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Eigenvalues upper bounds for the magnetic Schrödinger operator

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We study the eigenvalues $\lambda_k(H_{A,q})$ of the magnetic Schrödinger operator $H_{A,q}$ associated with a magnetic potential A and a scalar potential q, on a compact Riemannian manifold M, with Neumann boundary conditions if $\partial M \neq \emptyset$. We obtain various bounds on $\lambda_1(H_{A,q}), \lambda_2(H_{A,q})$ and, more generally on $\lambda_k(H_{A,q})$. Some of them are sharp. Besides the dimension and the volume of the manifold, the geometric quantities which plays an important role in these estimates are: the first eigenvalue $\lambda''_{1,1}(M)$ of the Hodge-de Rham Laplacian acting on co-exact 1-forms, the mean value of the scalar potential q, the L^2-norm of the magnetic field $B = dA$, and the distance, taken in L^2, between the harmonic component of A and the subspace of all closed 1-forms whose cohomology class is integral (that is, having integral flux around any loop). In particular, this distance is zero when the first cohomology group $H^1(M, \mathbf{R})$ is trivial. Many other important estimates are obtained in terms of the conformal volume, the mean curvature and the genus (in dimension 2). Finally, we also obtain estimates for sum of eigenvalues (in the spirit of Kröger estimates) and for the trace of the heat kernel.

*Our colleague and friend Ahmad El Soufi passed away on December 29, 2016.

1. Introduction

Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M, if non empty. Consider the trivial complex line bundle $M \times \mathbf{C}$ over M; its space of sections can be identified with $C^\infty(M, \mathbf{C})$, the space of smooth complex valued functions on M. Given a smooth real 1-form A on M we define a connection ∇^A on $C^\infty(M, \mathbf{C})$ as follows:

$$\nabla^A_X u = \nabla_X u - iA(X)u \tag{1}$$

for all vector fields X on M and for all $u \in C^\infty(M, \mathbf{C})$ (here ∇ denotes the Levi-Civita connection of (M, g)). The operator

$$\Delta_A = (\nabla^A)^\star \nabla^A \tag{2}$$

is called the *magnetic Laplacian* associated to the magnetic potential A, and the smooth two form

$$B = dA$$

is the associated *magnetic field*. In this paper, we are interested in magnetic Schrödinger operators of the form

$$H_{A,q} = \Delta_A + q$$

where q is a real valued continuous function on M. If $A = 0$, Δ_A is simply the usual Laplacian Δ on M. Note that we have

$$\Delta_A u = \Delta u + 2i\langle A, du\rangle + \left(|A|^2 + i\delta A\right) u \tag{3}$$

where $\delta A = d^\star A$ is the co-differential of A.

If the boundary of M is non empty, we will consider Neumann magnetic conditions, that is:

$$\nabla^A_N u = 0 \quad \text{on} \quad \partial M, \tag{4}$$

where N denotes the inner unit normal. Then, it is well-known that $H_{A,q}$ is self-adjoint, and admits a discrete spectrum

$$\lambda_1(H_{A,q}) \le \lambda_2(H_{A,q}) \le ... \to \infty.$$

Estimates of eigenvalues of such operators have received a great attention in the last decades, especially in the case where the underlying manifold is a bounded Euclidean domain with Dirichlet boundary conditions (see for instance [1, 4, 20, 22, 35, 39]) or with Neumann boundary conditions (see [3, 7–10, 13, 21, 25, 34, 36, 44]). Let us review some known results.

Among the important results in the Dirichlet case, we point out those concerning Euclidean domains with constant magnetic field: a Faber Krahn inequality [20] , a Berezin-Li-Yau inequality [22, 35] and finally the Polya conjecture in this case [26] , where the authors show that this conjecture is not true in presence of magnetic field, even for tiling domains.

Another interesting application of the magnetic Laplacian to mathematics has been done in the study of spectral minimal partitions (see [4] for a survey).

In the case of Euclidean planar domains Ω with Neumann boundary condition and constant magnetic field, there is no Szegö-Weinberger type inequality (that is, there is no upper bound for the first eigenvalue of Δ_A on Ω by the first eigenvalue of Δ_A on the ball B of the same area as Ω). A counter-example is given in [25], Remark 2.4. Even for simply connected domains, the question is open. In [14], the authors obtain a lower bound depending, in particular, on the first non-zero eigenvalue of the usual Laplacian with Neumann boundary condition (Theorem 5.1).

In the case of a general Riemannian manifold, a Cheeger inequality for the magnetic Laplacian was established in [36] (Theorem 7.4) and even a higher order Cheeger inequality (Theorem 7.7). In [13], in the case where the potential is closed, the authors show a converse, that is, a Buser inequality (Theorem 1.2 and 1.3). We remark also that a large attention has been devoted to the asymptotic behaviour of the first eigenvalue of a magnetic Laplacian for large values of the magnetic field (see for instance, Simon [45, 46], Fournais and Helffer ([23]), Raymond ([40]).

In this paper, we first give *upper bounds* for the spectrum of $H_{A,q}$ in terms of the harmonic part of the potential A, the magnetic field B, the integral $\int_M q$ and the geometry of M: see Theorems 3, 4, 5, 6, 7. These estimates are compatible with the Weyl law, and many of them are deduced from the fact that we have the relation (see (16) for a proof)

$$\lambda_k(H_{A,q}) \leq \lambda_k(H_{0,|A|^2+q}) = \lambda_k(\Delta + |A|^2 + q) \tag{5}$$

where $|A|$ denote the pointwise norm of A. We will also focus on the two first eigenvalues $\lambda_1(H_{A,q})$ and $\lambda_2(H_{A,q})$, where we can get more precise results.

In Theorems 3, 6 and 7, we observe that the geometry of the underling manifold (M, g) appears through the first nonzero eigenvalue $\lambda''_{1,1}$ of the

Hodge-de Rham Laplacian Δ_{HR} acting on coexact 1-forms (with absolute condition when $\partial\Omega$ is not empty). This lead us to collect in section 2.5 many known results where we have an explicit control of $\lambda''_{1,1}(M,g)$.

An important case of our study is when A is closed (i.e. the magnetic field is zero) and not exact. In this discussion, we assume for simplicity that q is zero. Then, it could still happen that $\lambda_1(\Delta_A)$ is positive: this occurs precisely when there is at least a closed curve c with the property that the flux of A around c is not an integer (this fact is due to Shigekawa [44], see Proposition 1 below). The positivity of the ground state for closed potentials is related to the so-called *Aharonov-Bohm effect*, an important phenomenon in physics. In this regard, interesting results have been obtained in the paper [34] for non-simply connected plane domains, and sharp lower bounds of $\lambda_1(\Delta_A)$ have recently been published in [8, 9].

The harmonic 1-forms which have integral flux around each closed curve form a lattice, which we denote by $\mathcal{L}_{\mathbf{Z}}$; therefore, $\lambda_1(\Delta_A) = 0$ if and only if $A \in \mathcal{L}_{\mathbf{Z}}$. We quantify the Aharonov-Bohm effect by giving a *sharp* upper bound for the first eigenvalue $\lambda_1(H_{A,q})$ in term of the distance of A to $\mathcal{L}_{\mathbf{Z}}$ (the distance is taken with respect to the L^2-norm of one-forms). We refer to Theorem 4 for further details. In some cases we can also discuss the equality case (see Theorems 4 and 5) .

In Sections 3 and 4 we will obtain many other important estimates. For the second eigenvalue, we mention, among other things, generalizations to the magnetic Laplacian of the known results concerning the Laplacian, like those in terms of conformal volume, the Reilly inequality, the Hersch inequality and, for surfaces, the estimates in terms of the genus. We also obtain estimates for higher eigenvalues, for the sum of eigenvalues (which generalize Kröger inequality) and for the trace of the heat Kernel (which generalizes Kac inequality) In the rest of the introduction we will recall some known facts and discuss the main results.

1.1. Preliminary facts and notation

First, we recall the absolute boundary conditions for a differential p-form A. A form A is said to be *tangential* if $i_N A = 0$ on ∂M, where N denote the exterior normal vector to the boundary; then, ω satisfies the *absolute boundary conditions* if ω and $d\omega$ are both tangential. We denote $\lambda_{1,p}$ the first eigenvalue of the Hodge Laplacian on p-forms (with absolute boundary conditions if ∂M non empty). As the Hodge Laplacian commutes with both

d and δ, each positive eigenspace splits as the direct sum of exact and co-exact eigenforms. We denote by $\lambda''_{1,p}$ (resp. $\lambda'_{1,p}$) the first eigenvalue when the Hodge Laplacian is restricted to co-exact (resp. exact) p-forms. It follows that

$$\lambda_{1,p} \leq \min\{\lambda'_{1,p}, \lambda''_{1,p}\}$$

and as $\lambda''_{1,p} = \lambda'_{1,p+1}$ (which is true by differentiating eigenfunctions) we see

$$\lambda''_{1,p} \geq \max\{\lambda_{1,p}, \lambda_{1,p+1}\}.$$

In particular,

$$\lambda''_{1,1} \geq \max\{\lambda_{1,1}, \lambda_{1,2}\}.$$

We recall now the variational definition of the spectrum. Let M be a compact manifold. If the boundary is non empty, we assume for $u \in C^\infty(M, \mathbf{C})$ the magnetic Neumann conditions, as in (4). Then one verifies that

$$\int_M (H_{A,q}u)\bar{u}v_g = \int_M (|\nabla^A u|^2 + q|u|^2)v_g,$$

and the associated quadratic form is then

$$Q_{A,q}(u) = \int_M (|\nabla^A u|^2 + q|u|^2)v_g.$$

We also introduce the Rayleigh quotient of a smooth function $u \neq 0$, defined by

(6) $$R_{A,q}(u) = \frac{Q_{A,q}(u)}{\|u\|^2}$$

The spectrum of $H_{A,q}$ admits the usual variational characterization:

(7) $$\lambda_1(H_{A,q}) = \min\left\{R_{A,q}(u)\ u \in C^1(M, \mathbf{C})/\{\mathbf{0}\}\right\}$$

and

(8) $$\lambda_k(H_{A,q}) = \min_{E_k} \max\left\{R_{A,q}(u) :\ u \in E_k/\{0\}\right\}$$

where E_k runs through the set of all k-dimensional vector subspaces of $C^1(M, \mathbf{C})$.

The following proposition recalls some well-known facts. If c is a closed curve (a loop), the quantity

$$\Phi_c^A = \frac{1}{2\pi} \oint_c A \tag{9}$$

is called the *flux* of A across c. We will not specify the orientation of the loop, so that the flux will only be defined up to sign. This will not affect any of the statements, definitions or results which we will prove in this paper.

Proposition 1. 1) *The spectrum of $H_{A,q}$ is equal to the spectrum of $H_{A+d\phi,q}$ for all smooth real valued functions ϕ; in particular, when A is exact, the spectrum of $H_{A,q}$ reduces to that of the classical Schrödinger operator with potential q acting on functions (with Neumann boundary conditions if ∂M is not empty).*

2) *Let A be $1-$form on M. Then, there exists a smooth real valued function ϕ on M such that the 1-form $\tilde{A} = A + d\phi$ is co-closed and tangential, that is:*

$$\delta\tilde{A} = 0, \ i_N\tilde{A} = 0. \tag{10}$$

3) *Set:*

$$\mathrm{Har}_1(M) = \Big\{ h \in \Lambda^1(M) : dh = \delta h = 0 \text{ on } M, \ i_N h = 0 \text{ on } \partial M \Big\}.$$

Assume that the 1-form A is co-closed and tangential. Then A can be decomposed

$$A = \delta\psi + h, \tag{11}$$

where ψ is a smooth tangential 2-form and $h \in \mathrm{Har}_1(M)$. Note that the vector space $\mathrm{Har}_1(M)$ is isomorphic to the first de Rham absolute cohomology space $H^1(M, \mathbf{R})$.

4) *We have $\lambda_1(H_{A,0}) \doteq \lambda_1(\Delta_A) = 0$ if and only A is closed (i.e. $B = 0$) and the cohomology class of A is an integer (that is $\Phi_c^A \in \mathbf{Z}$ for any loop c in M).*

Assertion (1) expresses the well-known *Gauge invariance* of the spectrum. Thanks to Assertion (2), in the study of the spectrum of the magnetic Laplacian, we can always assume that the potential A is co-closed and tangential.

Proof. 1) This comes from the fact that

$$\Delta_A e^{-i\phi} = e^{-i\phi}\Delta_{A+d\phi} \tag{12}$$

hence Δ_A and $\Delta_{A+d\phi}$ are unitarily equivalent.

2) Observe that the problem:

$$\begin{cases} \Delta\phi = -\delta A \quad \text{on} \quad M, \\ \dfrac{\partial\phi}{\partial N} = -A(N) \quad \text{on} \quad \partial M \end{cases}$$

has a unique solution (modulo an additive constant). It is immediate to verify that $\tilde{A} = A + d\phi$ is indeed co-closed and tangential.

3) We apply the Hodge decomposition to the 1-form A (see [43], Thm. 2.4.2), and get:

$$A = df + \delta\psi + h, \tag{13}$$

where f is a function which is zero on the boundary, ψ is a tangential 2-form and h is a 1-form satisfying $dh = \delta h = 0$ (in particular, h is harmonic). Now, as $\delta A = 0$ we obtain $\delta df = 0$ hence f is a harmonic function; since f is zero on the boundary, we get $f = 0$ also on M and we can write

$$A = \delta\psi + h. \tag{14}$$

Now, since both A and $\delta\psi$ are tangential, also h will be tangential.

4) This result was proved by Shigekawa [44] for closed manifolds; for Neumann boundary condition see also [34] (for Dirichlet boundary condition one can see [33]).

□

- *In the sequel, when we write the decomposition $A = \delta\psi + h$, it will be implicitly supposed that ψ is a tangential 2-form and h is a 1-form satisfying $dh = \delta h = 0$ and $i_N h = 0$.*

From definition (1) we see

$$|\nabla^A u|^2 = |du|^2 + |A|^2|u|^2 + 2\mathrm{Im}\langle A, \bar{u}du\rangle. \tag{15}$$

Since A is real, it is clear that if $u \in C^\infty(M, \mathbf{R})$ is a real valued function, then $\mathrm{Im}\langle A, \bar{u} du\rangle = 0$ and, then,

$$R_{A,q}(u) = \frac{\int_M \left(|du|^2 + (|A|^2 + q)|u|^2\right) v_g}{\int_M |u|^2 v_g}.$$

Since $C^1(M, \mathbf{R})$ is a subspace of $C^1(M, \mathbf{C})$, it follows that the eigenvalues of $H_{A,q}$ are dominated by those of the scalar Schrödinger operator $H_{0,|A|^2+q} = \Delta + |A|^2 + q$, that is

$$\lambda_k(H_{A,q}) \leq \lambda_k(H_{0,|A|^2+q}) = \lambda_k(\Delta + |A|^2 + q). \tag{16}$$

For the first eigenvalue of $H_{A,q}$, one also has a lower estimate by the first eigenvalue of the scalar Schrödinger operator $H_{0,q} = \Delta + q$; in other words:

$$\lambda_1(H_{A,q}) \geq \lambda_1(H_{0,q}). \tag{17}$$

This property can be seen as an immediate consequence of the so-called diamagnetic inequality (see for instance Theorem 2.1.1 in [24]) :

$$|(\nabla + q)|u|| \leq |(\nabla - iA)u + qu|$$

for a.e. $x \in M$.

The diamagnetic inequality expresses the fact that matter under an applied external magnetic field gains energy. Now, in quantum mechanics, the momentum operator is $i\nabla$ and the corresponding hamitonian is $(i\nabla)^2$; under a magnetic field with magnetic potential A the hamiltonian becomes $(i\nabla + A)^2$ and the diamagnetic inequality simply says that energy, in fact, increases. Mathematically, it is a consequence of Kato-type inequalities.

In the Appendix we provide an alternative proof of the inequality (17), based on Lavine-O'Carroll identity which allows to characterize the equality case. This proof is an immediate extension to the context of Riemannian manifolds of arguments used by Helffer in the case of bounded domains of $\mathbf{R}^n$ under Dirichlet boundary conditions (see Proposition 1.1 in [33]). Its extension to the context of compact Riemannian manifolds does not require new ideas. Here is the result. Our proof is self-contained and elementary, compared to the proof which uses the diamagnetic inequality.

Proposition 2. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) possibly with nonempty boundary. We have,* *under Dirichlet or Neumann boundary conditions if $\partial M \neq \emptyset$,*

$$\lambda_1(H_{A,q}) \geq \lambda_1(H_{0,q}). \tag{18}$$

Moreover, equality holds if and only if the magnetic field $B = dA$ vanishes and the cohomology class of the magnetic potential A is an integral multiple of 2π (that is, if and only if $A \in \mathcal{L}_{\mathbb{Z}}$).

Proof. See Appendix. □

1.2. Statement of results

Before stating the results, let us define a distance associated to the 1-form A which will play an important role in our estimates (see (19) below). Let $\mathcal{L}_{\mathbf{Z}}$ be the lattice in $\mathrm{Har}_1(M) \sim H^1(M, \mathbf{R})$ formed by the integral harmonic 1-forms (those having integral flux around any loop). Given $A \in \mathrm{Har}_1(M)$, we define its distance to the lattice $\mathcal{L}_{\mathbf{Z}}$ by the formula:

$$d(A, \mathcal{L}_{\mathbf{Z}})^2 = \min\left\{\|\omega - A\|^2,\, \omega \in \mathcal{L}_{\mathbf{Z}}\right\}, \tag{19}$$

where $\|\cdot\|$ denotes the L^2-norm of forms in M. Of course, when $H^1(M, \mathbf{R}) = 0$ any harmonic 1-forms is zero and by convention we set $d(A, \mathcal{L}_{\mathbf{Z}}) = 0$.

Theorem 3. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) of dimension n, where $A = \delta\psi + h$ is a potential as in (11). One has, under Neumann boundary conditions if $\partial M \neq \emptyset$:*

1)

$$\lambda_1(H_{A,q}) \leq \Gamma(M, A, q) := \frac{1}{|M|}\left(d(h, \mathcal{L}_{\mathbf{Z}})^2 + \frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M q v_g\right) \tag{20}$$

where $|M|$ denotes the volume of M and $\lambda''_{1,1}(M)$ is the first eigenvalue of the Hodge-de Rham Laplacian Δ_{HR} acting on co-exact 1-forms (with absolute boundary condition if $\partial M \neq \emptyset$).

2) *If the first absolute de Rham cohomology group vanishes : $H^1(M, \mathbf{R}) = 0$, then*

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|}\left(\frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M q v_g\right) \tag{21}$$

with equality if and only if $\Delta_{HR}(\delta\psi) = \lambda''_{1,1}(\delta\psi)$ *and* $|\delta\psi|^2 + q$ *is constant, equal to* $\lambda_1(H_{A,q})$.

The case when the potential A is closed (that is $B = 0$) is of special interest. We have

Theorem 4. *Let* $H_{A,q}$ *be a magnetic Schrödinger operator on a compact Riemannian manifold* (M, g) *of dimension* n*, where the potential* A *is closed, so that we can write* $A = h$ *as in (11). One has under Neumann boundary conditions if* $\partial M \neq \emptyset$*:*

$$\lambda_1(H_{A,q}) \leq \frac{d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_M q v_g}{|M|}. \tag{22}$$

In case of equality in (22), there exists an integer harmonic form $\omega \in \mathcal{L}_{\mathbf{Z}}$ *such that* $|A - \omega|^2 + q$ *is constant. In particular, if the potential* q *is constant,* (M, g) *carries a harmonic* 1*-form of constant length.*

We now discuss sharpness. First, we remark that the spectrum of a flat torus, endowed with a potential harmonic one-form, is computable (see Section 2.3). In that situation, we have equality in (22) for any constant potential function q. In dimension 2 we can also characterize flat tori among genus one surfaces attaining equality in (22) for constant potentials q. Precisely:

Theorem 5.
1) *When* (M, g) *is a flat torus, we have equality in (22) if and only if the potential* q *is constant.*
2) *When* M *is a two-dimensional torus (that is, a genus one surface) and* q *is constant we have equality in (22) if and only if* (M, g) *is a flat torus.*

Besides flat tori we don't know of other significant situations where equality is attained in (22). Explicit examples are hard to find also because the invariant $d(A, \mathcal{L}_{\mathbf{Z}})$ is difficult to compute.

In section 2 we will give applications of Theorem 3 for manifolds for which we have a good control of $\lambda''_{1,1}(M, g)$. First of all, using the Bochner formula, we show such control for closed manifolds with Ricci curvature bounded below by a positive constant; when the boundary is not empty, we have to impose that it is convex. Then, we extend such lower bound also when the inner curvature is not everywhere positive; for example, for convex domains in $\mathbf{R}^n$, and for hypersurfaces of manifolds with curvature operator with arbitrary sign, provided that the extrinsic curvatures are large enough. The general principle is that one still has a positive lower bound for $\lambda''_{1,1}$ if the positivity of the principal curvatures of the boundary compensate, in some sense, for the negativity of the inner curvature.

Let us now discuss the estimates involving the conformal volume, which is defined as follows. If $\phi : (M, g) \to (\mathbf{S}^n, \mathrm{can})$ is a conformal immersion, we first define the *conformal volume of* ϕ:

$$V_c(n, \phi) \doteq \sup\{\mathrm{vol}(\mathrm{M}, (\gamma \circ \phi)^\star \mathrm{can}),\ \phi \text{ is a conformal diffeomorphism of } \mathbf{S}^n\}$$

(this was first introduced by Gromov in [29] and called *visual volume*). Then, we take the infimum of $V_c(n, \phi)$ over all such conformal immersions ϕ to obtain the *n-dimensional conformal volume of* M, denoted $V_c(n, M)$: this invariant is non-increasing in n, hence we can finally define the conformal volume of M, defined by Li and Yau in [37] :

$$V_c(M) \doteq \lim_{n \to \infty} V_c(n, M).$$

Among other things, Li and Yau showed how the conformal volume is related to the Willmore conjecture. Then, El Soufi and Ilias in [16] showed the importance of the conformal volume in the study of minimal spherical submanifolds, and used it to obtain a generalization to higher dimensions of the Hersch inequality concerning the first eigenvalue of the Laplacian on simply connected surfaces. In [17] they also obtained an upper bound of the second eigenvalue of a Schrödinger operator in terms of the conformal volume. This inequality, in fact, extends to the magnetic Laplacian, as follows.

Theorem 6. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) of dimension n, where $A = \delta\psi + h$ is a potential as in (11). One has (under Neumann boundary conditions if $\partial M \neq \emptyset$):*

$$\lambda_2(H_{A,q}) \leq n \frac{V_c(M)}{|M|} + \Gamma(M, A, q) \tag{23}$$

with $\Gamma(M, A, q)$ as in (20).

In section 3, we will give applications of Theorem 6 in specific situations.

We then state an upper bound valid for all the eigenvalues.

Theorem 7. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a closed Riemannian manifold (M, g) of dimension n, where $A = \delta\psi + h$ is a potential as in (11).*

1) *There exists a constant* $c([g])$ *depending only on the conformal class of* g *such that*

$$\lambda_k(H_{A,q}) \le \Gamma(M,A,q) + c([g]) \left(\frac{k}{|M|}\right)^{2/n}. \tag{24}$$

2) *If* (M^n, g) *has a Ricci curvature bound* $\mathrm{Ric}(M,g) \ge -a^2(n-1)$ *and if* $|A|^2 + q \ge 0$ *(in particular, if* $q \ge 0$*), there exist positive constants* c_1, c_2, c_3 *depending only on the dimension* n *of* M *such that*

$$\lambda_k(H_{A,q})) \le c_1\Gamma(M,A,q) + c_2a^2 + c_3 \left(\frac{k}{|M|}\right)^{2/n}, \tag{25}$$

with $\Gamma(M,A,q)$ *as in (20).*

These result will be deduced from inequality (16) and from estimates for the Schrödinger Laplacian derived in [32] and [28]. Note that if $A = 0$, we recover the result of [6] for the usual Laplacian.

In the specific situation of an Euclidean domain, we get other estimates in Theorem 19 using Riesz means, as a corollary of Inequality (16) and of [15].

2. Upper bounds for the first eigenvalue of $H_{A,q}$

2.1. Proof of Theorem 3.

We recall that $A = \delta\psi + h$ denotes the potential, ψ is a smooth tangential 2-form, $h \in \mathrm{Har}_1(\Omega)$, $\lambda''_{1,1}(\Omega)$ denotes the first eigenvalue of the Laplacian acting on co-exact 1-forms, $B = dA$ is the curvature of the potential A, and $\mathcal{L}_{\mathbf{Z}}$ denotes the integral lattice of $H^1(M)$ formed by the integer harmonic 1-forms $\mathrm{Har}_1(M)$.

Let $\omega \in \mathcal{L}_{\mathbf{Z}}$. Fix a base point x_0 and define, for $x \in \Omega$:

$$\phi(x) \doteq \int_{x_0}^{x} \omega, \tag{26}$$

where on the right we mean integration of ω along any path joining x_0 with x. As ω is closed, $\phi(x)$ does not depend on the choice of two homotopic paths and since the flux of A across each c_j is an integer, $\phi(x)$ is multivalued and defined up to $2\pi\mathbf{Z}$. This implies that the function $u(x) = e^{i\phi(x)}$ is well defined. As $d\phi = \omega$ we see that $du = iu\omega$ and therefore

$$\nabla^A u = du - iuh - iu\delta\psi = iu(\omega - h - \delta\psi).$$

Since $|u| = 1$, we obtain:

$$|\nabla^A u|^2 = |\omega - h - \delta\psi|^2.$$

We use $u(x)$ as test-function for the first eigenvalue of Δ_A. Then, for each $\omega \in \mathcal{L}_{\mathbf{Z}}$, we have the relation

$$\lambda_1(H_{A,q}) \leq \frac{\int_M |\nabla^A u|^2 v_g + \int_M |u|^2 q v_g}{\int_M |u|^2 v_g} = \frac{\|\omega - h - \delta\psi\|^2}{|M|} + \frac{\int_M q v_g}{|M|} \tag{27}$$

As $\omega - h$ is harmonic, it is L^2-orthogonal to $\delta\psi$ and we get

$$\lambda_1(H_{A,q}) \leq \frac{\|\omega - h\|^2 + \|\delta\psi\|^2 + \int_M q v_g}{|M|} \tag{28}$$

Now observe that, since $\delta\psi$ is coexact and tangential, one has by the variational characterization of the eigenvalue $\lambda''_{1,1}(M, g)$:

$$\frac{\int_M |d\delta\psi|^2 v_g}{\int_M |\delta\psi|^2 v_g} \geq \lambda''_{1,1}(M, g)$$

As $d\delta\psi = B$, we have

$$\int_M |\delta\psi|^2 v_g \leq \frac{1}{\lambda''_{1,1}(M, g)} \|B\|^2. \tag{29}$$

Taking the infimum on the right-hand side of (28) over all $\omega \in \mathcal{L}_{\mathbf{Z}}$ we obtain, taking into account (29):

$$\lambda_1(H_{A,q}) \leq \frac{d(h, \mathcal{L}_{\mathbf{Z}})^2}{|M|} + \frac{\|B\|^2}{\lambda''_{1,1}(M)|M|} + \frac{1}{|M|} \int_M q v_g$$

as asserted.

When $H^1(M,\mathbf{R}) = 0$, we have immediately the relation

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|}\left(\frac{\|B\|^2}{\lambda''_{1,1}(M,g)} + \int_M q v_g\right).$$

In case of equality, we must have equality in all the step of the proof: in particular, we must have

$$\frac{\int_M |d\delta\psi|^2 v_g}{\int_M |\delta\psi|^2 v_g} = \lambda''_{1,1}(M,g)$$

which means that $\lambda''_{1,1}$ is an eigenvalue for the eigenfunction $\delta\psi$. For $u = 1$, Equation (3) becomes

$$\Delta_A u = |\delta\psi|^2$$

and the equation $H_{A,q}u = \lambda_1(H_{A,q})u$ becomes

$$|\delta\psi|^2 + q = \lambda_1(H_{A,q}).$$

as asserted.

2.2. Proof of Theorem 4

1) Inequality (22) is an immediate consequence of Inequality (20).
2) In order to investigate the equality case, we will derive the inequality using a different approach. Let $\omega \in \mathcal{L}_{\mathbf{Z}}$ and $u = e^{i\phi}$ the associated function on M as defined in (26). Recall that $|\cdot|$ denotes the pointwise norm, thus defining a smooth function on M.

   First we observe that

$$\Delta_A u = |A-\omega|^2 u. \tag{30}$$

   In fact recall that, as $\delta A = 0$:

$$\Delta_A u = \Delta u + |A|^2 u + 2i\langle du, A\rangle.$$

As $du = iu\omega$ one gets:

$$\Delta u = \delta du = \delta(iu\omega) = i(-\langle du,\omega\rangle + u\delta\omega) = -i\langle du,\omega\rangle = |\omega|^2 u$$

and (30) follows after an easy computation. In turn, one has:

$$H_{A,q}u = |A-\omega|^2 u + qu.$$

Using u as a test-function, and recalling that$|u|^2 = 1$, we have

$$\lambda_1(H_{A,q})\int_M |u|^2 v_g \leq \int_M \langle H_{A,q}u, u\rangle v_g = \|\omega - A\|^2 + \int_M q v_g. \tag{31}$$

In particular, if we choose ω so that $d(\omega, A)^2 = d(A, \mathcal{L}_{\mathbf{Z}})^2$ we recover inequality (22). But now, if equality holds, we see that u must be an eigenfunction for $\lambda_1(H_{A,q})$, that is

$$\lambda_1(H_{A,q})u = H_{A,q}u = \Delta_A u + qu = (|A-\omega|^2 + q)u.$$

So, we deduce that $|A-\omega|^2 + q = \lambda_1(H_{A,q})$ as asserted. In particular, if q is constant, $|A-\omega|$ is constant, and (M, g) carries a harmonic 1-form of constant length.

2.3. Spectrum of flat tori

In order to prove Theorem 5, we investigate the spectrum of flat tori. Let M be a flat n-dimensional torus, quotient of $\mathbf{R}^n$ by a lattice Γ. Recall that the dual lattice $\Gamma^\star$ is defined by

$$\Gamma^\star = \{v : \langle v, w\rangle \in \mathbf{Z} \quad \text{for all} \quad w \in \Gamma\}$$

On a flat torus any harmonic 1-form ξ is parallel, and then it has constant pointwise norm $|\xi|$. In particular

$$\|\xi\| = |M||\xi|.$$

The lattice $\mathcal{L}_{\mathbf{Z}}$ is an additive subgroup of the vector space of harmonic (hence parallel) 1-forms. If ω is one such consider the associated dual parallel vector field, $\omega^\sharp$. We remark that this induces an isomorphism of groups:

$$\mathcal{L}_{\mathbf{Z}} \cong 2\pi\Gamma^\star. \tag{32}$$

To prove that, associate to each $X \in \Gamma$ the curve $c_X : [0,1] \to M$ given by $c_X(t) = tX$. Note that c_X is a loop because M is $\Gamma-$invariant. The flux of

ω across c_X is easily seen to be

$$\Phi^{\omega}_{c_X} = \frac{1}{2\pi}\omega(X) = \frac{1}{2\pi}\langle \omega^\sharp, X\rangle.$$

Hence any such flux is an integer if and only if $\langle \omega^\sharp, X\rangle \in 2\pi\mathbf{Z}$. This is true for all $X \in \Gamma$ iff $\omega^\sharp \in 2\pi\Gamma^\star$, which proves (32).

Now if $\omega \in \mathcal{L}_{\mathbf{Z}}$, it is readily seen that the associated function u as in (26) is given by:

$$u(x) = e^{i\langle \omega^\sharp,\, x\rangle}$$

which is well-defined on $(M,g) = \mathbf{R}^n/\Gamma$. Hence, for each $\omega \in \mathcal{L}_{\mathbf{Z}}$, thanks to (30), we have:

$$\Delta_A u = |A-\omega|^2 u$$

and the constant $|A-\omega|^2$ is thus an eigenvalue of Δ_A associated to the eigenfunction u. Because of (32) the set

$$\{u(x) = e^{i\langle \omega^\sharp,\, x\rangle}, \omega \in \mathcal{L}_{\mathbf{Z}}\}$$

gives rise to a complete orthonormal basis of $L^2(M)$, hence we have found all the eigenvalues of Δ_A. In conclusion, we have the following fact.

Proposition 8. *Let Σ be a flat torus, quotient of $\mathbf{R}^n$ by the lattice Γ and let $\Gamma^\star$ denote the lattice dual to Γ. Let A be a harmonic 1-form. Then the spectrum of the magnetic Laplacian with potential A, that is, the operator $\Delta_A = H_{A,0}$, is given by*

$$\{|A-\omega|^2 : \omega \in \mathcal{L}_{\mathbf{Z}} \cong 2\pi\Gamma^\star\}$$

with associated eigenfunctions $\{u(x) = e^{i\langle \omega^\sharp,\, x\rangle}\}$. In particular

$$\lambda_1(\Delta_A) = \inf_{\omega\in\mathcal{L}_{\mathbf{Z}}} |A-\omega|^2.$$

2.4. Proof of Theorem 5

1. Now let M be a flat torus, $A = h$ a harmonic 1-form, and let ω_0 be an element in $\mathcal{L}_{\mathbf{Z}}$ such that

$$d(A,\mathcal{L}_{\mathbf{Z}})^2 = \|A-\omega_0\|^2 = |M||A-\omega_0|^2.$$

Inequality (22) takes the form:

$$\lambda_1(H_{A,q}) \leq |A-\omega_0|^2 + \frac{1}{|M|}\int_M q v_g$$

with equality if and only if the associated test-function $u(x) = e^{i\langle \omega_0^\sharp, x\rangle}$ (of constant modulus one) is an eigenfunction of $H_{A,q}$. As

$$H_{A,q}u = \Delta_A u + qu = (|A-\omega_0|^2 + q)u$$

we see indeed that we have equality in (22) if and only if

$$q = \frac{1}{|M|}\int_M q v_g$$

that is, iff q is constant.

2. Now assume that (M,g) is a genus one surface and q is constant. It remains to show that, if equality holds, M has to be flat.

Since q is constant, there exists a harmonic one form ξ with constant length by the second assertion of Theorem 4. We will apply Bochner formula to ξ. Let Δ_{HR} the Laplacian on 1-forms and ∇ the covariant derivative. Bochner's identity gives, for any 1-form α:

$$\langle \Delta_{HR}\alpha, \alpha\rangle = |\nabla\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + \mathrm{Ric}(\alpha,\alpha). \tag{33}$$

In dimension 2 one has $\mathrm{Ric} = Kg$, where K is the Gaussian curvature. As ξ is harmonic and of constant pointwise norm, we get

$$0 = \int_M |\nabla\xi|^2 v_g + |\xi|^2 \int_M K v_g.$$

As M has genus one we see $\int_M K v_g = 0$; this means that ξ must actually be parallel. But then $\star\xi$ must also be parallel ; by normalization, we have a global orthonormal basis $(\xi, \star\xi)$ of parallel one forms, which forces (M,g) to be flat.

2.5. A few consequences

We can now describe a few consequences of Theorem 3 in some specific situations where we are able to control the eigenvalue $\lambda''_{1,1}$ of the manifold M. Note that there are very few lower bounds of $\lambda''_{1,1}$ for general Riemannian manifolds. For example there is no Cheeger inequality available in this case. In [5, 38], there is a lower bound depending on the sectional curvature, but this bound depends also on the injectivity radius. Most of the known estimates depend on some positivity of the curvature (in particular, the Ricci curvature) or on some convexity of the boundary. The reason is that they are obtained through an application of formulae of Bochner type, as in (33).

2.5.1. Positive Ricci curvature. When the Ricci curvature of M is positive (and ∂M is convex if nonempty), then $H^1(M, \mathbf{R}) = \{0\}$. This implies that the harmonic part h in the decomposition (14) of the potential 1-form A vanishes, so that $A = \delta\psi$ for a tangential two-form ψ. Moreover, the constant $\lambda''_{1,1}(M)$ can be controlled in terms of a lower bound of the Ricci curvature of M. Indeed, we have the

Lemma 9. *Let (M, g) be a compact Riemannian manifold whose Ricci curvature satisfies*

$$Ric \geq c\ g$$

for some positive c. When $\partial M \neq \emptyset$, assume furthermore that ∂M is convex (i.e. its shape operator S is nonnegative). One has

$$\lambda''_{1,1}(M) \geq 2c$$

Moreover, the equality holds if and only if every co-exact eigenform α associated with $\lambda''_{1,1}(M)$ is such that $\alpha^\sharp$ is a Killing vector field which satisfies $Ric(\alpha^\sharp) = c\alpha^\sharp$ and, when $\partial M \neq \emptyset$, $S(\alpha^\sharp) = 0$.

Here $S : T\partial M \to T\partial M$ is the shape operator of ∂M, defined as follows: if N is the inner unit normal vector to the boundary, and $X \in T\partial M$, then $S(X) = -\nabla_X N$.

Proof. We use again the Bochner identity (33):

$$\langle \Delta_{HR}\alpha, \alpha\rangle = |\nabla\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + \mathrm{Ric}(\alpha, \alpha).$$

On the other hand, we have the following general inequality (see [27], Lemma 6.8 p. 270)

$$|\nabla\alpha|^2 \geq \frac{1}{2}|d\alpha|^2 + \frac{1}{n}|\delta\alpha|^2 \geq \frac{1}{2}|d\alpha|^2 \tag{34}$$

in which the equality holds if and only if $\nabla\alpha$ is anti-symmetric, that is $\alpha^\sharp$ is a Killing vectorfield (see [2] Theorem 1.81, p. 40). When α is a co-exact eigenform associated to $\lambda''_{1,1}(M)$, we have

$$\lambda''_{1,1}(M)|\alpha|^2 = \langle \Delta_{HR}\alpha, \alpha\rangle \geq \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\Delta|\alpha|^2 + c|\alpha|^2 \tag{35}$$

When M is closed, one has

$$\int_M \Delta|\alpha|^2 v_g = 0 \quad \text{and} \quad \int_M |d\alpha|^2 v_g = \lambda''_{1,1}(M)\int_M |\alpha|^2 v_g$$

and the result follows from (35) after integration, that is

$$\begin{aligned} \lambda''_{1,1}(M)\int_M |\alpha|^2 v_g &\geq \frac{1}{2}\int_M |d\alpha|^2 v_g + c\int_M |\alpha|^2 v_g \\ &= \left(\frac{\lambda''_{1,1}(M)}{2} + c\right)\int_M |\alpha|^2 v_g \end{aligned} \tag{36}$$

which implies $\lambda''_{1,1}(M) \geq 2c$.

If $\partial M \neq \emptyset$, we observe that, since $i_N\alpha = 0$ and $i_N d\alpha = 0$, the vector field $\alpha^\sharp$ is tangent along the boundary and $0 = d\alpha(N, \alpha^\sharp) = \nabla_N\alpha(\alpha^\sharp) - \nabla_{\alpha^\sharp}\alpha(N) = \frac{1}{2}N\cdot|\alpha|^2 - \langle S(\alpha^\sharp), \alpha^\sharp\rangle$. Thus Green formula gives

$$\int_M \Delta|\alpha|^2 v_g = \int_{\partial M} N\cdot|\alpha|^2 v_g = 2\int_{\partial M}\langle S(\alpha^\sharp), \alpha^\sharp\rangle v_g \geq 0$$

The rest of the proof is the same as above.

Let us discuss the equality case when ∂M is empty. Assume that a co-exact eigenform α satisfies : $\alpha^\sharp$ is a Killing vector field and $Ric(\alpha^\sharp, \alpha^\sharp) = c|\alpha|^2$. Under these conditions, the equality holds in the inequality (35) and, then, in (36), which implies $\lambda''_{1,1}(M) = 2c$.

Conversely, if $\lambda''_{1,1}$(M)$= 2c$, then, for any co-exact eigenform α, the equality holds in (36) and, then, in (35) and (34) which implies that $Ric(\alpha^\sharp, \alpha^\sharp) = c|\alpha|^2$ and that $D\alpha$ is anti-symmetric, that is $\alpha^\sharp$ is a Killing vector field.

When ∂M is not empty, the discussion of the equality case follows the same lines observing that since ∂M is convex, the equality $\langle S(\alpha^\sharp), \alpha^\sharp\rangle = 0$ occurs if and only if $S(\alpha^\sharp) = 0$. □

An immediate consequence of Theorem 3 is the

Corollary 10. *Under the circumstances of Theorem 3 and the assumption that the Ricci curvature of M satisfies $Ric \geq c\ g$ for some positive c, and* *that the boundary ∂M is convex (if nonempty), one has*

$$\lambda_1(H_{A,q}) \leq \frac{1}{|M|}\left(\frac{\|B\|^2}{2c} + \int_M q v_g\right) \tag{37}$$

where $B = dA$ is the magnetic field. The equality holds in (37) if and only if $A^\sharp$ is a Killing vector field with $Ric(A^\sharp) = cA^\sharp$, $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$ and, when $\partial M \neq \emptyset$, $S(A^\sharp) = 0$.

Recall that Bochner vanishing Theorem tells us that a non Ricci-flat manifold M with non-negative Ricci curvature and mean-convex boundary if $\partial M \neq \emptyset$, satisfies $H^1(M, \mathbf{R}) = \{0\}$. On the other hand, Bochner's identity gives for any Killing vector field A, $\Delta_{HR} A = 2\,\mathrm{Ric}(A)$.

The inequality (37) improves by a factor 2 the estimate obtained by Cruzeiro, Malliavin and Taniguchi ([11], Theorem 1.1) for $\lambda_1(H_{A,0})$ for closed manifolds (be careful, the magnetic Laplacian defined in [11] coincides with $\frac{1}{2}\Delta_A$).

An important special case we want to emphasize is the following

Corollary 11. *(i) Let $H_{A,q}$ be a magnetic Schrödinger operator on the standard n-dimensional sphere $\mathbb{S}^n$. One has*

$$\lambda_1(H_{A,q}) \leq \frac{1}{\sigma_n}\left(\frac{\|B\|^2}{2(n-1)} + \int_{\mathbb{S}^n} q v_g\right) \tag{38}$$

where $\sigma_n = (n+1)\omega_{n+1}$ is the volume of $\mathbb{S}^n$ and $B = dA$ is the magnetic field. The equality holds in (38) if and only if $A^\sharp$ is a Killing vector field of $\mathbb{S}^n$ and $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$.

(ii) Let $H_{A,q}$ be a magnetic Schrödinger operator on a spherical cap $C_r(x_0)$ of radius $r \leq \frac{\pi}{2}$ centered at x_0. One has

$$\lambda_1(H_{A,q}) \leq \frac{1}{v_n(r)}\left(\frac{\|B\|^2}{2(n-1)} + \int_{C_r} q v_g\right) \tag{39}$$

where $v_n(r) = \sigma_{n-1}\int_0^r (\sin t)^{n-1} dt$ is the volume of $C_r(x_0)$.

If $r < \frac{\pi}{2}$, then the equality holds if and only if $B = 0$ and q is constant. When $r = \frac{\pi}{2}$ (i.e for a hemisphere), the equality holds in (39) if and only if $A^\sharp$ is a Killing vector field which vanishes at x_0 and $|A^\sharp|^2 + q = \lambda_1(H_{A,q})$.

Indeed, a Killing vector field is tangent along $\partial C_r(x_0)$ if and only if it vanishes at x_0. Since $\partial C_r(x_0)$ is totally umbilical, the condition $S(A^\sharp) = 0$ implies that $A^\sharp = 0$ unless $S = 0$ which only occurs when $r = \frac{\pi}{2}$.

In particular, for a magnetic Laplacian $\Delta_A = H_{A,0}$ on $\mathbb{S}^n$, the inequality (38) reads

$$\lambda_1(H_{A,0}) \le \frac{1}{2(n-1)\sigma_n}\|B\|^2. \tag{40}$$

Notice that when n is even, there is no nonzero vector field of constant length on $\mathbb{S}^n$. Therefore, in the even dimensional case, the equality in (40) holds if and only if $A^\sharp = 0$ (or, equivalently, $B = 0$). If n is odd, then the equality in (40) implies that $A^\sharp$ is proportional to the vector field $J(x) = (-x_2, x_1, \cdots, -x_{n+1}, x_n)$ which is the only Killing vector field of constant length, up to a dilation (see [12]).

In dimension 2, the inequality (40) (i.e. $\lambda_1(H_{A,0}) \le \frac{1}{8\pi}\|B\|^2$) improves the upper bound obtained by Besson, Colbois and Courtois in [3].

2.5.2. Closed hypersurfaces. We now assume that M is a closed, immersed hypersurface of a Riemannian manifold M'. At any point $x \in M$ denote the principal curvatures of M (eigenvalues of the shape operator) by $k_1(x), \dots, k_n(x)$. Let I_p denote the set of p-multi-indices

$$I_p = \{(j_1, \dots, j_p) : 1 \le j_1 \le \cdots \le j_p \le n\},$$

and , for each $\alpha = (j_1, \dots, j_p) \in I_p$, consider the corresponding p-curvature

$$K_\alpha(x) = k_{j_1}(x) + \cdots + k_{j_p}(x).$$

Set $\star\alpha = \{1, \dots, p\} \setminus \{j_1, \dots, j_p\}$, and moreover

$$\begin{cases} \beta_p(x) = \dfrac{1}{p(n-p)} \inf\limits_{\alpha \in I_p} K_\alpha(x) K_{\star\alpha}(x) \\ \beta_p(\Sigma) = \inf\limits_{x \in \Sigma} \beta_p(x) \end{cases}$$

We then have the following lower bound (see Theorem 7 in [42]):

Theorem 12. *Let M^n be a closed immersed hypersurface of the Riemannian manifold M'^{n+1} having curvature operator bounded below by $\gamma_{M'} \in \mathbf{R}$. Then we have the following lower bound*

$$\lambda_{1,p}(M) \ge p(n-p+1)(\gamma_{M'} + \beta_p(M)).$$

Equality holds for geodesic spheres in constant curvature spaces. In particular

$$\lambda''_{1,1}(M) \ge \max\{2(n-1)(\gamma_{M'} + \beta_2(M)), n(\gamma_{M'} + \beta_1(M)\}.$$

We say that M is *p-convex* if all p-curvatures are non-negative; that is, $K_\alpha(x) \geq 0$ for all $\alpha \in I_p$ and for all $x \in M$. Clearly if M is p-convex then it is q-convex for all $q \geq p$. Then, 1-convexity is the usual convexity assumption and n-convex is equivalent to mean convexity.

Note that we could have a positive lower bound even when the curvature of M' is negative; it is enough to assume that the p-curvatures K_α are positive enough. For example, for a 2-convex hypersurface in hyperbolic space $\mathbf{H}^{n+1}$, (where $\gamma_{M'} = 1$) with 2-curvatures $K_\alpha(x)$ uniformly bounded below by $c > 2$, elementary algebra shows that $\beta_2(M) \geq c^2/4$ hence

$$\lambda''_{1,1} \geq 2(n-1)(\frac{c^2}{4} - 1) > 0$$

On the other hand, if M^n is a 2-convex hypersurface of the sphere $\mathbf{S}^{n+1}$ then $\beta_2(M) \geq 0$ and therefore

$$\lambda''_{1,1} \geq 2(n-1).$$

We finally remark the following estimate by P. Guerini ([30]): if M is a convex hypersurface of $\mathbf{R}^n$ then

$$\lambda_{1,p}(M) \geq \frac{p}{2e^3} \cdot \frac{1}{\operatorname{diam}(M)^2}.$$

2.5.3. Convex domains in Euclidean space. Assume now that M is a convex domain in $\mathbf{R}^n$. Then we know from [41] that for all $p = 1, \ldots, n$ one has $\lambda_{1,p} = \lambda'_{1,p}$; in particular :

$$\lambda''_{1,1} = \lambda_{1,2}.$$

Theorem 1.1 in [41] states that, for all $p = 1, \ldots, n$:

$$\frac{a_{n,p}}{D_p^2} \leq \lambda_{1,p} \leq \frac{a'_{n,p}}{D_p^2}$$

for explicit constants $a_{n,p}, a'_{n,p}$. Here D_p is the p-th largest principal axis of the ellipsoid of maximal volume included in M, also called *John ellipsoid* of

M. In particular,

$$\lambda''_{1,1} \geq \frac{4}{n^3 D_2^2}.$$

Accordingly we have an upper bound for the spectrum of the magnetic Laplacian:

$$\lambda_1(H_{A,0}) \leq \frac{1}{|M|}\left(\frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M qv_g\right) \leq \frac{n^3\|B\|^2 D_2^2}{4|M|} + \frac{1}{|M|}\int_M qv_g$$

For example, assume that $q = 0$ and

$$\frac{1}{|M|}\int_M \|B\|^2 \leq c.$$

We then see

$$\lambda_1(H_{A,0}) \leq \frac{cn^3}{4} D_2^2$$

2.5.4. Other estimates. We refer to [31] for a lower bound of $\lambda''_{1,1}$ of any compact manifold with boundary Ω, in terms of a lower bound $\gamma \in \mathbf{R}$ of the eigenvalues of the curvature operator of Ω, and the 2-curvatures of $\partial\Omega$: if the 2 curvatures are large enough, then the lower bound is positive (see Theorem 3.3 in [31]). We also remark that in certain cases it is possible to estimate from below the gap $\lambda_{1,p} - \lambda_{1,0}$ between the first eigenvalue for p-forms (absolute boundary conditions) and the first eigenvalue on functions (Neumann conditions). For example, for convex domains in $\mathbf{S}^n$ one has, for $p = 2, \dots, \frac{n}{2}$:

$$\lambda_{1,p} \geq \lambda_{1,0} + (p-1)(n-p)$$

which reduces to an equality when Ω is the hemisphere. In particular,

$$\lambda''_{1,1} \geq \lambda_{1,0} + n - 2$$

which often improves the bound $\lambda''_{1,1} \geq 2(n-1)$ considered in Corollary 15 above : in fact, $\lambda_{1,0}$ is the first positive eigenvalue for the Neumann Laplacian acting on functions, which can be very large (for example, for small geodesic balls).

3. Upper bounds for the second eigenvalue of $H_{A,q}$

Let us first give the proof of Theorem 6: it is a consequence of the observation (16) and of a previous result of El Soufi and Ilias [17]. By (16), we have

$$\lambda_2(H_{A,q}) \leq \lambda_2(H(0, |A|^2 + q)$$

which corresponds to the usual Laplacian Δ on (M, g) with the potential $|A|^2 + q$, and $A = \delta\psi + h$ as in (11). By [17], for any scalar potential W on M one has

$$\lambda_2(\Delta + W) \leq n \left(\frac{V_c(M)}{|M|}\right)^{\frac{2}{n}} + \frac{1}{|M|}\int_M W v_g,$$

where $V_c(M)$ is the Li-Yau conformal volume of the Riemannian manifold M. In our situation, $W = |\delta\psi + h|^2 + q$ and we have already seen that

$$\frac{1}{|M|}\int_M (|\delta\psi + h|^2 + q)v_g \leq \Gamma(M, A, q)$$
$$:= \frac{1}{|M|}\left(d(h, \mathcal{L}_{\mathbf{Z}})^2 + \frac{\|B\|^2}{\lambda''_{1,1}(M)} + \int_M q v_g\right)$$

which allows to conclude.

As for the first eigenvalue, we have a lot of consequences of this result in specific situations. For example, the conformal volume of the sphere $\mathbb{S}^n$ endowed with the conformal class of its standard metric *can* is equal to the volume $\sigma_n = |\mathbb{S}^n|_{can}$ of the standard metric. Hence, any domain $\Omega \subset \mathbb{S}^n$, endowed with a metric conformal to the standard one will satisfies $V_c(\Omega) \leq \sigma_n$.

Corollary 13. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a bounded domain $\Omega \subset \mathbf{R}^n$, endowed with a Riemannian metric g conformally equivalent to the Euclidean metric. One has, under Neumann boundary conditions,*

$$\lambda_2(H_{A,q}) \leq n \left(\frac{\sigma_n}{|\Omega|_g}\right)^{\frac{2}{n}} + \frac{1}{|\Omega|_g}\left(\frac{\|B\|^2}{\lambda''_{1,1}(\Omega)} + d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_\Omega q v_g\right). \tag{41}$$

This corollary applies of course when Ω is a domain of the Euclidean space, the hyperbolic space, and the sphere. Note that the equality holds in (41) when g is the spherical metric, $A = 0$, $q = 0$ and Ω is a ball whose Euclidean radius tends to infinity.

For a compact orientable surface M of genus γ, one has (see [37])

$$V_c(M) \leq 4\pi \left[\frac{\gamma+3}{2}\right] \tag{42}$$

and

$$\lambda''_{1,1}(M) = \mu(M)$$

where [] stands for the floor function and $\mu(M)$ is the first positive eigenvalue of the Laplacian of M acting on functions, with Neumann boundary conditions if $\partial M \neq \emptyset$. Thus, the inequality (23) leads to the following:

Corollary 14. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a domain Ω of a compact orientable Riemannian surface M of genus γ. One has*

$$\lambda_2(H_{A,q})|\Omega| \leq 8\pi \left[\frac{\gamma+3}{2}\right] + \frac{\|B\|^2}{\mu(\Omega)} + d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_\Omega q v_g. \tag{43}$$

where $\mu(\Omega)$ is the first positive eigenvalue of the Laplacian on functions, with Neumann b. c. if $\Omega \subsetneq M$.

The following corollary extends Hersch's inequality

Corollary 15. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact orientable Riemannian surface M of genus zero. One has*

$$\lambda_2(H_{A,q})|M| \leq 8\pi + \frac{\|B\|^2}{\mu(M)} + \int_M q v_g. \tag{44}$$

In [16], we have proved that if a Riemannian manifold M admits an isometric immersion in a Euclidean space whose components are first eigenfunctions of the Laplacian, then

$$\left(\frac{V_c(M)}{|M|}\right)^{\frac{2}{n}} = \frac{\mu(M)}{n}. \tag{45}$$

In particular, the equality (45) holds for any compact rank one symmetric space. Such a space is Einstein and satisfies $H^1(M, \mathbf{R}) = \{0\}$. Thus, combining with Lemma 9, we get the following

Corollary 16. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a domain Ω (with convex boundary if $\Omega \subsetneq M$) of a compact rank one symmetric space M of (real) dimension n. One has*

$$\lambda_2(H_{A,q}) \le \mu(M)\left(\frac{|M|}{|\Omega|}\right)^{\frac{2}{n}} + \frac{1}{|\Omega|}\left(\frac{\|B\|^2}{2c_M} + \int_\Omega qv_g\right) \tag{46}$$

where c_M is the Ricci curvature constant of M and $B = dA$ is the magnetic field.

When M is a closed immersed submanifold in a Riemannian space form of curvature $\kappa = -1,\ 0,\ +1$ it was established ([18, 19]) the following relationship between the second eigenvalue of a scalar Schrödinger operator $\Delta + W$ and the L^2-norm mean curvature h_M of M :

$$\lambda_2(\Delta + W) \le \frac{1}{|M|}\int_M \left(n|h_M|^2 + n\ \kappa + W\right) v_g$$

This inequality is known as Reilly inequality when $\kappa = 0$ and $W = 0$.

The same arguments as before enable us to obtain the following

Corollary 17. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a closed immersed submanifold M of a space-form of curvature $\kappa = -1,\ 0,\ +1$. One has*

$$\begin{aligned}\lambda_2(H_{A,q})|M| \le &\int_M \left(n|h_M|^2 + n\ \kappa\right) v_g \\ &+ \frac{1}{\lambda''_{1,1}(M)}\|B\|^2 + d(h, \mathcal{L}_{\mathbf{Z}})^2 + \int_M qv_g.\end{aligned} \tag{47}$$

4. Upper bounds for higher order eigenvalues of $\boldsymbol{H_{A,q}}$

In order to prove Theorem 7, we use again the relation (16)

$$\lambda_k(H_{A,q}) \le \lambda_k(H_{0,|A|^2+q}).$$

In order to prove the inequality (24), we use the recent [28] Theorem 1.1: for a scalar Schrödinger operator $\Delta + W$ on a compact Riemannian manifold without boundary. From this result, we deduce that

$$\lambda_k(\Delta + W) \leq \frac{1}{|M|}\int_M W v_g + c([g])\left(\frac{k}{|M|}\right)^{\frac{2}{n}}$$

where $c([g])$ is a constant depending only on the conformal class $[g]$ of g, and the conclusion follows as before because $W = |A|^2 + q$.

In order to prove Inequality (25), one can make use of the estimates obtained by A. Hassannezhad [32] for a scalar Schrödinger operator $\Delta + W$ on a compact Riemannian manifold: If $\lambda_1(\Delta + W) \geq 0$ (which is in particular the case if $W \geq 0$ as in our situation), then

$$\lambda_k(\Delta + W) \leq \frac{c_1}{|M|}\int_M W v_g + c_2\left(\frac{V([g])}{|M|}\right)^{\frac{2}{n}} + c_3\left(\frac{k-1}{|M|}\right)^{\frac{2}{n}}$$

where c_1, c_2 and c_3 are constants which depend only on the dimension n and $V([g])$ is the infimum of the volume of M with respect to all Riemannian metrics g_0 conformal to g and such that $\mathrm{Ric}_{g_0} \geq -(n-1)g_0$.

In particular, if $\mathrm{Ric}_g \geq -(n-1)a^2 g$ for some $a \neq 0$, then the metric $g_0 = a^2 g$ satisfies $\mathrm{Ric}_{g_0} \geq -(n-1)g_0$ and $|M|_{g_0} = a^n|M|_g$. Thus, $V([g]) \leq a^n|M|_g$.

So, we can conclude by observing that

$$\int_M W v_g = \int_M (|A|^2 + q)v_g \leq \Gamma(M, A, q).$$

As a corollary, on a compact orientable surface M of genus $\gamma \geq 2$, every Riemannian metric g is conformal to a hyperbolic metric g_0 which implies $V([g]) \leq |M|_{g_0} = 4\pi(\gamma - 1)$. The same observations as before lead to the following

Corollary 18. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact orientable surface of genus γ, then*

$$\lambda_k(H_{A,q})|M| \leq ak + b\gamma + c\Gamma(M, A, q).$$

where a, b and c are universal constants.

Let us now consider a magnetic Schrödinger operator $H_{A,q}$ on a bounded domain of an Euclidean space (here, as precised before, we consider Neumann condition on the boundary). The following estimates for the sum of eigenvalues (generalizing that of Kröger for $H_{0,0}$), for the Riesz means and for the trace of the magnetic heat kernel (generalizing that of Kac for $H_{0,0}$) are consequences of the considerations above and the estimates obtained in [15]. For convenience and as before, we use the notation $\Gamma(\Omega, A, q) := \frac{1}{|\Omega|}\left(d(h, \mathcal{L}_{\mathbf{Z}})^2 + \frac{\|B\|^2}{\lambda''_{1,1}(\Omega)} + \int_\Omega q v_g\right)$

Theorem 19. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a bounded domain Ω of $\mathbb{R}^n$. One has*
(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 1} (z - \lambda_j(H_{A,q}))_+ \geq \frac{2\,|\Omega|}{n+2}\,\mathcal{W}_n^{-\frac{n}{2}}\,(z - \Gamma(\Omega, A, q))_+^{1+\frac{n}{2}}\,, \tag{48}$$

where $\mathcal{W}_n = 4\pi^2/\omega_n^{\frac{2}{n}}$ is the Weyl constant.
(2) For all $k \geq 1$,

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j(H_{A,q}) \leq \frac{n}{n+2}\mathcal{W}_n\left(\frac{k-1}{|\Omega|}\right)^{\frac{2}{n}} + \Gamma(\Omega, A, q). \tag{49}$$

and, if $\sum_{j=1}^{k}\lambda_j(\Delta + q) \geq 0$,

$$\lambda_k(H_{A,q}) \leq \max\left(2\,(n+2)^{\frac{2}{n}}\,\mathcal{W}_n\left(\frac{k-1}{|\Omega|}\right)^{\frac{2}{n}}\ ,\ 2\Gamma(\Omega, A, q)\right). \tag{50}$$

(3) For all $t > 0$,

$$\sum_{j\geq 1} e^{-t\lambda_j(H_{A,q})} \geq \frac{|\Omega|}{(4\pi t)^{\frac{n}{2}}}e^{-t\Gamma(\Omega,A,q)}. \tag{51}$$

Proof. Taking $A = \delta\psi + h$. As we have seen before,

$$\lambda_k(H_{A,q}) \leq \lambda_k(\Delta + |\delta\psi + h|^2 + q). \tag{52}$$

In [15], the authors obtained estimates for the eigenvalues, their Riesz means, their sum and the heat trace of a general elliptic operator. For a scalar Schrödinger operator $\Delta + W$ on a bounded Euclidean domain $\Omega \subset \mathbb{R}^n$, these estimates take the following form :

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 1}(z-\lambda_j(\Delta+W))_+ \geq \frac{2\,|\Omega|}{n+2}\,\mathcal{W}_n^{-\frac{n}{2}}\left(z-\frac{1}{|\Omega|}\int_\Omega W dx\right)_+^{1+\frac{n}{2}}, \tag{53}$$

(2) For all $k \geq 1$,

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j(\Delta+W) \leq \frac{n}{n+2}\mathcal{W}_n\left(\frac{k-1}{|\Omega|}\right)^{\frac{2}{n}}+\frac{1}{|\Omega|}\int_\Omega W dx. \tag{54}$$

and if $\sum_{j=1}^{k}\lambda_j(\Delta+W) \geq 0$, then

$$\lambda_k(\Delta+W) \leq \max\left(2\,(n+2)^{\frac{2}{n}}\,\mathcal{W}_n\left(\frac{k-1}{|\Omega|}\right)^{\frac{2}{n}}\ ,\ \frac{2}{|\Omega|}\int_\Omega W dx\right). \tag{55}$$

(3) For all $t > 0$,

$$\sum_{j\geq 1}e^{-t\lambda_j(\Delta+W)} \geq \frac{|\Omega|}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{t}{|\Omega|}\int_\Omega W dx}. \tag{56}$$

To conclude the proof, we simply apply these inequalities to the Schrödinger operator $\Delta + W$ with $W = |\delta\psi + h|^2 + q$ and observe that, using the same arguments as before,

$$\frac{1}{|\Omega|}\int_\Omega W dx \leq \Gamma(\Omega, A, q).$$

□

Estimates such as (53) ... (56) are also available in [15] for a bounded domain Ω of a Riemannian manifold M. However, in this case the constants which involve the geometry of Ω are less explicit than in the Euclidean case. Therefore, we can deduce that there exist constants $c_1(\Omega), \cdots, c_4(\Omega)$, depending only on Ω such that, for all $z \in \mathbb{R}$, $k \geq 1$ and $t > 0$

$$\sum_{j\geq 1}(z-\lambda_j(H_{A,q}))_+ \geq c_1(\Omega)\,(z-\Lambda(A,q))_+^{1+\frac{n}{2}}, \tag{57}$$

$$\frac{1}{k}\sum_{j=1}^{k}\lambda_j(H_{A,q}) \leq c_2(\Omega)\left(\frac{k-1}{|\Omega|}\right)^{\frac{2}{n}}+\Gamma(\Omega,A,q), \tag{58}$$

anf if $\sum_{j=1}^{k} \lambda_j(\Delta + q) \geq 0$, then

$$\lambda_k(H_{A,q}) \leq \max \left(c_3(\Omega) \left(\frac{k-1}{|\Omega|} \right)^{\frac{2}{n}} , \ 2\Gamma(\Omega, A, q) \right), \tag{59}$$

$$\sum_{j\geq 1} e^{-t\lambda_j(H_{A,q})} \geq \frac{c_4(\Omega)}{t^{\frac{n}{2}}} e^{-t\Gamma(\Omega,A,q)}. \tag{60}$$

5. Appendix

Proposition 20. *Let $H_{A,q}$ be a magnetic Schrödinger operator on a compact Riemannian manifold (M, g) possibly with nonempty boundary. We have, under Dirichlet or Neumann boundary conditions if $\partial M \neq \emptyset$,*

$$\lambda_1(H_{A,q}) \geq \lambda_1(H_{0,q}). \tag{61}$$

Moreover, equality holds if and only if the magnetic field $B = dA$ vanishes and the cohomology class of the magnetic potential A is an integral multiple of 2π (that is, if and only if $A \in \mathcal{L}_{\mathbb{Z}}$).

Proof. If q is a constant potential, then the inequality is immediate since $\lambda_1(H_{A,q}) = \lambda_1(\Delta_A) + q \geq q$ and $\lambda_1(H_{0,q}) = \lambda_1(\Delta) + q = q$. Assume therefore that q is not constant and let u_0 be a positive first eigenfunction of $H_{0,q} = \Delta + q$. Consider the vector field $X = \nabla(\ln u_0) = \frac{\nabla u_0}{u_0}$. The proof of (61) relies on the following Lavine-O'Carroll type identity : For every $u \in C^\infty(M, \mathbb{C})$ (resp. $u \in C_0^\infty(M, \mathbb{C})$ for Dirichlet b.c.),

$$\int_M (|\nabla^A u|^2 + q|u|^2)v_g - \lambda_1(H_{0,q}) \int_M |u|^2 v_g = \int_M |\nabla^A u - uX|^2 v_g \geq 0. \tag{62}$$

Indeed, this identity implies that $R_{A,q}(u) \geq \lambda_1(H_{0,q})$ for any $u \in C^\infty(M, \mathbb{C})$ and, then, $\lambda_1(H_{A,q}) \geq \lambda_1(H_{0,q})$. Now, to prove (62), we first observe that

$$|\nabla^A u - uX|^2 = |\nabla^A u|^2 + |u|^2|X|^2 - 2\mathrm{Re}\langle \nabla^A u, uX\rangle.$$

Since

$$\langle \nabla^A u, uX\rangle = \langle \bar{u}\nabla u, X\rangle - i\ |u|^2 A(X),$$

we have

$$\mathrm{Re}\langle \nabla^A u, uX\rangle = \mathrm{Re}\langle \bar{u}\nabla u, X\rangle = \frac{1}{2}\langle \nabla|u|^2, X\rangle$$

and

$$\int_M |\nabla^A u - uX|^2 v_g = \int_M \left(|\nabla^A u|^2 + |u|^2|X|^2 - \langle \nabla |u|^2, X\rangle\right) v_g$$
$$= \int_M \left(|\nabla^A u|^2 + \left(|X|^2 - \delta X\right) |u|^2\right) v_g$$

where, if $\partial M \neq \emptyset$, the integration by parts is justified by the fact that u vanishes on the boundary for Dirichlet b. c., and $X = \frac{\nabla u_0}{u_0}$ is tangent to the boundary when Neumann boundary conditions are assumed. Now

$$\delta X = \Delta(\ln u_0) = \frac{\Delta u_0}{u_0} + \frac{|\nabla u_0|^2}{u_0^2} = |X|^2 + \lambda_1(H_{0,q}) - q.$$

Substituting in the above equation we get (62).

Now, if $B = 0$ and the cohomology class of A is an integer multiple of 2π and, using the gauge invariance property (12), one has $\mathrm{Spec}(H_{A,q}) = \mathrm{Spec}(H_{0,q})$ which clearly implies the equality in (61). Conversely, assume that $\lambda_1(H_{A,q}) = \lambda_1(H_{0,q})$ and let v_0 be a first eigenfunction of $H_{A,q}$. Applying (62) to v_0, we get $\nabla^A v_0 - v_0 X = 0$ which means that

$$\nabla v_0 = i\ v_0 A^\flat + v_0 X.$$

Setting $w = v_0/u_0$, one gets (with $\nabla u_0 = u_0 X$)

$$\nabla w = \frac{1}{u_0^2}\left(u_0 \nabla v_0 - v_0 \nabla u_0\right) = \cdots = i\ wA^\flat$$

and (as $\bar{A} = A$)

$$\nabla |w|^2 = w\nabla\bar{w} + \bar{w}\nabla w = -i\ |w|^2 A + i\ |w|^2 A = 0.$$

Therefore, $|w|^2$ is constant. In conclusion, one has $A = i\ \frac{dw}{w}$ which is clearly closed (i.e. $B = dA = 0$) and satisfies $\int_\Gamma A \in 2\pi\mathbb{Z}$ for any closed curve Γ. □

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