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An integral representation and decay for the wave equation in the extreme Kerr geometry

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We consider the Cauchy problem for the massless scalar wave equation in the extreme Kerr geometry with the smooth initial data compactly supported outside the event horizon. Firstly, we derive an integral representation of the solution as an infinite sum over the angular momentum modes, each of which is an integral of the energy variable ω on the real axis. This integral representation involves solutions of the radial and angular ordinary differential equations which arise in the separation of variables. Furthermore, based on this integral representation, we prove that the solution of every azimuthal mode decays pointwise in time in L^∞_{loc}.

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1. Introduction and main results

1.1. Brief introduction

Since the black hole stability problem has been intensively studied in the past few decades, dating back to Regge and Wheeler in 1957 [24], who firstly investigated spin-2 tensor fields on the Schwarzschild manifold, which indicates the linear stability of Schwarzschild black hole. In the late 1960s and early 1970s, Carter, Teukolsky and Chandrasekhar discovered that the equations describing scalar, Dirac, Maxwell and linearized gravitational fields in the Kerr geometry are separable into ordinary differential equations(ODEs) [10, 28, 29]. Large amounts of significant progress have been made concerning the long-time behaviors of the solutions of these equations, through both numerical and analytical methods. Here we briefly refer a few recent results on mode stability and linear stability of Kerr black hole and the references therein. Mode stability of sub-extreme Kerr black hole was established by Whiting in [30], which was the first significant breakthrough of Kerr stability problem. Quite recently, Andersson, Whiting and their collaborators generalized the mode stability to the real axis in [4] and for the special case(spin-0), the mode stability argument was revisited by Shlapentokh-Rothman in [26]. Linear stability of sub-extreme Kerr black hole under higher spin perturbations was given by Finster and Smoller very recently in [20]. We also refer some results in [3, 12, 16, 17] for scalar waves in sub-extreme Kerr geometry.

However, in contrast to many previous papers concerning sub-extreme Kerr black hole, there are few results for extremal ones until recently Aretakis found the horizon instabilities of extremal black holes [6, 7]. An important question regarding the dynamical behaviors of extremal black holes is worth studying. Aretakis proved the decay of axisymmetric solutions in the extreme Kerr black hole geometry under scalar perturbations [5] while Dain and Gentile de Austria extended to gravitational perturbations [13]. However, without symmetric conditions, we have to consider whether the solutions of the wave equations under extreme Kerr black hole background remain pointwise decay in the black hole exterior region.

Moreover, turning to near-horizon geometries of extremal black holes, there are also several interesting issues worth discussing, for example, employing the near-horizon geometry to determine the correct boundary conditions to extend black hole uniqueness problems [2, 11, 15] and show the horizon instability of extremal Kerr black hole under gravitational perturbations [23]. In addition, a major breakthrough made by Strominger and Vafa [27] was to use the string theory to supply a microscopic derivation of the entropy for certain five dimensional extremal black holes. Therefore, we are also interested in the near-horizon geometry of the extreme Kerr black hole.

1.2. Main difficulties and results

To start with the scalar wave equation in the extreme Kerr geometry

$$\square_g \psi = g^{\mu\nu}\nabla_\mu \nabla_\nu \psi = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g} g^{\mu\nu}\frac{\partial}{\partial x^\nu})\psi = 0 \tag{1.1}$$

where g denotes the determinant of the Kerr metric $g_{\mu\nu}$ and in Boyer-Lindquist coordinates (t, r, θ, φ) it reads

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \frac{\triangle}{U}(dt - a\sin^2\theta d\varphi)^2 - U(\frac{dr^2}{\triangle} + d\theta^2) - \frac{\sin^2\theta}{U}\Big(adt - (r^2+a^2)d\varphi\Big)^2 \end{aligned}$$

where $U = r^2 + a^2\cos^2\theta$ and $\triangle(r) = r^2 - 2Mr + a^2$.

Now we restrict our attention to the extreme case($a^2 = M^2$). As a consequence, the function $\triangle(r) = r^2 - 2Mr + a^2 = (r-M)^2$ has a double root $r_0 = M$, so that the Cauchy horizon coincides with the event horizon. We shall focus on the region $r > r_0$ outside the horizon and therefore $\triangle(r) > 0$. Then the wave equation in Boyer-Lindquist coordinates becomes

$$\begin{aligned} &\Big[-\frac{\partial}{\partial r}\triangle\frac{\partial}{\partial r} + \frac{1}{\triangle}\Big((r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \varphi}\Big)^2 - \frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} \\ &- \frac{1}{\sin^2\theta}(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\varphi})^2\Big]\psi = 0 \end{aligned} \tag{1.2}$$

Moreover, since the Kerr geometry is axisymmetric, we just focus on a fixed φ-mode and thus for a given $k \in \mathbb{Z}$, we make the ansatz

$$\psi(t, r, \theta, \varphi) = e^{-ik\varphi}R(t, r, \theta)$$

Furthermore, usually one can introduce the Regge-Wheeler coordinate $u \in \mathbb{R}$ by

$$\frac{du}{dr} = \frac{r^2+a^2}{\triangle} \tag{1.3}$$

and

$$\frac{\partial}{\partial r}=\frac{r^2+a^2}{\triangle}\frac{\partial}{\partial u} \tag{1.4}$$

and introduce the new function Φ by

$$\Phi(t,u,\theta)=\sqrt{r^2+a^2}R(t,r,\theta) \tag{1.5}$$

Then (1.2) takes the following form

$$T\Phi=0 \tag{1.6}$$

in (t,u,θ), where

$$\begin{aligned}T=&\frac{(r^2+a^2)^2}{\triangle}(\frac{\partial^2}{\partial u^2}-\frac{\partial_u^2\sqrt{r^2+a^2}}{\sqrt{r^2+a^2}})-\frac{1}{\triangle}\Big((r^2+a^2)\frac{\partial}{\partial t}-iak\Big)^2\\&+\frac{1}{\sin^2\theta}(a\sin^2\theta\frac{\partial}{\partial t}-ik)^2+\frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta}\end{aligned}$$

and the variable u ranges over $(-\infty,\infty)$ as r ranges over (r_0,∞).

Unfortunately, the energy density is indefinite inside the ergosphere(the ergoregion lies outside the event horizon and the vector field ∂_t is spacelike), making it impossible to introduce a positive definite conserved scalar product. What's worse, in the extreme case, the upper limit of the superradiant frequencies is in some sense marginally trapped on the horizon. Thus the standard energy estimates and the classical vector fields methods of wave equations fail here. Also, we find the near-horizon geometry of extreme Kerr is essentially different from the sub-extreme case.

Despite many difficulties mentioned above, following the same strategy and basic framework of [16, 17] and the improved ODE techniques in [18, 20], we try to establish an integral representation of the wave solution in extreme Kerr geometry and then prove that the solution of the fixed azimuthal mode decays pointwise. Here we also emphasize that the constructions and estimates for the radial solutions in the near-horizon part are original and novel, as well as a significant improvement over the sub-extreme case in [16, 17]. In addition, some estimates and proofs for the radial solutions can also be simplified compared with the higher perturbations in [20].

More specifically, making the ansatz

$$\Psi=\begin{pmatrix}\Phi\\ i\partial_t\Phi\end{pmatrix} \tag{1.7}$$

we could rewrite the equation in a Hamiltonian form as

$$i\partial_t \Psi = H\Psi \tag{1.8}$$

where H is the Hamiltonian(explicitly given below). Next, we decompose the initial data into Fourier series of azimuthal modes

$$\Psi_0(u,\theta,\varphi) = \sum_{k\in\mathbb{Z}} e^{-ik\varphi}\Psi_0^{(k)}(u,\theta)$$

By linearity, the solution is obtained by taking the sum of all the resulting solutions for each azimuthal mode k. Hence we can derive an integral representation for the solution in terms of each azimuthal mode and show that the solution decays pointwise as follows.

Theorem 1.1. *For any $k \in \mathbb{Z}$, there is a parameter $p > 0$ such that for any $t < 0$, the solution of the wave equation with initial data $\Psi_0 = e^{-ik\varphi}\Psi_0^{(k)}(u,\theta)$ $\in C_0^\infty(\mathbb{R}\times S^2, \mathbb{C}^2)$ has the integral representation*

$$\Psi(t,u,\theta,\varphi) = \\ -\frac{1}{2\pi i}e^{-ik\varphi}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{(\omega+3ic)^p}(R^-_{\omega,n}Q^\omega_n(H+3ic)^p\Psi_0^{(k)})(u,\theta)d\omega$$

Moreover, the integrals above all exist in the Lebesgue sense. Furthermore, for every $\varepsilon > 0$ and $u_\infty \in \mathbb{R}$, there is N such that for all $u < u_\infty$

$$\sum_{n=N}^{\infty}\int_{-\infty}^{\infty}\|\frac{1}{(\omega+3ic)^p}(R^-_{\omega,n}Q^\omega_n(H+3ic)^p\Psi_0^{(k)})(u)\|_{L^2(S^2)}d\omega < \varepsilon \tag{1.9}$$

Theorem 1.2. *The solution of the wave equation with initial data $\Psi_0 = e^{-ik\varphi}\Psi_0^{(k)}(u,\theta) \in C_0^\infty(\mathbb{R}\times S^2,\mathbb{C}^2)$ decays pointwise. More precisely, we have*

$$\lim_{t\to-\infty}\Psi(t,u,\theta,\varphi) = 0$$

in $L^\infty_{loc}(\mathbb{R}\times S^2)$.

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2. Resolvent estimates and completeness of contour integrals

In order to study the dynamics of the scalar wave, we could solve the Cauchy problem in the first order Hamiltonian form by the contour integrals over the resolvent. However, we need to arrange us in the Hilbert space with suitable inner products and ensure the existence of the resolvent as well as the completeness of contour integrals in our infinite dimensional setting.

The Hamiltonian H reads

$$H = \begin{pmatrix} 0 & 1 \\ A & \beta \end{pmatrix}$$

The corresponding functions and operators above read

$$\begin{cases} \rho = r^2 + a^2 - a^2 \sin^2\theta \frac{\triangle}{r^2+a^2} \\ \beta = -\frac{2ak}{\rho}(1 - \frac{\triangle}{r^2+a^2}) \\ A = \frac{r^2+a^2}{\rho}(-\frac{\partial^2}{\partial u^2} + \frac{\partial_u^2 \sqrt{r^2+a^2}}{\sqrt{r^2+a^2}}) + \frac{\triangle}{\rho(r^2+a^2)}(-\frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} + \frac{k^2}{\sin^2\theta}) \\ \quad -\frac{a^2k^2}{\rho(r^2+a^2)} \end{cases}$$

The operators A and β are symmetric in the Hilbert space $L^2(\mathbb{R}\times S^2, d\mu)^2$ with the measure

$$d\mu := \rho du d\cos\theta$$

It is immediately verified that the Hamiltonian is symmetric with respect to the bilinear form

$$< \Psi_1, \Psi_2 >= \int_{\mathbb{R}\times S^2} < \Psi_1, \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \Psi_2 >_{\mathbb{C}^2} d\mu \tag{2.1}$$

In our setting, however, the energy scalar product is not definitely positive can be understood from the fact that the operator A is not positive on $L^2(\mathbb{R}\times S^2, d\mu)^2$. Hence our strategy is to modify (2.1) in such a way that it becomes positive definite. More precisely, we introduce the following scalar product by

$$(\Psi_1, \Psi_2) = \int_{\mathbb{R}\times S^2} (\Psi_1, \begin{pmatrix} A+\delta & 0 \\ 0 & 1 \end{pmatrix} \Psi_2)_{\mathbb{C}^2} d\mu \tag{2.2}$$

where

$$\delta = \frac{1}{\rho}(r^2 + a^2 + \frac{a^2k^2}{r^2+a^2})$$

For a suitable constant $c > 0$, the functions ρ, β and δ obviously satisfy the bounds as follows

$$\frac{1}{c} \leq \frac{\rho}{r^2+a^2} \leq c, \qquad |\beta|, |\delta| \leq c \tag{2.3}$$

One could take the completion of the domain(we choose the smooth wave functions which are compactly supported outside the horizon)

$$\mathcal{D}(H) = C_0^\infty(\mathbb{R} \times S^2, \mathbb{C}^4) \tag{2.4}$$

which gives rise to a Hilbert space $(\mathcal{H}, (.,.))$. The corresponding Hilbert space norm is equivalent to the Sobolev norm on $(H^{1,2} \oplus L^2)(\mathbb{R} \times S^2, \mathbb{C}^4)$.

For a fixed k-mode, obviously, the Hamiltonian H is not symmetric on $(\mathcal{H}, (.,.))$ in terms of the modified scalar product (2.2), which implies that the Hamiltonian H is not a self-adjoint operator. However, we still can get a symmetric operator by modifying the Hamiltonian H to a new one H_+ which looks like

$$H_+ = \begin{pmatrix} 0 & 1 \\ A+\delta & \beta \end{pmatrix}$$

where we again choose the domain (2.4). Fortunately, the difference of H and H_+ is a bounded operator. That is to say

$$\|(H - H_+)\Psi\| = \left\| \begin{pmatrix} 0 & 0 \\ -\delta & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \right\| = \| - \delta\Psi_1\|_{L^2} \leq \sup_{\mathbb{R}\times S^2} |\delta| \|\Psi\|$$

Upon the detailed analysis for H and H_+, we can exactly follow the previous work in the sub-extreme Kerr case in [16, Lemma 4.1, Corollary 4.2] and [20, Lemma 4.1, Theorem 5.2] to obtain the resolvent estimates and the contour integral completeness as follows.

Lemma 2.1. *There are constants* $c_1, c_2 > 0$ *such that for all* $\Psi \in \mathcal{D}(H)$ *and* $\omega \in \mathbb{C}$*, we have*

$$\|(H-\omega)\Psi\| \geq \frac{1}{c_1}(|Im\,\omega| - \frac{c_2}{1+|Re\,\omega|})\|\Psi\|$$

Moreover, for every $\omega \in \mathcal{S} := \{\omega \in \mathbb{C} \mid |Im\,\omega| \geq \frac{c}{1+|Re\,\omega|}\}$*, the resolvent* $R_\omega = (H-\omega)^{-1}$ *exists and is bounded by*

$$\|R_\omega\| \leq \frac{c_1}{|Im\,\omega|} \tag{2.5}$$

Theorem 2.2. *Choosing the contour*

$$C = \{\omega \mid Im\,\omega = 2c\} \cup \{\omega \mid Im\,\omega = -2c\} \tag{2.6}$$

with counter-clockwise orientation, then for every $\Psi \in \mathcal{D}(H)$ *we have the following integral*

$$\Psi = -\frac{1}{2\pi i}\int_C (R_\omega \Psi + \frac{\Psi}{\omega+3ic})d\omega \tag{2.7}$$

Moreover, the Cauchy problem for (1.8) with initial data $\Psi_0 \in \mathcal{D}(H)$ *has a unique solution given by*

$$\Psi(t) = -\frac{1}{2\pi i}\int_C e^{-i\omega t}(R_\omega \Psi_0 + \frac{\Psi_0}{\omega+3ic})d\omega \tag{2.8}$$

Similarly as in [20, Corollary 5.3, Corollary 5.4], the extra counter terms which do not change the value of the contour integral in (2.7) make it possible to derive the alternative integral representations for any integer $p \geq 1$. For negative time $t < 0$, due to the dominated exponential decay term, the integral representation of the solution can even be further simplified by deforming the upper contour to the entire upper half plane. The integral representation we will use in the remainder of this article is given below.

Corollary 2.3. *For negative time* $t < 0$*, the Cauchy problem for (1.8) with initial data* $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ *has a unique solution given by*

$$\Psi(t) = -\frac{1}{2\pi i}\int_{\mathbb{R}-2ic} e^{-i\omega t}(R_\omega \Psi_0 + \frac{\Psi_0}{\omega+3ic})d\omega \tag{2.9}$$

Moreover, for any integer $p \geq 1$,

$$\Psi(t) = -\frac{1}{2\pi i}\int_{\mathbb{R}-2ic} e^{-i\omega t}\frac{1}{(\omega+3ic)^p}(R_\omega (H+3ic)^p \Psi_0)d\omega \tag{2.10}$$

3. Separation of the wave equation and global estimates of the Jost solutions

Roughly we have the contour integral for the wave solution above, however, it remains to give the detailed analysis for the resolvent. Before that, we firstly separate the wave equation into the radial and angular ODEs and later we could derive the detailed estimates for the separated resolvent and show the convergence over an infinite sum of angular modes. By making the usual multiplicative ansatz

$$\Phi(t,u,\theta) = e^{-i\omega t}\phi(u)\Theta(\theta)$$

the fixed k-mode equation (1.6) can be separated into a radial ODE

$$\Big(-\frac{\partial^2}{\partial u^2} + V(u)\Big)\phi(u) = 0 \tag{3.1}$$

with the potential

$$V(u) = -(\omega + \frac{ak}{r^2+a^2})^2 + \frac{\lambda_n \triangle}{(r^2+a^2)^2} + \frac{1}{\sqrt{r^2+a^2}}\partial_u^2\sqrt{r^2+a^2} \tag{3.2}$$

and an angular ODE

$$\mathcal{A}_{\omega,k}\Theta(\theta) = \lambda_n\Theta(\theta) \tag{3.3}$$

where the angular operator is also called the spheroidal wave operator. The separation constant λ_n is an eigenvalue of $\mathcal{A}_{\omega,k}$ and can thus be regarded as an angular quantum number.

Since the angular operator $\mathcal{A}_{\omega,k}$ as well as the angular ODE here is almost the same with those in sub-extreme Kerr case, we refer more details concerning the spectral decomposition of the angular operator and some estimates for the angular ODE in [18, 20, 21].

However, the radial ODE here is completely different from the one in sub-extreme Kerr case. In particular, the potential near the horizon is quite different because of the lack of the exponential decay. The near-horizon geometry of extreme Kerr is also totally different from that of the sub-extreme Kerr. In addition, the integral representation involves solutions of the radial ODE and indeed the radial ODE is the most difficult and important part if one wants to study the wave solution. Therefore we have to understand the radial solutions and derive the desired estimates for them.

In the remainder of this section, firstly, we will construct the Jost solutions $\acute{\phi}$ and $\grave{\phi}$. Secondly, in order to control the infinite sum of angular modes of the contour integrals uniformly in time, we will adopt some modified ODE techniques to estimate the global behaviors of the Jost solutions for large λ.

3.1. Construction of the Jost solutions

In this subsection we fix the angular quantum numbers k, n and consider the two fundamental solutions $\acute{\phi}$ and $\grave{\phi}$ of the radial ODE (3.1) which satisfy the following asymptotic boundary conditions on the horizon and at infinity, respectively,

$$\lim_{u\to-\infty} e^{-i\Omega u}\acute{\phi}(u) = 1, \quad \lim_{u\to-\infty}(e^{-i\Omega u}\acute{\phi}(u))' = 0 \tag{3.4}$$

$$\lim_{u\to\infty} e^{i\omega u}\grave{\phi}(u) = 1, \qquad \lim_{u\to\infty}(e^{i\omega u}\grave{\phi}(u))' = 0 \tag{3.5}$$

where $\Omega = \omega - \omega_0$ and $\omega_0 = -\frac{ak}{r_0^2+a^2}$.

In what follows we will construct $\acute{\phi}$ carefully in more details, which is completely different from the sub-extreme Kerr case in [17]. Firstly, we rewrite (3.1) as follows

$$(-\frac{d^2}{du^2} - \Omega^2)\acute{\phi}(u) = -W(u)\acute{\phi}(u) \tag{3.6}$$

with a potential $W(u) = \Omega^2 + V(u)$ which vanishes at $u = -\infty$. Note that, using the relation (1.3) between u and r, now the potential has only polynomial decay, which is completely different from the one in sub-extreme Kerr case. More precisely, in the extreme case, the Regge-Wheeler coordinate $u(r)$ reads as

$$u = r + 2Mln(r-M) - \frac{2M^2}{r-M} \tag{3.7}$$

Thus the potential

$$W(u) = \Omega^2 + V(u) \tag{3.8}$$

has the asymptotic decay behavior $|W(u)| \leq \frac{c}{u^2}$ at $u = -\infty$ (approaching the horizon). However, it has the similar asymptotic structure as $u \to \infty$.

As a consequence, we can also follow the similar method in [17, Lemma 3.3] and modify to choose the Green's function as

$$S(u,v) = \frac{1}{2i\Omega}\Big(e^{-i\Omega(u-v)}\mathbf{\Theta}(u-v) + e^{i\Omega(u-v)}\mathbf{\Theta}(v-u)\Big) \tag{3.9}$$

where $\Omega \neq 0$, and $\mathbf{\Theta}$ denotes the Heaviside function defined by $\mathbf{\Theta}(x) = 1$ if $x \geq 0$ and otherwise $\mathbf{\Theta}(x) = 0$. Then the iteration scheme

$$\begin{cases} \phi^{(0)}(u) & = e^{i\Omega u} \\ \phi^{(l+1)}(u) & = -\int_{-\infty}^{u} S(u,v)W(v)\acute{\phi}^{(l)}(v)dv \end{cases} \tag{3.10}$$

will be used to construct the Jost solutions as follows.

Theorem 3.1. *For every angular momentum number n, the solutions $\acute{\phi}$ are well-defined in $E_1 := \{\Omega \in \mathbb{C} \mid Im\,\Omega \leq 0 \text{ and } \Omega \neq 0\}$. They form a holomorphic family in the interior of E_1.*

Similarly as in [17, Lemma 3.4, Theorem 3.2], we can also analytically extend $\acute{\phi}$ across the real axis except for $\Omega = 0$ as follows.

Theorem 3.2. *For every angular momentum number n, there is an open set E containing E_1 such that the solutions $\acute{\phi}$ are well-defined for every $\Omega \in E$ and form a holomorphic family on E.*

So far we still make no statements about the behaviors of $\acute{\phi}$ at $\Omega = 0$. This is also the subtle difference in extreme Kerr, compared with the sub-extreme Kerr case. However, after suitable rescaling, $\acute{\phi}$ may have a well-defined continuous limit at $\Omega = 0$. More precisely, for any Ω in the set

$$F := \{\Omega \in \mathbb{C} \,|\, Im\,\Omega \leq 0 \ \text{ and } \ |\Omega| \leq \delta\} \tag{3.11}$$

we rewrite the radial ODE (3.1) as

$$\Big(-\frac{d^2}{du^2} + \frac{\nu^2 - \frac{1}{4}}{u^2} - \Omega^2\Big)\phi(u) = -W_1(u)\phi(u) \tag{3.12}$$

The new potential W_1 is continuous in Ω and bounded by

$$|W_1(u)| \leq \frac{c}{|u|^3} \tag{3.13}$$

for all $\Omega \in F$. The solutions of the unperturbed equation can be explicitly expressed with Bessel functions

$$\begin{cases} h_1(u) = \sqrt{\frac{\pi\Omega u}{2}} J_\nu(\Omega u) \\ h_2(u) = \sqrt{\frac{\pi\Omega u}{2}} Y_\nu(\Omega u) \end{cases}$$

They have the following asymptotics

$$\begin{cases} h_1(u) \sim \cos(\Omega u), & h_2(u) \sim \sin(\Omega u), & \text{if } |\Omega u| \gg 1 \\ h_1(u) \sim \frac{\sqrt{\pi}}{\Gamma(\nu+1)}(\frac{\Omega u}{2})^{\nu+\frac{1}{2}}, & h_2(u) \sim -\frac{\Gamma(\nu)}{\sqrt{\pi}}(\frac{\Omega u}{2})^{-\nu+\frac{1}{2}}, & \text{if } |\Omega u| \ll 1 \end{cases}$$

where

$$(\frac{\Omega u}{2})^{\nu+\frac{1}{2}} = e^{(\nu+\frac{1}{2})[ln(\frac{|\Omega u|}{2})+i\arg(\frac{\Omega u}{2})+2k\pi i]}$$

We can choose k such that $i\arg(\frac{\Omega u}{2}) + 2k\pi i \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$, then $(\frac{\Omega u}{2})^{\nu+\frac{1}{2}}$ is analytic in the fixed branch. Hence the branch cut does not affect the limit behavior if taking $\Omega \in \mathbb{R}$.

On the other hand, the Green's function can be expressed in terms of these two fundamental solutions h_1 and h_2 as

$$S(u,v) = \frac{1}{\omega(h_1,h_2)}[h_1(u)h_2(v)\mathbf{\Theta}(v-u) + h_1(v)h_2(u)\mathbf{\Theta}(u-v)] \tag{3.14}$$

where $\omega(h_1,h_2) = h_1'h_2 - h_1h_2' = -\Omega$ is the Wronskian. The perturbation series ansatz reads

$$\phi = \sum_{l=0}^{\infty} \phi^{(l)} \tag{3.15}$$

In what follows we choose the function $\phi^{(0)}$

$$\phi^{(0)}(u) = i(\frac{\Omega}{2})^{\nu-\frac{1}{2}}(h_1 + ih_2)(u) \tag{3.16}$$

such that its asymptotics at $u = -\infty$ is a multiple times the plane wave $e^{i\Omega u}$, whereas for $\Omega = 0$, it has the asymptotics (3.18). Also we give the integral equation for the iteration scheme as follows

$$\phi^{(l+1)}(u) = -\int_{-\infty}^{u} S(u,v)W_1(v)\phi^{(l)}(v)dv \tag{3.17}$$

Therefore we can proceed similarly as in [17, Lemma 3.6, Theorem 3.5] to obtain the continuous limit of $\acute{\phi}$ at $\Omega = 0$ as

Theorem 3.3. *For every angular momentum number n, there is a real solution ϕ_0 of the radial ODE (3.6) for $\Omega = 0 (\omega = \omega_0)$ with the asymptotics*

$$\lim_{u\to-\infty}(-u)^{\nu_0-\frac{1}{2}}\phi_0(u) = \frac{\Gamma(\nu_0)}{\sqrt{\pi}}, \quad \nu_0 := \sqrt{\lambda_n(\omega_0)+\frac{1}{4}} \tag{3.18}$$

This solution can be regarded as a limit of the solutions from Theorem 3.1 and Theorem 3.2, in the sense that for all $u \in \mathbb{R}$ and $\Omega \in E_1$,

$$\phi_0(u) = \lim_{\Omega\to 0}\Omega^{\nu-\frac{1}{2}}\acute{\phi}(u), \quad \phi_0'(u) = \lim_{\Omega\to 0}\Omega^{\nu-\frac{1}{2}}\acute{\phi}'(u), \quad \nu := (\lambda_n(\omega)+\frac{1}{4})^{\frac{1}{2}}$$

Here we just list the results for $\grave{\phi}$ because they are almost the same with those in sub-extreme Kerr case in [17, Theorem 3.2, Theorem 3.5].

Theorem 3.4. *For every angular momentum number n, there is an open set G containing the real axis except for the origin*

$$G \supset G_0 := \{\omega \in \mathbb{C} \mid Im\,\omega \leq 0 \; and \; \omega \neq 0\} \tag{3.19}$$

such that the solutions $\grave{\phi}$ are well-defined for all $\omega \in G$ and form a holomorphic family on G.

Theorem 3.5. *For every angular momentum number n, there is a real solution ϕ_1 of the radial ODE (3.1) for $\omega = 0$ with the asymptotics*

$$\lim_{u\to\infty}u^{\mu_0-\frac{1}{2}}\phi_1(u) = \frac{\Gamma(\mu_0)}{\sqrt{\pi}}, \quad \mu_0 := \sqrt{\lambda_n(0)+\frac{1}{4}} \tag{3.20}$$

This solution can be obtained as a limit of the solutions from Theorem 3.4, in the sense that for all $u \in \mathbb{R}$ and $\omega \in G_0$,

$$\phi_1(u) = \lim_{\omega\to 0}\omega^{\mu-\frac{1}{2}}\grave{\phi}(u), \;\; \phi_1'(u) = \lim_{\omega\to 0}\omega^{\mu-\frac{1}{2}}\grave{\phi}'(u), \;\; \mu := \sqrt{\lambda_n(\omega)-2ak\omega+\frac{1}{4}}$$

3.2. Global estimates of the Jost solutions

Since we will deform the contours up to the real axis as in Lemma 4.3 later, we just need to consider the real ω, which implies that the potentials of the separated radial and angular ODEs above are both real. Moreover, the angular part is almost the same with the sub-extreme case in [21], thus it suffices to study the radial part. Compared with the methods in sub-extreme

Kerr case in [17], we will take advantage of the modified ODE methods in [20, 21]. Also the behaviors of the fundamental solution $\acute{\phi}$ here are completely different from those in sub-extreme Kerr case.

Moreover, one could also simplify all the ODE methods in [20] to estimate the potential V and the fundamental solutions because here we work with a real ODE with a real potential V. More specifically, compared with the results in [20, Section 10], the invariant region estimates for $\acute{\phi}$ are different and thus we need to have some new estimates in different ranges. While for $\grave{\phi}$, one can directly use the similar arguments in [20] and just simplify all of them in terms of the scalar perturbation case(special case: the parameter $s = 0$).

We follow the same notational conventions as in [20, Section 9] to ensure that our estimates are uniform in the parameters ω and λ for fixed k. All constants are independent of ω and λ(but they may depend on k). The constants with small letters $\mathbf{c}_1, \mathbf{c}_2, \cdots$ are determined at the beginning and are fixed throughout, while the symbol $\lesssim$ for $\leq \mathbf{c}$ with a constant $\mathbf{c}$ which is independent of the capital constants $\mathcal{C}_l$. Also, we adopt the convention that the constant $\mathcal{C}_l$ may depend on all previous constants $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{l-1}$, but is independent of the subsequent constants $\mathcal{C}_{l+1}, \cdots$. In particular, we may choose the capital constants such that $\mathcal{C}_1 << \mathcal{C}_2 << \cdots$.

Firstly, we also restrict in the large λ and ω range:

$$\omega^2 \geq \mathcal{C}_6 \quad \text{and} \quad \lambda \geq \mathcal{C}_7 \tag{3.21}$$

Here our potential V in (3.2) can be expanded with respect to parameters λ and ω as

$$V = -\omega^2 + \frac{\lambda\triangle}{(r^2+a^2)^2} - \frac{2\omega ak}{r^2+a^2}(1 - \frac{2\triangle}{r^2+a^2}) + O(\omega^0) + O(\lambda^0) \tag{3.22}$$

Compared with the potential in sub-extreme Kerr case in [20, Section 9.1], similarly as in [20, Lemma 9.1], if the ω^2 term is large enough to dominate all the other terms, we can also directly use the WKB approximations to estimate both $\acute{\phi}$ and $\grave{\phi}$. Hence, it also suffices to estimate the solutions in the subrange

$$\lambda \geq \mathcal{C}_5|\omega|^{\frac{3}{2}} \tag{3.23}$$

where choosing $\mathcal{C}_5$ sufficiently large such that the large λ term dominates all the other lower order terms(except ω^2). Then one could work with the much simpler expansion as

$$V = -\omega^2 + \frac{\lambda \triangle}{(r^2+a^2)^2} + O(\omega) + O(\lambda^0) \tag{3.24}$$

Although we find it has the same form with that in [20, Equation (9.7)], keep in mind that the relation (1.4) will lead to the different asymptotical behaviors for the potential when approaching the horizon and only has polynomial decay instead of exponential decay. Therefore we also need to calculate the first and second u-derivatives of V carefully by making use of the relation (1.4)(here ω^2 acts as a constant and thus dominated by large λ), which shows that V also has a unique maximum but at a point $u_{\max}$ with $r(u_{\max}) = (1+\sqrt{2})M$ and V is also concave near $u_{\max}$. V is also monotone increasing on $(-\infty, u_{\max})$ while monotone decreasing on $(u_{\max}, \infty)$ and the maximal value is also bounded by λ. On $[u_{\max} - \frac{1}{2}, u_{\max} + \frac{1}{2}]$, there is a constant $\mathbf{c}$ such that

$$\frac{\lambda}{\mathbf{c}} \leq -V''(u) \leq \mathbf{c}\lambda \tag{3.25}$$

Moreover, if $V(u_{\max}) > 0$, we can also denote the unique zeros of V by u_0^L and u_0^R, where $u_0^L < u_{\max} < u_0^R$. Otherwise, we denote $u_0^{L/R} = u_{\max}$.

After the detailed analysis of the potential, we find that we can proceed similarly as in [20, Section 9, Section 10] to choose the WKB method and T-method to estimate the fundamental solutions since the different values of $V(u_{\max})$ and the zeros of V will arrange us in different cases and regions. The major difference is that we need to treat the potential V in the near-horizon region in a different way. The remaining estimates are almost the same with those in [20]. More precisely, we also set

$$\begin{cases} V(u_{\max}) < -\mathcal{C}_4\sqrt{\lambda} & \text{WKB case} \\ -\mathcal{C}_4\sqrt{\lambda} \leq V(u_{\max}) < \mathcal{C}_4\sqrt{\lambda} & \text{Parabolic Cylinder (PC) case} \\ V(u_{\max}) \geq \mathcal{C}_4\sqrt{\lambda} & \text{Airy case} \end{cases} \tag{3.26}$$

Furthermore, we introduce

$$u_-^L = \begin{cases} u_0^L & \text{in the WKB case} \\ u_0^L - \mathcal{C}_3\mathcal{C}_1^{-\frac{1}{6}}|\omega|^{-\frac{1}{2}} & \text{in the PC case} \\ u_0^L - u_{\text{Airy}} & \text{in the Airy case} \end{cases} \tag{3.27}$$

and

$$u_-^R = \begin{cases} u_0^R & \text{in the WKB case} \\ u_0^R + \mathcal{C}_3\mathcal{C}_1^{-\frac{1}{6}}|\omega|^{-\frac{1}{2}} & \text{in the PC case} \\ u_0^R + u_{\text{Airy}} & \text{in the Airy case} \end{cases} \tag{3.28}$$

where

$$u_{\text{Airy}} := \mathcal{C}_3\lambda^{\frac{1}{6}}|\omega|^{-\frac{1}{3}}\max\Big(|\omega|^{-\frac{2}{3}}, (\mathcal{C}_1 V(u_{\max}))^{-\frac{1}{6}}|\omega|^{-\frac{1}{3}}\Big)$$

In addition, in the Airy case we set

$$u_+^L = u_0^L + u_{\text{Airy}} \tag{3.29}$$

and

$$u_+^R = u_0^R - u_{\text{Airy}} \tag{3.30}$$

Hence we have the following regions:

$$\begin{cases} \text{WKB regions} & (-\infty, u_-^L),\ (u_-^R, \infty) \text{ in all cases} \\ \text{PC region} & (u_-^L, u_-^R) \text{ in the PC case} \\ \text{Airy regions} & (u_-^L, u_+^L),\ (u_+^R, u_-^R) \text{ in the Airy case} \\ \text{WKB region with } V > 0 & (u_+^L, u_+^R) \text{ in the Airy case} \end{cases} \tag{3.31}$$

We remark that the setting for u_-^L and u_+^L in the Airy case is completely different from that in [20], which implies the major difference in the following procedure for the global estimates of $\acute{\phi}$ we mentioned above.

3.2.1. WKB estimates. Firstly, considering the potential in (3.22), we can proceed exactly as in [20, Lemma 9.1] to derive the WKB estimates for both $\acute{\phi}$ and $\grave{\phi}$ in ω-dominated range as follows.

Proposition 3.6. *Given $\mathcal{C}_5$, assume that $\lambda < \mathcal{C}_5|\omega|^{\frac{3}{2}}$, then for any $\varepsilon > 0$, by choosing the constants $\mathcal{C}_6$ and $\mathcal{C}_7$ sufficiently large we can show that*

$$\begin{cases} \frac{|V'(u)|}{|V(u)|^{\frac{3}{2}}} \leq \varepsilon \\ \frac{|V''(u)|}{|V(u)|^2} \leq \varepsilon \end{cases} \tag{3.32}$$

for all $u \in \mathbb{R}$, uniformly in ω and λ. Hence, we can have the WKB approximations up to an arbitrarily small error as

$$\acute{\phi} \approx \frac{1}{\sqrt[4]{-V}} \exp(\pm i \int^u \sqrt{-V}), \quad \lim_{u \to -\infty} \pm\sqrt{-V} = \Omega \tag{3.33}$$

and

$$\grave{\phi} \approx \frac{1}{\sqrt[4]{-V}} \exp(\pm i \int^u \sqrt{-V}), \quad \lim_{u \to \infty} \pm\sqrt{-V} = -\omega \tag{3.34}$$

In the following two propositions, we will show that $\acute{\phi}$ can be estimated by WKB approximations in the WKB regions and the WKB region with $V > 0$, respectively.

Proposition 3.7. *In the WKB regions, for any $\varepsilon > 0$, by choosing the constants $\mathcal{C}_0, \cdots, \mathcal{C}_7$ sufficiently large we find that (3.32) still holds for all ω and λ in the range (3.23) and $u \in (-\infty, u_-^L)$. Hence, we can still use the WKB approximation to estimate $\acute{\phi}$ on $(-\infty, u_-^L)$.*

Proof. Firstly, we can proceed exactly as in [20, Proposition 9.4] to verify the WKB conditions in the WKB case and PC case. However, in the Airy case, there are two subcases in terms of different values of $V(u_{\max})$ and $u_0^{L/R}$ as quantified below:

(A) $\omega^2 > \mathcal{C}_1 V(u_{\max})$

(B) $\omega^2 \leq \mathcal{C}_1 V(u_{\max})$

In subcase **(A)**, omitting the imaginary part of V, it can be proved exactly as in [20, Proposition 9.4]. Now we go on to the subcase **(B)**, which is something different from $\acute{\phi}$ in sub-extreme Kerr case, but similar to $\grave{\phi}$ there. Similarly as in [20, Proposition 9.5], we can proceed with a minor change with index L and R, as well as the sign. Therefore the WKB conditions (3.32) hold in the WKB regions for all the cases and we also have the WKB approximation (3.33). □

Proposition 3.8. *In the WKB region with $V > 0$, there are constants $\mathcal{C}_{10}$ and $c = c(\lambda, \omega)$ such that $\acute{\phi}$ can be estimated by*

$$\frac{|\acute{\phi}(u)|}{\mathcal{C}_{10}} \leq \frac{c(\lambda,\omega)}{(V(u))^{\frac{1}{4}}} e^{\int_{u_+^L}^{u} \sqrt{V}} \leq \mathcal{C}_{10} |\acute{\phi}(u)|$$

on $[u_+^L, u_{max}]$.

Proof. Since we are in the Airy case, for subcase **(A)**, the proof can be done exactly as in [20, Proposition 9.4]. However, in subcase **(B)**, the proof is completely different. More precisely, the equation (3.29) is reduced to

$$u_+^L = u_0^L + \mathcal{C}_3 \lambda^{\frac{1}{6}} |\omega|^{-1} \tag{3.35}$$

Also we know that V has no zeros in a $\frac{1}{\sqrt{\mathcal{C}_1}}$-neighborhood of $u_{\max}$ and $V(u_{\max}) \gtrsim \frac{\lambda}{\mathcal{C}_1}$ is large enough to derive the desired estimates. Therefore we will separate into two subregions as follows

- **(a)** $(u_+^L, u_{\max} - \frac{1}{\sqrt{\mathcal{C}_1}})$
- **(b)** $|u - u_{\max}| \leq \frac{1}{\sqrt{\mathcal{C}_1}}$

In the subregion **(a)**, we can proceed similar as in the proof of Proposition 3.7 in subcase **(B)**. In the remaining subregion **(b)**, we can proceed exactly as in [20, Proposition 9.9] to verify the WKB conditions (3.32). Therefore, we can proceed exactly as in [20, Lemma 10.3] to conclude the proof. □

Next, for $\grave{\phi}$, omitting the imaginary part of V, we can proceed exactly as in [20, Proposition 9.5, Proposition 9.9, Lemma 10.6] to derive the WKB estimates as follows.

Proposition 3.9. *In the WKB regions, for any $\varepsilon > 0$, by choosing the constants $\mathcal{C}_0, \cdots, \mathcal{C}_7$ sufficiently large we find that (3.32) still holds for all ω and λ in the range (3.23) and $u \in (u_-^R, \infty)$. Hence, we can use the WKB approximation to estimate $\grave{\phi}$ on (u_-^R, ∞).*

Proposition 3.10. *In the WKB region with $V > 0$, there are constants $\mathcal{C}_{10}$ and $c = c(\lambda, \omega)$ such that $\grave{\phi}$ can be estimated by*

$$\frac{|\grave{\phi}(u)|}{\mathcal{C}_{10}} \leq \frac{c(\lambda,\omega)}{(V(u))^{\frac{1}{4}}} e^{-\int_{u_+^R}^{u} \sqrt{V}} \leq \mathcal{C}_{10} |\grave{\phi}(u)|$$

on $[u_{max}, u_+^R]$.

3.2.2. Invariant region estimates. In the PC region and the Airy region, we will adopt the invariant region estimates(T-method) as introduced in [19] to estimate the fundamental solutions and their associated Riccati equation solutions. Here we denote the corresponding solutions of the Riccati equation by

$$\acute{y}(u) := \frac{\acute{\phi}'(u)}{\acute{\phi}(u)}, \quad \grave{y}(u) := \frac{\grave{\phi}'(u)}{\grave{\phi}(u)}$$

Also we want to point out that the potential in our case is always real and thus we can perfectly simplify all the invariant region estimates in [20, Lemma 10.1, Lemma 10.2]. More precisely, the error terms $\{E_i, i = 2, 3, 4\}$ vanish, which implies $E = |E_1|$ and we can always choose the suitable function $g \equiv 0$. Now we could start with the specific estimates for $\acute{\phi}$ in both PC and Airy regions as follows.

Proposition 3.11. *In the PC region, there is a constant $\mathcal{C}_9$ such that the solutions $\acute{y}$ and $\acute{\phi}$ are bounded in terms of their values at u_-^L by*

$$|\acute{y}(u)| \leq \mathcal{C}_9 |\acute{y}(u_-^L)| \tag{3.36}$$

$$Im\ \acute{y}(u) \geq \frac{Im\ \acute{y}(u_-^L)}{\mathcal{C}_9} \tag{3.37}$$

$$\frac{|\acute{\phi}(u_-^L)|}{\mathcal{C}_9} \leq |\acute{\phi}(u)| \leq \mathcal{C}_9 |\acute{\phi}(u_-^L)| \tag{3.38}$$

on $[u_-^L, u_{max}]$.

Proof. Set $\gamma = \sup_{[u_-^L, u_{\max}]} |V|$ and note that V is real, compared with [20, Lemma 10.1], our error terms are bounded by

$$E = |E_1| \lesssim \sqrt{\gamma} + \frac{|V'|}{\gamma} \tag{3.39}$$

Thus the integral of E can be estimated by

$$\int_{u_-^L}^{u_{\max}} E \lesssim \sqrt{\gamma}(u_{\max} - u_-^L)(1 + \sup_{[u_-^L, u_{\max}]} \frac{|V'|}{\gamma^{\frac{3}{2}}})$$

On the other hand, we can proceed exactly as in [20, Lemma 9.12] to show the bound of $\sqrt{\gamma}(u_{\max} - u_-^L)$. Omitting the imaginary part of V, we can simplify the argument in [20, Lemma 10.1] to verify the hypothesis of [19, Theorem 3.3] and conclude the proof. □

Proposition 3.12. *In the Airy region, there is a constant* $\mathcal{C}_9$ *such that the solutions* $\acute{y}$ *and* $\acute{\phi}$ *are bounded in terms of their values at* u_-^L *by*

$$|\acute{y}(u)| \leq \mathcal{C}_9 |\acute{y}(u_-^L)|$$
$$Im\ \acute{y}(u) \geq \frac{Im\ \acute{y}(u_-^L)}{\mathcal{C}_9}$$
$$\frac{\acute{\phi}(u_-^L)|}{\mathcal{C}_9} \leq |\acute{\phi}(u)| \leq \mathcal{C}_9 |\acute{\phi}(u_-^L)|$$

on $[u_-^L, u_+^L]$.

Proof. Here we also set $\gamma = \sup_{[u_-^L, u_+^L]} |V|$ and thus the integral of E can also be estimated by

$$\int_{u_-^L}^{u_+^L} E \lesssim \sqrt{\gamma}(u_+^L - u_-^L)(1 + \sup_{[u_-^L, u_+^L]} \frac{|V'|}{\gamma^{\frac{3}{2}}}) \tag{3.40}$$

Since now we are in the Airy case, for subcase **(A)**, we can proceed exactly as in [20, Lemma 9.10] to derive the bound of $\sqrt{\gamma}(u_+^L - u_-^L)$. In the remaining subcase **(B)**, we can proceed similarly as in [20, Lemma 9.11] to derive the bound of $\sqrt{\gamma}(u_+^L - u_-^L)$. Therefore we can proceed exactly as in Proposition 3.11 to conclude the proof. □

Since the behaviors of the potential are similar between both sides, we can proceed the T-method exactly as in Proposition 3.11 and Proposition 3.12 by omitting the index to derive the estimates for $\grave{\phi}$ as follows.

Proposition 3.13. *In the PC region, there is a constant* $\mathcal{C}_9$ *such that the solutions* $\grave{y}$ *and* $\grave{\phi}$ *are bounded in terms of their values at* u_-^R *by*

$$|\grave{y}(u)| \leq \mathcal{C}_9 |\grave{y}(u_-^R)|$$
$$Im\ \grave{y}(u) \geq \frac{Im\ \grave{y}(u_-^R)}{\mathcal{C}_9}$$
$$\frac{|\grave{\phi}(u_-^R)}{\mathcal{C}_9} \leq |\grave{\phi}(u)| \leq \mathcal{C}_9 |\grave{\phi}(u_-^R)|$$

on $[u_{max}, u_-^R]$.

Proposition 3.14. *In the Airy region, there is a constant* $\mathcal{C}_9$ *such that the solutions* $\grave{y}$ *and* $\grave{\phi}$ *are bounded in terms of their values at* u_-^R *by*

$$|\grave{y}(u)| \leq \mathcal{C}_9 |\grave{y}(u_-^R)|$$
$$Im\ \grave{y}(u) \geq \frac{Im\ \grave{y}(u_-^R)}{\mathcal{C}_9}$$
$$\frac{|\grave{\phi}(u_-^R)|}{\mathcal{C}_9} \leq |\grave{\phi}(u)| \leq \mathcal{C}_9 |\grave{\phi}(u_-^R)|$$

on $[u_+^R, u_-^R]$.

3.2.3. Estimates for bounded $\boldsymbol{\omega}$. In the above subsections, the estimates of fundamental solutions in the range (3.21) have been studied. Now we need to consider that ω ranges in a bounded set, but still for large λ. Moreover, we exclude the cases $\omega = 0$ and $\omega = \omega_0(\Omega = 0)$, which have been considered in Theorem 3.3 and 3.5. Thus we focus on the case as follows

$$0 \neq \omega^2 < \mathcal{C}_6, \quad \Omega \neq 0 \quad \text{and} \quad \lambda \geq \mathcal{C}_7 \tag{3.41}$$

The potential is negative both at $u = \pm\infty$ with the similar asymptotics

$$V(u) = -\Omega^2 + \frac{\lambda}{u^2} + O(u^{-3}) \qquad \text{if } u \to -\infty \tag{3.42}$$

and

$$V(u) = -\omega^2 + \frac{\tilde{\lambda}}{u^2} + O(u^{-3}) \qquad \text{if } u \to \infty \tag{3.43}$$

where $\tilde{\lambda} := \lambda + 2ak\omega$. It is obvious that V has a unique maximum $u_{\max}$ and $V \geq 0$ on the interval (u_0^L, u_0^R). Fortunately, the potential looks qualitatively as in subcase **(B)** of the Airy case by the scaling argument. Therefore we can deal with both $\acute{\phi}$ and $\grave{\phi}$ in the similar way as that for $\grave{\phi}$ in [20, Section 10.3] and we summarize the estimates as follows.

The results of Proposition 3.7 remain true for all ω and λ in (3.41) so that the fundamental solution $\acute{\phi}$ satisfies the WKB approximation on $(-\infty, u_-^L)$ again with an arbitrarily small error. Additionally, on (u_-^L, u_+^L), it can be estimated as in Proposition 3.12. Finally, one can estimate the solution on $(u_+^L, u_{\max})$ exactly as in Proposition 3.8.

The results of Proposition 3.9 still hold for all ω and λ in (3.41). Consequently, on (u_-^R, ∞) the fundamental solution $\grave{\phi}$ satisfies the WKB approximation again with an arbitrarily small error. Moreover, on (u_+^R, u_-^R) it can be estimated as in Proposition 3.14. Finally, on $(u_{\max}, u_+^R)$, one can estimate the solution exactly as in Proposition 3.10.

4. Construction of the separated resolvent and contour deformations

Suppose that $\omega \notin \sigma(H)$ and $k \geq 0$(because otherwise we could reverse the sign of ω). If the solutions $\acute{\phi}$ and $\grave{\phi}$ are linearly dependent, they would give rise to a vector in the kernel of $H - \omega$, in contradiction to our assumption $\omega \notin \sigma(H)$. Thus the Wronskian

$$w(\acute{\phi}, \grave{\phi}) := \acute{\phi}' \grave{\phi} - \acute{\phi} \grave{\phi}' \tag{4.1}$$

is nonzero. Then we could construct the Green's function with respect to the radial ODE (3.1) for ω in the lower half plane as

$$s(u, u') := \frac{1}{w(\acute{\phi}, \grave{\phi})} \times \begin{cases} \acute{\phi}(u)\grave{\phi}(u'), & \text{if } u \leq u' \\ \grave{\phi}(u)\acute{\phi}(u'), & \text{if } u > u' \end{cases} \tag{4.2}$$

satisfies the distributional equation

$$\Big(- \frac{\partial^2}{\partial u^2} + V(u) \Big) s(u, u') = \delta(u - u')$$

In order to simplify the notations, we also regard $s(u, u')$ as the integral kernel of a corresponding operator s. That is to say

$$(s\phi)(u) := \int s(u, u')\phi(u')du'$$

Once we have the constructions of the Jost solutions of the radial ODE as well as the spectral decomposition and eigenvalues estimates of the angular ODE, we could proceed exactly as in [16, Lemma 5.3, Proposition 5.4] and [20, Theorem 7.1] to construct the separated resolvent as follows.

Theorem 4.1. *For $\omega \notin \sigma(H)$, we let Q_n^ω be a spectral projector of the angular operator $\mathcal{A}_\omega$. Then the resolvent of H has the representation*

$$Q_n^\omega R_{\omega,n} = Q_n^\omega T(\omega, \lambda)$$

where T is the operator with the integral kernel

$$T(u,\theta;u',\theta') = \delta(\cos\theta - \cos\theta')\delta(u-u')\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$+ (r^2+a^2)^{-\frac{1}{2}}\delta(\cos\theta - \cos\theta')g(u,u')\begin{pmatrix} \tau(u',\theta') & \sigma(u',\theta') \\ \omega\tau(u',\theta') & \omega\sigma(u',\theta') \end{pmatrix}$$

and the operator g is given in

$$g = \sum_{l=0}^{\infty}(-\mathcal{N})^l s^{l+1} \tag{4.3}$$

where $\mathcal{N}$ is the nilpotent matrix in the Jordan decomposition

$$\mathcal{A}_\omega Q_n^\omega = (\lambda\mathbb{I} + \mathcal{N})Q_n^\omega$$

and τ, σ are the functions

$$\begin{cases} \sigma = (r^2+a^2)^{-\frac{3}{2}}[(r^2+a^2)^2 - \triangle a^2\sin\theta] \\ \tau = 2ak(r^2+a^2)^{-\frac{3}{2}}[(r^2+a^2) - \triangle] + \omega\sigma \end{cases}$$

Then we could also decompose the resolvent into infinite angular modes in the explicit forms as follows.

Corollary 4.2. *For any ω on the contour C, the resolvent $R_\omega = (H-\omega)^{-1}$ has the representation*

$$R_\omega = \sum_{n=0}^{\infty} R_{\omega,n}Q_n^\omega$$

where the operators $R_{\omega,n}$ can be written as integral operators

$$R_{\omega,n}\Psi(u,\theta) = \int_{-\infty}^{\infty}\frac{\rho(v,\theta)}{r(v)^2+a^2}\mathcal{R}_{\omega,n}(u,v)\Psi(v,\theta)dv \tag{4.4}$$

with integral kernels given by

$$\mathcal{R}_{\omega,n}(u,v) = \frac{r^2+a^2}{\rho}\delta(u-v)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + g(u,v)\begin{pmatrix} \omega-\beta(v) & 1 \\ \omega(\omega-\beta(v)) & \omega \end{pmatrix} \tag{4.5}$$

Therefore now we can also rewrite the contour integral in Corollary 2.3 over an infinite sum of angular modes. Lack of the convergence of the summand, we firstly analyze the partial sums defined by

$$\Psi^N(t) = -\frac{1}{2\pi i}\sum_{n=0}^{N}\int_{\mathbb{R}-2ic} e^{-i\omega t}(R_{\omega,n}Q_n^\omega\Psi_0 + Q_n^\omega\frac{\Psi_0}{\omega+3ic})d\omega \tag{4.6}$$

and

$$\Psi^{N,p}(t) = -\frac{1}{2\pi i}\sum_{n=0}^{N}\int_{\mathbb{R}-2ic} \frac{e^{-i\omega t}}{(\omega+3ic)^p}(R_{\omega,n}Q_n^\omega(H+3ic)^p\Psi_0)d\omega \tag{4.7}$$

where again $p \geq 1$ and $t \leq 0$. Upon getting the global estimates of fundamental solutions in terms of large angular modes in the following section, we will be able to show that the limit($N \to \infty$) of the partial sums exists, both with the summation inside and outside the integral.

In next step we now can use the mode stability result of extreme Kerr black hole away from the real axis in [9, 22, 25] to move the contour for the partial sums up to the real axis in the same manner with [20, Lemma 8.1].

Lemma 4.3. *For any $\Psi_0 \in \mathcal{D}(H)$ and any integer $p \geq 1$, the partial sums (4.6) and (4.7) can be written for any $t \leq 0$ as*

$$\Psi^N(t) = -\frac{1}{2\pi i}\sum_{n=0}^{N}\lim_{\epsilon\to 0}\int_{\mathbb{R}-i\epsilon} e^{-i\omega t}(R_{\omega,n}Q_n^\omega\Psi_0 + Q_n^\omega\frac{\Psi_0}{\omega+3ic})d\omega \tag{4.8}$$

and

$$\Psi^{N,p}(t) = -\frac{1}{2\pi i}\sum_{n=0}^{N}\lim_{\epsilon\to 0}\int_{\mathbb{R}-i\epsilon} \frac{e^{-i\omega t}}{(\omega+3ic)^p}(R_{\omega,n}Q_n^\omega(H+3ic)^p\Psi_0)d\omega \tag{4.9}$$

5. Estimates for the large angular modes

Based on the global estimates of the Jost solutions above, we will have a-priori estimate for the Green's function as follows.

Lemma 5.1. *For any $u_\infty > 0$, there is a constant $C_L > 0$ and $n \in \mathbb{N}$ such that the kernels of the Green's function s and of the operator g in (4.2)* *and (4.3) satisfy the following bound*

$$|s(u,u')|,\ |g(u,u')| \leq C_L$$

for all $\omega \in \mathbb{R}$ *and* $\lambda > \mathcal{C}_7$ *uniformly for all* $u < u_\infty$ *and* $-u_\infty < u' < u_\infty$.

Proof. To start with $|\omega|$ in (3.21), if $\lambda < \mathcal{C}_5|\omega|^{\frac{3}{2}}$, the fundamental solutions $\acute{y}$ and $\grave{y}$ lie in different half planes and the WKB approximations in (3.33) and (3.34) imply that

$$|s(u,u')|,\ |g(u,u')| \lesssim \frac{1}{|\omega|} \tag{5.1}$$

Otherwise, for $|\omega|$ in (3.23), we consider the following three different cases in (3.26). In the WKB case, the WKB approximations give the same bound (5.1). In the PC case, the estimates of Proposition 3.11 and Proposition 3.13 show that (5.1) again holds. In the Airy case, from the estimates of Proposition 3.8, Proposition 3.12, Proposition 3.10 and Proposition 3.14, one sees that $\acute{\phi}$ is increasing exponentially in the WKB region with $V > 0$, whereas $\grave{\phi}$ is exponentially decaying. Therefore $|s(u,u')|$ and $|g(u,u'|$ decay for large λ uniformly in ω.

If ω is in a bounded set in (3.41), the estimates show that $\acute{\phi}$ and $\grave{\phi}$ behave similar as in the Airy case. As shown in Theorem 3.3 and 3.5, the fundamental solutions $\acute{\phi}$ and $\grave{\phi}$ are continuous at $\omega = 0$, ω_0, and the Wronskian is nonzero in the limit(see Lemma 6.1).

This concludes the proof for all $\omega \in \mathbb{R}$ and $\lambda > \mathcal{C}_7$. □

It follows from Lemma 5.1 that we can proceed similarly as in [20, Proposition 10.13, Corollary 10.14] to give the uniform control of the separated resolvent in terms of the large angular modes as follows.

Lemma 5.2. *For sufficiently large* p *and all* $u < u_\infty$, *the following estimate holds*

$$\frac{1}{|\omega + 3ic|^p}\|(R_{\omega,n}Q_n^\omega(H+3ic)^p\Psi_0)(u)\|_{L^2(S^2)} \leq \frac{c(u_\infty,\Psi_0)}{(n+1)^2(1+|\omega|)^2}$$

Therefore we have the convergence of the integral representation which involves the separated resolvent into large angular modes as follows.

Corollary 5.3. *For sufficiently large* p, *the solution of the Cauchy problem for the wave equation with initial data* $\Psi|_{t=0} = \Psi_0 \in \mathcal{D}(H)$ *can be written* *for any $t < 0$ as*

$$\Psi(t) = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\lim_{\epsilon\to 0}\int_{\mathbb{R}-i\epsilon}\frac{e^{-i\omega t}}{(\omega+3ic)^p}(R_{\omega,n}Q_n^{\omega}(H+3ic)^p\Psi_0)d\omega \tag{5.2}$$

Here the series absolutely converge in the sense that for any $\epsilon > 0$, there is N such that for all $t < 0$ and $u < u_\infty$,

$$\sum_{n=N}^{\infty}\|\lim_{\epsilon\to 0}(\int_{\mathbb{R}-i\epsilon}\frac{e^{-i\omega t}}{(\omega+3ic)^p}(R_{\omega,n}Q_n^{\omega}(H+3ic)^p\Psi_0)d\omega)(u)\|_{L^2(S^2)} < \epsilon \tag{5.3}$$

Proof. To start with the integral representation (2.10) in Corollary 2.3, then separating the resolvent in Corollary 4.2, applying the estimates of the large angular modes in Lemma 5.2 and deforming the contours(see Lemma 4.3), this gives the result. □

6. Ruling out the poles on the real axis

We know that all integrands of the integral representation in Corollary 5.3 are holomorphic for ω in the lower half plane, making it possible to move the contour arbitrarily close to the real axis. However, the remaining problem is that these integrands might have poles on the real axis. These so-called radiant modes are ruled out in two steps as follows.

Lemma 6.1. *For every angular mode, the kernels of the Green's function s and of the operator g in (4.2) and (4.3) are uniformly bounded in a neighborhood of $\omega = 0$ and $\Omega = 0$ (here again $Im\ \omega \leq 0$).*

Proof. In view of the continuity results of Theorem 3.2, 3.3, 3.4 and 3.5, it remains to show that choosing $\omega = 0$, the Wronskian is nonzero. In the case of $\omega_0 \neq 0$, we consider it in two separate subcases.

Firstly, choosing $\omega = 0$ and thus $\Omega = \omega - \omega_0 \neq 0$, the Wronskian $W = W(\acute{\phi}, \phi_1)$. Since $\acute{\phi}$ is complex and ϕ_1 is real, obviously the Wronskian is nonzero.

Secondly, choosing $\Omega = 0$ and thus $\omega = \Omega + \omega_0 \neq 0$, the Wronskian $W = W(\phi_0, \grave{\phi})$. Since ϕ_0 is real and $\grave{\phi}$ is complex, again the Wronskian is nonzero.

Otherwise, in the case of $\omega_0 = 0$(i.e. $\omega = \Omega$), choosing $\omega = \Omega = 0$, the Wronskian $W = W(\phi_0, \phi_1)$. Noting that in this case the potential V in (3.2) is everywhere positive, hence solutions of the radial equation (3.1) are convex. Also, the asymptotics in (3.18) and (3.20) implies that the solutions ϕ_0 and ϕ_1 do not coincide, and thus their Wronskian is non-zero. □

In what follows we prove that the separated resolvent has no poles on the real axis of ω. The causality method is the same as in [20, Section 11.2], which is an improvement of the method first developed in [17, Section 7], but here we need to exclude two kinds of small neighborhoods of $\omega = 0$ and $\omega = \omega_0$.

Lemma 6.2. *For any $n \in \mathbb{N}_0$, the separated resolvent $R_{\omega,n}$ in (4.4) is holomorphic in the lower half plane $\{Im\, \omega < 0\}$. Moreover, it is continuous up to the real axis. That is to say that the limit of the resolvent reads*

$$R^-_{\omega,n}\Psi := \lim_{\epsilon\to 0}(R_{\omega-i\epsilon,n}\Psi) \tag{6.1}$$

for all $\omega \in \mathbb{R}$.

Proof. We want to show that $R_{\omega,n}$ is continuous at $\omega_1 \in \mathbb{R}$. In the case $\omega_1 = 0$ and $\omega_1 = \omega_0$, the result follows immediately from Lemma 6.1. In the remaining case, we can exactly proceed the proof as in [20, Proposition 11.2]. □

7. Integral representation and decay of wave solutions

Now we can give the proofs for our main theorems as follows.

Proof of Theorem 1.1:

Proof. In Corollary 5.3, we have (5.2) and (5.3). Then, we apply Lemma 6.2 to move the contour up to the real axis. This concludes the proof. □

Proof of Theorem 1.2:

Proof. Given $\varepsilon > 0$, we can choose N such that (1.9) holds, which controls the sums of the integral representation in terms of the large angular modes. For the remaining finite angular modes $n = 0, \cdots, N-1$, the Riemann-Lebesgue lemma gives the pointwise decay locally uniformly in the spatial variables as $t \to -\infty$. Therefore we conclude that $\Psi(t)$ decays in $L^2_{loc}(\mathbb{R}\times S^2, \mathbb{C}^2)$. Differentiating the equation with respect to t, we conclude that all time derivatives $\partial_t^m\Psi(t)$ decays in $L^2_{loc}(\mathbb{R}\times S^2, \mathbb{C}^2)$. Using the equation (1.8) and applying the Sobolev embedding theorem, we obtain that the solution decays pointwise in $L^\infty_{loc}(\mathbb{R}\times S^2)$. □

Remark 7.1. For clarity, we point out that, applying Theorem 1.2 with the reversed time direction, one can also get the decay of the solution for $t \to +\infty$.

8. Conclusions

Firstly, we derive the integral representation which involves the fundamental solutions of the ODEs arising in the separation of variables, which is a starting point for a detailed analysis of the dynamics of the extreme Kerr black hole. Based on the integral representation, we also prove that the solution of each azimuthal mode pointwise decays in time in L^{∞}_{loc}.

The results above cannot address the regularity and exact decay rate for the wave solutions. In fact, we hope that we can derive the quantitative decay estimates for the radial solutions in terms of parameters k in the future. Then we can derive the convergence of infinite azimuthal modes as well as the decay and regularity in some weighted Sobolev norm sense.

Moreover, Aretakis, Lucietti and Reall's works [5–7, 23] found the horizon instabilities of extremal black holes and showed that extremal Kerr black holes are linearly unstable. From our constructions of the radial solutions, we can see that the purely real zero-mode solution of $\acute{\phi}$ and the polynomial decay rate of the potential W approaching the horizon are related to the horizon instabilities, which is in contrast to the sub-extreme Kerr case. Indeed, $\acute{\phi}$ cannot have a holomorphic extension in a neighborhood of $\Omega = 0$ and we just get a continuous limit ϕ_0 which has a branch cut. Quite recently, Zimmerman and his co-authors [9] exactly associated the enhanced growth rate of the transverse derivatives of such mode solution(has a singular branch point in the Green's function at the superradiant bound frequency) on the horizon with Aretakis's results and even extended the results under higher spin perturbations [22], which also agree with our results of radial Jost solutions.

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