Weighted extremal Kähler metrics and the Einstein–Maxwell geometry of projective bundles

VESTISLAV APOSTOLOV, GIDEON MASCHLER,
AND CHRISTINA W. TØNNESEN-FRIEDMAN

We study the existence of weighted extremal Kähler metrics in the sense of [4, 32] on the total space of an admissible projective bundle over a Hodge Kähler manifold of constant scalar curvature. Admissible projective bundles have been defined in [5], and they include the projective line bundles [29] and their blow-downs [31], thus providing a most general setting for extending the existence theory for extremal Kähler metrics pioneered by a seminal construction of Calabi [12]. We obtain a general existence result for weighted extremal metrics on admissible manifolds, which yields many new examples of conformally Kähler, Einstein–Maxwell metrics of complex dimension $m > 2$, thus extending the recent constructions of [30, 38] to higher dimensions. For each admissible Kähler class on an admissible projective bundle, we associate an explicit function of one variable and show that if it is positive on the interval $(-1, 1)$, then there exists a weighted extremal Kähler metric in the given class, whereas if it is strictly negative somewhere in $(-1, 1)$, there is no Kähler metrics of constant weighted scalar curvature in that class. We also relate the positivity of the function to a notion of weighted K-stability, thus establishing a Yau–Tian–Donaldson type correspondence for the existence of Kähler metrics of constant weighted scalar curvature in the rational admissible Kähler classes on an admissible projective bundle. Weighted extremal orthotoric metrics are examined in an appendix.

1. Introduction

1.1. Motivation and Background

There has been a great deal of interest recently in studying the relation between the existence of extremal Kähler metrics, especially constant scalar curvature Kähler metrics (CSCK), on a compact complex m-dimensional manifold and various notions of algebro-geometric stability: this is the so-called Yau–Tian–Donaldson (YTD) conjecture.

The YTD conjecture takes its origins in a formal GIT picture in which the formal (infinite dimensional) Kähler manifold is the Fréchet space $\mathcal{K}(M, \omega)$ of all complex structures J on M, compatible with a given symplectic form ω, whereas the formal complex structure $\mathbf{J}$ and Kähler form $\mathbf{\Omega}$ on $\mathcal{K}(M, \omega)$ are defined by

$$(1) \qquad \mathbf{J}_J(\dot{J}) = J\dot{J}, \quad \mathbf{\Omega}_J(\dot{J}_1, \dot{J}_2) = \int_M \mathrm{tr}\left(J\dot{J}_1\dot{J}_2 \right) \frac{\omega^m}{m!},$$

where $\dot{J}, \dot{J}_1, \dot{J}_2 \in \Gamma(\mathrm{End}(TM))$ are tangent vectors at $J \in \mathcal{K}(M, \omega)$, viewed as smooth fields of endomorphisms of TM which anti-commute with J. The main observation due to Fujiki [23] and Donaldson [18] is that the infinite dimensional Lie group $\mathrm{Ham}(M, \omega)$ of hamiltonian transformations of (M, ω) acts on $\mathcal{K}(M, \omega)$ in a hamiltonian way, with moment map

$$(2) \qquad \boldsymbol{\mu}(J) = \overset{\circ}{Scal}(g_J) = Scal(g_J) - \underline{S}(\omega),$$

where $Scal(g_J)$ is the scalar curvature of the Kähler metric $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$, $\underline{S}(\omega)$ is its average, and the momentum map $\boldsymbol{\mu}(J)$ at J is viewed as an element of the Lie algebra $\mathrm{Lie}(\mathrm{Ham}(M, \omega)) \cong (C^\infty(M))_0$ of smooth functions

with zero mean with respect to the measure $\omega^m/m!$, via the Ad-invariant inner product

$$(3) \qquad \langle \varphi_1, \varphi_2 \rangle = \int_M \varphi_1 \varphi_2 \frac{\omega^m}{m!}.$$

Thus, finding CSCK metrics in $\mathcal{K}(M, \omega)$ corresponds to finding zeroes of the momentum map $\boldsymbol{\mu}$ whereas finding extremal Kähler metrics in $\mathcal{K}(M, \omega)$ corresponds to finding critical points of the Calabi functional $\mathbf{Cal}(J) = ||\boldsymbol{\mu}(J)||^2 = \int_M (Scal(g_J) - \underline{S})^2 \omega^m / m!$.

It is natural to expect that for other geometric problems in Kähler geometry, for which we have a similar formal GIT interpretation, a suitable YTD type correspondence holds true. This is the case for a class of problems, motivated in [7] and further developed in [34], which can be fit into the above GIT picture by fixing a torus $\mathbb{T} \subset \mathrm{Ham}(M, \omega)$ with momentum map $z : M \to \mathfrak{t}^*$, and image a compact polytope $P \subset \mathfrak{t}^*$ (here $\mathfrak{t}$ denotes the Lie algebra of $\mathbb{T}$ and $\mathfrak{t}^*$ its dual) and two positive smooth functions $u, v : P \to \mathbb{R}$. Then, the space $\mathcal{K}^{\mathbb{T}}(M, \omega)$ of $\mathbb{T}$-invariant ω-compatible complex structures on M carries a formal Kähler structure $(\mathbf{J}, \boldsymbol{\Omega}^u)$ with

$$(4) \qquad \boldsymbol{\Omega}_J^u(\dot{J}_1, \dot{J}_2) = \int_M \mathrm{tr}\Big(J \dot{J}_1 \dot{J}_2 \Big)(u \circ z) \frac{\omega^m}{m!},$$

which is invariant under the action of the group $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ of hamiltonian transformation commuting with $\mathbb{T}$. Furthermore, the action of $\mathrm{Ham}^{\mathbb{T}}(M, \omega)$ is hamiltonian. By considering the Ad-invariant inner product

$$(5) \qquad \langle \varphi_1, \varphi_2 \rangle_v = \int_M \varphi_1 \varphi_2 (v \circ z) \frac{\omega^m}{m!},$$

on the Lie algebra $\mathrm{Lie}\big(\mathrm{Ham}(M, \omega)^{\mathbb{T}}\big) \cong \big(C^\infty(M)\big)_{0,v}^{\mathbb{T}}$ of $\mathbb{T}$-invariant smooth functions of zero mean with respect to the measure $(v \circ z)\omega^m/m!$, one identifies linear functionals on $\mathrm{Lie}\big(\mathrm{Ham}(M, \omega)^{\mathbb{T}}\big)$ with $\mathbb{T}$-invariant smooth functions, and denote by

$$(6) \qquad \overset{\circ}{Scal}_{u,v,\mathbb{T}}(g_J) := \boldsymbol{\mu}_{u,v,\mathbb{T}}(J)$$

the corresponding moment map. One thus expects that the problem of finding $\mathbb{T}$-invariant Kähler metrics in a given Kähler class, for which $\overset{\circ}{Scal}_{u,v,\mathbb{T}}(g_J) = 0$ or, more generally, for which $\mathrm{grad}_{g_J}(\overset{\circ}{Scal}_{u,v,\mathbb{T}}(g_J))$ is a holomorphic vector field, shares many common features with the well-established theory of CSCK and extremal Kähler metrics. We shall refer

to such special Kähler metrics as (u, v)-*CSCK* and (u, v)-*extremal* Kähler metrics, respectively.

There are a number of geometric situations which reduce to the above formal GIT setting for particular choices of the functions u and v. These include the problem of the existence of admissible CSCK metrics on rigid semi-simple toric bundles [6], and on manifolds with free multiplicity [20].

In this paper, we shall focus on the geometric problem introduced in [7] and [4], which fits into the above context by choosing $\xi \in \mathfrak{t}$ and $\boldsymbol{a} > 0$ such that $\pi_\xi(P) + \boldsymbol{a} > 0$ (where $\pi_\xi : \mathfrak{t}^* \to \mathfrak{t}_\xi^* = \mathbb{R}$ is the projection dual to the inclusion $\mathbb{R} \cdot \xi \subset \mathfrak{t}$), and letting $u := (\pi_\xi + \boldsymbol{a})^{-(\mathrm{p}+1)}$ and $v := (\pi_\xi + \boldsymbol{a})^{1-\mathrm{p}}$ for $\mathrm{p} \in \mathbb{R}$. As observed in [4, 33], the corresponding momentum map (6) then becomes (up to an additive constant depending only on the deRham class $[\omega]$)

$$(7) \qquad Scal_{f,\mathrm{p}}(g) = f^2 Scal(g) - 2(\mathrm{p}-1)f\Delta_g f - \mathrm{p}(\mathrm{p}-1)|df|_g^2,$$

where $f := \langle z, \xi \rangle + \boldsymbol{a}$ is a positive Killing potential of the Kähler metric g. The smooth function $Scal_{f,\mathrm{p}}(g)$ is referred to as *the (f, p)-scalar curvature* of g, and we are interested in finding Kähler metrics of constant (f, p)-scalar curvature, which we call (f, p)-*CSCK metrics* or, more generally, Kähler metrics for which $Scal_{f,\mathrm{p}}(g)$ is a Killing potential, which we refer to as (f, p)-*extremal Kähler metrics*. The case $\mathrm{p} = 2m$ has been studied in [3, 7, 21, 22, 30, 32, 33, 37, 38], having the geometric meaning that $Scal_{f,2m}(g)$ computes the scalar curvature of the Hermitian metric $h = f^{-2}g$. Thus, a Kähler metric g for which $Scal_{f,2m}(g)$ is constant corresponds to a Hermitian metric $h = f^{-2}g$ which is *conformally Kähler, Einstein–Maxwell*, see [3, 7, 36]. Another geometrically interesting case is when $\mathrm{p} = m + 2$, which is related to the study of Levi–Kähler quotients [4] and extremal Sasaki metrics [1].

1.2. Main Results

The main results of this paper concern the existence and obstruction theory of (f, p)-extremal Kähler metrics and (f, p)-CSCK metrics on certain holomorphic projective bundles of the form $M = P(E_0 \oplus E_\infty) \to S$, where E_0 and E_∞ are projectively flat bundles (of arbitrary rank) over a compact CSCK manifold S. Such complex manifolds, introduced and studied in [5] are called *admissible*. Any admissible manifold M is endowed with a

natural $\mathbb{S}^1$-action, induced by the natural $\mathbb{S}^1$-action by fibre-wise multiplications on the bundle E_0. Following [5], M admits a family of $\mathbb{S}^1$-invariant symplectic structures ω (called *admissible*) with associated momentum image $z(M) = [-1, 1] \subset \mathbb{R}$. With this normalization, for any real constants $\boldsymbol{a} > 1$ and p, we consider the problem of finding an ω-compatible $(z + \boldsymbol{a}, \mathrm{p})$-CSCK metric or, more generally, an ω-compatible Kähler metric for which $Scal_{(z+\boldsymbol{a}),\mathrm{p}}$ is a Killing potential (called a $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric). The construction of $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metrics on admissible complex manifolds naturally extends the well-known constructions, going back to Calabi [12], of explicit extremal Kähler metrics on $\mathbb{C}P^1$-bundles over a CSCK base and their blow-downs (see [5] and the references therein) and involves a smooth function of a single variable, defined on the momentum image $[-1, 1] \subset \mathbb{R}$. More precisely, we show (see Proposition 2.2) that for *any* admissible Kähler class $\Omega = [\omega]$ on the admissible manifold M, there exists an explicitly defined smooth function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ which gives rise to an explicit $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in Ω (given by the Calabi ansatz), provided that $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ is strictly positive on $(-1, 1)$. This gives a *sufficient condition* for Ω to admit a $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric, and leads to many new examples. A sample is provided by the following

Theorem 1. *(see Theorem 3.1 below) Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold over a compact CSCK manifold S. Then, for any $\boldsymbol{a} > 1$ and $\mathrm{p} \in \mathbb{R}$, M admits (admissible) $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metrics in some admissible Kähler classes. If, moreover, the Kähler manifold S is a local product of nonnegative CSCK metrics, then for any $\boldsymbol{a} > 1$ and $\mathrm{p} \in \mathbb{R}$, every admissible Kähler class contains an (admissible) $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric.*

Similarly to the extremal case studied [5], the smooth function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ also allows one to *compute* the vanishing of the $(z + \boldsymbol{a}, \mathrm{p})$-Futaki invariant associated to the admissible Kähler class Ω, which in turn is the obstruction found in [7, 32] for a $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in Ω to be actually $(z + \boldsymbol{a}, \mathrm{p})$-CSCK (see Proposition 2.5). Specializing to the case $\mathrm{p} = 2m$, this leads to many new explicit examples of conformally Kähler, Einstein–Maxwell metrics (see Propositions 4.3 and 4.4), thus extending the constructions in [21, 30, 37, 38] to higher dimensions.

Another aspect of the theory in [5], which we partially extend in this paper to the $(z + \boldsymbol{a}, \mathrm{p})$-extremal case, consists of showing that the positivity of the function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ on the interval $(-1, 1)$ is also a *necessary condition* for the existence of a $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in Ω. This was

achieved in [5] by using the following two deep results concerning extremal Kähler manifolds: (a) the boundedness of the relative Mabuchi energy, and, (b) the uniqueness of the extremal Kähler metrics modulo the action of the automorphism group of M. In their greatest generality, these results have been established in [10], after the contribution of many other mathematicians. Neither of these results is yet available in the general (f, p)-extremal setting, but A. Lahdili [33] has recently established the analogue of (a) in the (f, p)-extremal case, assuming that Ω is rational and that the corresponding (f, p)-Futaki invariant vanishes. Expressing the relative $(K, \boldsymbol{a}, \mathrm{p})$-Mabuchi energy (Definition 2.3) of an admissible manifold in terms of the function $F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z)$ (Propositions 2.6 and 2.7), we thus obtain

Theorem 2. *(see Theorem 3.2) Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible Kähler manifold over a compact CSCK manifold S. Suppose that the admissble Kähler class Ω is a positive multiple of an element in $H^2(M, \mathbb{Z})$, and that the corresponding smooth function $F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z)$ is strictly negative somewhere on $(-1, 1)$. Then Ω does not admit a Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature.*

We note that in the case when M is a geometrically ruled complex surface over a compact curve of genus ≥ 2 and $\mathrm{p} = 4$, the above result was further strengthened in [33], where the rationality assumption on Ω was dropped and the non-existence of a $(z + \boldsymbol{a}, 4)$-CSCK metric was shown to also hold if $F_{\Omega, \boldsymbol{a}, 4}(z)$ vanishes somewhere on $(-1, 1)$.

The final and perhaps most original theme of this paper concerns the link of the above existence and obstruction results with the algebro-geometric notion of (relative) $(\hat{\beta}, \mathrm{p})$-K-stability, proposed in [7]: Given a polarized compact smooth projective variety (M, L) and a quasi-periodic holomorphic vector field $\hat{K} \in \mathrm{Lie}(\mathrm{Aut}(M, L))$, the considerations in [7, Sect. 7.1] lead to the definition of a (relative) $(\hat{K}, \mathrm{p})$-Donaldson–Futaki invariant associated to a certain sub-class of test configurations compatible with $(M, L, \hat{K})$, see Section 2.5 below. However, one caveat of this definition is that it involves transcendental quantities when p is not a negative integer, leading to difficulties reminiscent to the ones involved in the definition of the L^p-norm of a test configuration for positive real values of p, see the discussion at the end of [19]. Indeed, to the best of our knowledge, no direct link is established so far between the existence of an (f, p)-CSCK metric in $c_1(L)$ and the sign of the $(\hat{K}, \mathrm{p})$-Donaldson–Futaki invariant of a compatible normal test configuration, beyond the toric context considered in [7].

Thus motivated, on an admissible manifold $M = P(E_0 \oplus E_\infty) \to S$ polarized by an ample line bundle L whose Chern class is a multiple of an admissible Kähler class Ω, we endow the total space L with a quasi-periodic real holomorphic vector field $\hat{\beta}_{\boldsymbol{a}}$, obtained by lifting the generator of the $\mathbb{S}^1$-action on M with the help of its momentum map $(z + \boldsymbol{a})$. We further consider the 1-parameter family of ($\hat{\beta}_{\boldsymbol{a}}$-compatible) test-configurations corresponding to the degeneration to the normal cone of the infinity section $P(0 \oplus E_\infty) \subset M$, see [44], and compute the corresponding relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant, by adapting the arguments of [47] and [5] to our setting. The upshot is that the relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant of such test-configurations, which we call *admissible*, is given by a positive multiple of the function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta)$ alluded to above, where $\zeta \in (-1, 1) \cap \mathbb{Q}$ parametrizes the admissible test configurations. We thus say that on (M, L), the *relative* version of $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistability/$(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stability/analytically $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stability holds on admissible test configurations if $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) \geq 0$ on $(-1, 1)$/ $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) > 0$ on $(-1, 1) \cap \mathbb{Q}$/ $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) > 0$ on $(-1, 1)$, respectively. On the other hand, (M, L) is said to be $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistable/$(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable/analytically $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable on admissible test configurations if the above holds and, additionally, the $(z + \boldsymbol{a}, \mathrm{p})$-Futaki invariant of the admissible Kähler class Ω vanishes. Our results then can be summarized in the following YTD type correspondence.

Theorem 3. *(see Theorem 3.3) Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold, and L an ample holomorphic line bundle on M, which defines, up to a positive multiple, an admissible Kähler class Ω.*

- *If for some $\boldsymbol{a} > 1$, (M, L) is analytically relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable (resp. analytically $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable) with respect to admissible test configurations, then there exists an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in Ω (resp. an admissible Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature).*

- *If Ω admits a Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature, then (M, L) is $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistable with respect to admissible test configurations.*

Conjecturally, the second claim should be improved to showing that the existence of a Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature in Ω implies that (M, L) is analytically $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-stable with respect to admissible test configurations. Using the results in [30, 33] we are able to establish this

in the case of a geometrically ruled complex surface over a curve of genus ≥ 2 and p = 4:

Theorem 4. *(see Theorem 3.4) Let $M = P(\mathcal{O} \oplus E) \to \Sigma$ be a ruled complex surface, where E is a line bundle of positive degree over a complex curve Σ of genus ≥ 2, L a polarization of M, which, up to a positive multiple, corresponds to an admissible Kähler class Ω. Then the following conditions are equivalent*

(i) Ω *admits a $(z + \boldsymbol{a}, 4)$-CSCK metric, or, equivalently, a conformally Kähler, Einstein–Maxwell metric;*

(ii) (M, L) *is $(\hat{\beta}_{\boldsymbol{a}}, 4)$-K-stable on admissible test configurations;*

(iii) (M, L) *is analytically $(\hat{\beta}_{\boldsymbol{a}}, 4)$-K-stable on admissible test configurations.*

1.3. Structure of the paper

Section 2 contains the main technical body of the paper. After a brief review of the admissible setting of [5], we summarize in Proposition 2.2 the main tools allowing us to extend the theory of [5] from extremal to weighted extremal metrics. With this in place, we compute in Proposition 2.5 the weighted Futaki invariant, and in Proposition 2.6 the relative weighted Mabuchi functional, associated to admissible Kähler metrics. The upshot is Proposition 2.7 which links the function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ of Theorem 2 to the boundedness from below of the weighted Mabuchi functional. In Section 2.5, we explore the notion of weighted K-stability proposed in [7] and compute, using the method of [47] and [5], the (relative) weighted Donaldson–Futaki invariant of an admissible test configuration (a special case of the degeneration to the normal cone studied in [44]). In Section 3, we present the proofs of the main results from the Introduction. In the final Section 4, we specialize our existence results to the case of conformally Kähler, Einstein–Maxwell metrics. In particular, we obtain a large family of new examples in any real dimension ≥ 6, see Example 4.1, Propositions 4.3, 4.4, and 4.5. In Section 4.5, we show that some of the Einstein–Maxwell metrics (which have positive constant scalar curvature) are not Yamabe minimizers even though they satisfy the strict Aubin–Schoen inequality. Finally, local existence and uniqueness results for *orthotoric* (f, p)-extremal metrics are given in an appendix.

2. Admissible manifolds and metrics

In this section we review a well-known construction, originally used by Calabi [12] to construct extremal Kähler metrics on the Hirzebruch complex surfaces, and generalized in many subsequent works [26, 28, 29, 31, 45, 48] to the case of $\mathbb{C}P^1$-bundles over a CSCK base and their blow-downs. We follow closely the notation in [5] which combines in the same formalism the momentum profile construction of [29] and the blow-down construction of [31, 45].

2.1. Admissible manifolds

Let (S, g_S, ω_S) be a compact Kähler manifold covered by the product of simply connected Kähler manifolds $(S_a, \pm g_a, \pm \omega_a), a \in \mathcal{A} \subset \mathbb{Z}^+$, where $\mathcal{A} = \{1, \dots, N\}$ is a finite index set and $(\pm g_a, \pm \omega_a)$ are the Kähler structures with the usual sign ambiguity allowing for g_a and ω_a to be negative definite tensors (in which case we write $(-g_a, -\omega_a)$ for the Kähler structure on S_a). In all our applications, each $\pm g_a$ is assumed to have constant scalar curvature, i.e. (g_S, ω_S) is a constant scalar curvature metric (CSCK) on S. The real dimension of each component is denoted $2d_a$, while the scalar curvature $Scal(\pm g_a)$ of $\pm g_a$ is written as $\pm 2d_a s_a$, where $\pm s_a$ is the *normalized scalar curvature* of $\pm g_a$.

Let E_0, E_∞ be projectively flat hermitian holomorphic vector bundles over S, of ranks $d_0 + 1$ and $d_\infty + 1$, respectively, satisfying the condition

(8)
$$c_1(E_\infty)/(d_\infty + 1) - c_1(E_0)/(d_0 + 1) = \sum_{a \in \mathcal{A}} [\omega_a/2\pi].$$

Then, following [5], the total space of the projectivization $M = P(E_0 \oplus E_\infty) \xrightarrow{p} S$ is called an *admissible complex manifold*.

A useful observation is the following diagram of holomorphic maps

(9)
$$
\begin{array}{ccc}
\hat{M} = P\left(\hat{E}_0 \oplus \hat{E}_\infty\right) & \xrightarrow{\hat{p}} & \hat{S} = P(E_0) \times_S P(E_\infty) \\
b \downarrow & & \pi \downarrow \\
M = P\left(E_0 \oplus E_\infty\right) & \xrightarrow{p} & S,
\end{array}
$$

where $\hat{E}_0 = \mathcal{O}(-1)_{P(E_0)}$ and $\hat{E}_\infty = \mathcal{O}(-1)_{P(E_\infty)}$ are line bundles over $\hat{S}$. The latter in turn is a CSCK manifold covered by the Kähler product $\mathbb{C}P^{d_0} \times \left(\prod_{a \in \mathcal{A}} S_a\right) \times \mathbb{C}P^{d_\infty}$ where $d_0 = \mathrm{rk}(E_0) - 1$ and $d_\infty = \mathrm{rk}(E_\infty) - 1$.

Note that the complex dimension of M is $m = \sum_{a \in \hat{\mathcal{A}}} d_a + 1$, where $\hat{\mathcal{A}} \subset \mathbb{N} \cup \{\infty\}$ is the extended index set defined as follows:

- $\hat{\mathcal{A}} = \mathcal{A}$, if $d_0 = d_\infty = 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{0\}$, if $d_0 > 0$ and $d_\infty = 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{\infty\}$, if $d_0 = 0$ and $d_\infty > 0$.
- $\hat{\mathcal{A}} = \mathcal{A} \cup \{0\} \cup \{\infty\}$, if $d_0 > 0$ and $d_\infty > 0$.

Correspondingly, we consider on $\hat{S}$ the CSCK metric covered by the product of the Kähler metrics $(\pm g_a, \pm \omega_a), a \in \hat{\mathcal{A}}$ with (g_0, ω_0) and $(-g_\infty, -\omega_\infty)$ being the Fubini–Study structures with scalar curvatures $2d_0(d_0 + 1)$ and $2d_\infty(d_\infty + 1)$ on the factors $\mathbb{C}P^{d_0}$ and $\mathbb{C}P^{d_\infty}$, respectively.

We will consider the $\mathbb{C}^*$-action on M (resp. on $\hat{M}$), defined by diagonal multiplication on E_0 (resp. on $\hat{E}_0$) and denote by M_0 (resp. $\hat{M}_0$) the open dense subset of regular points of the action. It is not hard to see that the first vertical map in the diagram is a $\mathbb{C}^*$-biholomorphism from $\hat{M}_0$ to M_0, and is referred to in [5] as the *blow-down* map. In the sequel, we shall tacitly identify M_0 and $\hat{M}_0$; in particular, M_0 has the structure of a principal $\mathbb{C}^*$-bundle over the (stable) quotient under the $\mathbb{C}^*$-action of M and, in our case, using the blow down map b in (9), it corresponds to the $\mathbb{C}^*$-bundle over $\hat{S}$, obtained from the $\mathbb{C}P^1$-bundle $\hat{p} : \hat{M} \to \hat{S}$ by deleting the divisors $\hat{e}_\infty = P(0 \oplus \hat{E}_\infty)$ and $\hat{e}_0 = P(\hat{E}_0 \oplus 0)$.

2.2. Admissible metrics

A particular type of Kähler metric on M, also called *admissible*, will now be described as smooth tensors on M_0.

Using the assumption that E_0 and E_∞ are projectively flat and (8), there exist hermitian metrics h_0 on E_0 and h_∞ on E_∞ whose respective Chern connections have curvatures $\Omega_0 \otimes \mathrm{Id}_{E_0}$ and $\Omega_\infty \otimes \mathrm{Id}_{E_\infty}$, with Ω_0 and Ω_∞ being 2-forms on S satisfying (when pull-backed to the universal cover of S)

$$\Omega_\infty - \Omega_0 = \sum_{a \in \mathcal{A}} \omega_a.$$

The hermitian metrics h_0 and h_∞ induce hermitian metrics $\hat{h}_0$ and $\hat{h}_\infty$ on the line bundles $\hat{E}_0 = \mathcal{O}(-1)_{P(E_0)} \to \hat{S}$ and $\hat{E}_\infty = \mathcal{O}(-1)_{P(E_\infty)} \to \hat{S}$, with curvatures 2-forms $-\omega_0 + \Omega_0$ and $\omega_\infty + \Omega_\infty$, respectively. We denote respectively by K_0 and K_∞ the generators of the circle action on $\hat{E}_0$ and $\hat{E}_\infty$, and by r_0 and r_∞ the corresponding fibre-wise norm functions. Using the Chern

connections of $(\hat{E}_0, \hat{h}_0)$ and $(\hat{E}_\infty, \hat{h}_\infty)$, we let $\hat{\theta}_0$ and $\hat{\theta}_\infty$ be the connection 1-forms defined on the corresponding unitary bundles, i.e. satisfying

$$\hat{\theta}_0(K_0) = 1, \quad d\hat{\theta}_0 = \omega_0 - \Omega_0;$$
$$\hat{\theta}_\infty(K_\infty) = 1, \quad d\hat{\theta}_\infty = -\omega_\infty - \Omega_\infty.$$

Thus, the fibres-wise euclidean structures (viewed as tensors on the total spaces $\hat{E}_0$ and $\hat{E}_\infty$) take the following momentum/angular form

$$\hat{g}_0 = \frac{dz_0 \otimes dz_0}{2z_0} + 2z_0(\hat{\theta}_0 \otimes \hat{\theta}_0), \quad \hat{g}_\infty = \frac{dz_\infty \otimes dz_\infty}{2z_\infty} + 2z_\infty(\hat{\theta}_\infty \otimes \hat{\theta}_\infty),$$

where $z_0 := r_0^2/2, z_\infty := r_\infty^2/2$ are the fibre-wise momentum coordinates.

For each $a \in \mathcal{A}$ let $|x_a| < 1$ be fixed real numbers such that $x_a g_a$ is positive definite, and $x_0 := 1$, $x_\infty := -1$. We then consider the smooth positive semidefinite tensor on the total space $\hat{E}_0 \oplus \hat{E}_\infty$

$$\hat{g} = \sum_{a \in \hat{\mathcal{A}}} \frac{(1 + x_a)z_0 + (1 - x_a)z_\infty}{2x_a} g_a + \hat{g}_0 + \hat{g}_\infty.$$

Considering the "Kähler quotient" for $\hat{g}$ with respect to the $\mathbb{S}^1$-action generated by $K_0 + K_\infty$ at the level set $z_0 + z_\infty = 2$ on $\hat{E}_0 \oplus \hat{E}_\infty$, we denote by g_c the smooth (possibly degenerate) tensor field induced on $\hat{M} = P(\hat{E}_0 \oplus \hat{E}_\infty)$ and by $\omega = g_c J_c$ the corresponding smooth $(1,1)$-form, where J_c is the induced (canonical) complex structure on $\hat{M}$. Letting $z := (z_0 - z_\infty)/2 \in [-1, 1]$, the degenerate Kähler structure (g_c, ω) is written on $\hat{M}_0$ as:

(10)
$$g_c = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} g_a + \frac{dz^2}{\Theta_c(z)} + \Theta_c(z)\theta^2,$$
$$\omega = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} \omega_a + dz \wedge \theta,$$

where $\Theta_c(z) = 1 - z^2$ and $\theta := \hat{\theta}_0 - \hat{\theta}_\infty$ satisfies

(11)
$$d\theta = \sum_{a \in \hat{\mathcal{A}}} \omega_a.$$

We notice that z is the momentum map with respect to ω of the induced $\mathbb{S}^1$-action on $\hat{M}$ corresponding to multiplication on $\hat{E}_0$ or, equivalently, the $\mathbb{S}^1$-action induced by the push forward of $K = (K_0 - K_\infty)/2$ to the quotient

space $\hat{M}$. Thus, $\hat{e}_\infty = z^{-1}(-1), \hat{e}_0 = z^{-1}(1)$, and $\hat{M}_0 = z^{-1}(-1,1)$. It follows that (g_c, ω) defines a Kähler metric on $\hat{M}_0$, which degenerates over $\hat{e}_0 \cup \hat{e}_\infty$ when $\hat{\mathcal{A}} \neq \mathcal{A}$. Nevertheless, it is shown in [5] that (g_c, ω) gives rise to a genuine, non-degenerate, smooth Kähler metric on $M = P(E_0 \oplus E_\infty)$ where we identify M_0 with $\hat{M}_0$ via (9). Then, z is the momentum map with respect to ω of the $\mathbb{S}^1$-action on M by multiplication on E_0. The formulae (10) and (11) describe the pull-back of (g_c, ω) to $\hat{M}$ via the map b in (9).

It was observed in [5] (the argument actually goes back to [29]) that if instead of $\Theta_c(z)$ we take in (10) any smooth function $\Theta(z)$ on $[-1, 1]$, satisfying

(12) $(i)\ \Theta(z) > 0, \quad -1 < z < 1, \quad (ii)\ \Theta(\pm 1) = 0, \quad (iii)\ \Theta'(\pm 1) = \mp 2.$

then the formulea

(13) $g = \displaystyle\sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} g_a + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2, \quad \omega = \sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} \omega_a + dz \wedge \theta,$

and (11) introduce a smooth $\mathbb{S}^1$-invariant Kähler metric on M, compatible with the same symplectic form ω. The corresponding complex structure is then given on $\hat{M}_0 = M_0$ by the horizontal lift of the base complex structure on $\hat{S}$ (with respect to the chosen Chern connections on $\hat{E}_0$ and $\hat{E}_\infty$) along with the requirement $Jdz = \Theta\theta$ on the fibres. Such Kähler metrics on M are called *admissible Kähler metrics*, and we denote by $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ the space of all admissible Kähler metrics associated to a given choice of $x_a, a \in \mathcal{A}$. Thus, $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ is identified with a Fréchet space consisting of all smooth functions $\Theta(z)$ on $[-1, 1]$ satisfying (12). For fixed values $x_a \in (-1, 1)$ (and g_a), we let $\Omega_x := [\omega]$ be the corresponding deRham class on M, which we refer to as an *admissible Kähler class* on M. Note that from Section 1.3 in [5], it follows that Ω_x is a positive multiple of an element in $H^2(M, \mathbb{Z})$ precisely when $x_a \in \mathbb{Q}$ for all $a \in \mathcal{A}$.

The space $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ (associated to a given data (x_a, g_a)) can also be equally parametrized by the fibre-wise symplectic potentials $u(z)$, where $u(z)$ is defined up to an affine-linear function of z by $u''(z) = \frac{1}{\Theta(z)}$. It is shown in [5, p. 566] that the fibre-wise Legendre transform $\mathcal{T}$ maps $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ to the space $\mathcal{K}(M, J_c, \Omega) = \{\varphi \in C^\infty(M) : \omega + dd^c\varphi > 0\}$ of J_c-compatible Kähler metrics in the class $\Omega_x = [\omega]$, and has differential given by

(14) $(\mathbf{d}\mathcal{T})_g(\dot{u}) \to -\dot{\varphi}.$

Remark 1. The parametrization of the space $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ of admissible Kähler metrics in terms of $(x_a, g_a), a \in \hat{\mathcal{A}}$ and $\Theta(z)$ is not effective. Indeed, any admissible Kähler metric $(g, \omega) \in \mathcal{K}^{\mathrm{adm}}(M, \omega)$ of the form (13) can also be obtained by changing g_a to $-g_a$ for $a \in \hat{A}$, z to $-z$ and θ to $-\theta$. Geometrically, this corresponds simply to changing the rôles of the vector bundles E_0 and E_∞ or, equivalently, changing the generator K of the $\mathbb{S}^1$-action by multiplications on E_0 with $-K$. Notice that $-K$ is the generator of the $\mathbb{S}^1$-action on $P(E_0 \oplus E_\infty)$ corresponding to multiplications on E_∞.

In order to simplify various curvature computations, it is useful to introduce the function

$$(15) \qquad F(z) := \Theta(z)p_{\mathrm{c}}(z),$$

where $p_{\mathrm{c}}(z) = \prod_{a \in \hat{\mathcal{A}}}(1 + x_a z)^{d_a}$ and $d_a = \dim_{\mathbb{C}}(S_a)$ with $S_0 := \mathbb{C}P^{d_0}$ and $S_\infty = \mathbb{C}P^{d_\infty}$. The conditions (12) imply

$$(16) \qquad (i)\ F(z) > 0, \quad -1 < z < 1, \quad (ii)\ F(\pm 1) = 0,$$
$$(iii)\ F'(\pm 1) = \mp 2p_{\mathrm{c}}(\pm 1).$$

When $\mathcal{A} = \hat{\mathcal{A}}$, (16) is equivalent to (12).

Letting $s_a := \frac{\pm Scal(\pm g_a)}{2d_a}$ be the normalized scalar curvatures of S_a for $a \in \mathcal{A}$ and $s_0 := d_0 + 1, s_\infty := -(d_\infty + 1)$, we recall the following facts (see e.g. [2]).

Lemma 2.1. *For any admissible metric g, if $S(z)$ is a smooth function of z, then*

$$(17) \qquad \Delta_g S = -[F(z)S'(z)]'/p_c(z),$$

where Δ_g is the Laplacian of g, whereas the scalar curvature of g is given by

$$(18) \qquad Scal(g) = \sum_{a \in \hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z} - \frac{F''(z)}{p_{\mathrm{c}}(z)}.$$

2.3. Admissible $(z + a, \mathrm{p})$-extremal Kähler metrics

We now make the assumption that g in (7) is admissible and that the positive Killing potential f is of the form $f = |z + \boldsymbol{a}|$ for some constant $\boldsymbol{a} \in \mathbb{R}$ such

that $|\boldsymbol{a}| > 1$. In view of Remark 1, we shall assume (without loss of general-
ity) that $\boldsymbol{a} > 1$. It follows from (18) and Lemma 2.1 that the $(z + \boldsymbol{a}, \mathrm{p})$-scalar
curvature is given by

(19)
$$Scal_{z+\boldsymbol{a},\mathrm{p}}(g) = \frac{-(z+\boldsymbol{a})^2 F''(z) + 2(\mathrm{p}-1)(z+\boldsymbol{a})F'(z) - \mathrm{p}(\mathrm{p}-1)F(z)}{p_c(z)}$$
$$+ (z+\boldsymbol{a})^2 \sum_{a\in\hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z}.$$

Thus, g is $(z + \boldsymbol{a}, \mathrm{p})$-extremal in the sense of [4, 7], provided that

(20)
$$- (z+\boldsymbol{a})^2 F''(z) + 2(\mathrm{p}-1)(z+\boldsymbol{a})F'(z) - \mathrm{p}(\mathrm{p}-1)F(z)$$
$$= (A_1 z + A_2)p_c(z) - p_c(z)(z+\boldsymbol{a})^2 \sum_{a\in\hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z}.$$

Notice that $A_1 = 0$ in (20) is equivalent to g having a constant $(z + \boldsymbol{a}, \mathrm{p})$
scalar curvature.

Next we will view the equation (20) in an alternative way that will
enable us to ensure the existence of a unique solution satisfying the boundary
conditions (ii) and (iii) of (16). On closer inspection of the left hand side of
(20) we see that it equals

$$-(z+\boldsymbol{a})^{\mathrm{p}+1} \frac{d^2}{dz^2} \left[\frac{F(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}} \right]$$

and hence (20) is equivalent to

(21) $$\frac{d^2}{dz^2} \left[\frac{F(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}} \right] = \frac{p_c(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}} \sum_{a\in\hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z} - \frac{(A_1 z + A_2)p_c(z)}{(z+\boldsymbol{a})^{\mathrm{p}+1}}$$

Letting

(22) $$G(z) := \frac{F(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}} = \frac{\Theta(z)p_c(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}}$$

and

(23) $$Q(z) = \frac{p_c(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}} \left(\sum_{a\in\hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z} \right) - \frac{(A_1 z + A_2)p_c(z)}{(z+\boldsymbol{a})^{\mathrm{p}+1}},$$

we obtain the ODE

(24) $$G''(z) = Q(z)$$

with boundary conditions

(25) $$(i)\ G(\pm 1) = 0, \quad (ii)\ G'(\pm 1) = \mp \frac{2p_c(\pm 1)}{(\boldsymbol{a} \pm 1)^{\mathrm{p}-1}}.$$

It is not hard to see that (24)-(25) has a solution if and only if

(26) $$\frac{4p_c(-1)}{(\boldsymbol{a} - 1)^{\mathrm{p}-1}} + \int_{-1}^{1} Q(t)(1 - t)\, dt = 0,$$
$$\frac{2p_c(-1)}{(\boldsymbol{a} - 1)^{\mathrm{p}-1}} + \int_{-1}^{1} Q(t)\, dt = -\frac{2p_c(1)}{(\boldsymbol{a} + 1)^{\mathrm{p}-1}},$$

in which case the solution is

(27) $$G(z) = \frac{2p_c(-1)}{(\boldsymbol{a} - 1)^{\mathrm{p}-1}}(z + 1) + \int_{-1}^{z} Q(t)(z - t)\, dt.$$

It is now just a matter of technical detailing to see that the necessary and sufficient conditions (26) for the existence of the solution $G(x)$ above in fact determine the constants A_1 and A_2, via the linear system

(28) $$\alpha_{1,-(1+\mathrm{p})} A_1 + \alpha_{0,-(1+\mathrm{p})} A_2 = 2\beta_{0,(1-\mathrm{p})}$$
$$\alpha_{2,-(1+\mathrm{p})} A_1 + \alpha_{1,-(1+\mathrm{p})} A_2 = 2\beta_{1,(1-\mathrm{p})},$$

where

(29) $$\alpha_{r,q} = \int_{-1}^{1} (t + \boldsymbol{a})^q t^r p_c(t)\, dt$$
$$\beta_{r,q} = \int_{-1}^{1} \Big(\sum_{a \in \hat{\mathcal{A}}} \frac{x_a d_a s_a}{1 + x_a t} \Big) t^r p_c(t)(t + \boldsymbol{a})^q dt$$
$$+ \big((-1)^r (\boldsymbol{a} - 1)^q p_c(-1) + (1 + \boldsymbol{a})^q p_c(1)\big).$$

Since $\alpha_{1,q}^2 < \alpha_{0,q}\alpha_{2,q}$, (28) has a unique solution. It follows (once M and hence each s_a is fixed) that the functions $Q(z)$ given by (23) and $G_{x,\boldsymbol{a},\mathrm{p}}(z) := G(z)$ with G given by (27) are entirely determined from the data $(x_a, a \in \mathcal{A}, \boldsymbol{a}, \mathrm{p})$, and thus are invariants of the admissible Kähler class $\Omega = \Omega_x$ on M.

We now note that if $F(z)$ satisfies (21), or equivalently, $G(z) = F(z)/(z + \boldsymbol{a})^{\mathrm{p}-1}$ satisfies (27), then $G''(z) = p_c'(z)\Upsilon(z)$ with $\Upsilon(-1) = 2(d_0 + 1)/(\boldsymbol{a} - 1)^{\mathrm{p}-1}$, if $d_0 > 0$, and $\Upsilon(1) = -2(d_\infty + 1)/(\boldsymbol{a} + 1)^{\mathrm{p}-1}$, if $d_\infty > 0$. Hence, assuming (25), we have $G'(z) = p_c(z)\Psi(z)$ with $\Psi(-1) = 2(d_0 + 1)/(\boldsymbol{a} - 1)^{\mathrm{p}-1}$ and $\Psi(1) = -2(d_\infty + 1)/(\boldsymbol{a} + 1)^{\mathrm{p}-1}$, and $\Theta(\pm 1) = 0$. Now by l'Hôpital's rule, $\Theta'(\pm 1) = \mp 2$, showing that if the function $F_{x,\boldsymbol{a},\mathrm{p}}(z) := (z + \boldsymbol{a})^{\mathrm{p}-1}G_{x,\boldsymbol{a},\mathrm{p}}(z)$ satisfies the positivity condition (16) (i), then $\Theta(z) = F_{x,\boldsymbol{a},\mathrm{p}}(z)/p_c(z)$ gives rise to an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric. We summarize this construction in the following

Proposition 2.2. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold, $\Omega = \Omega_x$ an admissible Kähler class corresponding to the admissible data $(x_a, g_a), a \in \mathcal{A}$, and $\boldsymbol{a} > 1$ and p two real parameters. We denote by A_1, A_2 the unique solution of (28) and let*

$$F_{\Omega,\boldsymbol{a},\mathrm{p}}(z) = F_{x,\boldsymbol{a},\mathrm{p}}(z) = (z + \boldsymbol{a})^{\mathrm{p}-1}G_{x,\boldsymbol{a},\mathrm{p}}(z),$$

with $G_{x,\boldsymbol{a},\mathrm{p}}(z)$ given by (27) for $Q(z)$ given by (23), be the smooth function satisfying (21) and (16) (ii)-(iii). If $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ satisfies (i) of (16), then we have a corresponding admissible Kähler metric g on M which is $(z + \boldsymbol{a}, \mathrm{p})$-extremal with respect to the Killing potential $f(z) = z + \boldsymbol{a}$. Furthermore, g is $(z + \boldsymbol{a}, \mathrm{p})$-CSCK if, moreover, $A_1 = 0$.

2.4. The (K, a, p)-Mabuchi energy and the (K, a, p)-Futaki invariant of an admissible manifold

We start by recalling the general setting of [7, 32]. Let (M, J) be a connected, compact $2m$-dimensional Kähler manifold endowed with a Kähler class Ω. Suppose K denotes a real holomorphic vector field with zeroes which is quasi-periodic, i.e. whose flow generates a (real) torus. We fix a positive constant $\boldsymbol{b} > 0$ and, for any K-invariant Kähler metric $\omega \in \Omega$, let $f_{\omega,K,\boldsymbol{b}}$ be the Killing potential of K with respect to ω, normalized by $\int_M f_{\omega,K,\boldsymbol{b}}\omega^m/m! = \boldsymbol{b}$. It is not hard to see that with this normalization, the image $f_{\omega,K,\boldsymbol{b}}(M)$ is an interval independent of the choice of ω in Ω. We further require that $\boldsymbol{b}$ is chosen so that $f_{\omega,K,\boldsymbol{b}} > 0$.

Let $\mathbb{T}$ be a maximal torus in the reduced automorphism group $\mathrm{Aut}^r(M, J)$ of (M, J) with $K \in \mathrm{Lie}(\mathbb{T})$, and $\mathcal{K}^{\mathbb{T}}(M, J, \Omega)$ denote the space of $\mathbb{T}$-invariant Kähler metrics in Ω, viewed as an affine space modelled on the vector space $C^\infty(M, \mathbb{R})^{\mathbb{T}}/\mathbb{R}$ of $\mathbb{T}$-invariant smooth functions modulo constants. At each

point $\omega \in \mathcal{K}^{\mathbb{T}}(M, J, \Omega)$, we identify the corresponding tangent space

$$T_\omega \mathcal{K}^{\mathbb{T}}(M, J, \Omega) \cong \left\{ \dot\varphi \in C^\infty(M, \mathbb{R})^{\mathbb{T}} \mid \int_M \dot\varphi f_{\omega, K, \boldsymbol{b}}^{-(\mathrm{p}+1)} \frac{\omega^m}{m!} = 0 \right\}.$$

Furthermore, we denote by $P_g^{\mathbb{T}}(M, \mathbb{R})$ the finite dimensional space of Killing potentials with respect to $g = -J\omega$ of the vector fields in $\mathrm{Lie}(\mathbb{T})$, and for a smooth function $\varphi \in C^\infty(M, \mathbb{R})^{\mathbb{T}}$, we let $\Pi^{g, K, \boldsymbol{b}, \mathrm{p}, \mathbb{T}}(\varphi)$ denote its orthogonal projection to $P_g^{\mathbb{T}}(M, \mathbb{R})$ by using the inner product

$$(30) \qquad \langle \varphi, \psi \rangle_{\omega, K, \boldsymbol{b}, \mathrm{p}} = \int_M \varphi \psi f_{\omega, K, \boldsymbol{b}}^{-(\mathrm{p}+1)} \frac{\omega^m}{m!}$$

on $C^\infty(M, \mathbb{R})^{\mathbb{T}}$. We shall use the following definition from [32].

Definition 2.3. The *relative $(K, \boldsymbol{b}, \mathrm{p})$-Mabuchi energy* is a functional

$$\mathcal{M}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})}^{\mathbb{T}} : \mathcal{K}^{\mathbb{T}}(M, J, \Omega) \to \mathbb{R},$$

defined, up to an additive constant, by the property

$$\left(\mathbf{d}\mathcal{M}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})}^{\mathbb{T}} \right)_\omega(\dot\varphi) = -\int_M (Scal_{f, \mathrm{p}}(g))^{\perp_g} \dot\varphi f^{-(\mathrm{p}+1)} \frac{\omega^m}{m!},$$

where $f = f_{\omega, K, \boldsymbol{b}}$ is the Killing potential of K with respect to ω and $Scal_{f, \mathrm{p}}(g)^{\perp_g} := Scal_{f, \mathrm{p}}(g) - \Pi^{g, K, \boldsymbol{b}, \mathrm{p}, \mathbb{T}}(Scal_{f, \mathrm{p}}(g))$ is *the reduced (f, p)-scalar curvature*. The Killing vector field $Z = J\mathrm{grad}_g \left(\Pi^{g, K, \boldsymbol{b}, \mathrm{p}, \mathbb{T}}(Scal_{f, \mathrm{p}}(g)) \right)$ is independent of $\omega \in \mathcal{K}^{\mathbb{T}}(M, J, \Omega)$ and is called *the $(K, \boldsymbol{b}, \mathrm{p})$-extremal vector field* associated to $(M, J, \Omega, \mathbb{T})$. It vanishes if and only if the $(K, \boldsymbol{b}, \mathrm{p})$-*Futaki invariant* $\mathfrak{F}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})} : \mathrm{Lie}(\mathbb{T}) \to \mathbb{R}$ defined in [7] and [32, Def. 4] is zero.

Remark 2. It turns out that in the case when the $(K, \boldsymbol{b}, \mathrm{p})$-extremal vector field of $(M, J, \Omega, \mathbb{T})$ vanishes, one can also express the differential of $\mathcal{M}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})}^{\mathbb{T}}$ as

$$\left(\mathbf{d}\mathcal{M}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})}^{\mathbb{T}} \right)_\omega(\dot\varphi) = -\int_M \left(Scal_{f, \mathrm{p}}(g) - c_{(\Omega, K, \boldsymbol{b}, \mathrm{p})} \right) \dot\varphi f^{-(\mathrm{p}+1)} \frac{\omega^m}{m!},$$

where the constant

$$c_{(\Omega, K, \boldsymbol{b}, \mathrm{p})} = \frac{\int_M Scal_{f, \mathrm{p}}(g) f^{-(\mathrm{p}+1)} \frac{\omega^m}{m!}}{\int_M f^{-(\mathrm{p}+1)} \frac{\omega^m}{m!}}$$

is independent of $\omega \in \mathcal{K}^{\mathbb{T}}(M, J, \Omega)$, see [7, 32]. Thus, in this case, $\mathcal{M}^{\mathbb{T}}_{(\Omega, K, \boldsymbol{b}, \mathrm{p})}$ reduces to the Mabuchi functional introduced in [32, 33].

We now specialize to the case when M is an admissible manifold with CSCK base, and K is the generator of the natural $\mathbb{S}^1$-action. It is shown in [5, Prop. 5] that an admissible Kähler metric (g, ω, J) is invariant under a common maximal compact connected subgroup $G \subset \mathrm{Aut}^r(M, J)$ with $K \in \mathrm{Lie}(G)$. We thus fix a maximal torus $\mathbb{T} \subset G$ with $K \in \mathrm{Lie}(\mathbb{T})$. Notice that a Killing potential of K with respect to g is given by $z + \boldsymbol{a}$ for some $\boldsymbol{a} \in \mathbb{R}$. The constant $\boldsymbol{a}$ here is essentially the real constant $\boldsymbol{b}$ in the above general setting: indeed, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linked by an affine-linear expression which is independent of the choice of $g \in \mathcal{K}^{\mathrm{adm}}(M, \omega)$. For this reason, in the admissible context, we shall use $\boldsymbol{a}$ instead of $\boldsymbol{b}$, thus referring to the relative $(K, \boldsymbol{a}, \mathrm{p})$-Mabuchi energy and $(K, \boldsymbol{a}, \mathrm{p})$-Futaki invariant for the corresponding quantities defined for a fixed $\boldsymbol{a} > 1$ (and a maximal torus $\mathbb{T}$ as above).

We shall first compute the reduced scalar curvature $Scal_{z+\boldsymbol{a}, \mathrm{p}}(g)^{\perp_g}$, as defined in Definition 2.3 (similarly to [5, Prop. 6]). The formula (19) reads as

$$(31) \quad Scal_{z+\boldsymbol{a}, \mathrm{p}}(g) = (z + \boldsymbol{a})^2 \Big(\sum_a \frac{2 d_a s_a x_a}{1 + x_a z} \Big) - \frac{(z + \boldsymbol{a})^{\mathrm{p}+1}}{p_c(z)} \Big(\frac{F(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} \Big)''.$$

For A_1, A_2 given by (28), integration by parts of (31) shows that $Scal_{z+\boldsymbol{a}, \mathrm{p}}(g) - A_1 z - A_2$ is L^2-orthogonal to 1 and z with respect to the measure $p_c(z)(z + \boldsymbol{a})^{-(\mathrm{p}+1)} dz$. Geometrically, this means that $Scal_{z+\boldsymbol{a}, \mathrm{p}}(g) - A_1 z - A_2$ is $\langle \cdot, \cdot \rangle_{\omega, K, \boldsymbol{a}, \mathrm{p}}$-orthogonal to the Killing potentials z and 1, where $\langle \cdot, \cdot \rangle_{\omega, K, \boldsymbol{a}, \mathrm{p}}$ stands for the inner product (30) corresponding to the Killing potential $f_{\omega, K, b} = z + \boldsymbol{a}$. By [5, Prop. 2], any other Killing potential for a vector field in $\mathrm{Lie}(\mathbb{T})$ has the form $\sum_a (1 + x_a z) f_a$ where f_a is a Killing potential for the base factor S_a, which we can assume without loss is of zero mean with respect to $(\hat{S}, g_{\hat{S}})$. Formula (31) then shows that $Scal_{z+\boldsymbol{a}, \mathrm{p}}(g) - A_1 z - A_2$ will be $\langle \cdot, \cdot \rangle_{\omega, K, \boldsymbol{a}, \mathrm{p}}$-orthogonal to such Killing potentials. In particular, $A_1 K$ is the $(\Omega, K, \boldsymbol{a}, \mathrm{p})$-extremal vector field of $(M, J, \Omega, K, \boldsymbol{a})$, computed with respect to a maximal torus $\mathbb{T} \subset \mathrm{Aut}^r(M, J)$ (see [7] and [32, Def. 7]).

Using that $F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z)$ is a solution of (21), we have

Lemma 2.4. *Let (M, J, g, ω) be an admissible Kähler manifold over a CSCK base and $\mathbb{T}$ a maximal torus in the isometry group of (g, ω). If g is*

parameterized by the function $F(z)$ given in (15), then the reduced $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature $Scal_{z+\boldsymbol{a},\mathrm{p}}(g)^{\perp_g}$ is given by

$$(32) \qquad Scal_{z+\boldsymbol{a},\mathrm{p}}(g)^{\perp_g} = -A_1 z - A_2 + (z + \boldsymbol{a})^2 \Big(\sum_a \frac{2 d_a s_a x_a}{1 + x_a z} \Big)$$

$$- \frac{(z + \boldsymbol{a})^{\mathrm{p}+1}}{p_c(z)} \Big(\frac{F(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} \Big)''$$

$$= \frac{(z + \boldsymbol{a})^{\mathrm{p}+1}}{p_c(z)} \Big(\frac{F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} - \frac{F(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} \Big)'',$$

where $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ is the smooth function defined in Proposition 2.2 in terms of $(\Omega, \boldsymbol{a}, \mathrm{p})$. Furthermore, the $(K, \boldsymbol{a}, \mathrm{p})$-extremal vector field is $A_1 K$.

A direct corollary is the following

Proposition 2.5. *Let (M, J, g, ω) be an admissible Kähler manifold over a CSCK base, and $\mathbb{T}$ a maximal torus in the isometry group of (g, ω). Then, the corresponding $(K, \boldsymbol{a}, \mathrm{p})$-Futaki invariant $\mathfrak{F}_{([\omega], K, \boldsymbol{a}, \mathrm{p})}$ restricted to $\mathrm{Lie}(\mathbb{T})$ vanishes iff $A_1 = 0$. The latter condition is equivalent to $\mathfrak{F}_{[\omega], K, \boldsymbol{a}, \mathrm{p}}(K) = 0$.*

We now give an explicit form for the relative $(K, \boldsymbol{a}, \mathrm{p})$-Mabuchi energy in the admissible case, following the similar construction in [5, Prop. 7]. To this end, we use the parametrization of $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ in terms of fibre-wise symplectic potentials $u(z)$ (defined by $u''(z) = \frac{p_c(z)}{F(z)}$, see Sect. 2.2) and (14).

Proposition 2.6. *Let (M, J, g, ω) be an admissible Kähler manifold over a CSCK base, and $\mathbb{T}$ a maximal torus in the isometry group of (g, ω). Then, the relative $(K, \boldsymbol{a}, \mathrm{p})$-Mabuchi energy associated to $\Omega = [\omega]$ and $\mathbb{T}$, restricted to the space of admissible Kähler metrics $\mathcal{K}^{\mathrm{adm}}(M, \omega)$ is given (up to an additive constant) by a positive multiple of the functional*

$$\mathcal{M}_{g_c} : u(z) \longmapsto \int_{-1}^1 \frac{F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} \big(u''(z) - u_c''(z) \big) dz$$

$$- \int_{-1}^1 \frac{p_c(z)}{(z + \boldsymbol{a})^{\mathrm{p}-1}} \log \Big(\frac{u''(z)}{u_c''(z)} \Big) dz,$$

where $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ is the smooth function defined in Proposition 2.2, and $u_c(z)$ is the fibre-wise symplectic potential for some fixed ω-compatible admissible Kähler metric $g_c \in \mathcal{K}^{\mathrm{adm}}(M, \omega)$.

Proof. Using Lemma 2.4, the proof is identical to the one of [5, Prop. 7]. $\quad\square$

The proof of [5, Cor. 3] yields

Proposition 2.7. *Let (M, J, g, ω) be an admissible Kähler manifold over a CSCK base, and $\mathbb{T}$ a maximal torus in the isometry group of (g, ω). If the function $F_{[\omega], \boldsymbol{a}, \mathrm{p}}(z)$ is strictly negative somewhere on $(-1, 1)$, then the relative $(K, \boldsymbol{a}, \mathrm{p})$-Mabuchi energy of $(M, J, [\omega], \mathbb{T})$ is unbounded from below.*

2.5. The $(\hat{\beta}_a, \mathrm{p})$-Donaldson–Futaki invariant

In [7, Sect. 5.1], a quantized version of the $(f, 2m)$-Donaldson–Futaki invariant was proposed, which leads to a notion of $(\beta, 2m)$-K-stability, where $\hat{\beta}$ is a fixed $\mathbb{C}^*$-subgroup in the automorphism group $\mathrm{Aut}(M, L)$ of a smooth compact polarized variety (M, L). This was further developed and generalized in [33] for arbitrary weights p and for quasi-periodic vector fields $\hat{K} \in \mathrm{Lie}(\mathrm{Aut}(M, L))$. We first briefly recall the general setting of [7, 33].

Let (M, L) be a smooth compact polarized projective variety and $\Omega = 2\pi c_1(L)$ the corresponding Kähler class. Let $\hat{\beta}$ be a $\mathbb{C}^*$-subgroup of $\mathrm{Aut}(M, L)$, which covers a $\mathbb{C}^*$-subgroup β of the reduced automorphisms group $\mathrm{Aut}^r(M, J) \cong \mathrm{Aut}(M, L)/\{\mathbb{C}^* \cdot \mathrm{Id}_L\}$. We denote by $\hat{K}_\beta$ (resp. K_β) the generator of the corresponding $\mathbb{S}^1$-action on L (resp. on M), and by B_k the infinitesimal generators for the induced linear $\mathbb{C}^*$-actions on $H^0(M, L^k), k \geq 1$. We use the following normalization for B_k: for any holomorphic section $s \in H^0(M, L^k)$ and $x \in M$,

$$(33) \qquad (B_k \cdot s)(x) := i \frac{d}{dt}_{|t=0} \left(\varphi_t^{\hat{K}_\beta} \left(s(\varphi_{-t}^{K_\beta}(x)) \right) \right),$$

where $\varphi_t^{\hat{K}_\beta}$ and $\varphi_t^{K_\beta}$ are the flows of $\hat{K}_\beta$ and K_β, respectively. Notice that B_k is hermitian (and therefore semi-simple with real eignvalues) with respect to the L^2 inner product defined by any $\hat{K}_\beta$-invariant hermitian metric on L. We further assume that the $\mathbb{C}^*$-action $\hat{\beta}$ on L is such that B_k have positive eigenvalues for any k large enough: this is equivalent with the property that for any $\hat{K}_\beta$-invariant hermitian product h on L, the induced momentum map f_β for K_β with respect to the curvature form $\omega_h \in \Omega$ is positive. Indeed, if $\nabla^{h,k}$ is the corresponding Chern connection on L^k, it is well-known (see e.g. [25]) that

$$(34) \qquad B_k = -i\nabla_{K_\beta}^{h,k} + kf_\beta.$$

It then follows that each eigenvalue λ of B_k/k equals $f_\beta(x_0)$ for a point x_0 of maxima of $|s|_{h,k}^2$ of an eigensection s corresponding to λ.

We now consider a *normal $\hat{\beta}$-compatible test configuration* (of exponent r) $(\mathcal{X}, \mathcal{L})$ associated to (M, L). By this, we mean that $(\mathcal{X}, \mathcal{L})$ is a normal polarized variety of complex dimension $(m + 1)$, endowed with a $\mathbb{C}^*$-equivariant map $\pi : \mathcal{X} \to \mathbb{C}$, such that

- $(M_t = \pi^{-1}(t), L_t = \mathcal{L}_{|_{M_t}}) \cong (M, L^r)$ for $t \neq 0$, and

- π is flat (and therefore the dimensions of $H^0(M_t, L_t^k)$ stay unchanged for $t \in \mathbb{C}$ and k large enough),

- π is $\hat{\beta}$-invariant.

It follows that the central fibre (M_0, L_0) of such a test configuration is (in general) a polarized projective scheme, endowed with two commuting $\mathbb{C}^*$-actions, $\hat{\alpha}$ and $\hat{\beta}$. We denote respectively by A_k, B_k the corresponding infinitesimal generators for the induced linear $\mathbb{C}^*$-actions on $H^0(M_0, L_0^k)$. For each $q \in \mathbb{R}$, we expect to have expansions

$$(35) \qquad \begin{aligned} k^{-m+1-q}\mathrm{Tr}(B_k^q) &= kb_0^{q,0}(\hat{\beta}) + b_1^{q,0}(\hat{\beta}) + O(k^{-1}), \\ k^{-m-q}\mathrm{Tr}(A_k B_k^q) &= kb_0^{q,1}(\hat{\beta}, \hat{\alpha}) + b_1^{q,1}(\hat{\beta}, \hat{\alpha}) + O(k^{-1}). \end{aligned}$$

Remark 3. To the best of our knowledge, such Hilbert expansions hold for $q \in \mathbb{N}$ (see [19]) and, for any q, if we assume that M_0 is smooth (this follows from the considerations in [33, 34]) or that M_0 is a polarized toric variety [7]. We shall exhibit below another situation (in which (M_0, L_0) is a singular normal polarized variety) where the expansions (35) hold.

Assuming that (35) do hold on (M_0, L_0), we define the $(\hat{\beta}, \mathrm{p})$-Donaldson–Futaki invariant of the test configuration $(\mathcal{X}, \mathcal{L})$ as follows.

Definition 2.8. The $(\hat{\beta}, \mathrm{p})$-*Donaldson–Futaki invariant* of a normal $\hat{\beta}$-compatible test configuration $(\mathcal{X}, \mathcal{L})$ associated to (M, L) such that (35) holds true on the central fibre (M_0, L_0) is defined to be
$$(36)$$
$$\mathrm{DF}_{(\hat{\beta}, \mathrm{p})}(\mathcal{X}, \mathcal{L}) := \frac{b_1^{(1-\mathrm{p}),1}(\hat{\beta}, \hat{\alpha})b_0^{-(\mathrm{p}+1),0}(\hat{\beta}) - b_0^{-(\mathrm{p}+1),1}(\hat{\beta}, \hat{\alpha})b_1^{(1-\mathrm{p}),0}(\hat{\beta})}{b_0^{-(\mathrm{p}+1),0}(\hat{\beta})}.$$

The polarized variety (M, L) is called $(\hat{\beta}, \mathrm{p})$-*K-semistable* if for any normal, $\hat{\beta}$-compatible test configuration $(\mathcal{X}, \mathcal{L})$ for (M, L) as above,

$$(37) \qquad\qquad\qquad \mathrm{DF}_{(\hat{\beta}, \mathrm{p})}(\mathcal{X}, \mathcal{L}) \geq 0.$$

(M, L) is $(\hat{\beta}, \mathrm{p})$-*K-stable* if, furthermore, equality in (37) holds if and only if $\mathcal{X} = M \times \mathbb{C}, \mathcal{L} = L \otimes \mathcal{O}_{\mathbb{C}}$ is a product test configuration (see Remark 4 part (2) below).

Remark 4. (1) Note that the definition of $(\hat{\beta}, \mathrm{p})$-K-stability makes sense for any positive multiple λ of $\hat{\beta}$, i.e. taking λB_k instead of B_k. This will introduce an overall positive factor in the computation of $\mathrm{DF}_{(\lambda\hat{\beta},\mathrm{p})}$. More generally, one can define $(\hat{K}, \mathrm{p})$-K-stability with respect to a quasi-periodic vector field $\hat{K} \in \mathrm{Lie}(\mathrm{Aut}(M, L))$, by considering the linear operators B_k acting on $H^0(M_0, L_0)$ via (34) for a $\hat{K}$-invariant hermitian product h on L, see [33].

(2) In the special case of the product test configuration $\mathcal{X} = M \times \mathbb{C}, \mathcal{L} = L \otimes \mathcal{O}_{\mathbb{C}}$, the expression on the rhs of (36) can be regarded as defining a numerical invariant $\mathrm{DF}_{(\hat{\beta},\mathrm{p})}(\hat{\alpha})$ associated to any $\mathbb{C}^*$-subgroup $\hat{\alpha} \subset \mathrm{Aut}(M, L)$. It is possible to see (this follows essentially from [33]) that the expansions (35) hold and $\mathrm{DF}_{(\hat{\beta},\mathrm{p})}(\hat{\alpha})$ coincides, up to a positive multiplicative constant, with the differential-geometric $(f_{\hat{\beta}}, \mathrm{p})$-Futaki invariant $\mathfrak{F}_{\Omega, f_{\hat{\beta}}, \mathrm{p}}$ evaluated on the $\mathbb{S}^1$-generator of α. (Recall that $f_{\hat{\beta}}$ is the Killing potential of β determined from $\hat{\beta}$, see (34).) In particular, $\mathrm{DF}_{(\hat{\beta},\mathrm{p})}(\hat{\alpha})$ does not depend on the lift $\hat{\alpha}$ of the $\mathbb{C}^*$-action $\alpha \subset \mathrm{Aut}^r(M, J)$. At times we will even replace α in this notation by the corresponding $\mathbb{S}^1$-generator.

Following [47], one can also define a *relative* version of the $(\hat{\beta}, \mathrm{p})$-Donaldson–Futaki invariant. To this end, we suppose that (M, L) is a polarized variety, $\mathbb{T} \subset \mathrm{Aut}^r(M, J)$ a fixed maximal torus, $\beta \subset \mathbb{T}^c$ with a lift $\hat{\beta} \subset \mathrm{Aut}(M, L)$ as before. Here $\mathbb{T}^c$ denotes the complexification of $\mathbb{T}$. For any two $\mathbb{C}^*$-actions $\gamma', \gamma'' \subset \mathbb{T}^c$, we suppose that there exist an expansion

$$(38) \qquad k^{-m-q-2}\mathrm{Tr}(C_k' C_k'' B_k^q) = c_0^{q,0}(\hat{\beta}, \hat{\gamma}', \hat{\gamma}'') + O(k^{-1}),$$

where C_k' and C_k'' are the corresponding generators on the space $H^0(M, L^k)$ for some lifts $\hat{\gamma}'$ and $\hat{\gamma}''$ of γ' and γ'' to $\mathrm{Aut}(M, L)$. We then define the $(\hat{\beta}, \mathrm{p})$-weighted product for the $\mathbb{C}^*$-actions γ', γ'' by

$$(39) \quad \langle \gamma', \gamma'' \rangle_{(\hat{\beta},\mathrm{p})} := c_0^{-(1+\mathrm{p}),0}(\hat{\beta}, \hat{\gamma}', \hat{\gamma}'') - b_0^{-(1+\mathrm{p}),0}(\hat{\beta}, \hat{\gamma}')b_0^{-(1+\mathrm{p}),0}(\hat{\beta}, \hat{\gamma}'').$$

The above definition does not depend on the lifts of γ' and γ'' to $\mathrm{Aut}(M, L)$: it computes the zero order coefficient of the expansion

$$k^{-m+\mathrm{p}-1}\mathrm{Tr}(\mathring{C}_k' \mathring{C}_k'' B_k^{-(1+\mathrm{p})}) = \langle \gamma', \gamma'' \rangle_{(\hat{\beta},\mathrm{p})} + O(k^{-1}),$$

where $\mathring{C}_k = C_k - c_k k \mathrm{Id}$ for a constant c_k, uniquely determined by the condition (see (35))

$$k^{-m+\mathrm{p}}\mathrm{Tr}(\mathring{C}_k B_k^{-(1+\mathrm{p})}) = O(k^{-1}).$$

Using the fact that the eigenvalues of B_k/k are in the interval

$$[(f_\beta)_{\min}, (f_\beta)_{\max}] \subset (0, \infty),$$

one sees that (39) defines (by linearity) an inner product on $\mathfrak{t} = \mathrm{Lie}(\mathbb{T})$.

We now consider a normal test configuration $(\mathcal{X}, \mathcal{L})$ associated to a smooth polarized variety (M, L) as before, and also assume that it is compatible with a fixed maximal torus $\mathbb{T} \subset \mathrm{Aut}^r(M, J)$ (with $\beta \subset \mathbb{T}^c$ as before). Thus, $\mathbb{T}$ is subtorus of a maximal torus $\mathbb{T}_0 \subset \mathrm{Aut}(M_0, L_0)/\{\mathbb{C}^* \cdot \mathrm{Id}_{L_0}\}$ of the reduced autmorphism group of the central fibre (M_0, L_0), and we have an embedding $\mathfrak{t} \subset \mathfrak{t}_0$ of the corresponding Lie algebras. We denote by $\gamma_{\mathrm{ex}} \in \mathfrak{t} \subset \mathfrak{t}_0$ the element corresponding to the $K_{(\hat{\beta},\mathrm{p})}$-extremal vector field of $(M, \Omega, \mathbb{T})$, where $\Omega = 2\pi c_1(L)$. We also assume that the inner product $\langle \cdot, \cdot \rangle_{(\hat{\beta},\mathrm{p})}$ is well-defined on $\mathfrak{t}_0$, i.e. the expansions (38) hold on (M_0, L_0). We then define (by using Remark 4 part (2))

Definition 2.9. The relative $(\hat{\beta}, \mathrm{p})$-Donaldson–Futaki invariant $\mathrm{DF}^{\gamma_{\mathrm{ex}}}_{(\hat{\beta},\mathrm{p})}(\mathcal{X}, \mathcal{L})$ of a test configuration $(\mathcal{X}, \mathcal{L})$ as above is defined by

$$\mathrm{DF}^{\gamma_{\mathrm{ex}}}_{(\hat{\beta},\mathrm{p})}(\mathcal{X}, \mathcal{L}) := \mathrm{DF}_{(\hat{\beta},\mathrm{p})}(\mathcal{X}, \mathcal{L}) - \left[\frac{\langle \alpha, \gamma_{\mathrm{ex}} \rangle_{(\hat{\beta},\mathrm{p})}}{\langle \gamma_{\mathrm{ex}}, \gamma_{\mathrm{ex}} \rangle_{(\hat{\beta},\mathrm{p})}}\right] \mathrm{DF}_{(\hat{\beta},\mathrm{p})}(\gamma_{\mathrm{ex}}).$$

The polarized variety (M, L) is said to be *relative $(\hat{\beta}, \mathrm{p})$-K-semistable* if for each $\mathbb{T}$ compatible normal test configuration $(\mathcal{X}, \mathcal{L})$ as above, $\mathrm{DF}^{\gamma_{\mathrm{ex}}}_{(\hat{\beta},\mathrm{p})}(\mathcal{X}, \mathcal{L}) \geq 0$. It is *relative $(\hat{\beta}, \mathrm{p})$-K-stable* if, furthermore, equality in the latter inequality holds if and only if $(\mathcal{X}, \mathcal{L})$ is a product configuration.

We now explore Definitions 2.8 and 2.9 in the case when Ω is an admissible Kähler class on $M = P(E_0 \oplus E_\infty)$, which is also a rational multiple of a polarization, i.e. $r\Omega = 2\pi c_1(L)$ for a holomorphic line bundle L over M and a positive integer r. As the theory is homogeneous in r, we shall assume without loss $r = 1$ and refer to L as an *admissible polarization*.

In [5, 47] (following [44]) a 1-parameter family of test configurations associated to an admissible polarized variety (M, L) is constructed as follows: let $\mathcal{X}$ be the *degeneration to the normal cone* of the divisor $e_\infty := P(0 \oplus E_\infty) \subset M$. Thus, $\pi : \mathcal{X} \to \mathbb{C}$ is a (smooth) polarized variety obtained by

blowing up $M \times \mathbb{C}$ along $e_\infty \times \{0\}$. Notice that $M_0 = \pi^{-1}(0)$ consists of two (smooth) varieties: the exceptional divisor P in $\mathcal{X}$ and the blow-up $\hat{M}$ of M along e_∞, intersecting at the exceptional divisor $E = P(\nu_{e_\infty})$ of $\hat{M}$, where $\nu_{e_\infty} := TM_{|e_\infty}/Te_\infty \to e_\infty$ is the normal bundle of e_∞.

We denote by $\hat{\alpha}$ the induced $\mathbb{C}^*$-action on (M_0, L_0). It is shown in [44] that the Seshadri constant of e_∞ with respect to L is 2, which means that there is a 1-parameter family of polarizations $\mathcal{L}_c = p^*(L) \otimes \mathcal{O}(-cP)$, $c \in (0,2) \cap \mathbb{Q}$ of $\mathcal{X}$, where $p : \mathcal{X} \to M$ is induced by the natural projection from $M \times \mathbb{C}$ to M, with $(M_t, (\mathcal{L}_c)_{|M_t}) \cong (M, L)$, thus giving rise to the family $(\mathcal{X}, \mathcal{L}_c)$ of test-configurations associated to (M, L). Letting $\zeta := c - 1$ (this is a formal substitution), we have, for k sufficiently large, the following $\hat{\alpha}$-invariant decomposition of $H^0(M_0, L_0^k)$ (see [5, 47])

$$
\begin{aligned}
H^0(M_0, L_0^k) = & \bigoplus_{i=0}^{(1-\zeta)k} H^0(e_\infty, L_{|e_\infty}^k \otimes S^{2k-i}\nu_\infty^*) \\
(40) \qquad\qquad & \bigoplus_{j=1}^{(1+\zeta)k} H^0(e_\infty, L_{|e_\infty}^k \otimes S^{(1+\zeta)k-j}\nu_\infty^*) \\
= & \bigoplus_{i=0}^{2k} H^0(e_\infty, L_{|e_\infty}^k \otimes S^{2k-i}\nu_\infty^*).
\end{aligned}
$$

It is shown in [5, 47] that (40) gives rise to the eigenspace decomposition of the generator A_k for the action of $\hat{\alpha}$ on $H^0(M_0, L_0^k)$ as follows: each summand of the first line corresponds to the eigenvalue 0 whereas (with the normalization (33) for A_k) the summands on the second line correspond to eigenvalues j. In the above equalities, ν_∞^* denotes the dual of the normal bundle of e_∞.

Let us now endow the admissible manifold $M = P(E_0 \oplus E_\infty)$ with an admissible Kähler metric (g, ω) in the admissible Kähler class $\Omega = 2\pi c_1(L)$: we can take for instance the canonical Kähler metric (10). We denote by K the vector field on M generating the $\mathbb{S}^1$-action $\beta(e^{i\varphi}) \cdot [a, b] = [e^{i\varphi}a, b] = [a, -e^{i\varphi}b]$. Recall that (g, ω) is K-invariant by construction, and z is the momentum map of K with respect to ω whereas $e_\infty = z^{-1}(-1)$. We denote by h the hermitian metric on L whose curvature form is ω and use its Chern connection to lift K to a holomorphic vector field $\hat{K} = K^H - (z + \boldsymbol{a})T$ on the total space of L, where K^H is the horizontal lift and T is the vector field generating the multiplications by $e^{i\varphi}$ on each fibre of L. It is well-known (see e.g. [25]) that for suitable values of $\boldsymbol{a}$, $\hat{K}$ generates an $\mathbb{S}^1$-subgroup in $\mathrm{Aut}(M, L)$. Restricting $\hat{K}$ to $L_{|e_\infty}$ (where $z = -1$ and $K^H = 0$), we see that

these values are $\boldsymbol{a} \in \mathbb{Z}$. We denote by $\hat{\beta}_{\boldsymbol{a}}$ the corresponding lift of the $\mathbb{S}^1$-action β to L, and by $B_{k,\boldsymbol{a}}$ the generator for the action on $H^0(M_0, L_0^k)$. Notice that $\hat{\beta}_{\boldsymbol{a}}$ acts fibre-wise with weight $(-\boldsymbol{a}+1)$ on $L_{|e_\infty}$, and with weight 1 on the normal bundle ν_∞ of e_∞: The latter follows for instance by computing the eigenvalues of the hessian of z with respect to g (or equivalently of $dJdz = d(1-z^2)\theta$ with respect to ω) along $e_\infty = z^{-1}(-1)$, by using the explicit form of the metric (10) and (11). Thus, under the normalization (33), $B_{k,\boldsymbol{a}}$ acts on $H^0(e_\infty, L_{|e_\infty}^k \otimes S^{uk+v}\nu_\infty^*)$ as $(k(u+\boldsymbol{a}-1)+v)\mathrm{Id}$.

By Remark 4 part (1), in relation to questions of stability, we can consider more generally the quasi-periodic vector fields $\hat{K} = K^H - (z+\boldsymbol{a})T$ with $\boldsymbol{a} \in]1, +\infty)$. With a small abuse of notation, we shall continue to refer to the corresponding $(\hat{K}, \mathrm{p})$-Futaki–Donaldson invariant as $\mathrm{DF}_{(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})}$ and to the $(\hat{K}, \mathrm{p})$-K-stability notion as $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stability. We now prove the following.

Proposition 2.10. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold over a CSCK base, and L an admissible polarization of M, which defines, up to a scale, an admissible Kähler class Ω with $\mathfrak{F}_{\Omega, K, \boldsymbol{a}, \mathrm{p}}(K) = 0$, see Proposition 2.5. Then, the $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant of the test configuration $(\mathcal{X}, \mathcal{L}_\zeta)$ with parameter $\zeta \in \mathbb{Q} \cap (-1, 1)$, corresponding to the degeneration of the normal cone of the infinity section $e_\infty = P(0 \oplus E_\infty)$, is given up to a positive scale by $F_{\Omega, \boldsymbol{a}, \mathrm{p}}(\zeta)$.*

Proof. The computations are essentially identical to those in [5, pp. 589–591], so we shall be brief. The dimension of $H^0(e_\infty, L_{|e_\infty}^k \otimes S^{uk+v}(\nu_\infty^*))$ is computed in [5] to be (up to a common positive constant which we shall ignore)

$$(41) \qquad k^{m-1}p_c^s(u - 1 + v/k) + O(k^{m-3}),$$

where $p_c^s(t) := \prod_{a \in \hat{\mathcal{A}}}(1 + x_a(t + s_a/2k))^{d_a}$.

In order to compute $\mathrm{Tr}(B_{k,\boldsymbol{a}})^q$, we shall use that for any smooth function on the interval $[0, R]$, $R \in \mathbb{Q}_{>0}$, we have

$$(42) \qquad \sum_{i=\varepsilon}^{Rk} f\left(\frac{i}{k}\right) = k \int_0^R f(t)dt + \frac{1}{2}\left(f(R) + (-1)^\varepsilon f(0)\right) + O(k^{-1}),$$

where $\varepsilon = 0, 1$. The above estimate is established in [44] for a polynomial (see also [5, Lemma 9]), but it also holds for any smooth function f, see [27, 49].

We have already shown that $(B_k/k)^q$ acts on $H^0(e_\infty, L^k_{|e_\infty} \otimes S^{2k-i}(\nu^*_\infty))$ as $(1 + \boldsymbol{a} - i/k)^q \mathrm{Id}$. We thus obtain (by using (41) and (42))

$$
\begin{aligned}
k^{-q-m+1}\mathrm{Tr}(B_{k,\boldsymbol{a}})^q = \Big(&\sum_{i=0}^{2k} \big((1 + \boldsymbol{a}) - i/k\big)^q p^s_c(1 - i/k)\Big) + O(k^{-1}) \\
= &\; k \int_0^2 (1 + \boldsymbol{a} - t)^q p^s_c(1 - t)dt \\
&+ \frac{1}{2}\big((\boldsymbol{a} - 1)^q p^s_c(-1) + (1 + \boldsymbol{a})^q p^s_c(1)\big) + O(k^{-1}) \\
= &\; k \int_{-1}^1 (t + \boldsymbol{a})^q p_c(t)dt \\
&+ \frac{1}{2}\int_{-1}^1 \Big(\sum_{a\in\hat{\mathcal{A}}} \frac{x_a d_a s_a}{1 + x_a t}\Big) p_c(t)(t + \boldsymbol{a})^q dt \\
&+ \frac{1}{2}\big((\boldsymbol{a} - 1)^q p_c(-1) + (1 + \boldsymbol{a})^q p_c(1)\big) + O(k^{-1}) \\
= &\; k\alpha_{0,q} + \frac{1}{2}\beta_{0,q} + O(k^{-1}),
\end{aligned}
$$

(43)

where $\alpha_{r,q}$ and $\beta_{r,q}$ are defined by (29).

Similarly, we have

(44)

$$
\begin{aligned}
k^{-q-m}\mathrm{Tr}(A_k B^q_{k,\boldsymbol{a}}) = &\sum_{j=1}^{(1+\zeta)k} (j/k)\big((\zeta + \boldsymbol{a}) - j/k\big)^q p^s_c(\zeta - j/k)\big) + O(k^{-2}) \\
= &\; k\int_0^{1+\zeta} t\big((\zeta + \boldsymbol{a}) - t\big)^q p^s_c(\zeta - t)dt \\
&+ \frac{1}{2}(1 + \zeta)(\boldsymbol{a} - 1)^q p_c(-1) + O(k^{-1}) \\
= &\; k\int_{-1}^\zeta (\zeta - t)(t + \boldsymbol{a})^q p_c(t)dt \\
&+ \frac{1}{2}\int_{-1}^\zeta \Big(\sum_{a\in\hat{\mathcal{A}}} \frac{d_a s_a x_a}{1 + x_a t}\Big)(\zeta - t)(t + \boldsymbol{a})^q p_c(t)dt \\
&+ \frac{1}{2}(1 + \zeta)(\boldsymbol{a} - 1)^q p_c(-1) + O(k^{-1}).
\end{aligned}
$$

Notice that (43)-(44) show that the expansions (35) hold for our test configurations (for any value of the parameter $\zeta \in (-1, 1) \cap \mathbb{Q}$). Our remaining task is to compute the corresponding $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariants.

By (43), we obtain

$$(45) \qquad b_0^{-(\mathrm{p}+1),0}(\hat{\beta}_{\boldsymbol{a}}) = \alpha_{0,-(\mathrm{p}+1)}, \qquad b_1^{1-\mathrm{p},0}(\hat{\beta}_{\boldsymbol{a}}) = \frac{1}{2}\beta_{0,1-\mathrm{p}}.$$

whereas (44) yields

$$b_0^{-(\mathrm{p}+1),1}(\hat{\beta}_{\boldsymbol{a}}, \hat{\alpha}) = \int_{-1}^{\zeta} (\zeta - t)(t + \boldsymbol{a})^{-(\mathrm{p}+1)} p_c(t) dt$$

$$b_1^{(1-\mathrm{p}),1}(\hat{\beta}_{\boldsymbol{a}}, \hat{\alpha}) = \frac{1}{2}\int_{-1}^{\zeta} \Big(\sum_{a \in \hat{\mathcal{A}}} \frac{d_a s_a x_a}{1 + x_a t} \Big)(\zeta - t)(t + \boldsymbol{a})^{1-\mathrm{p}} p_c(t) dt$$

$$+ \frac{1}{2}(1 + \zeta)(\boldsymbol{a} - 1)^{1-\mathrm{p}} p_c(-1).$$

Substituting in (36), and using also $A_1 = 0$ (so that, by the first relation in (28) $2\beta_{0,-\mathrm{p}+1} = \alpha_{0,-(1+\mathrm{p})}A_2$), (23) and (27), we get

$$\mathrm{DF}_{(\hat{\beta}_{\boldsymbol{a}},\mathrm{p})}(\mathcal{X}, \mathcal{L}_\zeta) = b_1^{(1-\mathrm{p}),1}(\hat{\beta}_{\boldsymbol{a}}, \hat{\alpha}) - \Big(\frac{b_1^{1-\mathrm{p},0}(\hat{\beta}_{\boldsymbol{a}})}{b_0^{-(1+\mathrm{p}),0}(\hat{\beta}_{\boldsymbol{a}})} \Big) b_0^{-(\mathrm{p}+1),1}(\hat{\beta}_{\boldsymbol{a}}, \hat{\alpha})$$

$$= \frac{1}{4}\int_{-1}^{\zeta} \Big(\sum_{a \in \hat{\mathcal{A}}} \frac{2 d_a s_a x_a}{1 + x_a t} \Big)(\zeta - t)(t + \boldsymbol{a})^{1-\mathrm{p}} p_c(t) dt$$

$$+ \frac{1}{2}(1 + \zeta)(\boldsymbol{a} - 1)^{1-\mathrm{p}} p_c(-1)$$

$$- \frac{1}{4}\int_{-1}^{\zeta} (\zeta - t)(t + \boldsymbol{a})^{-(\mathrm{p}+1)} A_2 p_c(t) dt$$

$$= \frac{1}{4}G_{x,\boldsymbol{a},\mathrm{p}}(\zeta) = \frac{1}{4}\Big((\zeta + \boldsymbol{a})^{1-\mathrm{p}} F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) \Big).$$

The claim follows.   □

Proposition 2.11. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold over a CSCK base, and L an admissible polarization of M which corresponds, up to a scale, to an admissible Kähler class Ω. If for some $\boldsymbol{a} > 1$ (M, L) is $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable, then $\mathfrak{F}_{\Omega,K,\boldsymbol{a},\mathrm{p}}(K) = 0$ and $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z) > 0$ on $(-1, 1) \cap \mathbb{Q}$.*

Proof. We first prove that if an admissible polarized manifold (M, L) is $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable, then $\mathfrak{F}_{\Omega,K,\boldsymbol{a},\mathrm{p}}(K) = 0$, i.e. $A_1 = 0$, see Proposition 2.5. To this end, we consider the product test configuration $\mathcal{X} = M \times \mathbb{C}$ with $\pi : \mathcal{X} \to \mathbb{C}$ being the projection to the $\mathbb{C}$-factor, and the polarization $\mathcal{L} = L \otimes$

$\mathcal{O}_{\mathbb{C}}$. We endow $\mathcal{X}$ with the $\mathbb{C}^*$-action given by β on M and the standard $\mathbb{C}^*$-action on $\mathbb{C}$, and consider, via β_0, the lifted action on $\mathcal{L}$. Thus, the central fibre of this test configuration is (M, L) with induced $\mathbb{C}^*$-action $\alpha = \beta_0$. A computation similar to (43) shows that (35) hold true with

$$b_0^{-(1+\mathrm{p}),1}(\hat{\beta}_{\boldsymbol{a}}, \alpha) = \alpha_{1,-(1+\mathrm{p})}, \qquad b_1^{1-\mathrm{p},1}(\hat{\beta}_{\boldsymbol{a}}, \alpha) = \frac{1}{2}\beta_{1,(1-\mathrm{p})}.$$

and (45). It follows from the definition of $\mathrm{DF}_{(\hat{\beta}_{\boldsymbol{a}},\mathrm{p})}$ and $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stability that

$$\begin{aligned}\mathrm{DF}_{(\hat{\beta}_{\boldsymbol{a}},\mathrm{p})}(\mathcal{X}, \mathcal{L}) &= \frac{1}{2}\Big(\beta_{1,(1-\mathrm{p})} - \frac{\alpha_{1,-(1+\mathrm{p})}}{\alpha_{0,-(1+\mathrm{p})}}\beta_{0,(1-\mathrm{p})}\Big) \\ &= \Big(\frac{\alpha_{0,-(1+\mathrm{p})}\alpha_{2,-(1+\mathrm{p})} - \alpha_{1,-(1+\mathrm{p})}^2}{4\alpha_{0,-(1+\mathrm{p})}}\Big)A_1 \\ &= 0.\end{aligned}$$

Thus, we have $A_1 = 0$ or, equivalently, $\mathfrak{F}_{\Omega,K,\boldsymbol{a},\mathrm{p}}(K) = 0$, see Proposition 2.5. By Proposition 2.10, the corresponding function satisfies $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) > 0$ for $\zeta \in (-1, 1) \cap \mathbb{Q}$. □

By Lemma 2.4, the $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-extremal vector field of an admissible polarized manifold (M, L) (endowed with a maximal torus $\mathbb{T} \subset \mathrm{Aut}(M, L)$ covering a maximal torus of the isometry group of an admissible Kähler metric) is $A_1 K$. As the definition of the relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant given in Definition 2.9 does not change if we replace γ_{ex} by a non-zero multiple, and the inner product (39) does not depend on the chosen lift of the action to L, we can assume that γ_{ex} corresponds the the $\mathbb{S}^1$-action β_0. Calculations similar to the ones in the proofs of Propositions 2.10 and 2.11 allow us to compute the relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant of an admissible test configuration

Proposition 2.12. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold over a CSCK base, and L an admissible polarization of M corresponding, up to a scale, to an admissible Kähler class Ω. Then, the relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-Donaldson–Futaki invariant with respect to a maximal torus $\mathbb{T} \subset \mathrm{Aut}(M, L)$ covering a maximal torus of isometries of an admissible Kähler metric on M of the test configuration $(\mathcal{X}, \mathcal{L}_\zeta)$ with parameter $\zeta \in \mathbb{Q} \cap (-1, 1)$ corresponding to the degeneration of the normal cone of the infinity section $e_\infty = P(0 \oplus E_\infty)$, is a positive multiple of $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta)$. In particular, if (M, L) is relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable, then $F_{\Omega,\boldsymbol{a},\mathrm{p}}(\zeta) > 0$ on $(-1, 1) \cap \mathbb{Q}$.*

3. Proof of the main results

3.1. Existence results for admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metrics

In this section, we use Proposition 2.2 to construct admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler manifolds.

Theorem 3.1. *Suppose that $M = P(E_0 \oplus E_\infty) \to S$ is an admissible manifold over a CSCK base S. Then, for every choice of $\boldsymbol{a}, \mathrm{p} \in \mathbb{R}$ such that $\boldsymbol{a} > 1$, M admits an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in the admissible Kähler class Ω_x if the parameters $x = (x_a, a \in \mathcal{A})$ are sufficiently small. If, furthermore, S is a local product of non-negative CSCK metrics, then any admissible Kähler class on M contains an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal metric.*

Proof. It is not hard to check that the linear system in (28) does not degenerate as $x \to 0$ (meaning $x_a \to 0$ for all $a \in \mathcal{A}$). In particular, $\lim_{x \to 0} A_1$ and $\lim_{x \to 0} A_2$ both exist. Further, the limit for $x \to 0$ of the right hand side (23) of (24) is a function of z that has

- at most one zero in $(-1, 1)$ if $d_0 = d_\infty = 0$.

- at most two zeroes in $(-1, 1)$ if either d_0 or d_∞ are non-zero, but not both.

- at most three zeroes in $(-1, 1)$ if both d_0 and d_∞ are non-zero.

Combined with the end point conditions (25) in each of these three cases, $\lim_{x \to 0} G(z)$ will not have enough inflection points to have any zeroes over $-1 < z < 1$. Hence we can conclude that for $|x_a|$ sufficiently small for all $a \in \mathcal{A}$, $F_{x,\boldsymbol{a},\mathrm{p}}(z)$ satisfies (i) of (16) and the claim follows from Proposition 2.2.

Assuming that S is a local product of non-negative CSCK metrics, we shall adapt the root counting argument due to Hwang [28] and Guan [26] (see also Proposition 11 in [5]) to check that the function $F_{x,\boldsymbol{a},\mathrm{p}}(z)$ in Proposition 2.2 verifies (i) of (16) for any admissible data $x_a, a \in \mathcal{A}$, thus defining an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal metric in any admissible Kähler class.

The idea is to interpret (21) in a form similar to Equations (4) and (5) of [5, Prop. 1]. Indeed, (21) can be rewritten as follows

$$(46) \qquad (z + \boldsymbol{a})^{\mathrm{p}+1} G''_{x,\boldsymbol{a},\mathrm{p}}(z) = (\prod_{a \in \hat{\mathcal{A}}} (1 + x_a z)^{d_a - 1}) P(z),$$

where $P(z)$ is a polynomial of degree $\leq \#\hat{\mathcal{A}} + 1$, satisfying for all $a \in \hat{\mathcal{A}}$,

$$(47) \qquad P(-1/x_a) = 2d_a s_a x_a (-1/x_a + \boldsymbol{a})^2 \prod_{b \in \hat{\mathcal{A}} \setminus \{a\}} \left(1 - \frac{x_b}{x_a} \right).$$

Now, the positivity (i) of (16) is equivalent to positivity of $G_{x,\boldsymbol{a},\mathrm{p}}(z) = \frac{F_{x,\boldsymbol{a},\mathrm{p}}(z)}{(z+\boldsymbol{a})^{\mathrm{p}-1}}$ over $(-1,1)$. Thus our task is to check that, under the assumptions in the theorem, $G_{x,\boldsymbol{a},\mathrm{p}}(z)$ is positive for $-1 < z < 1$. Note that for $-1 < z < 1$, the sign of $G''_{x,\boldsymbol{a},\mathrm{p}}(z)$ equals the sign of $(z + \boldsymbol{a})^{\mathrm{p}+1} G''_{x,\boldsymbol{a},\mathrm{p}}(z)$, and hence also the sign of $P(z)$ defined by (46)-(47). Using the boundary conditions (25) for $G_{x,\boldsymbol{a},\mathrm{p}}(z)$, the proof now essentially follows the proof of [5, Prop. 11] with some minor modifications. $\qquad \square$

3.2. A non-existence result for $(z + \boldsymbol{a}, \mathrm{p})$-CSCK metrics

In this section we establish the following

Theorem 3.2. *Let M be an admissible Kähler manifold over a CSCK base. Suppose that Ω is an admissible Kähler class which is a positive multiple of an element in $H^2(M, \mathbb{Z})$, and the function $F_{\Omega,\boldsymbol{a},\mathrm{p}}(z)$ defined in Proposition 2.2 has negative values on $(-1,1)$. Then, M does not admit a $(z + \boldsymbol{a}, \mathrm{p})$-CSCK metric in Ω.*

Proof. It was observed in [7, 32] that the vanishing of the $(K, \boldsymbol{b}, \mathrm{p})$-Futaki invariant $\mathfrak{F}_{(\Omega,K,\boldsymbol{b},\mathrm{p})}$ is a necessary condition for the existence of an (f, p)-CSCK metric in Ω. In the admissible setting, by using Proposition 2.5, this corresponds to the condition that A_1 given by (28) vanishes. In the remainder of the argument, we can therefore assume that $\mathfrak{F}_{(\Omega,K,\boldsymbol{b},\mathrm{p})} = 0$. A. Lahdili proved in [33, Thm. 1] that if $(M, J, \Omega, K, \mathbb{T})$ is a compact Kähler manifold as in Section 2.4, such that the Kähler class $\frac{1}{2\pi}\Omega \in H^2(M, \mathbb{Z})$ and the $(K, \boldsymbol{b}, \mathrm{p})$-Futaki invariant $\mathfrak{F}_{(\Omega,K,\boldsymbol{b},\mathrm{p})}$ vanishes on $\mathrm{Lie}(\mathbb{T})$, then the boundedness from below of the relative $(K, \boldsymbol{b}, \mathrm{p})$-Mabuchi functional $\mathcal{M}^{\mathbb{T}}_{(\Omega,K,\boldsymbol{b},\mathrm{p})}$ is a necessary condition for the existence of an (f, p)-CSCK metric in Ω. As all of the conditions are invariant under a positive scale of Ω, combining Lahdili's result with Proposition 2.7 concludes the proof. $\qquad \square$

Notice that in the admissible setting, the assumption that a positive multiple of Ω_x belongs to $H^2(M, \mathbb{Z})$ corresponds to admissible data $x = (x_a, a \in \mathcal{A})$ such that $x_a \in \mathbb{Q}, a \in \mathcal{A}$.

3.3. A Yau–Tian–Donaldson type correspondence

Because of Propositions 2.5, 2.11 and 2.12, we give the following

Definition 3.1. Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold over a CSCK base, L an admissible polarization of M, which defines, up to a scale, an admissible Kähler class Ω.

  (a) We say that (M, L) is $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistable/$(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable/*analytically* $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable on *admissible test configurations* if A_1 given by (28) vanishes and, respectively, $F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z) \geq 0$ on $(-1, 1)/F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z) > 0$ on $(-1, 1) \cap \mathbb{Q}/F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z) > 0$ on $(-1, 1)$.

  (b) Similarly, (M, L) is said to be relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistable/relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable/analytically relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$−K-stable on admissible test configurations if $F_{\Omega, \boldsymbol{a}, \mathrm{p}} \geq 0$ on $(-1, 1)/F_{\Omega, \boldsymbol{a}, \mathrm{p}}(z) > 0$ on $(-1, 1) \cap \mathbb{Q}/F_{\Omega, \boldsymbol{a}, \mathrm{p}} > 0$ on $(-1, 1)$.

Our discussion from the previous sections can be summarized in the following

Theorem 3.3. *Let $M = P(E_0 \oplus E_\infty) \to S$ be an admissible manifold, and L an admissible polarization of M which defines, up to a scale, an admissible Kähler class Ω.*

  • *If (M, L) is analytically relative $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable (resp. analytically $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stable) with respect to admissible test configurations, then there exists an admissible $(z + \boldsymbol{a}, \mathrm{p})$-extremal Kähler metric in Ω (resp. an admissible Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature).*

  • *If Ω admits a Kähler metric of constant $(z + \boldsymbol{a}, \mathrm{p})$-scalar curvature, then (M, L) is $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-semistable with respect to admissible test configurations.*

Proof. The first statement follows from Proposition 2.2. The second statement follows from [7, Cor. 2], Proposition 2.5 and Theorem 3.2. □

Remark 5. In the previous theorem *analytic* stability implied the existence of a distinguished metric. We do not expect that the same is implied just by $(\hat{\beta}_{\boldsymbol{a}}, \mathrm{p})$-K-stability, see [5] for an example with $\boldsymbol{a} = +\infty$. Below, we prove no such counterexample exists if $p = 2m = 4$ and $A_1 = 0$.

Theorem 3.4. *Let $(M, J) = P(\mathcal{O} \oplus E) \to \Sigma$ be a ruled complex surface over a compact complex curve, where E is a line bundle of positive degree*

over Σ, L a polarization of (M, J), which, up to a positive scale, corresponds to an admissible Kähler class Ω_x with $x \in (0, 1)$, and $\mathbf{a} > 1$ a real number. Then the following conditions are equivalent

(i) *Ω_x admits an admissible Kähler metric which is conformally Einstein–Maxwell with conformal factor $(z + \mathbf{a})^{-2}$;*

(ii) *Ω_x admits a Kähler metric which is conformally Einstein–Maxwell with conformal factor $(z + \mathbf{a})^{-2}$;*

(iii) *(M, L) is $(\hat{\beta}_{\mathbf{a}}, 4)$-K-stable on admissible test configurations;*

(iv) *(M, L) is analytically $(\hat{\beta}_{\mathbf{a}}, 4)$-K-stable on admissible test configurations.*

Proof. We first notice that in this case, $F_{\Omega_x, \mathbf{a}, 4}(z) = F_{x, \mathbf{a}, 4}(z)$ is a polynomial of degree ≤ 4. (This follows from Proposition 2.2 and (27), see also [30].) As shown in [33, Cor. 2], the construction of [30] combined with Proposition 2.7 above and the stability under deformation result in [32], yield the equivalences

$$(\text{i}) \Longleftrightarrow (\text{ii}) \Longleftrightarrow (\text{iv}).$$

Noting that clearly (iv) $\Longrightarrow$ (iii), we thus need to establish the implication (iii) $\Longrightarrow$ (iv), i.e. that

$$F_{x, \mathbf{a}, 4}(z) > 0 \text{ on } (-1, 1) \cap \mathbb{Q} \Longrightarrow F_{x, \mathbf{a}, 4}(z) > 0 \text{ on } (-1, 1)$$

under the assumption $\mathfrak{F}_{\Omega_x, \mathbf{a}, 4}(K) = 0 = A_1$ (with A_1 computed by (28)).

Suppose for contradiction that $F_{x, \mathbf{a}, 4}(z) \geq 0$ on $(-1, 1)$ and has double irrational root. By Theorem 3.1 this implies that the genus of Σ is at least two. By the results in [30], the condition $A_1 = 0$ implies that $\mathbf{a} = \mathbf{a}(x)$ is a root of

$$(48) \qquad\qquad x = 2\mathbf{a}/(1 + \mathbf{a}^2)$$

which is determined uniquely by the requirement $|\mathbf{a}| > 1$. On the other hand, the condition that $F_{x, \mathbf{a}, 4}(z)$ has a double root on $(-1, 1)$, together with (16)(ii), implies the vanishing of the discriminant of the second order polynomial $F_{x, \mathbf{a}, 4}(z)/(1 - z^2)$. It then follows (see [30, Sect. 3.1]) that x must coincide with the a root in $(0, 1)$ of

$$(49) \qquad D_s(x) = 12 + 12sx - 19x^2 - 12sx^3 + (7 + s^2)x^4$$
$$+ 6(2 + 2sx - 2x^2 - sx^3)\sqrt{1 - x^2} = 0,$$

with $s = 2(1 - \mathbf{g})/d$ (here $\mathbf{g}$ stands for the genus of Σ and d for the degree of L). We denote the above two roots by $(x_0, \boldsymbol{a}_0)$.

Now, we can take $x = x_0$ to be rational, as the polarization assumption implies that Ω_x has rational coefficients. Our first goal is to show $\boldsymbol{a}_0$ must also be rational. A careful look at (49) reveals that either $\sqrt{1 - x_0^2}$ is rational, and then so must $\boldsymbol{a}_0 = (1 + \sqrt{1 - x_0^2})/x_0$ be, or else $f_1(x_0) = f_2(x_0) = 0$, where $f_1 = 12 + 12sx - 19x^2 - 12sx^3 + (7 + s^2)x^4$ and $f_2 = 6(2 + 2sx - 2x^2 - sx^3)$. However, the latter cannot hold since $f_2 = 0$ implies $s = \frac{2(1-x^2)}{x(-2+x^2)}$ and substituting this into $f_1 - f_2 = 0$ gives $\frac{(1-x^2)x^4(-12+7x^2)}{(-2+x^2)^2} = 0$ which cannot be zero for $0 < x < 1$. Thus $\boldsymbol{a}_0$ is rational.

As both $\boldsymbol{a}_0$ and x_0 are rational, it follows from the explicit computations in [30] that $F_{x_0, \boldsymbol{a}_0, 4}(z)$ has rational coefficients. As ± 1 are roots of $F_{x_0, \boldsymbol{a}_0, 4}(z)$ by (16)(ii)-(iii), any double root of $F_{x_0, \boldsymbol{a}_0, 4}(z)$ must then be rational too, a contradiction. $\qquad\square$

4. Conformally Kähler, Einstein–Maxwell metrics

While Proposition 2.2 gives an existence result for the boundary value problem consisting of (20) together with (ii) and (iii) of (16), if we are aiming for $(|z + \boldsymbol{a}|, \mathrm{p})$-CSCK solutions, we need to set $A_1 = 0$, which in turn yields an equation for $\boldsymbol{a}$ in terms of the admissible data that may or may not have a solution with $|\boldsymbol{a}| > 1$. Of course, if we are successful in finding such a solution, we still need to ensure that (i) of (16) holds. We can often be more specific about the form of the solution when $A_1 = 0$, if we consider particular values of p.

Indeed, supposing $\mathrm{p} \neq 0, 1, ..., m, m + 1$ then it is easy to see that any solution to (20) is of the form

$$(50) \qquad F(z) = c_\mathrm{p}(z + \boldsymbol{a})^\mathrm{p} + c_{\mathrm{p}-1}(z + \boldsymbol{a})^{\mathrm{p}-1} + \sum_{k=0}^{m} c_k (z + \boldsymbol{a})^k,$$

where $c_0, ..., c_m$ depend on A_1, A_2 and the admissible data. Thus, $F_{x, \boldsymbol{a}, \mathrm{p}}(z)$ must be given by (50) for a unique choice of the coefficients A_1, A_2, c_p, and $c_{\mathrm{p}-1}$ (whose existence is guaranteed by Proposition 2.2 so that $F(z)$ satisfies (ii) and (iii) of (16)). In particular, if $\mathrm{p} > m + 1$ is an integer, then $F(z) = F_{x, \boldsymbol{a}, \mathrm{p}}(z)$ is a polynomial.

In this section, we will explore further the case $p = 2m$ ($m > 1$). Then (20) becomes

$$(51) \quad -(z + \boldsymbol{a})^2 F''(z) + 2(2m - 1)(z + \boldsymbol{a})F'(z) - 2m(2m - 1)F(z)$$
$$= (A_1 z + A_2)p_c(z) - p_c(z)(z + \boldsymbol{a})^2 \sum_{a \in \hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1 + x_a z}.$$

Imposing $A_1 = 0$ and (i) of (16) in (51) will thus produce an Einstein–Maxwell metric $h = \frac{1}{(z+\boldsymbol{a})^2} g$ where g is given by (13) with $\Theta(z) = F(z)/p_c(z)$, see [7].

4.1. Conformally Kähler, Einstein–Maxwell metrics over the product of two Riemann surfaces

Let Σ_a ($a = 1, 2$) be compact Riemann surfaces with CSCK metrics $(\pm g_a, \pm \omega_a)$ and let M be $P(\mathcal{O} \oplus E) \to \Sigma_1 \times \Sigma_2$ where $E = E_1 \otimes E_2$ for E_a being pullbacks of line bundles on Σ_a with $c_1(E_a) = [\omega_a/2\pi]$. Let $\pm 2s_a$ be the scalar curvature of $\pm g_a$. Note that if $\deg E_a = n_a$, then $s_a = 2(1 - \mathbf{g}_a)/n_a$, where $\mathbf{g}_a$ denotes the genus of Σ_a. In this case, with slight abuse of notation, we will also write $M = P(\mathcal{O} \oplus \mathcal{O}(n_1, n_2)) \to \Sigma_1 \times \Sigma_2$.

Equation (51) now takes the form

$$(52) \quad -(z + \boldsymbol{a})^2 F''(z) + 10(z + \boldsymbol{a})F'(z) - 30F(z)$$
$$= (A_1 z + A_2)(1 + x_1 z)(1 + x_2 z)$$
$$+ (z + \boldsymbol{a})^2 \left(2s_1 x_1(1 + x_2 z) + 2s_2 x_2(1 + x_1 z)\right),$$

and we know that its solution $F_{x,\boldsymbol{a},6}(z)$ is of the form

$$F_{x,\boldsymbol{a},6}(z) = c_6(z + \boldsymbol{a})^6 + c_5(z + \boldsymbol{a})^5 + c_3(z + \boldsymbol{a})^3$$
$$+ c_2(z + \boldsymbol{a})^2 + c_1(z + \boldsymbol{a}) + c_0.$$

Plugging $F_{x,\boldsymbol{a},6}(z)$ into (52) tells us that

$$c_0 = \frac{\boldsymbol{a}(\boldsymbol{a}x_1 - 1)(\boldsymbol{a}x_2 - 1)A_1 - (\boldsymbol{a}x_1 - 1)(\boldsymbol{a}x_2 - 1)A_2}{30}$$
$$c_1 = \frac{(-1 + 2\boldsymbol{a}x_1 + 2\boldsymbol{a}x_2 - 3\boldsymbol{a}^2 x_1 x_2)A_1 + (-x_1 - x_2 + 2\boldsymbol{a}x_1 x_2)A_2}{20}$$

$$c_2 = \frac{(-x_1 - x_2 + 3\boldsymbol{a}x_1x_2)A_1 - x_1x_2A_2 + 2(s_1x_1(1 - \boldsymbol{a}x_2) + s_2x_2(1 - \boldsymbol{a}x_1))}{12}$$

$$c_3 = \frac{-x_1x_2A_1 + 2x_1x_2(s_1 + s_2)}{6}$$

On the other hand, (ii) of (16) is equivalent to

$$c_5 = \frac{-(2\boldsymbol{a}(3+\boldsymbol{a}^2)(1+3\boldsymbol{a}^2)c_0 + (\boldsymbol{a}^2-1)(1+10\boldsymbol{a}^2+5\boldsymbol{a}^4)c_1 + 4\boldsymbol{a}(\boldsymbol{a}^2-1)^2(\boldsymbol{a}^2+1)c_2 + (\boldsymbol{a}^2-1)^3(1+3\boldsymbol{a}^2)c_3)}{(1-\boldsymbol{a}^2)^5}$$

$$c_6 = \frac{(1+10\boldsymbol{a}^2+5\boldsymbol{a}^4)c_0 + 4\boldsymbol{a}(\boldsymbol{a}^4-1)c_1 + (\boldsymbol{a}^2-1)^2(1+3\boldsymbol{a}^2)c_2 + 2\boldsymbol{a}(\boldsymbol{a}^2-1)^3c_3}{(1-\boldsymbol{a}^2)^5},$$

so c_5 and c_6 are determined by c_0, c_1, c_2, c_3 (and $\boldsymbol{a}$). With this established, (iii) of (16) is equivalent to the following two equations
(53)
$$(3 + 12\boldsymbol{a} + 30\boldsymbol{a}^2 + 20\boldsymbol{a}^3 + 15\boldsymbol{a}^4)c_0 + 2(\boldsymbol{a}^2 - 1)(1 + 5\boldsymbol{a} + 5\boldsymbol{a}^2 + 5\boldsymbol{a}^3)c_1$$
$$+ 2(\boldsymbol{a}^2 - 1)^2(1 + 2\boldsymbol{a} + 3\boldsymbol{a}^2)c_2 + (\boldsymbol{a}^2 - 1)^3(1 + 3\boldsymbol{a})c_3$$
$$= (\boldsymbol{a} - 1)(\boldsymbol{a} + 1)^5(x_1 - 1)(x_2 - 1)$$

$$(3 - 12\boldsymbol{a} + 30\boldsymbol{a}^2 - 20\boldsymbol{a}^3 + 15\boldsymbol{a}^4)c_0 + 2(\boldsymbol{a}^2 - 1)(-1 + 5\boldsymbol{a} - 5\boldsymbol{a}^2 + 5\boldsymbol{a}^3)c_1$$
$$+ 2(\boldsymbol{a}^2 - 1)^2(1 - 2\boldsymbol{a} + 3\boldsymbol{a}^2)c_2 + (\boldsymbol{a}^2 - 1)^3(-1 + 3\boldsymbol{a})c_3$$
$$= (\boldsymbol{a} - 1)^5(\boldsymbol{a} + 1)(1 + x_1)(1 + x_2).$$

Using the formulas for $c_0, ..., c_3$ above this yields a linear system of two equations with the two unknowns A_1 and A_2. The linear system has coefficients that depend on the admissible data (s_1, s_2, x_1, x_2) as well as $\boldsymbol{a}$ in a rather unwieldy way.

One may check, aided by Mathematica, that if $x_1, x_2 > 0$ then there exists $\boldsymbol{a} > 1$ such that $A_1 = 0$. In general, it appears very to be quite difficult to write this $\boldsymbol{a} > 1$ solution explicitly, so it is non-trivial to test the final condition (i) of (16). But we were able to describe an explicit example below.

Example 4.1. Let $x_1 = 1/2$, $x_2 = 1/3$, $\boldsymbol{a} = 5$. This gives

$$A_1 = \frac{20(9840 - 4502s_1 + 1203s_2)}{24073}$$

$$A_2 = \frac{20(7836 + 4442s_1 + 2883s_2)}{3439}$$

$$c_0 = -\frac{12(314 + 2978s_1 + 787s_2)}{24073}$$

$$c_1 = \frac{2(-135 + 1294s_1 + 279s_2)}{1267}$$

$$c_2 = \frac{7640 - 14198s_1 - 2697s_2}{15204}$$

$$c_3 = \frac{-98400 + 69093s_1 + 12043s_2}{433314}$$

$$c_5 = \frac{415 - 83s_1 - 13s_2}{30408}$$

$$c_6 = \frac{-21912 + 2862s_1 + 433s_2}{13866048}.$$

When we solve for $A_1 = 0$ we get $s_2 = \frac{2(2251s_1 - 4920)}{1203}$ and then compute $Scal(h) = \frac{240}{401}(148s_1 - 153)$ and

$$F_{x,\boldsymbol{a},6}(z) = \frac{(1 - z^2)}{86616}\Big(3(26078 + 22965z + 7553z^2 + 1095z^3 + 53z^4) \\ + s_1(1 - z^2)(1181 + 465z + 28z^2)\Big).$$

Notice that $F_{x,\boldsymbol{a},6}(z)$ satisfies (i) of (16) when $s_1 > 0$, so we get a family of conformally Kähler, Einstein–Maxwell metrics on

$$M = P(\mathcal{O} \oplus \mathcal{O}(n_1, n_2)) \to \mathbb{CP}^1 \times \Sigma_2,$$

where $n_1 \in \mathbb{Z}^+$ is arbitrary and $\mathbf{g}_2, n_2 \in \mathbb{Z}^+$ are such that

$$\frac{(4502 - 4920n_1)}{1203n_1} = \frac{(1 - \mathbf{g}_2)}{n_2}.$$

We can take for instance $n_2 = 1203n_1$ and $\mathbf{g}_2 = 4920n_1 - 4501$ in order to satisfy the above relation. With this choice, $Scal(h) > 0$ for $n_1 = 1$, but for $n_1 > 1$ $Scal(h) < 0$.

Since $F_{x,a,6}(z)$ satisfies (i) of (16) when $s_1 = 0$, we also have conformally Kähler, Einstein–Maxwell metrics on $M = P(\mathcal{O} \oplus \mathcal{O}(n_1, n_2)) \to T^2 \times \Sigma_2$, where $n_1 \in \mathbb{Z}^+$ is arbitrary and $\mathbf{g}_2, n_2 \in \mathbb{Z}^+$ are such that

$$\frac{(-1640)}{401} = \frac{(1 - \mathbf{g}_2)}{n_2}.$$

A specific solution is $n_2 = 401, \mathbf{g}_2 = 1641$.

Finally, if $s_1 < 0$ is sufficiently close to zero, (i) of (16) still holds and we will have some conformally Kähler, Einstein–Maxwell metrics on $M = P(\mathcal{O} \oplus \mathcal{O}(n_1, n_2)) \to \Sigma_1 \times \Sigma_2$, where the genus of Σ_1 and Σ_2 are both at least two. (On the other hand, (16)-(i) fails as $s_1 \to -\infty$.)

By [5, Thm. 8], none of the manifolds above admits a CSCK metric.

The next example is inspired by the construction of Kähler–Einstein admissible metrics by Koiso and Sakane [31, 45].

Example 4.2. We consider $M = P(\mathcal{O} \oplus \mathcal{O}(1, -1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Thus, we assume that $s_1 = 2$ and $s_2 = -2$, $0 < x_1 < 1$ and $-1 < x_2 < 0$. The pair $(x_1, x_2) \in (0, 1) \times (-1, 0)$ determines the admissible Kähler class and, up to rescaling, this exhausts the entire Kähler cone. Notice that when $x_1 = 1/2 = -x_2$, the corresponding Kähler class admits a Kähler–Einstein admissible metric which was first discovered by Koiso–Sakane. Moreover, for $x_2 = -x_1$ or $x_2 = -1 + x_1$, the corresponding Kähler class admits a CSCK admissible metric (see e.g. [5, Thm. 9]).

As in Section 4.2, this is a case that is included by Theorem 3.1. For a given pair $(x_1, x_2) \in (0, 1) \times (-1, 0)$, by Proposition 2.2, we have an admissible metric associated to a conformally Kähler Einstein-Maxwell metric whenever there is a solution $|\mathbf{a}| > 1$ of $A_1 = 0$. Here we calculate that $A_1 = 0$ if and only if

$$\begin{aligned}
q(x_1, x_2, \mathbf{a}) :=\ & -3(x_1 + x_2)(x_1 - x_2 + x_1 x_2) \\
& + 3\left(2x_1 + 3x_1^2 - 2x_2 + 8x_1 x_2 + 2x_1^2 x_2 + 3x_2^2 - 2x_1 x_2^2 + 2x_1^2 x_2^2\right) \mathbf{a} \\
& - 3(x_1 + x_2)(15 - 2x_1 + 2x_2 + 17x_1 x_2)\mathbf{a}^2 \\
& + (60 - 10x_1 + 45x_1^2 + 10x_2 + 240x_1 x_2 - 18x_1^2 x_2 \\
& \quad + 45x_2^2 + 18x_1 x_2^2 + 90x_1^2 x_2^2)\mathbf{a}^3 \\
& - 5(x_1 + x_2)(33 + 4x_1 - 4x_2 + 45x_1 x_2)\mathbf{a}^4 \\
& + (72 + 34x_1 + 123x_1^2 - 34x_2 + 408x_1 x_2 + 50x_1^2 x_2 \\
& \quad + 123x_2^2 - 50x_1 x_2^2 + 90x_1^2 x_2^2)\mathbf{a}^5
\end{aligned}$$

$$
\begin{aligned}
&- (x_1 + x_2)(159 - 2x_1 + 2x_2 + 105x_1 x_2)\boldsymbol{a}^6 \\
&+ (60 - 30x_1 + 15x_1^2 + 30x_2 + 96x_1 x_2 - 38x_1^2 x_2 \\
&\quad + 15x_2^2 + 38x_1 x_2^2 + 6x_1^2 x_2^2)\boldsymbol{a}^7 \\
&+ 15(-1 + x_1 - x_2)(x_1 + x_2)\boldsymbol{a}^8 = 0.
\end{aligned}
$$

Now, $q(x_1, x_2, 1) = 192(-1 + x_1)^2(-1 + x_2)^2 > 0$ and $q(x_1, x_2, -1) = -192(1 + x_1)^2(1 + x_2)^2 < 0$. If the leading coefficient of $q(x_1, x_2, \boldsymbol{a})$, i.e. $15(x_1 + x_2)(-1 + x_1 - x_2)$, is nonzero, it follows that there is an $|\boldsymbol{a}| > 1$ such that $q(x_1, x_2, \boldsymbol{a}) = 0$. On the other hand, when $x_2 = -x_1$, we have

$$
\begin{aligned}
q(x_1, x_2, \boldsymbol{a})/\boldsymbol{a} = {}& 96(1 - x_1^2)^2 \\
&+ 16(1 - x_1^2)(12 - 2x_1 + 9x_1^2)(\boldsymbol{a}^2 - 1) \\
&+ 2\left((1 - x_1)(59 + 31x_1 - 59x_1^2 - 27x_1^3) + 4\right)(\boldsymbol{a}^2 - 1)^2 \\
&+ \left((1 - x_1)^2(28 + 26x_1 - 9x_1^2 - 6x_1^3) + 2 + 6x_1^5\right)(\boldsymbol{a}^2 - 1)^3,
\end{aligned}
$$

and so in this case $q(x_1, x_2, \boldsymbol{a}) = 0$ has no solutions for $|\boldsymbol{a}| > 1$. In a similar, but slightly more tedious fashion one can verify that when $x_2 = -1 + x_1$, there are no solutions $|\boldsymbol{a}| > 1$ to $q(x_1, x_2, \boldsymbol{a}) = 0$. However, in those cases there are admissible CSCK metrics (see [5]), which are of course conformally Kähler, Einstein–Maxwell metrics with constant conformal factor (in the setting of this paper, they can be thought as $((z + \boldsymbol{a}), 2m)$-CSCK metrics with $\boldsymbol{a} = \infty$), so we can formulate the following general existence result.

Proposition 4.3. *Every Kähler class on $M = P(\mathcal{O} \oplus \mathcal{O}(1, -1)) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ has an admissible Kähler metric, conformal to an Einstein–Maxwell metric.*

4.2. Conformally Kähler, Einstein–Maxwell metrics on admissible projective bundles over a CSCK 4-manifold

We now assume that the base S is a Hodge Kähler manifold of real dimension 4 with non-negative constant scalar curvature $4s$ (we drop the index a when $|\mathcal{A}| = 1$). Note that $0 \le s \le 3$ (the latter inequality follows from the Fujita inequality for Hodge manifolds, see [24]). By [5, Thm 7], there are no admissible CSCK metrics in this case.

Given $0 < x < 1$ (again, without loss of generality we may assume x is positive as long as we also do not assume upfront that $\boldsymbol{a} > 1$), Theorem 3.1 and Proposition 2.2 tells us that an admissible metric associated to a $(|z + \boldsymbol{a}|, \mathrm{p})$-CSCK solution is equivalent to a solution $|\boldsymbol{a}| > 1$ of $A_1 = 0$.

From the linear system (28) and a simple calculation, A_1 is given by

$$A_1 = \frac{8\left(-x\boldsymbol{a}^2 + 2\boldsymbol{a} - x\right)}{45(\boldsymbol{a}-1)^{10}(1+\boldsymbol{a})^{10}}q(\boldsymbol{a}, x),$$

where

$$
\begin{aligned}
q(\boldsymbol{a}, x) = {} & 96(1-x)^3 \\
& + 32(1-x)^2(12 - 9x - sx)(\boldsymbol{a}-1) \\
& + 8(1-x)(87 - 120x - 16sx + 39x^2 + 10sx^2)(\boldsymbol{a}-1)^2 \\
& + 8(1-x)(93 - 81x - 29sx + 18x^2 + 3sx^2)(\boldsymbol{a}-1)^3 \\
& + 2(243 - 315x - 104sx + 99x^2 + 70sx^2 - 15x^3 + 22sx^3)(\boldsymbol{a}-1)^4 \\
& + 2(90 - 63x - 45sx + 14sx^2 - 3x^3 + 15sx^3)(\boldsymbol{a}-1)^5 \\
& + 5(6 - 2s + s(2+x)(1-x)^2)(\boldsymbol{a}-1)^6
\end{aligned}
$$

We always get at least one solution of $A_1 = 0$ with $|\boldsymbol{a}| > 1$ from the factor

$$-x\boldsymbol{a}^2 + 2\boldsymbol{a} - x = 0,$$

namely

$$(54) \qquad\qquad \boldsymbol{a}_0(x) := \frac{1 + \sqrt{1 - x^2}}{x}.$$

Notice that $\boldsymbol{a}_0(x) > 1$ and $\boldsymbol{a}_0(x)$ is a decreasing function of $0 < x < 1$ with $\lim_{x \to 0} \boldsymbol{a}_0(x) = +\infty$ and $\lim_{x \to 1} \boldsymbol{a}_0(x) = 1$. Any additional solutions of $A_1 = 0$ would come from solutions of the equation $q(\boldsymbol{a}, x) = 0$ satisfying $|\boldsymbol{a}| > 1, x \in (0, 1)$.

Proposition 4.4. *Let (S, g_S, ω_S) be a compact CSCK Hodge 4-manifold with non-negative scalar curvature, and E be a holomorphic line bundle such that $c_1(E) = [\omega_S/2\pi]$. Then, in each admissible Kähler class on $M = P(\mathcal{O} \oplus E) \to S$ there exists at least one admissible Kähler metric conformal to an Einstein–Maxwell metric.*

Remark 6. Over the interval $1 < s \le 3$, the expression $2s/(1 + s^2)$ is a decreasing function surjecting onto $[3/5, 1)$. If $x = 2s/(1 + s^2)$, then we observe

that

$$q(\boldsymbol{a}, x) = \frac{2(\boldsymbol{a} - s)^2}{(1 + s^2)^3} \left(15 - 2s^2 + 3s^4 - 32s \left(3 + s^2 \right) \boldsymbol{a} + 2 \left(9 + 130s^2 + 5s^4 \right) \boldsymbol{a}^2 \right.$$
$$\left. - 32s \left(3 + 5s^2 \right) \boldsymbol{a}^3 + 5 \left(3 + 6s^2 + 7s^4 \right) \boldsymbol{a}^4 \right) = 0$$

has a double root at $\boldsymbol{a} = s$ and, moreover, $\boldsymbol{a}_0(x) = s$. Thus $A_1 = 0$ has a triple root at $\boldsymbol{a}_0 = s$.

If, on the other hand, $2s/(1 + s^2) < x < 1$, then $\boldsymbol{a}_0(x) < s$ so that

$$q(\boldsymbol{a}_0(x), x) = \frac{6\boldsymbol{a}_0(\boldsymbol{a}_0^2 - 1)^4 \left(5\boldsymbol{a}_0^2 - 1 \right) (\boldsymbol{a}_0 - s)}{\left(\boldsymbol{a}_0^2 + 1 \right)^3}$$

(with $\boldsymbol{a}_0 = \frac{1 + \sqrt{1 - x^2}}{x}$) is negative whereas $q(1, x)$ and $\lim_{\boldsymbol{a} \to +\infty} q(\boldsymbol{a}, x)$ are positive. We conclude that in this case, for each $x \in (0, 1)$ $q(\boldsymbol{a}, x) = 0$ has (at least) two additional solutions, $\boldsymbol{a}_\pm(x)$ with $1 < \boldsymbol{a}_-(x) < \boldsymbol{a}_0(x) < \boldsymbol{a}_+(x)$.

Remark 7. Assuming more generally that $(S, \pm g_S, \pm \omega_S)$ is a Hodge Kähler manifold of complex dimension d with non-negative constant scalar curvature, it seems from experimental data (letting d take various values ≥ 3) that we always have the solution $\boldsymbol{a}_0(x)$ of $A_1 = 0$ defined in (54) but a direct proof of this seems out of reach at the moment.

Conjecture 1. Let (S, g_S, ω_S) be a compact CSCK Hodge $2(m - 1)$-manifold with non-negative scalar curvature and E be a holomorphic line bundle such that $c_1(E) = [\omega_S/2\pi]$. Then, in each admissible Kähler class on $M = P(\mathcal{O} \oplus E) \to S$ there exists at least one admissible Kähler metric which is conformal to an Einstein–Maxwell metric.

4.3. Conformally Kähler, Einstein metrics

We recall here the constructions going back to Page [43] and Bérard-Bergery [11] of admissible Kähler manifolds which are conformally Einstein. These of course are special cases of the conformally Kähler, Einstein–Maxwell metrics discussed in this paper. By the results of [14, 15], any compact Kähler manifold (M, J, g, ω) of real dimension $2m \geq 6$, which is conformally Einstein is isometric to one of these examples. We use the computations of [2, Sect. 5.6] in order to recast the construction of [11, 14] in the admissible setting of this paper. Indeed, according to [2, Sect. 1.4 & 5.6], for an admissible Kähler metric g of the form (13) to be conformally Einstein with a conformal factor $(z + \boldsymbol{a})$, we must have that

- $|\mathcal{A}| = 1$;

- $M = P(\mathcal{O} \oplus E) \to S$, where (S, g_S, ω_S) is a compact Kähler–Einstein manifold of positive scalar curvature $Scal(g_S) = 2(m-1)s$, and E is a holomorphic line bundle over S with $c_1(E) = [\omega_S/2\pi]$; here $s = \frac{c_1(S) \cdot c_1(E)^{m-2}}{c_1(E)^{m-1}}$ is the normalized scalar curvature of g_S;

- there exists an admissible parameter $x = x_e \in (0,1)$ and a real constant $\boldsymbol{a}_e > 1$, such that $F(z) = F_{x_e, \boldsymbol{a}_e, 2m}(z)$ is given by

$$(55) \quad \frac{F_{x_e, \boldsymbol{a}_e, 2m}(z)}{x_e^{m-1}} = \sum_{j=1}^{m} \frac{j}{m} \binom{2m}{m+j} \left[\lambda_+ \left(\boldsymbol{a}_e - \frac{1}{x_e} \right)^{m-j} \left(z + \frac{1}{x_e} \right)^{m+j} \right.$$
$$\left. - \lambda_- \left(\boldsymbol{a}_e - \frac{1}{x_e} \right)^{j-1} \left(z + \frac{1}{x_e} \right)^{m-j} + \frac{s}{m} \left(z + \frac{1}{x_e} \right)^{m} \right],$$

where λ_+, λ_- are real constants.

The point is that $F(z) = F_{x_e, \boldsymbol{a}_e, 2m}(z)$ automatically verifies (20) and $A_1 = 0$ (because the metric $(z + \boldsymbol{a}_e)^{-2}g$ is Einstein and therefore g has constant $(z + \boldsymbol{a}_e, 2m)$-scalar curvature), so we are left with the 4 boundary conditions (16)-(ii) &(iii). These in turn place 4 algebraic relations for the real constants $(\lambda_+, \lambda_-, x_e, \boldsymbol{a}_e)$. The upshot of the constructions in [11, 15] is that if $s > 1$, then these relations determine the 4 constants, up to a two-fold ambiguity, i.e. there exists a unique $x_e \in (0,1)$ and a pair $\boldsymbol{a}_e = \boldsymbol{a}_{\pm} > 1$ for which $F_{x_e, \boldsymbol{a}_+, 2m}(z) = F_{x_e, \boldsymbol{a}_-, 2m}(z)$ satisfies (55) (see [2, 15] for the geometric meaning of this). Notice that the positivity condition (16)(i) is then automatically satisfied by Theorem 3.1.

It is clear from the setting above that we can weaken the Kähler–Einstein assumption for (S, g_S, ω_S) and assume instead that (S, g_S, ω_S) is a CSCK Hodge manifold with normalized scalar curvature $s > 1$. Then the solutions $(x_e, \boldsymbol{a}_e = \boldsymbol{a}_{\pm})$ will correspond to two Einstein–Maxwell metrics $h_{\pm} = (z + \boldsymbol{a}_{\pm})^{-2}g$ in the conformal class of the admissible metric corresponding to $F(z) = F_{x_e, \boldsymbol{a}_{\pm}, 2m}(z)$. We thus have the following existence result (related to Conjecture 1).

Proposition 4.5. *Let (S, g_S, ω_S) be a compact Hodge Kähler $2(m-1)$-manifold of constant scalar curvature $Scal(g_S) > 2(m-1)$, and E a holomorphic line bundle such that $c_1(E) = [\omega_S/2\pi]$. Then, $M = P(\mathcal{O} \oplus E) \to S$ admits an admissible Kähler metric conformal to an Einstein–Maxwell metric.*

Notice that the constraint $Scal(g_S) > 2(m-1)$ in the above proposition is equivalent to

$$c_1(S) \cdot c_1(E)^{m-2} > c_1(E)^{m-1},$$

which in turn limits the choice for the line bundles E on a given S. By the Fujita inequality [24], the number of such line bundles cannot exceed $(m-1)$.

4.4. Conformally Kähler, Einstein–Maxwell metrics and the Yamabe functional

On a compact manifold M of real dimension $2m$, the *normalized Einstein–Hilbert functional* on the set of Riemannian metrics is defined by

$$\mathfrak{S}(g) := \frac{\int_M Scal(g)\, dv_g}{\left(\int_M dv_g\right)^{\frac{m-1}{m}}},$$

where dv_g denotes the volume form of g. The restriction of $\mathfrak{S}$ to a conformal class $[g]$ of Riemannian metrics on M is known as the *Yamabe functional*. It is a deep result that the Yamabe functional attains a minimum $Y_{[g]}$ on $[g]$ (see e.g. [9, 39, 46]). Any metric $h \in [g]$ for which $\mathfrak{S}(h) = Y_{[g]}$ is called a *Yamabe minimizer* of $[g]$. It is well-known that any Yamabe minimizer h has constant scalar curvature and, if $Y_{[g]} \leq 0$, any metric in $[g]$ which has constant scalar curvature must be homothetic to the (unique up to scaling) Yamabe minimizer in $[g]$. For $Y_{[g]} > 0$, the Yamabe minimizers are not necessarily homothetic and, furthermore, a constant scalar curvature metric in $[g]$ is not necessarily a Yamabe minimizer. Thus, one can ask

Question 4.6. Given a constant scalar curvature metric $h \in [g]$ such that $\mathfrak{S}(h) > 0$, is h a Yamabe minimizer?

It is known (see e.g. [9]) that if a constant scalar curvature representative h of $[g]$ is a Yamabe minimizer, it must satisfy the inequality

(56) $$\mathfrak{S}(h) \leq 2m(2m-1)Vol(\mathbb{S}^{2m})^{1/m},$$

where $\mathbb{S}^{2m}$ denote the unit sphere in $\mathbb{R}^{2m+1}$. Furthermore, by [46], the inequality (56) is strict if $(M, [g])$ is not conformal to $\mathbb{S}^{2m}$. Notice that for e.g. $m = 2$, the right hand side of this inequality is equal to $8\sqrt{6}\pi$.

In what follows, we shall investigate Question 4.6 for some of the Einstein–Maxwell metrics (which are of constant scalar curvature by definition, see [7]) that we found in the conformal classes of admissible Kähler metrics.

As we have already mentioned, some of the Einstein–Maxwell metrics we found have negative scalar curvature (see Example 4.1), so they are Yamabe minimizers by the above general remarks. Other such examples are the Einstein metrics discussed in Section 4.3, which have positive constant scalar curvature and are Yamabe minimizers by virtue of the Obata theorem [41].

We shall now give examples for which the Einstein–Maxwell metrics are not Yamabe minimizers. To this end, we use the following general remarks. Suppose we have an admissible metric g as defined in (13). Then for any $t > 1$ we consider the conformal metric $h_t := (z+t)^{-2}g$ with scalar curvature equal to

$$
\begin{aligned}
Scal(h_t) \quad &= \quad \frac{-(z+t)^2 F''(z)+2(2m-1)(z+t)F'(z)-2m(2m-1)F(z)}{p_c(z)} \\
&+ \quad (z+t)^2 \sum_{a \in \hat{\mathcal{A}}} \frac{2d_a s_a x_a}{1+x_a z},
\end{aligned}
$$

(57)

and volume form $(z+t)^{-2m}\omega^m/m!$. It follows from [21] (this can be checked directly in the admissible setting) that the function $f(t) = \mathfrak{S}(h_t)$ does not depend on the choice of $F(z)$, i.e. on the particular choice of admissible representative in the given admissible Kähler class. Moreover, as follows from Theorem 2.3 (b) in [21], (or can be checked directly in the admissible setting) the critical values of $f(t)$ correspond exactly to the values $t = \boldsymbol{a}$ where A_1 from (20) (or equivalently the Futaki invariant $\mathfrak{F}_{([\omega],K,\boldsymbol{a},2m)}$, see Proposition 2.5) vanishes. Thus, from Proposition 2.2 it follows that any critical value $t = \boldsymbol{a}$ of $f(t)$ corresponds to a conformally Kähler, Einstein–Maxwell metric $h_{\boldsymbol{a}} = (z+\boldsymbol{a})^{-2}g$ provided that $F_{[\omega],\boldsymbol{a},2m}(z)$ satisfies (i) of (16). Notice that the latter condition is automatic on the manifolds described in Theorem 3.1. Further, for such an $h_{\boldsymbol{a}}$ to be a Yamabe minimizer, it is necessary (albeit not sufficient) that $f(t)$ has a minimum at $t = \boldsymbol{a}$.

4.5. Einstein–Maxwell metrics on the first Hirzebruch surface which are not Yamabe minimizers

We now restrict ourselves to the case $M = P(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{C}P^1$ which has been studied in [30, 38]. We let g be an admissible Kähler metric given by (13). In accordance with [30], we simplify the notation from Section 2 by dropping the index a and noting that the normalized scalar curvature of

$S = \mathbb{C}P^1$ is 1. Thus, (57) becomes

$$Scal(h_t) = \frac{-(z+t)^2 F''(z) + 6(z+t)F'(z) - 12F(z) + 4x(z+t)^2}{(1+xz)}$$

whereas the volume form of $h_t = (z+t)^{-2}g$ is

$$dv_{h_t} = (z+t)^{-4}(1/x + z)\omega_{\mathbb{C}P^1} \wedge dz \wedge \theta.$$

Using that $f(t)$ is independent of the choice of $F(z) = \Theta(z)p_c(z)$, we can take $F_c(z) = (1 - z^2)(1 + xz)$ (see (10)), and compute

$$f(t) = \frac{\int_M Scal(h_t)v_{h_t}}{\sqrt{\int_M dv_{h_t}}} = 4\pi\sqrt{6}\frac{\left(1 - 2x - 2xt + (1+2x)t^2\right)}{\sqrt{x\left(1 - 4xt + 3t^2\right)\left(t^2 - 1\right)}}$$

and

$$f'(t) = 16\pi\sqrt{6}\frac{x(xt^2 - 2t + x)((1-x)t^2 - xt + x)}{(x(t^2-1)(3t^2 - 4xt + 1))^{3/2}}.$$

It is not hard to check that if $x \le 4/5$, $t = \boldsymbol{a}_0(x)$ with $\boldsymbol{a}_0(x)$ defined by (54) is the only critical point of $f(t)$ for $|t| > 1$, and it is a minimum. If, on the other hand, $4/5 < x < 1$, then we find three critical points for $f(t)$: $\boldsymbol{a}_0(x) = \frac{1+\sqrt{1-x^2}}{x}$, $\boldsymbol{a}_+(x) = \frac{x+\sqrt{x(5x-4)}}{2(1-x)}$, and $\boldsymbol{a}_-(x) = \frac{x-\sqrt{x(5x-4)}}{2(1-x)}$ (all greater than 1 with $\boldsymbol{a}_-(x) < \boldsymbol{a}_0(x) < \boldsymbol{a}_+(x)$). As noticed in [38], the values $\boldsymbol{a}_\pm(x)$ give rise to the same admissible Kähler metric, i.e. $F_{x,\boldsymbol{a}_+(x),4}(z) = F_{x,\boldsymbol{a}_-(x),4}(z)$. We denote by g_0 the admissible Kähler metric corresponding to $(x, \boldsymbol{a}_0(x))$, by g the admissible Kähler metric corresponding to $(x, \boldsymbol{a}_\pm(x))$, and by $h_0 = \frac{1}{(z+\boldsymbol{a}_0(x))^2}g_0, h_\pm = \frac{1}{(z+\boldsymbol{a}_\pm(x))^2}g$ the corresponding Einstein–Maxwell metrics. Even though the function $f(t)$ is the same for all of these cases, we are dealing with two different conformal structures, $[g_0]$ and $[g]$.

Now, as the only critical points of $f(t)$ for $t > 1$ are $\boldsymbol{a}_0(x), \boldsymbol{a}_\pm(x)$, they cannot all occur as minima. Indeed, it is not hard to see that $f(t)$ has a relative maximum at $t = \boldsymbol{a}_0(x)$ (and relative minima at $t = \boldsymbol{a}_\pm(x)$). Thus, the conformally Kähler, Einstein–Maxwell metric h_0 is not a Yamabe minimizer of $[g_0]$, and this despite the fact that for $4/5 < x < 1$ we have

$$f(\boldsymbol{a}_0(x)) = 4\pi\sqrt{6}\frac{(1+2x)\sqrt{1-x^2} + (1+2x-x^2)}{\sqrt{x(6 - 5x^2 + (6 - 2x^2)\sqrt{1-x^2})}} < 8\pi\sqrt{6},$$

i.e. h_0 does not violate the estimate (56).

Remark 8. Otoba [42] produced a different family of conformally Kähler, constant scalar curvature metrics on all Hirzebruch surfaces, and in particular proved that (on each Hirzebruch surface) some of these are not Yamabe minimizers.

Remark 9. If we return to the explicit example in Remark 6, where $S = \mathbb{CP}^2$, $s = 3$, and $x = 4/5$, one may also observe that $\mathfrak{S}(h_t)$ has a local maximum at $t = \boldsymbol{a}_0(x)$ and local minima at $t = \boldsymbol{a}_\pm(x)$. Further, $\mathfrak{S}(h_0) < 30\pi \left(\frac{16}{15}\right)^{\frac{1}{3}}$, so, similarly to the first Hirzebruch surface, we have a constant scalar curvature Einstein–Maxwell metric which is not a Yamabe minimizer but satisfies (56).

Appendix A. Orthotoric $(\boldsymbol{f}, \mathrm{p})$-extremal metrics

The bundle geometry examined so far in this paper is related to the theory of hamiltonian 2-forms of order $\ell = 1$, see [2]. In this appendix, we will describe local examples of (f, p)-extremal metrics which admit a hamiltonian 2-form of order $\ell = m$, that is, orthotoric Kähler metrics. Presumably, similar explicit constructions hold for Kähler metrics admitting a hamiltonian 2-form of any order $1 \leq \ell \leq m$, but these will not be investigated in this paper.

Recall from [2] that a Kähler $2m$-manifold (M, g, J, ω) is orthotoric if it is equipped with m Poisson-commuting Killing potentials $\sigma_1, \ldots, \sigma_m$ such that on a dense open set, the roots ξ_j of $\sum_{r=0}^{m}(-1)^r \sigma_r t^{m-r}$, with $\sigma_0 = 1$, are smooth, with linearly independent, orthogonal gradients. The functions $(\xi_1, \ldots, \xi_m)$ together with the angular coordinates $(t_1, \ldots, t_m)$ for the momenta $(\sigma_1, \ldots, \sigma_m)$ form a coordinate system on that open dense subset, and are called *orthotoric coordinates*. It is shown in [2] that with respect to the orthotoric coordinates the metric is given by

$$
\begin{aligned}
g &= \sum_{j=1}^{m} \frac{\Delta_j}{\Theta_j(\xi_j)} d\xi_j^2 + \sum_{j=1}^{m} \frac{\Theta_j(\xi_j)}{\Delta_j} \left[\sum_{r=1}^{m} \sigma_{r-1}(\widehat{\xi}_j) dt_r\right]^2, \\
\omega &= \sum_{j=1}^{m} d\xi_j \wedge \left(\sum_{r=1}^{m} \sigma_{r-1}(\widehat{\xi}_j) dt_r\right),
\end{aligned}
$$
(A.1)

where $\Delta_j = \prod_{j \neq k}(\xi_j - \xi_k)$, each $\Theta_j(z)$ is a function of one variable, and $\sigma_{r-1}(\widehat{\xi}_j)$ is the $r - 1$-th elementary symmetric function of the remaining ξ_k's after ξ_j is removed.

It is easily deduced from (A.1) that the following formulas hold

$$g(\nabla\sigma_k, \nabla\sigma_l) = \sum_{j=1}^{m}\sigma_{k-1}(\hat{\xi}_j)\sigma_{l-1}(\hat{\xi}_j)\Theta_j(\xi_j)/\Delta_j,$$

(A.2) $$\Delta_g\sigma_k = -\sum_{j=1}^{m}\sigma_{k-1}(\hat{\xi}_j)\Theta'_j(\xi_j)/\Delta_j,$$

$$Scal(g) = -\sum_{j=1}^{m}\Theta''_j(\xi_j)/\Delta_j,$$

where $\nabla\sigma_k$, $\Delta_g\sigma_k$ are the gradient and Laplacian of σ_k, respectively, $Scal(g)$ is the scalar curvature of g, and the primes denote differentiation with respect to ξ_j. Indeed, the last two formulas are obtained from (78),(79) in [2], after noting that $\partial\sigma_k/\partial\xi_j = \sigma_{k-1}(\hat{\xi}_j)$. The first follows from (54) in [2], by noting that

(A.3) $$|\nabla\xi_j|_g^2 = \Theta_j(\xi_j)/\Delta_j.$$

We now recall equation (7) in the form

$$Scal_{f,\mathrm{p}}(g) = f^2 Scal(g) - 2(\mathrm{p}-1)f\Delta_g f - \mathrm{p}(\mathrm{p}-1)|\nabla f|_g^2.$$

We notice that σ_k is a Killing potential for the Killing vector field $\partial/\partial t_k$ of (A.1), and any Killing potential f of a Killing vector field commuting with $\partial/\partial t_j, j = 1,\ldots, m$ is necessarily an affine function in the σ_k's. Thus, for such an f, the above formulas show that g is (f,p)-extremal if and only if there exist constants a_k, b_k, $k = 0,\ldots, m$ such that

(A.4) $$-\Big(\sum_{k=0}^{m} a_k\sigma_k\Big)^2 \sum_{j=1}^{m}\Theta''_j(\xi_j)/\Delta_j$$

$$+ 2(\mathrm{p}-1)\Big(\sum_{k=0}^{m} a_k\sigma_k\Big)\sum_{k,j=1}^{m} a_k\sigma_{k-1}(\hat{\xi}_j)\Theta'_j(\xi_j)/\Delta_j$$

$$- \mathrm{p}(\mathrm{p}-1)\sum_{k,l,j=1}^{m}\Big(a_k a_l\sigma_{k-1}(\hat{\xi}_j)\sigma_{l-1}(\hat{\xi}_j)\Theta_j(\xi_j)/\Delta_j\Big) = \sum_{k=0}^{m} b_m\sigma_k.$$

We now consider some special cases.

A.1. Bochner-flat orthotric metrics are $(f, m+2)$-extremal

It has been observed in [7] for $m = 2$ and in [1] for $m \geq 2$ that a Bochner–flat metric is $(f, m+2)$-extremal for any positive Killing potential f. The metric (A.1) is Bochner-flat iff $\Theta_j(z) = P(z)$ are all equal to a j-independent

polynomial of degree $\leq m + 2$, see [2, Prop. 17] and the references therein. Thus, in this case, we get a solution of (A.4) with p $= m + 2$ for any choice of $a_0, \ldots, a_m$.

A.2. Flat orthotoric metrics which are (f, p)-extremal

Recall from [2, Prop. 17] that (A.1) is flat iff $\Theta_j(z) = P(z)$ for a (j-independent) polynomial P of degree $\leq m$. In this case, we show

Proposition A.1. *Let (g, ω) be a flat orthotoric metric in the form* (A.1) *with $\Theta_j(z) = P(z)$ for a polynomial $P(z)$ of degree $\leq m$, and $f = \sum_{r=0}^m a_r \sigma_r$ be a positive Killing potential. Then, (g, ω) is (f, p)-extremal for any* p.

Proof. Using the Vandermonde identitity (see [2, App. A])

$$(A.5) \qquad \sum_{j=1}^m \frac{\xi_j^{m-s} \sigma_{r-1}(\hat{\xi}_j)}{\Delta_j} = (-1)^{s-1} \delta_{rs}, \ r, s = 1, \ldots, m,$$

with $r = 1$ yields that the first term in the LHS of (A.4) is identically zero when $\Theta_j(z) = P(z)$ for a polynomial P of degree $\leq m$. Similarly, (A.5) shows that the second term is an affine-linear function in $\sigma_1, \ldots, \sigma_m$. We thus conclude that the metric (g, ω) is (f, p)-extremal if p $= 0, 1$, and for p $\neq 0, 1$ it is (f, p)-extremal iff

$$\sum_{k,l,j=1}^m \left(a_k a_l \sigma_{k-1}(\hat{\xi}_j) \sigma_{l-1}(\hat{\xi}_j) \frac{P(\xi_j)}{\Delta_j} \right)$$

is an affine-linear function in $(\sigma_1, \ldots, \sigma_m)$. Notice that the latter condition does not depend on p, and it does hold for p $= m + 2$ by the discussion in Sect. A.1. Thus, we conclude that (g, ω) is (f, p)-extremal for any value of p. $\qquad \square$

In some special cases of Proposition A.1, we can find explicitly the relationship between the coefficients $a_0, \ldots, a_m$ of f, those of $P(z)$ and $b_0, \ldots, b_m$ in the LHS. For example, let us take $f = a_0 + a_1 \sigma_1$ with $a_1 \neq 0$ and $\Theta_j(z) =$

$P(z) = \sum_{k=0}^{m} c_k z^{m-k}$. By the Vandermonde identities (see [2, App. B])

(A.6)
$$\sum_{j=1}^{m} \frac{\xi_j^{m-s}}{\Delta_j} = \delta_{s1}, s = 1, \ldots, m;$$
$$\sum_{j=1}^{m} \frac{\xi_j^m}{\Delta_j} = \sigma_1.$$

the LHS of (A.4) now reduces to

$$- (a_0 + a_1\sigma_1)^2 \sum_{j=1}^{m} \frac{P''(\xi_j)}{\Delta_j}$$
$$+ (\mathrm{p} - 1)\left[2(a_0 + a_1\sigma_1)a_1 \sum_{j=1}^{m} \frac{P'(\xi_j)}{\Delta_j} - \mathrm{p}a_1^2 \sum_{j=1}^{m} \frac{P(\xi_j)}{\Delta_j}\right]$$
$$= (\mathrm{p} - 1)\left[2m(a_0 + a_1\sigma_1)a_1 c_0 - \mathrm{p}a_1^2 c_0 \sigma_1 - \mathrm{p}a_1^2 c_1\right]$$
$$= b_0 + b_1\sigma_1$$

with

$$b_1 = a_1^2 c_0(\mathrm{p} - 1)(2m - \mathrm{p}), \quad b_0 = (\mathrm{p} - 1)(2m a_0 a_1 c_0 - \mathrm{p}a_1^2 c_1).$$

We conclude in this case that (A.1) is a flat Kähler metric which has constant $(a_0 + a_1\sigma_1, \mathrm{p})$-scalar curvature iff $c_0 = 0$ or $\mathrm{p} = 1, 2m$. In particular, we get an $(m + 2)$-dimensional family (parametrized by $a_0, c_0, \ldots, c_m$) of conformally-Kähler, Einstein–Maxwell metrics for which the Kähler metric is flat. We also notice that, more generally, if $\mathrm{p} \neq 0, 1$ the coefficients b_0 and b_1 uniquely determine the coefficients c_0 and c_1 of P, but not its other coefficients.

A.3. Orthotoric metrics which are (σ_m, p)-extremal

We now consider case $f = \sigma_m = \prod_{i=1}^{m} \xi_i$. In this case we will exhibit solutions extending some of the ambitoric examples discussed in [1, 7] to higher dimensions.

Proposition A.2. *Let (g, ω) be an orthotoric metric of the form (A.1). Then g is (σ_m, p)-extremal metric with $m \geq 2$ and $\mathrm{p} \neq 1, \ldots, m + 1$ iff each function $\Theta_j(z)$ is a sum of a polynomial $P(z)$ of degree $\leq m$, whose coefficients are independent of j, and an expression of the form $b_{1j}z^{\mathrm{p}-1} + b_{2j}z^{\mathrm{p}}$, for arbitrary constants $b_{ij}, i = 1, 2, j = 1, \ldots, m$. Furthermore, in this case $Scal_{\sigma_m, \mathrm{p}}(g)$ is a linear combination of σ_{m-1} and σ_m, i.e. the coefficients*

$b_i = 0$ *for* $i < m - 1$. *In particular,* $Scal_{\sigma_m, \mathrm{p}}(g)$ *is constant iff it vanishes, which happens iff* $P(0) = P'(0) = 0$.

For $\mathrm{p} \in \{1, \ldots, m+1\}$, we can find solutions $\Theta_j(\xi_j)$ of a similar form but the first summand will contain logarithmic terms. Note also that if at least one of the b_{ij}'s is nonzero, g is not flat.

Proof. For $f = \sigma_m$, Equation (A.4) becomes

$$(A.7) \quad -\sigma_m^2 \sum_{j=1}^{m} \Theta_j''(\xi_j)/\Delta_j + (\mathrm{p}-1)\Big[2\sigma_m \textstyle\sum_{j=1}^{m}(\Theta_j'(\xi_j)/\Delta_j)\sigma_{m-1}(\hat{\xi}_j)$$

$$-\mathrm{p}\textstyle\sum_{j=1}^{m}(\Theta_j(\xi_j)/\Delta_j)\sigma_{m-1}^2(\hat{\xi}_j)\Big] = \textstyle\sum_{k=0}^{m} b_k \sigma_k,$$

with $\sigma_{m-1}(\hat{\xi}_j) = \sigma_m/\xi_j$.

We first check that $\Theta_j(x) = P(z) + b_{1j}z^{\mathrm{p}-1} + b_{2j}z^{\mathrm{p}}$ as in the proposition give a solution. To this end, we re-write (A.7) as

(A.8)

$$\sigma_m^2 \sum_{j=1}^{m} \left(\frac{-\xi_j^2 \Theta_j''(\xi_j) + 2(\mathrm{p}-1)\xi_j \Theta_j'(\xi_j) - \mathrm{p}(\mathrm{p}-1)\Theta_j(\xi_j)}{\xi_j^2 \Delta_j} \right) = \sum_{k=0}^{m} b_k \sigma_k.$$

We notice that for each j, the term $b_{1j}z^{\mathrm{p}-1} + b_{2j}z^{\mathrm{p}}$ is the solution of the homogeneous ODE at the LHS of (A.8), so it is enough to compute it with $\Theta_j(z) = P(z)$ being a j-independent polynomial of degree $\leq m$. To this end, we use the Vandermonde identities (A.6) and

(A.9)
$$\sum_{j=1}^{m} \frac{\xi_j^{s-2}}{\Delta_j} = (-1)^{m-1}\frac{\delta_{s1}}{\sigma_m}, s = 1, \ldots, m;$$

$$\sum_{j=1}^{m} \frac{\xi_j^{-2}}{\Delta_j} = (-1)^{m-1}\frac{\sigma_{m-1}}{\sigma_m^2},$$

which follow from (A.6) written for $(1/\xi_i)$'s instead of the ξ_i's. The claim in the proposition follows easily from (A.6) and (A.9). This computation also shows that $Scal_{\sigma_m, \mathrm{p}}(g)$ is a linear combination of σ_{m-1} and σ_m, i.e. $b_0 = \cdots = b_{m-2} = 0$, as well as the condition for the vanishing of $Scal_{\sigma_m, \mathrm{p}}(g)$. As mentioned above, by the classification in [2], the orthotoric metric g is not flat provided that at least one of the constants b_{ij} is nonzero.

We now turn to the necessity of the conditions. Denote by $\Delta = (-1)^{m(m-1)/2} \prod_{i<j}(\xi_i - \xi_j)$ the Vandermont determinant of $\xi_1, \ldots, \xi_m$ and

by $\Delta(\hat{\xi}_j)$ the Vandermonde determinant of ξ_k with $k \neq j$. Notice that $\Delta(\hat{\xi}_j)$ is, up to sign, Δ/Δ_j.

We can now rewrite (A.7) in the form

$$\sum_{j=1}^{m} \left[\pm \sigma_{m-1}^2(\hat{\xi}_j)\Delta(\hat{\xi}_j)[-\xi_j^2\Theta_j'' + 2(\mathrm{p}-1)\xi_j\Theta_j' - \mathrm{p}(\mathrm{p}-1)\Theta_j] \right]$$
$$= \Delta \sum_{k=0}^{m} b_k\sigma_k,$$

obtained by multiplying both sides of (A.8) by Δ and rearranging the result, with the signs $\pm$ left unspecified, as they will not matter for the rest of the argument. This equation has the form

$$(A.10) \qquad \sum_{j=1}^{m} F_j(\hat{\xi}_j)H_j(\xi_j) = G,$$

where H_j is the expression in the inner square brackets, $F_j(\hat{\xi}_j)$ the rest of the j-th summand on the LHS (which does not depend on ξ_j), and G a polynomial function of all the ξ_j's.

We now count degrees. For G, we notice that the combination of σ_k's has all terms of degree at most one in ξ_j, whereas Δ has degree $m-1$ in each ξ_j, so G has degree at most m in each ξ_j. Also, $F_j(\hat{\xi}_j)$ has degree $2 + m - 2 = m$ in each ξ_k, $k \neq j$. Differentiating equation (A.10) m times with respect to, say, ξ_1, yields

$$F_1(\hat{\xi}_1)H_1^{(m)}(\xi_1) + \sum_{j=2}^{m} F_j^{(m)}(\hat{\xi}_1, \hat{\xi}_j)H_j(\xi_j) = k,$$

Where '(m)' denotes this m-th partial derivative, $F_j^{(m)}(\hat{\xi}_1, \hat{\xi}_j)$ does not depend on ξ_1 or ξ_j and k is a constant. Separation of variables yields that $H_1^{(m)}(\xi_1)$ is constant, so that, H_1, and similarly each $H_j(\xi_j)$, $j = 1, \ldots, m$, is a polynomial of degree at most m in ξ_j. Now for $\mathrm{p} \neq 1, \ldots, m+1$, the solutions of

$$(A.11) \qquad H(\xi_j) = -\xi_j^2\Theta_j''(\xi_j) + 2(\mathrm{p}-1)\xi_j\Theta_j'(\xi_j) - \mathrm{p}(\mathrm{p}-1)\Theta_j(\xi_j)$$
$$= \sum_{k=0}^{m} c_k\xi_j^k$$

have the form $\Theta_j(\xi_j) = P_j(\xi_j) + b_{1j}\xi_j^{\mathrm{p}-1} + b_{2j}\xi_j^{\mathrm{p}}$, where $P_j(z)$ is a polynomial of degree at most m.

To show the independence from j of the coefficients of $P_j(\xi_j)$, we consider another form for equation (A.7), obtained by multiplying it by Δ_k for some fixed k.

$$
\begin{aligned}
-\sigma_m^2 &\left(\Theta_k'' + \sum_{j \neq k}(\Delta_k/\Delta_j)\Theta_j''\right) \\
&+ 2(\mathrm{p}-1)\sigma_m\left(\sigma_{m-1}(\hat{\xi}_k)\Theta_k' + \sum_{j \neq k}^m ((\Delta_k/\Delta_j)\sigma_{m-1}(\hat{\xi}_j)\Theta_j'\right) \\
&- \mathrm{p}(\mathrm{p}-1)\left(\sigma_{m-1}^2(\hat{\xi}_k)\Theta_k + \sum_{j \neq k}(\Delta_k/\Delta_j)\sigma_{m-1}^2(\hat{\xi}_j)\Theta_j\right) \\
&= (\sum_{l=1}^m b_m \sigma_l)\Delta_k,
\end{aligned}
$$

Setting $\xi_k = \xi_{j_0}$ for some fixed $j = j_0 \neq k$ in this equation, we note that in a non-empty open set $\Delta_k/\Delta_j\big|_{\xi_k = \xi_{j_0}} = -\delta_{jj_0}$, so that we obtain

$$
\begin{aligned}
-\sigma_m^2\big|_{\xi_k = \xi_{j_0}} &\left(\Theta_k''(\xi_{j_0}) - \Theta_{j_0}''(\xi_{j_0})\right) \\
&+ 2(\mathrm{p}-1)\sigma_m\big|_{\xi_k = \xi_{j_0}} \left(\sigma_{m-1}(\hat{\xi}_k)\Theta_k'(\xi_{j_0}) - \sigma_{m-1}(\hat{\xi}_{j_0})\big|_{\xi_k=\xi_{j_0}}\Theta_{j_o}'(\xi_{j_0})\right) \\
&- \mathrm{p}(\mathrm{p}-1)\left(\sigma_{m-1}^2(\hat{\xi}_k)\Theta_k(\xi_{j_0}) - \sigma_{m-1}^2(\hat{\xi}_{j_0})\big|_{\xi_k=\xi_{j_0}}\Theta_{j_0}(\xi_{j_0})\right) = 0.
\end{aligned}
$$

After dividing by the common factor $\sigma_{m-1}^2(\hat{\xi}_k)$ this simplifies to

$$
\begin{aligned}
-\xi_{j_0}^2 &\left(\Theta_k''(\xi_{j_0}) - \Theta_{j_0}''(\xi_{j_0})\right) + 2(\mathrm{p}-1)\xi_{j_0}\left(\Theta_k'(\xi_{j_0}) - \Theta_{j_o}'(\xi_{j_0})\right) \\
&- \mathrm{p}(\mathrm{p}-1)\left(\Theta_k(\xi_{j_0}) - \Theta_{j_0}(\xi_{j_0})\right) = 0.
\end{aligned}
$$

Denoting the coefficients of $P_j(z)$ by $a_{j\ell}$, it follows from the known form of Θ_j and the last equation that

$$
(a_{k\ell} - a_{j_0\ell})[-\mathrm{p}(\mathrm{p}-1) + 2(\mathrm{p}-1)\ell - \ell(\ell-1)] = 0, \qquad \ell = 2 \dots m,
$$

Since the factor in the square brackets vanishes only when $\ell = \mathrm{p}$ or $\ell = \mathrm{p}-1$, and $\mathrm{p} \neq 2, \dots, m+1$, it follows that $a_{k\ell} = a_{j_0\ell}$, $\ell = 2, \dots, m$. $\qquad\square$

We notice that, similarly to the example discussed after Proposition A.1, the arguments in first part of the proof of Proposition A.2 show that b_{m-1} and b_m depend only on the affine part of $P(z)$ whereas its other coefficients, as well as the real constants b_{1j} and b_{2j}, are arbitrary. This gives rise to a $(3m-1)$-dimensional family of orthotoric Kähler metrics of constant (σ_m, p)-scalar curvature.

References

[1] V. Apostolov and D. M. J. Calderbank, *The CR geometry of weighted extremal Kähler and Sasaki metrics*, Math. Ann. **379** (2021), 1047–1088.

[2] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, *Hamiltonian 2-forms in Kähler geometry*, I *General theory*, J. Differential Geom. **73** (2006), 359–412.

[3] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, J. Reine Angew. Math. **721** (2016), 109–147.

[4] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and E. Legendre, *Levi-Kähler reduction of CR structures, products of spheres, and toric geometry*, Math. Res. Lett., **27** (2020), 1565–1629.

[5] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry*, III *Extremal Metrics and Stability*, Invent. Math. **173** (2008) 547–601.

[6] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, *Extremal Kähler metrics on projective bundles over a curve.* Adv. Math. **227** (2011), 2385–2424.

[7] V. Apostolov and G. Maschler, *Conformally Kähler Einstein-Maxwell geometry*, J. Eur. Math. Soc (JEMS), **21** (2019), 1319–1360.

[8] V. Apostolov and C. Tønnesen-Friedman, *A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces*, Bull. London Math. Soc. **38** (2006), 494–500.

[9] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, *Springer Monographs in Mathematics*, Springer-Verlag Berlin-Heidelberg 1998.

[10] R. Berman and B. Berndtsson, *Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics*, J. Amer. Math. Soc. **30** (2017), 1165–1196.

[11] L. Bérard-Bergery, *Sur des nouvelles variétés riemanniennes d'Einstein*, Publ. Institut E. Cartan, No 4 (1982), 1–60.

[12] E. Calabi, *Extremal Kähler metrics* II, in: Differential Geometry and Complex Analysis (eds. I. Chavel and H.M. Farkas), Springer, Berlin, 1985.

[13] X. X. Chen and G. Tian, *Geometry of Kähler metrics and foliations by holomorphic discs.* Publ. Math. Inst. Hautes Études Sci. **107** (2008), 1–107.

[14] A. Derdzinski and G. Maschler, *Local classification of conformally Einstein Kähler metrics in higher dimension*, Proc. London Math. Soc. (3) **87** (2003), 779–819.

[15] A. Derdzinski and G. Maschler, *Special Kähler-Ricci potentials on compact Kähler manifolds*, J. Reine Angew. Math. **593** (2006), 73–116.

[16] R. Dervan, *Relative K-stability for Kähler manifolds*, Math. Ann., **372** (2018), 859–889.

[17] R. Dervan and J. Ross, *K-stability for Kähler manifolds*, Math. Res. Lett. **24** (2017), 689–739.

[18] S. K. Donaldson, *Remarks on gauge theory, complex geometry and 4-manifold topology*, Fields Medallists lectures, 384–403, World Sci. Ser. 20th Century Math., 5, World Sci. Publishing, River Edge, NJ, 1997.

[19] S. Donaldson, *Lower bounds of the Calabi functional*, J. Differential Geom. **70** (2006), 453–472.

[20] S. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry.* in Handbook of geometric analysis. No. 1, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

[21] A. Futaki and H. Ono, *Volume minimization and Conformally Kähler Einstein–Maxwell Geometry*, J. Math. Soc. Japan, **70** (2018), 1493–1521.

[22] A. Futaki and H. Ono, *Conformally Einstein–Maxwell Kähler metrics and structure of the automorphism group*, Math. Z. (2019) **292**, 571–589.

[23] A. Fujiki, *Moduli space of polarized algebraic manifolds and Kähler metrics*, [translation of Sugaku **42**, no. 3 (1990), 231–243], Sugaku Expositions **5**, no. 2 (1992), 173–191.

[24] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*, in 'Algebraic Geometry' (Sendai, 1985), 167–178, Adv. Stud. in Pure Math. **10**, North Holland, 1987.

[25] P. Gauduchon, Calabi's extremal Kähler metrics: An elementary introduction, book available upon request.

[26] D. Guan, *Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends*, Trans. Am. Math. Soc. **347** (1995), 2255–2262.

[27] V. Guillemin and S. Sternberg, *Riemann sums over polytopes*, Ann. Inst. Fourier (Grenoble) **57** (2007), 2183–2195.

[28] A. D. Hwang, *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math. **31** (1994), 561–595.

[29] A. D. Hwang and M. A. Singer, *A momentum construction for circle-invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), 2285–2325.

[30] C. Koca and C. Tønnesen-Friedman, *Strongly Hermitian Einstein–Maxwell solutions on ruled surfaces*, Annals of Global Analysis and Geometry, **50** (2016), 29-46.

[31] N. Koiso and Y. Sakane, *Nonhomogeneous Kähler–Einstein metrics on compact complex manifolds*, in Curvature and Topology of Riemannian Manifolds (Kataka, 1985), Lecture Notes in Math. **1201**, Springer, Berlin, 1986, 165–179.

[32] A. Lahdili, *Automorphisms and deformations of conformally Kähler, Einstein–Maxwell metrics*, J. Geom. Anal. **29** (2019), 542–568.

[33] A. Lahdili, *Conformally Kähler, Einstein–Maxwell metrics and boundedness of the modified Mabuchi functional*, Intern. Math. Res. Notices, **2020** (2020), 8418–8442.

[34] A. Lahdili, *Kähler metrics with constant weighted scalar curvature and weighted K-stability*, Proc. London Math. Soc. **119** (2019), 1065–1114.

[35] C. LeBrun, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. **5** (1997), 535–553.

[36] C. LeBrun, *The Einstein–Maxwell equations, extremal Kähler metrics, and Seiberg–Witten theory*, in: The Many Facets of Geometry, Oxford Univ. Press. Oxford 2010, pp. 17–33.

[37] C. LeBrun, *The Einstein–Maxwell equations, Kähler metrics, and Hermitian geometry*, Journal of Geometry and Physics **91** (2015) 163–171.

[38] C. LeBrun, *The Einstein-Maxwell equations and conformally Kähler geometry*, Commun. Math. Phys. (2016) **344**, 621–653.

[39] J. Lee and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987), 37–91.

[40] G. Maschler and C. Tønnesen-Friedman, *Generalizations of Kähler-Ricci Solitons on Projective Bundles*, Math. Scand. **108** (2011), 161–176.

[41] M. Obata, *Certain conditions for a Riemannian manifold to be isometric to the sphere*, J. Math. Soc. Japan **14** (1962), 333–340.

[42] N. Otoba, *Constant scalar curvature metrics on Hirzebruch surfaces*, Annals of Global Analysis and Geometry **46** (2014), 197–223.

[43] D. Page, *A compact rotating gravitational instanton*, Phys. Lett. B **79** (1979), 235–238.

[44] J. Ross and R. Thomas, *An obstruction to the existence of constant scalar curvature Kähler metrics*, J. Differential Geom. **72** (2006), 429–466.

[45] Y. Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*. Osaka J. Math. **23** (1986), 585–616.

[46] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), 478–495.

[47] G. Székelyhidi, *Extremal metrics and K-stability*, Bull. London Math. Soc. **39** (2007), 76–84.

[48] C. Tønnesen-Friedman, *Extremal Kähler metrics on minimal ruled surfaces*, J. Reine Angew. Math. **502** (1998), 175–197.

[49] S. Zelditch, *Bernstein polynomials, Bergman kernels and toric Kähler varieties*, J. Symplectic Geom. **7** (2009), 51–76.

Département de Mathématiques, UQAM
C.P. 8888, Succ. Centre-ville, Montréal (Québec), Canada
and Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. Georgi Bonchev Str. Bl. 8, 1113 Sofia, Bulgaria
E-mail address: `apostolov.vestislav@uqam.ca`

Department of Mathematics, Clark University
Worcester, MA 01610, USA
E-mail address: `gmaschler@clarku.edu`

Department of Mathematics, Union College
Schenectady, New York 12308, USA
E-mail address: `tonnesec@union.edu`