)$ denote an arbitrary Riemannian submersion between Riemannian manifolds M and Σ. Recall that the vertical distribution $\mathcal{V} = \mathrm{Ker}(\mathrm{pr}_*)$ is given by the tangent spaces to the pr-fibers, and the horizontal distribution $\mathcal{H} = \mathcal{V}^\perp$ is the orthogonal complement.

The geometry of the submersion pr is governed by the two $(1,2)$-tensor fields on M, called the A-tensor and T-tensor, given by

$$\begin{aligned}\mathsf{A}(X, Y) &= (\nabla_{X^{\mathrm{Hor}}} Y^{\mathrm{Ver}})^{\mathrm{Hor}} + (\nabla_{X^{\mathrm{Hor}}} Y^{\mathrm{Hor}})^{\mathrm{Ver}}\\ \mathsf{T}(X, Y) &= (\nabla_{X^{\mathrm{Ver}}} Y^{\mathrm{Ver}})^{\mathrm{Hor}} + (\nabla_{X^{\mathrm{Ver}}} Y^{\mathrm{Hor}})^{\mathrm{Ver}}\end{aligned}$$

for $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection on TM, and where $\mathrm{Ver}\colon TM \to \mathcal{V}$ and $\mathrm{Hor}\colon TM \to \mathcal{H}$ are the projections onto the vertical and horizontal distributions, respectively.

Note that $\mathsf{A} \equiv 0$ if and only if the horizontal distribution $\mathcal{H}$ is integrable. Indeed, $\mathsf{A}(X, Y) = \frac{1}{2}[X, Y]^{\mathrm{Ver}}$ for $X, Y \in \Gamma(\mathcal{H})$. Meanwhile, the T-tensor is essentially the second fundamental form $\mathrm{I\!I}$ of the pr-fibers. Indeed, $\mathsf{T}(X, Y) = \mathrm{I\!I}(X, Y)$ for $X, Y \in \Gamma(\mathcal{V})$.

Finally, we point out that $\mathsf{A}(X, \cdot) = 0$ for all $X \in \Gamma(\mathcal{V})$, and similarly $\mathsf{T}(X, \cdot) = 0$ for all $X \in \Gamma(\mathcal{H})$. Thus, the A- and T-tensors are recovered, respectively, from the knowledge of $\mathsf{A}(X, \cdot)$ for $X \in \Gamma(\mathcal{H})$ and $\mathsf{T}(X, \cdot)$ for $X \in \Gamma(\mathcal{V})$. For more information, see [3].

4.5.2. Geometric interpretation of the torsion functions. We now return to our usual setting, which is that of a cohomogeneity-two nearly-Kähler 6-manifold (M^6, g) with coisotropic principal orbits.

Let $\mathrm{pr}\colon (M^6, g) \to \Sigma$ denote the projection to the orbit space. As in Lemma 4.3, we equip Σ with the Riemannian metric $g_\Sigma = \eta \circ \overline{\eta}$, so that pr is a Riemannian submersion. We claim that:

Proposition 4.6:

(a) The torsion functions h_1, h_4 determine the A-tensor, and conversely.

(b) The torsion functions $p_1, p_2, p_3, p_4, q_1, q_2, r, h_2, h_3$ determine the T-tensor, and conversely.

To see this, let $F_{\mathrm{SO}(6)}M$ denote the oriented orthonormal frame bundle of the metric g. By the Fundamental Lemma of Riemannian Geometry, there is a unique 1-form $\psi \in \Omega^1(F_{\mathrm{SO}(6)}M; \mathfrak{so}(6))$, the Levi-Civita connection, such that

$$d\omega = -\psi \wedge \omega.$$

One can check that, using the notation of (4.2), the Levi-Civita connection (restricted to P) is

$$\psi = \left(\begin{array}{cc|cccc} 0 & \alpha_3 + \frac{1}{2}\omega^3 & -\alpha_2 - \frac{1}{2}\omega^2 & -\beta_{11} & -\beta_{12} - \frac{1}{2}\omega^6 & -\beta_{13} + \frac{1}{2}\omega^5 \\ -\alpha_3 - \frac{1}{2}\omega^3 & 0 & \alpha_1 + \frac{1}{2}\omega^1 & -\beta_{12} + \frac{1}{2}\omega^6 & -\beta_{22} & -\beta_{23} - \frac{1}{2}\omega^4 \\ \hline \alpha_2 + \frac{1}{2}\omega^2 & -\alpha_1 - \frac{1}{2}\omega^1 & 0 & -\beta_{13} - \frac{1}{2}\omega^5 & -\beta_{23} + \frac{1}{2}\omega^4 & -\beta_{33} \\ \beta_{11} & \beta_{12} - \frac{1}{2}\omega^6 & \beta_{13} + \frac{1}{2}\omega^5 & 0 & \alpha_3 - \frac{1}{2}\omega^3 & -\alpha_2 + \frac{1}{2}\omega^2 \\ \beta_{12} + \frac{1}{2}\omega^6 & \beta_{22} & \beta_{23} - \frac{1}{2}\omega^4 & -\alpha_3 + \frac{1}{2}\omega^3 & 0 & \alpha_1 - \frac{1}{2}\omega^1 \\ \beta_{13} - \frac{1}{2}\omega^5 & \beta_{23} - \frac{1}{2}\omega^4 & \beta_{33} & \alpha_2 - \frac{1}{2}\omega^2 & -\alpha_1 + \frac{1}{2}\omega^1 & 0 \end{array}\right).$$

Thus, letting ∇ denote the corresponding covariant derivative operator on TM, we have

$$\begin{aligned} (\nabla_X e_i)^{\mathrm{Hor}} &= \psi_i^1(X)e_1 + \psi_i^2(X)e_2 \\ (\nabla_X e_i)^{\mathrm{Ver}} &= \psi_i^3(X)e_3 + \psi_i^4(X)e_4 + \psi_i^5(X)e_5 + \psi_i^6(X)e_6 \end{aligned}$$

where $(e_1, \ldots, e_6)$ is any local $\mathrm{O}(2)$-frame field. Using these formulas, together with (4.7), one can compute the A- and T-tensors.

For example, a calculation shows that

$$\begin{aligned} &\mathsf{A}(e_1 + ie_2, e_1) = i\left[(h_1 + \tfrac{1}{2})e_3 - h_4e_6\right] && \mathsf{A}(e_1 + ie_2, e_4) = 0 \\ &\mathsf{A}(e_1 + ie_2, e_2) = -i\left[(h_1 + \tfrac{1}{2})e_3 - h_4e_6\right] && \mathsf{A}(e_1 + ie_2, e_5) = 0 \\ &\mathsf{A}(e_1 + ie_2, e_3) = -i(h_1 + \tfrac{1}{2})\,(e_1 + ie_2) && \mathsf{A}(e_1 + ie_2, e_6) = ih_4\,(e_1 + ie_2) \end{aligned}$$

where we have extended A to be $\mathbb{C}$-bilinear. In particular, we observe that the 2-plane field $\mathrm{Ker}(\omega^3, \omega^4, \omega^5, \omega^6)$ in M normal to the principal G-orbits is integrable if and only if $\mathsf{A} = 0$, or equivalently

$$h_1 = -\tfrac{1}{2} \quad \text{and} \quad h_4 = 0. \tag{4.12}$$

Computations by the author suggest that this integrability cannot happen (locally), but the details require closer examination.

We now exhibit the second fundamental form $\mathrm{I\!I}$ of the principal orbits. This is the normal bundle-valued quadratic form $\mathrm{I\!I} \in \Gamma(\mathrm{Sym}^2(T^*M) \otimes NM)$ given by

$$\mathrm{I\!I} = h^1_{ij}\,\omega^i\omega^j \otimes e_1 + h^2_{ij}\,\omega^i\omega^j \otimes e_2$$

where

$$(h^1_{ij}) = \begin{bmatrix} -\mathrm{Im}(p_1) & -\mathrm{Im}(q_1)+h_2 & \mathrm{Re}(q_1) & -\mathrm{Im}(p_2) \\ -\mathrm{Im}(q_1)+h_2 & -3\,\mathrm{Re}(p_4)-\mathrm{Im}(r) & -\mathrm{Im}(p_4)+\mathrm{Re}(r) & -\mathrm{Re}(q_2)-h_3 \\ \mathrm{Re}(q_1) & -\mathrm{Im}(p_4)+\mathrm{Re}(r) & -\mathrm{Re}(p_4)+\mathrm{Im}(r) & -\mathrm{Im}(q_2)-\frac{1}{2} \\ -\mathrm{Im}(p_2) & -\mathrm{Re}(q_2)-h_3 & -\mathrm{Im}(q_2)-\frac{1}{2} & -\mathrm{Re}(p_3) \end{bmatrix}$$

$$(h^2_{ij}) = \begin{bmatrix} \mathrm{Re}(p_1) & \mathrm{Re}(q_1) & \mathrm{Im}(q_1)+h_2 & \mathrm{Re}(p_2) \\ \mathrm{Re}(q_1) & -\mathrm{Im}(p_4)+\mathrm{Re}(r) & -\mathrm{Re}(p_4)+\mathrm{Im}(r) & -\mathrm{Im}(q_2)+\frac{1}{2} \\ \mathrm{Im}(q_1)+h_2 & -\mathrm{Re}(p_4)+\mathrm{Im}(r) & -3\,\mathrm{Im}(p_4)-\mathrm{Re}(r) & \mathrm{Re}(q_2)-h_3 \\ \mathrm{Re}(p_2) & -\mathrm{Im}(q_2)+\frac{1}{2} & \mathrm{Re}(q_2)-h_3 & -\mathrm{Im}(p_3) \end{bmatrix}.$$

Conversely, one can invert these formulas to recover $p_1, p_2, p_3, p_4, q_1, q_2, r$, h_2, h_3 in terms of h^1_{ij}, h^2_{ij}. Indeed,

$$\begin{aligned} p_1 &= h^2_{11} - ih^1_{11} \qquad 2q_1 = 2h^1_{13} + i(h^2_{13} - h^1_{12}) \qquad 2h_2 = h^1_{12} + h^2_{13} \\ p_2 &= h^2_{14} - ih^1_{14} \qquad 2q_2 = (h^2_{34} - h^1_{24}) - i(1 + 2h^1_{34}) \qquad 2h_3 = -h^1_{24} - h^2_{34} \\ p_3 &= -h^1_{44} - ih^2_{44} \\ 4p_4 &= -(h^1_{22} + h^1_{33}) - i(h^1_{23} + h^2_{33}) \qquad 4r = (3h^1_{23} - h^2_{33}) + i(3h^1_{33} - h^1_{22}) \end{aligned}$$

illustrating that these torsion functions are simply affine-linear combinations of second fundamental form coefficients. We also note that the mean curvature of the principal orbits is

$$H = -(\mathrm{Im}(p_1) + \mathrm{Re}(p_3) + 4\,\mathrm{Re}(p_4))\, e_1 + (\mathrm{Re}(p_1) - \mathrm{Im}(p_3) - 4\,\mathrm{Im}(p_4))\, e_2,$$

so that a principal orbit is minimal if and only if $p_1 + ip_3 + 4ip_4 = 0$. Finally, we point out that the presence of the $\frac{1}{2}$ terms in (h^1_{ij}) and (h^2_{ij}) implies that they cannot vanish simultaneously, so that none of the principal orbits can be totally-geodesic.

4.5.3. Descent to M. We caution the reader that the 1-forms $\eta, \overline{\eta}, \omega^3$, $\theta, \overline{\theta}, \omega^6$ and torsion functions p_j, q_j, r, h_j are defined on the bundle P, not on the base manifold M. However, O(2)-invariant combinations of these will descend to be well-defined (possibly up to sign) on M.

For example, the quadratic forms $\eta \circ \overline{\eta}$, $(\omega^3)^2$, $\theta \circ \overline{\theta}$, and $(\omega^6)^2$ descend to M, and the differential forms $\eta \wedge \overline{\eta}$, ω^3, $\theta \wedge \overline{\theta}$, and ω^6 descend to be well-defined *up to sign*. Similarly, the norms of the torsion functions $|p_1|$, $|p_2|$, $|p_3|$, $|p_4|$, $|q_1|, |q_2|$, $|r|$, $|h_1|, |h_2|, |h_3|, |h_4|$ are well-defined on M, while the

1-forms, quadratic forms, and cubic forms

$$\overline{p}_1\eta, \qquad \overline{p}_2\eta, \qquad \overline{p}_3\eta, \qquad \overline{p}_4\eta, \qquad \overline{q}_1\,\eta\circ\eta, \qquad \overline{q}_2\,\eta\circ\eta, \qquad \overline{r}\,\eta\circ\eta\circ\eta$$

descend to be well-defined up to sign.

5. Local existence and generality

We continue with the setup of §4, which we reiterate for clarity. We let M be a nearly-Kähler 6-manifold acted upon by a connected Lie group G with cohomogeneity-two. We suppose that this G-action preserves the SU(3)-structure (J, Ω, Υ) and that the principal G-orbits are coisotropic. Conventions 4.1 and 4.2 (stated in §4.2) remain in force.

We continue to work on the principal O(2)-bundle $\pi\colon P \to M$, defined in §4.2 as a frame adaptation. On P, we work with either of the global coframings $(\omega^1, \ldots, \omega^6, \phi)$ or $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \phi)$, recalling that their exterior derivatives are given by (4.3) and (4.7). Finally, the intrinsic torsion of the O(2)-structure has been encoded as a function

$$T = (p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4)\colon P \to \mathbb{C}^7 \oplus \mathbb{R}^4$$

which satisfies the O(2)-equivariance described in (4.8).

Our primary objective is to prove a local existence/generality theorem for nearly-Kähler 6-manifolds of cohomogeneity-two (always assuming the principal orbits are coisotropic) by appealing to Cartan's Third Theorem (Theorem 2.2). Concretely, this means satisfying the integrability conditions

$$d(d\eta) = d(d\overline{\eta}) = 0$$

$$d(d\omega^3) = d(d\theta) = d(d\overline{\theta}) = d(d\omega^6) = 0 \tag{5.1}$$

$$d(d\phi) = 0 \tag{5.2}$$

$$d(dp_i) = d(dq_i) = d(dh_i) = 0 \tag{5.3}$$

$$d(dr) = 0, \tag{5.4}$$

as well as ensuring the involutivity and correct dimension of the tableau of free derivatives. Fortunately, the equations $d(d\eta) = d(d\overline{\eta}) = 0$ are already satisfied (by Lemma 4.3).

By contrast, the integrability conditions (5.1) are quite complicated, consisting of $4\binom{6}{3} = 80$ quadratic equations on 55 real-valued functions: the 18 real and imaginary parts of the torsion functions, their 36 "directional derivatives" in the two directions normal to the G-orbits, and the Gauss curvature K of the orbit space Σ. Thus, arranging for (5.1) will occupy us for some time.

5.0.1. The three types. We begin by solving two of the simpler quadratic equations arising in (5.1). Namely, we calculate

$$0 = d(d\theta) \wedge \eta \wedge \overline{\eta} \wedge \theta = -8\,[q_1(h_1 + 3 + ih_3) - q_2(h_2 + ih_4)]\,\omega^{123456}$$
$$0 = d(d\theta) \wedge \eta \wedge \overline{\eta} \wedge \overline{\theta} = -4i\,[q_1\overline{q}_2 + \overline{q}_1 q_2 - 2\,(h_1h_2 + 3h_2 + h_3h_4)]\,\omega^{123456}$$

yielding the quadratics

$$q_1(h_1 + 3 + ih_3) - q_2(h_2 + ih_4) = 0 \tag{5.5a}$$
$$q_1\overline{q}_2 + \overline{q}_1 q_2 = 2\,(h_1h_2 + 3h_2 + h_3h_4). \tag{5.5b}$$

To solve this system, we introduce the $\mathbb{C}$-valued functions

$$s_1 = (h_1 + 3) + ih_3$$
$$s_2 = h_2 + ih_4.$$

For $z, w \in \mathbb{C}$, we let $\langle z, w\rangle = \mathrm{Re}(z\overline{w})$ denote the euclidean inner product on $\mathbb{C} \simeq \mathbb{R}^2$, and let $\|z\| = \sqrt{z\overline{z}}$ denote the euclidean norm. Then equations (5.5a)-(5.5b) are simply

$$q_1 s_1 - q_2 s_2 = 0 \tag{5.6a}$$
$$\langle q_1, q_2\rangle = \langle s_2, s_1\rangle. \tag{5.6b}$$

The solution to (5.6a)-(5.6b) is provided by the following geometric fact.

Lemma 5.1: Let $a, b, c, d \in \mathbb{C}$ be complex numbers satisfying both

$$ad - bc = 0$$
$$\langle a, b\rangle = \langle c, d\rangle.$$

Then exactly one of the following holds:
(i) $a = b = c = d = 0$.
(ii) $\langle a, b\rangle = \langle c, d\rangle \neq 0$ and $\|a\| = \|c\|$ and $\|b\| = \|d\|$.
(iii) $\langle a, b\rangle = \langle c, d\rangle = 0$ and $(a, b, c, d) \neq (0, 0, 0, 0)$.

Definition: Let M be a nearly-Kähler 6-manifold of cohomogeneity-two with coisotropic principal orbits. We say that a point $m \in M$ is of:

- *Type I* if $q_1 = q_2 = s_1 = s_2 = 0$ at m.
- *Type II* if $\langle q_1, q_2\rangle = \langle s_2, s_1\rangle \neq 0$ and $\|q_1\| = \|s_2\|$ and $\|q_2\| = \|s_1\|$ at m.
- *Type III* if $\langle q_1, q_2\rangle = \langle s_2, s_1\rangle = 0$ and $(q_1, q_2, s_1, s_2) \neq (0, 0, 0, 0)$ at m.

Definition: We say that M is of Type I (resp., II, III) if every point of M is of Type I (resp., II, III).

Remark: Although the functions q_1, q_2, s_1, s_2 are defined on P, the Type conditions are O(2)-invariant. Thus, it makes sense to speak of points of M as being of "Type I," etc. It is conceivable for a nearly-Kähler structure on M to be of (say) Type I at some points of M and be of Type II at others. □

Remark: Our partitioning of Types II and III entails a somewhat *ad hoc* choice, motivated by a pragmatic desire to solve (5.1). Namely, a point of M may satisfy both $\langle q_1, q_2\rangle = \langle s_2, s_1\rangle = 0$ as well as $\|q_1\| = \|s_2\|$ and $\|q_2\| = \|s_1\|$, and we have declared that such a point is of Type III. This choice eases our treatment of (5.1), playing a role in the proof in the technical Lemma 5.6(a). □

In the sequel, we study each Type of cohomogeneity-two nearly-Kähler structure separately. In each case, the primary challenge will be solving the 80 quadratic equations (5.1). Once this is done, we will solve (5.2), (5.3), (5.4) and draw conclusions. We will see that the algebra involved in solving (5.1) is fairly simple for Type I structures, but is significantly more labor intensive in the Type II case, and even more so in the Type III case.

5.1. Type I

In this section, we study nearly-Kähler structures of Type I. In particular, we prove a local existence/generality result (Theorem 5.4) for these structures. We then show that for this Type, the acting Lie group G is nilpotent (Proposition 5.5), and hence the underlying metrics are incomplete.

5.1.1. The integrability conditions.

Our first task is to make explicit the integrability conditions (5.1), which amount to quadratic equations on both the torsion functions p_i, q_i, r, h_i and on their first derivatives. In preparation, we note that since p_4 and r are G-invariant and O(2)-equivariant (recall (4.8)), their exterior derivatives take the form

$$
\begin{aligned}
dp_4 &= p_4'\,\eta + p_4''\,\overline{\eta} + ip_4\phi \\
dr &= r'\,\eta + r''\,\overline{\eta} + 3ir\phi
\end{aligned}
$$

for some functions $p_4', p_4'', r', r'' \colon P \to \mathbb{C}$.

Lemma 5.2: Let M be a nearly-Kähler manifold of Type I.

(a) On the O(2)-coframe bundle P, the following algebraic relations hold:

$$
\begin{aligned}
(p_1, p_2, p_3, p_4) &= (-4ip_4, 0, 4p_4, p_4) \\
(q_1, q_2) &= (0, 0) \\
(h_1, h_2, h_3, h_4) &= (-3, 0, 0, 0).
\end{aligned} \tag{5.7}
$$

Thus, the torsion can be expressed in terms of the functions p_4 and r.

(b) On the O(2)-coframe bundle P, the following differential relations hold:

$$
\begin{aligned}
p_4' &= -6|p_4|^2 - \tfrac{3}{2} & \qquad r' &= -5ip_4^2 - \overline{p}_4 r \\
p_4'' &= -5p_4^2 + i\overline{p}_4 r & \qquad K &= 2(6 + |r|^2 - |p_4|^2).
\end{aligned} \tag{5.8}
$$

Proof: The equations $(q_1, q_2) = (0, 0)$ and $(h_1, h_2, h_3, h_4) = (-3, 0, 0, 0)$ are immediate from the definition of "Type I." For the others, we calculate (using (4.3) and (4.7))

$$
\begin{aligned}
0 &= d(d\theta) \wedge \theta \wedge \overline{\theta} \wedge \omega^6 = 8p_2\, \omega^{123456} \\
0 &= d(d\theta) \wedge \eta \wedge \overline{\theta} \wedge \omega^6 = 6(p_1 + ip_3)\, \omega^{123456} \\
0 &= d(d\overline{\theta}) \wedge \eta \wedge \theta \wedge \omega^6 = 8(p_1 + 4ip_4)\, \omega^{123456}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= d(d\omega^3) \wedge \overline{\eta} \wedge \omega^3 \wedge \theta = 16\big(p_4' + 6|p_4|^2 + \tfrac{3}{2}\big)\, \omega^{123456} \\
0 &= d(d\omega^3) \wedge \eta \wedge \theta \wedge \omega^3 = 16\big(p_4'' + 5p_4^2 - i\overline{p}_4 r\big)\, \omega^{123456} \\
0 &= d(d\theta) \wedge \omega^3 \wedge \overline{\theta} \wedge \omega^6 = 2\big(12 + 2|r|^2 - 2|p_4|^2 - K\big)\, \omega^{123456} \\
0 &= d(d\theta) \wedge \omega^3 \wedge \theta \wedge \omega^6 = 4\big(ir' + p_4''\big)\, \omega^{123456}
\end{aligned}
$$

from which the result follows. $\Diamond$

A calculation using MAPLE shows that if the equations (5.7)-(5.8) of Lemma 5.2 hold, then the integrability conditions (5.1), (5.2), and (5.3) are all satisfied, and that

$$d(dr) = \big(F\,\eta\wedge\overline{\eta} - 4ir''\phi\wedge\overline{\eta}\big) + dr''\wedge\overline{\eta} \tag{5.9a}$$

where

$$F = -\big(13|p_4|^2 r + \tfrac{39}{2}r + 3r|r|^2 + 50ip_4^3 - \overline{p}_4 r''\big). \tag{5.9b}$$

We summarize our discussion so far.

Summary 5.3: Nearly-Kähler structures of Type I are encoded by augmented coframings $((\eta,\overline{\eta},\omega^3,\ \theta,\overline{\theta},\omega^6,\phi),(p,r),r'')$ on P satisfying the following structure equations:

$$\begin{aligned} d\eta &= i\phi\wedge\eta \\ d\phi &= i(6+|r|^2-|p|^2)\,\eta\wedge\overline{\eta} \end{aligned} \tag{5.10a}$$

$$\begin{aligned} d\theta &= i\phi\wedge\theta + \big(ir\,\overline{\theta}\wedge\overline{\eta} - \overline{p}\,\theta\wedge\eta\big) - 2p\,\mathrm{Re}(\theta\wedge\overline{\eta}) - 4i\omega^6\wedge\eta \\ d\omega^3 &= \tfrac{5}{2}i\,\overline{\eta}\wedge\eta + \tfrac{3}{2}i\,\overline{\theta}\wedge\theta + 4\,\mathrm{Re}(p\,\omega^3\wedge\overline{\eta}) + 8\,\mathrm{Re}(p\,\omega^6\wedge\overline{\theta}) \\ d\omega^6 &= \tfrac{3}{2}i\,\overline{\eta}\wedge\theta - \tfrac{3}{2}i\,\eta\wedge\overline{\theta} - 4\,\mathrm{Re}(p\,\omega^6\wedge\overline{\eta}) \end{aligned}$$

and

$$\begin{aligned} dp &= -\big(6|p|^2+\tfrac{3}{2}\big)\,\eta - \big(5p^2 - i\overline{p}r\big)\,\overline{\eta} + ip\phi \\ dr &= -\big(5ip^2+\overline{p}r\big)\,\eta + r''\overline{\eta} + 3ir\phi \end{aligned} \tag{5.10b}$$

where for ease of notation, we have set $p = p_4$.

Augmented coframings satisfying the structure equations (5.10a)-(5.10b) will satisfy the integrability conditions (5.1), (5.2), (5.3), as well as (5.9a)-(5.9b). In the language of §2.4, the functions p and r are the "primary invariants," while r'' is the "free derivative."

Remark: The formulas of §4.5.2 simplify considerably in the Type I setting. In particular, we remark that the principal G-orbits in M have mean curvature vector

$$H = -4\,(\mathrm{Re}(p)\,e_1 + \mathrm{Im}(p)\,e_2)$$

and have scalar curvature

$$\mathrm{Scal} = -\tfrac{9}{4} - 16\,|p|^2 < 0.$$

Thus, p is essentially the mean curvature (or scalar curvature) of the principal orbits. □

Remark: Comparing (4.12) with (5.7), we see that for Type I nearly-Kähler structures, the 2-plane distribution normal to the principal G-orbits in M is never integrable. □

5.1.2. Local existence/generality. We are now ready to state a local existence and generality theorem for Type I structures.

Theorem 5.4: Nearly-Kähler structures of Type I exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any $x \in \mathbb{R}^6$ and $(a_0, b_0) \in \mathbb{C}^2 \times \mathbb{C}$, there exists a Type I nearly-Kähler structure on an open neighborhood $U \subset \mathbb{R}^6$ of x and an $\mathrm{O}(2)$-coframe $f_x \in P|_x$ at x for which

$$(p, r)(f_x) = a_0 \quad \text{and} \quad r''(f_x) = b_0.$$

Remark: In a certain sense [8], the space of diffeomorphism classes of k-jets of Type I nearly-Kähler structures has dimension $2k+4$. □

Proof: The discussion in §5.1.1 shows that hypotheses (2.4) and (2.5) of Cartan's Third Theorem (Theorem 2.2) are satisfied. It remains to examine the tableau of free derivatives. At a point $(u, v) \in \mathbb{R}^4 \times \mathbb{R}^2$, this is the vector subspace $A(u, v) \subset \mathrm{Hom}(\mathbb{R}^7; \mathbb{R}^4) \cong \mathrm{Mat}_{4\times 7}(\mathbb{R})$ given by

$$A(u,v) = \left\{ \left(\begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & y & 0 & 0 & 0 & 0 & 0 \\ y & -x & 0 & 0 & 0 & 0 & 0 \end{array} \right) : x, y \in \mathbb{R} \right\}.$$

Since $A(u, v)$ is independent of the point $(u, v) \in \mathbb{R}^4 \times \mathbb{R}^2$, we can write $A = A(u, v)$ without ambiguity. We observe that A is 2-dimensional and has Cartan characters $\widetilde{s}_1 = 2$ and $\widetilde{s}_k = 0$ for $k \geq 2$. One can check that A is an involutive tableau, meaning that its prolongation $A^{(1)}$ satisfies $\dim(A^{(1)}) =$

$2 = \widetilde{s}_1 + 2\widetilde{s}_2 + \cdots + 7\widetilde{s}_7$. Thus, Cartan's Third Theorem applies, and we conclude the result. $\Diamond$

Remark: The complex characteristic variety of the tableau A is

$$\begin{aligned}\Xi_A^{\mathbb{C}} &= \{[\xi] \in \mathbb{P}(\mathbb{C}^7) \colon w \otimes \xi \in A \text{ for some } w \in \mathbb{R}^4, w \neq 0\} \\ &= \{[\xi] \in \mathbb{P}(\mathbb{C}^7) \colon (\xi_1 + i\xi_2)(\xi_1 - i\xi_2) = 0,\ \xi_3 = \cdots = \xi_7 = 0\}.\end{aligned}$$

The fact that the local generality of Type I structures is 2 functions of 1 variable, with complex characteristic variety consisting of two complex conjugate points, strongly suggests the possibility of a holomorphic interpretation of these structures. $\square$

5.1.3. Incompleteness. Nearly-Kähler structures of Type I cannot arise from a complete metric, as we now show. Recall that the real Heisenberg group is the (non-compact) Lie group

$$\mathrm{H}_3 = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\} \leq \mathrm{GL}_3(\mathbb{R}).$$

Proposition 5.5: If M is of Type I, then the universal cover of the acting Lie group G is $\widetilde{G} = \mathrm{H}_3 \times \mathbb{R}$. In particular, the metric on M is incomplete.

Proof: As in the proof of Proposition 4.5, we identify G with an integral 4-fold of the ideal $\mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle$. Under this identification, $\{\omega^3, \omega^4, \omega^5, \omega^6\}$ is a basis of left-invariant 1-forms on G. From (5.10a), their exterior derivatives (mod $\mathcal{I}_G$) are given by

$$\begin{aligned} d\omega^3 &\equiv -3\omega^{45} - 8\,\mathrm{Re}(p)\,\omega^{46} - 8\,\mathrm{Im}(p)\,\omega^{65} \\ d\omega^4 &\equiv d\omega^5 \equiv d\omega^6 \equiv 0. \end{aligned}$$

Let $\{X_3, X_4, X_5, X_6\}$ be a basis of $\mathfrak{g} = \{\text{left-invariant vector fields on } G\}$ whose dual basis is $\{\omega^3, \omega^4, \omega^5, \omega^6\}$. Let $Y = \frac{8}{3}\mathrm{Im}(p)\,X_4 - \frac{8}{3}\mathrm{Re}(p)X_5 + X_6$. Then $\{X_3, X_4, X_5, Y\}$ is a basis of $\mathfrak{g}$ with

$$\begin{aligned} [X_4, X_5] &= 3X_3 & [X_3, X_4] &= 0 \\ [X_4, Y] &= 0 & [X_3, X_5] &= 0 \\ [X_5, Y] &= 0 & [X_3, Y] &= 0. \end{aligned}$$

This exhibits $\mathfrak{g}$ as the Lie algebra of the Lie group $\mathrm{H}_3 \times \mathbb{R}$, and so the universal cover of G is $\widetilde{G} = \mathrm{H}_3 \times \mathbb{R}$. Thus, Proposition 3.3 implies that the underlying metric is incomplete. $\Diamond$

5.2. Type II

We now examine nearly-Kähler structures of Type II. The integrability conditions for Type II structures are significantly more complicated than those for Type I. To satisfy them, we will make a further frame adaptation and a change-of-variable.

Ultimately, we will draw two conclusions. First, we obtain (Theorem 5.8) a local existence and generality theorem for Type II structures. Second, will show that the Lie group G is solvable (Proposition 5.9), and hence that the underlying metrics are incomplete.

5.2.1. A frame adaptation. By definition, Type II structures are those with $\|q_1\| = \|s_2\|$ and $\|q_2\| = \|s_1\|$ and $\langle q_1, q_2\rangle = \langle s_1, s_2\rangle \neq 0$. Thus, the $\mathrm{O}(2)$-equivariant function $\frac{q_1}{s_2} = \frac{q_2}{s_1} \colon P \to \mathbb{C}$ maps into the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Accordingly, we may adapt frames as follows: define the $\mathbb{Z}_2$-subbundle

$$P_1 = \{u \in P \colon q_1(u) = is_2(u)\} \subset P.$$

We refer to elements of P_1 as *$\mathbb{Z}_2$-coframes*. For the remainder of §5.2, we work on P_1.

The price we pay for this adaptation is the presence of additional torsion functions. Indeed, on P_1 the 1-form ϕ is no longer a connection form, but rather

$$\phi = \ell_1\omega^1 + \ell_2\omega^2$$

for some new G-invariant functions $\ell_1, \ell_2 \colon P_1 \to \mathbb{R}$.

5.2.2. The integrability conditions. We now move to solve the integrability conditions (5.1). For this, we make the following change-of-variables. Rather than work with $p_1, p_2, p_3, p_4, r \colon P_1 \to \mathbb{C}$, we will work with $t_1, \ldots, t_8$, $r_1, r_2 \colon P_1 \to \mathbb{R}$ defined by:

$$\begin{aligned}
t_1 &= \mathrm{Re}(p_1 + 4ip_4) & t_5 &= \mathrm{Im}(p_1 + 4ip_4) & r_1 &= \mathrm{Re}(r)\\
t_2 &= \tfrac{1}{24}\,\mathrm{Re}(p_3 + 4p_4) & t_6 &= \mathrm{Im}(p_3 + 4p_4) & r_2 &= \mathrm{Im}(r)\\
t_3 &= \mathrm{Re}(p_2) & t_7 &= \mathrm{Im}(p_2) & &\\
t_4 &= \mathrm{Im}(p_4) & t_8 &= \mathrm{Re}(p_4). & &
\end{aligned}$$

The factor of $\frac{1}{24}$ appearing in t_2 is merely for the sake of clearing future denominators. Since each t_i, h_i, and ℓ_i is G-invariant, we can write their exterior derivatives as

$$dt_i = t_{i1}\omega^1 + t_{i2}\omega^2 \qquad dh_i = h_{i1}\omega^1 + h_{i2}\omega^2 \qquad d\ell_i = \ell_{i1}\omega^1 + \ell_{i2}\omega^2.$$

We now state the Type II analogue of Lemma 5.2.

Lemma 5.6: Let M be a nearly-Kähler manifold of Type II.

(a) On the $\mathbb{Z}_2$-coframe bundle P_1, the following 12 algebraic equations hold:

$$\begin{aligned}
&\mathrm{Re}(q_1) = -h_4 && t_5 = 0 && h_2 = -4t_2t_3 \\
&\mathrm{Im}(q_1) = h_2 && t_6 = t_1 + 8t_4 - 64t_1t_2^2 && h_3 = -4t_1t_2 \\
&\mathrm{Re}(q_2) = -h_3 && t_7 = 0 && r_1 = \ell_1 + t_4 \\
&\mathrm{Im}(q_2) = h_1 + 3 && t_8 = -t_2(2h_1 + 3) && r_2 = \ell_2 + t_8 + 24t_2
\end{aligned} \tag{5.11}$$

Thus, the torsion is expressible in terms of the 8 real-valued functions

$$t_1, t_2, t_3, t_4 \qquad \text{and} \qquad h_1, h_4, \ell_1, \ell_2.$$

(b) The integrability conditions $d(d\omega^i) = 0$ are equivalent to the 12 algebraic equations (5.11) together with the equations

(5.12)

$$\begin{aligned}
t_{11} &= t_1(\ell_2 - 32h_1t_2) & h_{11} &= H_{11}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{12} &= t_1^2(64t_2^2 + 2) + t_1(4t_4 - \ell_1) + 2t_3^2 & h_{12} &= H_{12}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
&\quad + 6(h_1 + 3) & & \\
t_{21} &= 4t_2^2(2h_1 - 9) - t_2\ell_2 - \tfrac{1}{2} & h_{41} &= H_{41}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{22} &= t_2\ell_1 & h_{42} &= H_{42}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{31} &= 16t_2(h_4t_1 - h_1t_3) + \ell_2t_3 & \ell_{11} &= u_1 + \tfrac{1}{2}G_1(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{32} &= t_3(192t_1t_2^2 - 12t_4 - \ell_1) - 6h_4 & \ell_{12} &= u_2 + \tfrac{1}{2}G_2(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{41} &= T_{41}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) & \ell_{21} &= u_2 - \tfrac{1}{2}G_2(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{42} &= T_{42}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) & \ell_{22} &= -u_1 + \tfrac{1}{2}G_1(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2)
\end{aligned}$$

where we set $(u_1, u_2) = \left(\frac{1}{2}(\ell_{11} - \ell_{22}), \frac{1}{2}(\ell_{12} + \ell_{21})\right)$, and where the functions T_{41}, T_{42} and H_{11}, H_{12}, H_{41}, H_{42} and G_1, G_2 appearing on the right-hand sides of (5.12) are polynomial functions (of degree ≤ 5) whose explicit formulas are listed in the appendix.

Proof: (a) The left-most equations for q_1 and q_2 define our frame adaptation $P_1 \subset P$. For the remaining eight equations, a calculation shows:

$$
\begin{aligned}
&(5.13a) && 0 = d(d\omega^4) \wedge \omega^{126} && \Longrightarrow && (h_1+3)t_7 + h_4 t_5 = 0\\
&(5.13b) && 0 = d(d\omega^5) \wedge \omega^{126} && \Longrightarrow && h_2 t_5 - h_3 t_7 = 0.
\end{aligned}
$$

We rewrite (5.13a)-(5.13b) as

$$\begin{pmatrix} h_1+3 & h_4 \\ -h_3 & h_2 \end{pmatrix}\begin{pmatrix} t_7 \\ t_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since M is of Type II, we have $(h_1+3)h_2 + h_3 h_4 = \langle s_1, s_2\rangle \neq 0$, from which it follows that $t_5 = t_7 = 0$. Similarly, one can compute

$$
\begin{aligned}
&(5.14a) && 0 = d(d\omega^3) \wedge \omega^{126} && \Longrightarrow && 4t_1t_2 + h_3 = 0\\
&(5.14b) && 0 = d(d\omega^3) \wedge \omega^{123} && \Longrightarrow && 4t_2t_3 + h_2 = 0\\
&(5.15a) && 0 = d(d\omega^3) \wedge \omega^{236} && \Longrightarrow && 2h_1t_2 + 3t_2 + t_8 = 0\\
&(5.15b) && 0 = d(d\omega^5) \wedge \omega^{123} && \Longrightarrow && 16h_3t_2 + t_1 + 8t_4 - t_6 = 0
\end{aligned}
$$

$$
\begin{aligned}
&(5.16a) && 0 = d(d\omega^4) \wedge \omega^{235} && \Longrightarrow && h_4(\ell_1 - r_1 + t_4) = 0\\
&(5.16b) && 0 = d(d\omega^5) \wedge \omega^{235} && \Longrightarrow && h_2(\ell_1 - r_1 + t_4) = 0\\
&(5.16c) && 0 = d(d\omega^4) \wedge \omega^{135} && \Longrightarrow && h_4(\ell_2 - r_2 + 24t_2 + t_8) = 0\\
&(5.16d) && 0 = d(d\omega^5) \wedge \omega^{135} && \Longrightarrow && h_2(\ell_2 - r_2 + 24t_2 + t_8) = 0.
\end{aligned}
$$

Equations (5.14a)-(5.14b) now give the formulas for h_2 and h_3, while (5.15a)-(5.15b) give the formulas for t_6 and t_8. Finally, since M is of Type II, we have $(h_2)^2 + (h_4)^2 = \|s_2\|^2 \neq 0$. Thus, equations (5.16a)-(5.16d) give the remaining two equations.

(b) This is a direct check of the equations remaining in $d(d\omega^i) = 0$. $\Diamond$

A calculation using Maple shows that if the equations (5.11)-(5.12) of Lemma 5.6 hold, then $d(dt_i) = 0$ and $d(dh_i) = 0$ are also satisfied, and that

$$
\begin{aligned}
d(d\ell_1) &= F_1\,\omega^{12} + \big(du_1 \wedge \omega^1 + du_2 \wedge \omega^2\big)\\
d(d\ell_2) &= F_2\,\omega^{12} + \big(du_2 \wedge \omega^1 - du_1 \wedge \omega^2\big),
\end{aligned} \tag{5.17}
$$

where F_1, F_2 are certain polynomial functions (of degree ≤ 8) of t_1, t_2, t_3, t_4, h_1, h_4, ℓ_1, ℓ_2 and u_1, u_2 whose explicit formulas we will not list here.

Summary 5.7: Nearly-Kähler structures of Type II are encoded by augmented coframings $((\omega^1, \ldots, \omega^6), (t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2), (u_1, u_2))$ on the $\mathbb{Z}_2$-bundle $P_1 \to M$ satisfying the structure equations

$$
\begin{aligned}
d\omega^1 &= -\ell_1\, \omega^{12} \\
d\omega^2 &= -\ell_2\, \omega^{12}
\end{aligned} \tag{5.18a}
$$

and

$$
\begin{aligned}
d\omega^3 = {} & (2h_1 + 1)\, \omega^{12} - 4t_8\omega^{13} + 2h_4\omega^{15} - (t_1 + 4t_4)\, \omega^{23} \\
& - 2h_2\omega^{25} - t_3\omega^{26} - t_3\omega^{35} + 2h_2\omega^{36} - 3\omega^{45} \\
& - 24t_2\omega^{46} - t_6\omega^{56} \\
d\omega^4 = {} & (\ell_2 + 4t_8 + 24t_2)\, \omega^{14} - 2\ell_1\omega^{15} + 2h_4\omega^{23} - \ell_1\omega^{24} \\
& - 2(\ell_2 + 12t_2)\, \omega^{25} + 2(h_1 + 1)\, \omega^{26} + 2(h_1 + 3)\, \omega^{35} \\
& - 8(t_8 - 3t_2)\, \omega^{36} + 2h_4\omega^{56} \\
d\omega^5 = {} & -(\ell_2 + 24t_2)\, \omega^{15} + 4\omega^{16} - 2h_2\omega^{23} - 24t_2\omega^{24} \\
& + (\ell_1 + 4t_4)\, \omega^{25} + 2h_3\omega^{26} + 2h_3\omega^{35} \\
& + (t_6 - t_1 - 8t_4)\, \omega^{36} - 2h_2\omega^{56} \\
d\omega^6 = {} & -2h_4\omega^{12} + (2h_1 + 3)\, \omega^{15} - 4(t_8 - 6t_2)\, \omega^{16} - t_3\omega^{23} \\
& + 3\omega^{24} + 2h_3\, \omega^{25} + (t_6 - 4t_4)\, \omega^{26} + t_1\omega^{35} \\
& - 2h_3\omega^{36} - t_3\omega^{56}
\end{aligned} \tag{5.18b}
$$

where t_6, t_8, h_2, h_3 are given by (5.11), and

$$
dt_i = t_{i1}\omega^1 + t_{i2}\omega^2 \qquad dh_i = h_{i1}\omega^1 + h_{i2}\omega^2 \qquad d\ell_i = \ell_{i1}\omega^1 + \ell_{i2}\omega^2 \tag{5.18c}
$$

where $t_{11}, \ldots, t_{42}$ and $h_{11}, h_{12}, h_{41}, h_{42}$ and $\ell_{11}, \ell_{12}, \ell_{21}, \ell_{22}$ are given by (5.12).

Augmented coframings satisfying the structure equations (5.18a)-(5.18c) and (5.11)-(5.12) will satisfy $d(d\omega^i) = 0$ and $d(dt_i) = d(dh_1) = d(dh_4) = 0$ and (5.17). In the language of §2.4, the functions $t_1, t_2, t_3, t_4,\ h_1, h_4,\ \ell_1, \ell_2$ are the "primary invariants," while u_1, u_2 are the "free derivatives."

5.2.3. Local existence/generality.

We may now state the corresponding local existence/generality result for Type II structures.

Theorem 5.8: Nearly-Kähler structures of Type II exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any $x \in \mathbb{R}^6$ and any $(a_0, b_0) \in \mathbb{R}^8 \times \mathbb{R}^2$, there exists a Type II nearly-Kähler structure on an open neighborhood $U \subset \mathbb{R}^6$ of x and a $\mathbb{Z}_2$-coframe $f_x \in P_1|_x$ at x for which

$$(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2)(f_x) = a_0 \quad \text{and} \quad (u_1, u_2)(f_x) = b_0$$

Proof: The discussion in §5.2.2 shows that hypotheses (2.4) and (2.5) of Cartan's Third Theorem (Theorem 2.2) are satisfied. It remains to examine the tableau of free derivatives. At a point $(u, v) \in \mathbb{R}^8 \times \mathbb{R}^2$, this is the vector subspace $A(u, v) \subset \mathrm{Hom}(\mathbb{R}^6; \mathbb{R}^8) \cong \mathrm{Mat}_{8\times 6}(\mathbb{R})$ given by

$$A(u,v) = \left\{ \left(\begin{array}{cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & y & 0 & 0 & 0 & 0 \\ y & -x & 0 & 0 & 0 & 0 \end{array} \right) : x, y \in \mathbb{R} \right\}.$$

Since $A(u, v)$ is independent of the point $(u, v) \in \mathbb{R}^8 \times \mathbb{R}^2$, we can write $A = A(u, v)$ without ambiguity. We observe that A is 2-dimensional and has Cartan characters $\widetilde{s}_1 = 2$ and $\widetilde{s}_k = 0$ for $k \geq 2$. One can also check that A is an involutive tableau, meaning that its prolongation $A^{(1)}$ satisfies $\dim(A^{(1)}) = 2 = \widetilde{s}_1 + 2\widetilde{s}_2 + \cdots + 6\widetilde{s}_6$. Thus, from Cartan's Third Theorem, we conclude the result. $\lozenge$

5.2.4. Incompleteness. As in the Type I setting, the non-compactness of the Lie group G will prevent metrics of Type II from being complete.

Proposition 5.9: If M is of Type II, then the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is solvable. In particular, the metric on M is incomplete.

Proof: As in the proof of Proposition 4.5, we identify G with an integral 4-fold of the differential ideal $\mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle$. Under this identification, $\{\omega^3, \omega^4, \omega^5, \omega^6\}$ is a basis of $\mathfrak{g}^* = \{\text{left-invariant 1-forms on } G\}$.

Let $\zeta = \omega^5 + 8t_2\omega^6$, so that $\{\omega^3, \omega^4, \omega^6, \zeta\}$ is also a basis for $\mathfrak{g}^*$. One can check that their exterior derivatives (mod $\langle \omega^1, \omega^2 \rangle$) are

$$\begin{aligned} d\omega^3 &\equiv -t_3\,\omega^3 \wedge \zeta - 3\,\omega^4 \wedge \zeta + t_6\,\omega^6 \wedge \zeta \qquad & d\omega^6 &\equiv t_1\,\omega^3 \wedge \zeta + t_3\,\omega^6 \wedge \zeta \\ d\omega^4 &\equiv 2(h_1 + 3)\,\omega^3 \wedge \zeta - 2h_4\,\omega^6 \wedge \zeta & d\zeta &\equiv 0, \end{aligned}$$

where we recall $t_6 = t_1 + 8t_4 - 64t_1t_2^2$.

Let $\{X_3, X_4, X_5, Z\}$ be a basis of $\mathfrak{g} = \{\text{left-invariant vector fields on } G\}$ whose dual basis is $\{\omega^3, \omega^4, \omega^6, \zeta\}$. Their Lie brackets are then

$$\begin{aligned} [X_3, Z] &= t_3X_3 - 2(h_1+3)X_4 - t_1X_6 & [X_3, X_4] &= 0 \\ [X_4, Z] &= 3X_4 & [X_3, X_6] &= 0 \\ [X_6, Z] &= -t_6X_3 + 2h_4X_4 - t_3X_6 & [X_4, X_6] &= 0. \end{aligned}$$

From this, it is clear that $[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] = 0$, so that $\mathfrak{g}$ is solvable. (Note, however, that $\mathfrak{g}$ is not nilpotent in general.) Thus, Proposition 3.3 implies that the underlying metric is incomplete. $\Diamond$

5.3. Type III

We now consider nearly-Kähler structures of Type III. This is perhaps the most interesting case, as there is the possibility for complete metrics to exist in this class. Unfortunately, the integrability conditions (5.1)-(5.4) are even more complicated than those of Type II.

Examining these conditions leads us to several changes-of-variable (§5.3.1, §5.3.2). The upshot is that the intrinsic torsion

$$(p_1, p_2, p_3, p_4, q_1, q_2, r; h_1, h_2, h_3, h_4)\colon P \to \mathbb{C}^7 \oplus \mathbb{R}^4$$

will be recast as a function

$$(u, v, z, r;\, t_0, t_1, t_2)\colon P \to \mathbb{C}^4 \oplus \mathbb{R}^3.$$

Even with this repackaging, however, we find it difficult to solve (5.1) in general. As such, we will impose the ansatz that $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ (which by Proposition 3.3 is the case of most interest anyway). This will let us normalize the function t_2, which simplifies (5.1) further.

Finally, after this normalization, we are able to solve all the integrability conditions by means of a computer algebra system (Lemma 5.13), thus yielding a local existence/generality result (Theorem 5.14) for the nearly-Kähler structures with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

5.3.1. A first change-of-variable. By definition, Type III structures are those with $\langle q_1, q_2\rangle = \langle s_2, s_1\rangle = 0$ and q_1, q_2, s_1, s_2 not all zero. Recalling also that $q_1 s_1 - q_2 s_2 = 0$, we have:

$$\operatorname{rank}\begin{pmatrix} q_1 & q_2 \\ s_2 & s_1 \end{pmatrix} = 1, \text{ and}$$
$$q_1\overline{q}_2 \text{ and } s_2\overline{s}_1 \text{ both pure imaginary.}$$

This leads us to factor

$$\begin{pmatrix} q_1 & q_2 \\ s_2 & s_1 \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} \begin{pmatrix} it_1 & t_2 \end{pmatrix} = \begin{pmatrix} it_1 z & t_2 z \\ it_1 w & t_2 w \end{pmatrix},$$

where $z, w \colon P \to \mathbb{C}$ and $t_1, t_2 \colon P \to \mathbb{R}$. Note that by definition of Type III, we cannot have $(z, w) = (0, 0)$, nor can we have $(t_1, t_2) = (0, 0)$.

Remark: We caution that the functions z, w, t_1, t_2 are not uniquely defined: we may replace (z, w, it_1, t_2) with $(cz, cw, it_1/c, t_2/c)$ for any non-vanishing function $c : P \to \mathbb{R}$. $\square$

5.3.2. A second change-of-variable. We now solve $4[\binom{4}{3} + \binom{3}{3} + \binom{3}{3}] = 24$ of the $4\binom{6}{3} = 80$ equations arising in (5.1). Namely, we will solve $d(d\nu) \wedge \eta \wedge \overline{\eta} = 0$ and $d(d\nu) \wedge \eta \wedge \omega^{36} = 0$ and $d(d\nu) \wedge \overline{\eta} \wedge \omega^{36} = 0$ for $\nu \in \{\omega^3, \theta, \overline{\theta}, \omega^6\}$. This is accomplished in the following lemma:

Lemma 5.10: Let M be a nearly-Kähler manifold of Type III.

(a) There exist functions $u, v, \widehat{v} \colon P \to \mathbb{C}$ such that:

$$p_1 + 4ip_4 = 6t_2 u \qquad\qquad p_3 + 4p_4 = 24v.$$
$$p_2 = -6t_1 u \qquad\qquad -p_3 + 4p_4 = 24\widehat{v}.$$

(b) On the O(2)-coframe bundle P, the following algebraic relations hold:

$$\operatorname{Im}(w) = -24\operatorname{Re}(u\overline{v}) \tag{5.19}$$
$$12\widehat{v} = z(t_1^2\overline{u} + 4it_2\overline{v}) - iw(t_1^2 u - 4it_2 v) - 3it_2 u \tag{5.20}$$

Proof: The existence of $v, \widehat{v}$ is immediate. Let us set $y = p_1 + 4ip_4$ and expand the identities $d(d\omega^3) \wedge \eta \wedge \overline{\eta} = 0$ and $d(d\omega^6) \wedge \eta \wedge \overline{\eta} = 0$. This yields, for example,

$$\text{(5.21a)} \quad d(d\omega^6) \wedge \eta \wedge \overline{\eta} \wedge \theta = 0 \quad \Longrightarrow \quad is_2 y - q_1 \overline{y} - s_1 p_2 - iq_2 \overline{p}_2 = 0$$
$$\text{(5.21b)} \quad d(d\omega^6) \wedge \eta \wedge \overline{\eta} \wedge \overline{\theta} = 0 \quad \Longrightarrow \quad \overline{q}_1 y + i\overline{s}_2 \overline{y} - i\overline{q}_2 p_2 + \overline{s}_1 \overline{p}_2 = 0$$

and

$$\text{(5.22a)} \quad d(d\omega^3) \wedge \eta \wedge \overline{\eta} \wedge \omega^3 = 0 \quad \Longrightarrow \quad 4(\overline{p}_2 v + p_2 \overline{v}) = -(s_2 + \overline{s}_2)$$
$$\text{(5.22b)} \quad d(d\omega^6) \wedge \eta \wedge \overline{\eta} \wedge \omega^3 = 0 \quad \Longrightarrow \quad 4(\overline{y} v + y \overline{v}) = i(s_1 - \overline{s}_1)$$
$$\text{(5.23)} \quad d(d\omega^6) \wedge \eta \wedge \overline{\eta} \wedge \omega^6 = 0 \quad \Longrightarrow \quad \overline{p}_2 y - p_2 \overline{y} = 0.$$

In light of (5.23), we see that in order for the linear system (5.22a)-(5.22b) to have solutions, we must have

$$\text{(5.24)} \qquad i(s_1 - \overline{s}_1) \begin{pmatrix} \overline{p}_2 \\ p_2 \end{pmatrix} + (s_2 + \overline{s}_2) \begin{pmatrix} \overline{y} \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Regard the equations (5.21a), (5.21b), and (5.24) as a homogeneous linear system:

$$\begin{pmatrix} is_2 & -q_1 & -s_1 & -iq_2 \\ \overline{q}_1 & i\overline{s}_2 & -i\overline{q}_2 & \overline{s}_1 \\ s_2 + \overline{s}_2 & 0 & i(s_1 - \overline{s}_1) & 0 \\ 0 & s_2 + \overline{s}_2 & 0 & i(s_1 - \overline{s}_1) \end{pmatrix} \begin{pmatrix} y \\ \overline{y} \\ p_2 \\ \overline{p}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions to this system are of the form

$$\begin{pmatrix} y \\ \overline{y} \\ p_2 \\ \overline{p}_2 \end{pmatrix} = 6u \begin{pmatrix} t_2 \\ 0 \\ -t_1 \\ 0 \end{pmatrix} + 6\overline{u} \begin{pmatrix} 0 \\ t_2 \\ 0 \\ -t_1 \end{pmatrix}$$

for some $u\colon P \to \mathbb{C}$. This proves (a). The only equations left in $d(d\nu) \wedge \eta \wedge \overline{\eta} = 0$, $d(d\nu) \wedge \eta \wedge \omega^{36} = 0$ and $d(d\nu) \wedge \overline{\eta} \wedge \omega^{36} = 0$ for $\nu \in \{\omega^3, \theta, \overline{\theta}, \omega^6\}$ are exactly those in the statement of (b). $\Diamond$

Bookkeeping 5.11: We pause to unwind the notational changes. By definition of Type III and by Lemma 5.10(a), our torsion functions (p_i, q_i, r, s_i)

are now expressed as follows:

$$\begin{aligned}
&p_1 = 6t_2u - 12i(v+\widehat{v}) && p_3 = 12(v-\widehat{v}) && r = r && q_1 = it_1z && s_1 = t_2w\\
&p_2 = -6t_1u && p_4 = 3(v+\widehat{v}) && && q_2 = t_2z && s_2 = it_1w.
\end{aligned}$$

That is, the torsion is expressed in terms of

$$u, v, \widehat{v}, z, w, r\colon P \to \mathbb{C} \quad \text{and} \quad t_1, t_2\colon P \to \mathbb{R}.$$

By Lemma 5.10(b), the functions $\widehat{v}$ and $\mathrm{Im}(w)$ can be expressed in terms of the others. Hence, setting $t_0 = \mathrm{Re}(w)$, we regard the torsion as a function

$$(u, v, z, r;\, t_0, t_1, t_2)\colon P \to \mathbb{C}^4 \oplus \mathbb{R}^3.$$

One can check that these functions are O(2)-equivariant. Indeed, modulo $\langle \omega^1, \omega^2 \rangle = \langle \eta, \overline{\eta} \rangle$:

$$\begin{aligned}
&du \equiv iu\phi && dz \equiv 2iz\phi && dt_1 \equiv 0\\
&dv \equiv iv\phi && dr \equiv 3ir\phi && dt_2 \equiv 0\\
& && && dt_0 \equiv 0.
\end{aligned} \tag{5.25}$$

5.3.3. The Ansatz $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Having solved the 24 of the 80 equations arising in (5.1), we aim to solve the remaining 56 of them. To this end, we restrict attention to the case where G is a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$. This ansatz imposes inequalities on the torsion which allow us to normalize t_2.

Lemma 5.12: Let M be of Type III. If $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, then the (real) 1-form

$$\sigma_6 := 3\left(iu\overline{z} - \overline{uw}\right)\theta - 3\left(i\overline{u}z + uw\right)\overline{\theta} + \left(|w|^2 - |z|^2\right)\omega^6$$

is non-vanishing, and the (real) symmetric matrix

$$Q := \begin{pmatrix} \frac{1}{3}t_2 & -2t_1\,\mathrm{Im}(u) & -2t_1\,\mathrm{Re}(u)\\ -2t_1\,\mathrm{Im}(u) & 48\,\mathrm{Im}(u)\,\mathrm{Re}(v) - \mathrm{Im}(z) + t_0 & 24\,\mathrm{Re}(uv) - \mathrm{Re}(z)\\ -2t_1\,\mathrm{Re}(u) & 24\,\mathrm{Re}(uv) - \mathrm{Re}(z) & -48\,\mathrm{Re}(u)\,\mathrm{Im}(v) + \mathrm{Im}(z) + t_0 \end{pmatrix}$$

is positive-definite or negative-definite. In particular, t_2 is nowhere-vanishing.

Proof: As in the proof of Proposition 4.5, we identify G with an integral 4-fold of the differential ideal $\mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle$. Under this identification,

$\{\omega^3, \omega^4, \omega^5, \omega^6\}$ is a basis of $\mathfrak{g}^* = \{$left-invariant 1-forms on $G\}$.

Suppose that $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Then $\mathfrak{g}$ has a non-zero center, so there exists a non-zero element of $\mathfrak{g}^*$ which is closed. A calculation shows that the only elements of $\mathfrak{g}^*$ which are closed are multiples of

$$\sigma_6 := 3\left(iu\overline{z} - \overline{uw}\right)\theta - 3\left(i\overline{u}z + uw\right)\overline{\theta} + \left(|w|^2 - |z|^2\right)\omega^6$$

Thus, σ_6 is non-vanishing.

We now observe that $\{\sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ is a basis for $\mathfrak{g}^*$, where we are defining

$$(\sigma_3, \sigma_4, \sigma_5) := (t_2\omega^3 - t_1\omega^6,\, \mathrm{Im}(\chi),\, \mathrm{Re}(\chi)) \quad \chi := -3(2t_1 u\,\omega^3 - \theta - 8iv\,\omega^6).$$

One can calculate that modulo $\mathcal{I}_G$,

$$d\begin{pmatrix}\sigma_3\\ \sigma_4\\ \sigma_5\end{pmatrix} \equiv Q\begin{pmatrix}\sigma_4 \wedge \sigma_5\\ \sigma_5 \wedge \sigma_3\\ \sigma_3 \wedge \sigma_4\end{pmatrix}$$
$$d\sigma_6 \equiv 0.$$

Since $\{\sigma_3, \sigma_4, \sigma_5\}$ is a basis of $\mathfrak{su}(2)^*$, this coefficient matrix Q must be positive-definite or negative-definite, and hence t_2 is nowhere-vanishing. $\Diamond$

5.3.4. Local existence/generality. We now move to solve the integrability conditions (5.1) in the case of $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. By Lemma 5.12, the function t_2 is nowhere-vanishing. Recalling that z, w, t_1, t_2 are only defined up to scaling by a nowhere-vanishing function $c\colon P \to \mathbb{R}$, we shall choose c so that

$$t_2 = 1.$$

Thus, the torsion of the O(2)-structure is now encoded by $(u, v, z, r; t_0, t_1) : P \to \mathbb{C}^4 \oplus \mathbb{R}^2$. Since u, v, z, r, t_0, t_1 are G-invariant and O(2)-equivariant (by (5.25)), we may express their exterior derivatives as

$$\begin{aligned} du &= u'\eta + u''\overline{\eta} + iu\phi & dz &= z'\eta + z''\overline{\eta} + 2iz\phi & dt_1 &= t_1'\eta + t_1''\overline{\eta} \\ dv &= v'\eta + v''\overline{\eta} + iv\phi & dr &= r'\eta + r''\overline{\eta} + 3ir\phi & dt_0 &= t_0'\eta + t_0''\overline{\eta}. \end{aligned}$$

The Type III analogue of Lemma 5.2(b) and Lemma 5.6(b) is the following:

Lemma 5.13: Let M be a Type III nearly-Kähler structure with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. With the equations of Lemma 5.10 imposed and with the normalization $t_2 = 1$ in place, the integrability conditions (5.1) are equivalent to

$$
\begin{aligned}
(5.26)\quad u' &= f_1(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & u'' &= f_2(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1)\\
v' &= f_3(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & v'' &= f_4(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1)\\
z' &= f_5(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & z'' &= f_6(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1)\\
r' &= f_7(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & &\\
t_0' &= f_8(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & t_0'' &= f_9(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1)\\
t_1' &= f_{10}(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1) & t_1'' &= f_{11}(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1)\\
K &= f_{12}(u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1).
\end{aligned}
$$

for certain functions $f_1,\ldots,f_{12}$ of $u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1$.

Remark: The explicit expressions for $f_1,\ldots,f_{12}$ are sufficiently cumbersome that we will not list them here. They turn out to be polynomial functions of degree ≤ 10. □

Proof: With the equations of Lemma 5.10 imposed, and with $t_2 = 1$ imposed, there are $56 = 80 - 24$ polynomial equations remaining in (5.1) involving the functions

$$
\begin{aligned}
&u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1 \quad \text{and}\\
&u',u'',\overline{u}',\overline{u}'',v',v'',\overline{v}',\overline{v}'',z',z'',\overline{z}',\overline{z}'',r',r'',\overline{r}',\overline{r}'',t_0',t_0'',t_1',t_1'',K.
\end{aligned}
$$

A direct application of a computer algebra system (we used MAPLE) will solve these 56 equations, yielding lengthy explicit formulas for $u,u'',\ldots,t_1'$, t_1'',K in terms of $u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r},t_0,t_1$. ◊

As in the Type I and Type II settings, a calculation using MAPLE now shows that if the equations (5.26) of Lemma 5.13 hold, then $d(d\phi) = 0$ and $d(du) = d(dv) = d(dz) = d(dt_0) = d(dt_1) = 0$ are all satisfied, and that

$$
d(dr) = \big(F\,\eta\wedge\overline{\eta} - 4ir''\phi\wedge\overline{\eta}\big) + dr''\wedge\overline{\eta}
$$

where F is a certain polynomial function (of degree 14) of $u,\overline{u},v,\overline{v},z,\overline{z},r,\overline{r}$, t_0,t_1 whose explicit formula we will not list here.

The upshot of this discussion is that the integrability conditions (2.4) and (2.5) of Cartan's Third Theorem are finally satisfied. In particular, we obtain the following local existence/generality result:

Theorem 5.14: Nearly-Kähler structures (of Type III) for which G is a finite quotient of $\mathrm{SU}(2) \times \mathrm{U}(1)$ exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any $x \in \mathbb{R}^6$ and $(a_1, a_2, a_3, a_4, a_5, a_6, b_0) \in \mathbb{C}^4 \times \mathbb{R}^2 \times \mathbb{C}$ with $(a_3, a_5, \mathrm{Re}(a_1\overline{a}_2)) \neq (0,0,0)$, there exists a (Type III) nearly-Kähler structure with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ on an open neighborhood $U \subset \mathbb{R}^6$ of x and an $\mathrm{O}(2)$-coframe $f_x \in P|_x$ at x for which

$$\begin{aligned}&(u, v, z, r)(f_x) = (a_1, a_2, a_3, a_4) \quad \text{and} \\ &(t_0, t_1)(f_x) = (a_5, a_6) \quad \text{and} \quad r''(f_x) = b_0.\end{aligned}$$

Remark: The unusual looking requirement $(a_3, a_5, \mathrm{Re}(a_1\overline{a}_2)) \neq (0,0,0)$ is simply the condition $(z, \mathrm{Re}(w), \mathrm{Im}(w)) \neq (0,0,0)$ mentioned at the start of §5.3.1. That is, it is exactly the condition $(q_1, q_2, s_1, s_2) \neq (0,0,0,0)$ forming part of the definition of "Type III." $\square$

Proof: It remains only to examine the tableau of free derivatives. This proceeds exactly as in the cases of Types I and II, so we omit the details. $\Diamond$

6. Appendix

In Lemma 5.7(b), a calculation using MAPLE shows that the polynomial functions T_{41}, T_{42}, $H_{11}, H_{12}, H_{41}, H_{41}$, and G_1, G_2 are:

$$\begin{aligned}T_{41} &= -t_1\big(64t_2^3(2h_1+9) + 16t_2^2\ell_2 - 4t_2(h_1-2)\big) + t_2(6\ell_1 - 4h_4t_3) + \ell_2 t_4 \\ T_{42} &= 8t_2^2\left(2t_1(\ell_1+12t_4) + 4t_1^2 + 6h_1 + 27\right) - t_4(4t_1 + \ell_1 + 12t_4) \\ &\quad - \frac{1}{2}\left(t_1^2 + t_3^2 + 3\right) + 6\ell_2 t_2\end{aligned}$$

and

$$\begin{aligned}H_{11} &= 8t_2(t_1\ell_1 - 4h_1^2 - 12h_1) + 2\ell_2(h_1+3) \\ H_{12} &= 2t_1(16t_2^2(2h_1+9) + 4\ell_2t_2 + h_1 + 2) - 2\ell_1(h_1+3) - 2t_3h_4 \\ H_{21} &= -8t_2(t_3\ell_1 + 4h_1h_4 + 6h_4) + 2\ell_2h_4 \\ H_{22} &= -2t_3(16t_2^2(2h_1+9) + 4\ell_2t_2 + h_1 + 2) + 2h_4(64t_1t_2^2 - 8t_4 - t_1 - \ell_1)\end{aligned}$$

and

$$
\begin{aligned}
G_1 &= 256t_1t_2^3(2h_1+9) + 8t_2(2h_1t_1 - 2h_4t_3 - 15\ell_1 + 4t_1) + 64\ell_2t_1t_2^2 \\
G_2 &= 96t_2^2(2h_1-9) + 4t_2\ell_2(2h_1-21) - 3(\ell_1^2+\ell_2^2) - 4(h_1^2+h_4^2) \\
&\quad - 4\ell_1t_4 - 12h_1 - 12.
\end{aligned}
$$

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