

On positive scalar curvature cobordisms and the conformal Laplacian on end-periodic manifolds

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We show that the periodic η -invariant of Mrowka, Ruberman and Saveliev provides an obstruction to the existence of cobordisms with positive scalar curvature metrics between manifolds of dimensions 4 and 6. Our proof combines the end-periodic index theorem with a relative version of the Schoen–Yau minimal surface technique. As a result, we show that the bordism groups $\Omega_{n+1}^{\text{spin},+}(S^1 \times BG)$ are infinite for any non-trivial group G which is the fundamental group of a spin spherical space form of dimension $n = 3$ or 5 .

1. Introduction

A classic problem in global differential geometry is to determine when a given manifold admits a metric of positive scalar curvature ('psc') and if so, to say something about the classification of such metrics. Fundamental tools for this study make use of index theory [15] and minimal submanifolds [23], with many extensions of these methods over the years; see for instance the surveys [21, 25]. Because the space $\mathcal{P}(X)$ of psc metrics on a manifold X is (if non-empty) infinite dimensional, classification *per se* is not reasonable, and one considers such metrics up to deformation, or isotopy. This leads to the study of the homotopy groups of $\mathcal{P}(X)$ and of the associated moduli space $\mathfrak{M}^+(X) = \mathcal{P}(X)/\text{Diff}(X)$, and index theoretic methods show that these homotopy groups can be non-trivial [13].

Botvinnik and Gilkey [4, 5] used the Atiyah–Patodi–Singer index theorem [2] to show that, for any odd-dimensional non-simply connected spherical space form Y of dimension at least 5, the moduli space $\mathfrak{M}^+(Y)$ has

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infinitely many path components. They further strengthened this result by introducing a psc spin bordism group $\Omega_n^{\text{spin},+}(BG)$, whose definition will be recalled in Section 3. They showed that the group $\Omega_n^{\text{spin},+}(BG)$ is infinite for every odd $n \geq 5$ and G the fundamental group of a non-simply connected spherical space form of dimension n . They also extended these results to some non-orientable even-dimensional manifolds.

The second and third authors, together with T. Mrowka, extended [19] the Atiyah–Patodi–Singer theorem to the setting of even-dimensional manifolds with periodic ends. As a consequence, the *isotopy* results of [4, 5] continue to hold for many even dimensional orientable manifolds, for instance, the product of a spherical space form with a circle. (The results in dimension 4 are somewhat more limited, because [18] psc metrics are unique up to isotopy in dimension 3). It is natural to ask if these results on classification up to isotopy actually hold up to bordism.

In this paper, we combine the end-periodic index theory [19] with the relative version of the Schoen–Yau minimal surface technique due to Botvinnik–Kazaras [3] to show that the Botvinnik–Gilkey *bordism* results continue to hold for orientable manifolds in low even dimensions. The statement of our result will include a group-theoretic constant $r_n(G)$ defined in [5] via the representation theory of a finite group G .

Theorem 1. *Let $n = 4$ or 6 . Let G be a finite group and set $\Gamma = \mathbb{Z} \times G$. When $n = 4$, additionally assume that G is the fundamental group of a non-simply connected 3-dimensional spherical space form, while if $n = 6$ we assume that $r_5(G) > 0$. Then $\Omega_n^{\text{spin},+}(B\Gamma)$ contains infinitely many elements, represented by maps $f : M \rightarrow B\Gamma$ where M are connected manifolds with $\pi_1(M) \cong \Gamma$ that support psc metrics.*

Following the remark after [5, Theorem 0.1], $r_5(G) > 0$ if G contains an element g which is not conjugate to g^{-1} . This holds for example in any odd order group.

It is worth noting that the distinct bordism classes in $\Omega_6^{\text{spin},+}(B\Gamma)$ are obtained by constructing different psc metrics on the same underlying smooth manifold. In dimension 4, the conclusion is a bit weaker: given an integer N , there is a 4-manifold that supports at least N bordism classes of psc metrics. The restriction to relatively low dimensions in Theorem 1 has to do with possible singularities in an area-minimizing hypersurface in dimensions greater than 7; it is conceivable that this restriction could be removed using techniques recently developed in [16, 24]. Theorem 1 recovers an earlier result of Leichtnam–Piazza [14, Theorem 0.1] in the case of $n = 6$, but is new

for $n = 4$. The approach in [14] utilizes the higher eta-invariants, see [17], and appears quite different from ours.

Theorem 1 is a consequence of the following result about the periodic $\tilde{\xi}$ -invariants which were introduced in [19].

Theorem 2. *Let X_0 and X_1 be closed oriented Riemannian spin manifolds of dimension $n = 4$ or 6 with positive scalar curvature and a choice of primitive cohomology classes $\gamma_0 \in H^1(X_0; \mathbb{Z})$ and $\gamma_1 \in H^1(X_1; \mathbb{Z})$ and unitary representations $\alpha_0 : \pi_1(X_0) \rightarrow U(k)$ and $\alpha_1 : \pi_1(X_1) \rightarrow U(k)$. Suppose that X_0 is bordant to X_1 via a positive scalar curvature cobordism and that both the cohomology classes γ_0, γ_1 and the representations α_0 and α_1 extend to this cobordism. Then*

$$\tilde{\xi}_{\alpha_0}(X_0, \mathcal{D}^+) = \tilde{\xi}_{\alpha_1}(X_1, \mathcal{D}^+).$$

We prove Theorem 2 by applying the index theorem of [19] to an end-periodic manifold Z_∞ constructed from the psc-cobordism between X_0 and X_1 . The manifold Z_∞ has two periodic ends modeled on the infinite cyclic covers of X_0 and X_1 with metrics conformally equivalent to end-periodic metrics. The ‘middle portion’ of Z_∞ is, roughly speaking, a minimal hypersurface with free boundary as constructed in Botvinnik–Kazaras [3]; this is the crucial geometric ingredient of the proof. Though this middle portion comes equipped with a psc metric, it does not smoothly glue to the given metrics on the covers of X_0 and X_1 . To produce a smooth metric on Z_∞ , we introduce a transition region which may initially have negative scalar curvature. The main analytic result of the paper, Lemma 3.2, conformally changes this initial metric on Z_∞ to one of positive scalar curvature, without dramatically disturbing it far away from the transition region.

The paper is organized as follows. In Section 2, we recall the definition of the invariants $\tilde{\xi}_\alpha$ in all even dimensions, together with the formula of [19] expressing $\tilde{\xi}_\alpha$ in terms of the Dirac index on end-periodic manifolds. We further extend this formula to a class of end-periodic manifolds with metrics which are conformally equivalent to end-periodic metrics but are not end-periodic themselves. Theorem 2 is proved in Section 3 assuming Lemma 3.2. The proof of Lemma 3.2 then occupies the entire Section 4. Finally, Theorem 1 is proved in Section 5.

After this paper was accepted for publication, the preprint [20] appeared that gives a different approach to the main theorems of this paper.

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2. Periodic $\tilde{\xi}$ -invariants

Let X be a compact even-dimensional spin manifold with a choice of a primitive cohomology class $\gamma \in H^1(X; \mathbb{Z})$ and a Riemannian metric g of positive scalar curvature. Associated with this data is the spin Dirac operator $\mathcal{D}^+ = \mathcal{D}^+(X, g)$ and, given a representation $\alpha : \pi_1(X) \rightarrow U(k)$, the twisted Dirac operator $\mathcal{D}_\alpha^+ = \mathcal{D}_\alpha^+(X, g)$. The periodic $\tilde{\xi}$ -invariant was defined in [19, Section 8.1] by the formula

$$\tilde{\xi}_\alpha(X, \mathcal{D}^+) = \frac{1}{2} (\eta(X, \mathcal{D}_\alpha^+) - k \cdot \eta(X, \mathcal{D}^+))$$

using the periodic η -invariants of [19]. We briefly recall the definition. Choose a smooth function $f : X \rightarrow S^1$ such that $[df] = \gamma$. Since the metric g has positive scalar curvature, the twisted Dirac operators $\mathcal{D}_z^\pm = \mathcal{D}^\pm - \ln z \cdot df$ have zero kernels on the unit circle $|z| = 1$, and we define

$$\eta(X, \mathcal{D}^+) = \frac{1}{\pi i} \int_0^\infty \oint_{|z|=1} \text{Tr} \left(df \cdot \mathcal{D}_z^+ e^{-t \mathcal{D}_z^- \mathcal{D}_z^+} \right) \frac{dz}{z} dt.$$

The definition of $\eta(X, \mathcal{D}_\alpha^+)$ is similar. Of most importance to us is the fact that the periodic $\tilde{\xi}$ -invariant can be expressed in index theoretic terms, which is done as follows.

Let us consider an end-periodic manifold with the end modeled on the infinite cyclic cover $\tilde{X} \rightarrow X$ corresponding to γ . To be precise, let $Y \subset X$ be a hypersurface Poincaré dual to γ , and let W be the cobordism from Y to itself obtained by cutting X open along Y . We will assume that Y is spin bordant to zero, since this is the only case relevant to this paper, and choose a spin null-cobordism Z . The end-periodic manifold in question is then of the form

$$(1) \quad Z_\infty = Z \cup W \cup W \cup \dots$$

where we write $\partial W = Y^- \cup Y^+$ and identify each Y^+ with Y^- in the subsequent copy of W , and also identify the boundary of Z with Y^- in the

first copy of W . Let $f : Z_\infty \rightarrow \mathbb{R}$ be any smooth function with the property that $f(\tau(x)) = f(x) + 1$ over the end of Z_∞ , where τ stands for the covering translation.

Let g_∞ be any metric on Z_∞ which matches over the periodic end the lift of the metric g from X to its infinite cyclic cover. Denote by g_Z the induced metric on Z . The positive scalar curvature condition then ensures that the Dirac operators $\mathcal{D}^+(Z_\infty, g_\infty)$ and $\mathcal{D}_\alpha^+(Z_\infty, g_\infty)$ are uniformly invertible at infinity in the sense of Gromov and Lawson [12] and, in particular, their L^2 closures are Fredholm. Of course, the operator $\mathcal{D}_\alpha^+(Z_\infty, g_\infty)$ is only well defined if the pull-back of α to $\pi_1(Y)$ extends to a representation of $\pi_1(Z)$, which we will assume from now on.

Proposition 2.1. *Let Z_∞ be an end-periodic manifold whose end is modeled on the infinite cyclic cover \tilde{X} . Then*

$$(2) \quad \tilde{\xi}_\alpha(X, \mathcal{D}^+) = k \cdot \text{ind } \mathcal{D}^+(Z_\infty, g_\infty) - \text{ind } \mathcal{D}_\alpha^+(Z_\infty, g_\infty).$$

Proof. Apply the index theorem [19, Theorem A] to the Dirac operators $\mathcal{D}^+(Z_\infty, g_\infty)$ and $\mathcal{D}_\alpha^+(Z_\infty, g_\infty)$ to obtain

$$\begin{aligned} \text{ind } \mathcal{D}^+(Z_\infty, g_\infty) &= \int_Z \mathbf{I}(\mathcal{D}^+(Z, g_Z)) - \int_Y \omega + \int_X df \wedge \omega - \frac{1}{2} \eta(X, \mathcal{D}^+), \\ \text{ind } \mathcal{D}_\alpha^+(Z_\infty, g_\infty) &= \int_Z \mathbf{I}(\mathcal{D}_\alpha^+(Z, g_Z)) - \int_Y \omega_\alpha + \int_X df \wedge \omega_\alpha - \frac{1}{2} \eta(X, \mathcal{D}_\alpha^+), \end{aligned}$$

where $\mathbf{I}(\mathcal{D}^+(Z, g_Z)) = \hat{A}(Z, g_Z)$ and $\mathbf{I}(\mathcal{D}_\alpha^+(Z, g_Z)) = \hat{A}(Z, g_Z) \text{ch}(V_\alpha)$ are the local index forms, and ω and ω_α are transgressed classes such that $d\omega = \mathbf{I}(\mathcal{D}^+(X, g))$ and $d\omega_\alpha = \mathbf{I}(\mathcal{D}_\alpha^+(X, g))$. The local index forms are related by $\mathbf{I}(\mathcal{D}_\alpha^+(Z, g_Z)) = k \cdot \mathbf{I}(\mathcal{D}^+(Z, g_Z))$ hence the transgressed classes can be chosen so that $\omega_\alpha = k \cdot \omega$. Now, subtracting k copies of the first formula from the second gives the desired result. \square

We now wish to prove that a formula similar to (2) holds as well for certain metrics on Z_∞ which are conformally equivalent to the end-periodic metric g_∞ but which do not need to be end-periodic themselves. The metrics in question will be of the form $g' = \sigma^2 g_\infty$, where $\sigma : Z_\infty \rightarrow \mathbb{R}$ is a positive smooth function such that

- (a) the scalar curvature of g' is uniformly positive on Z_∞ , and
- (b) both σ and σ^{-1} are bounded functions on Z_∞ .

An example of such a function is the function $\sigma = u^{2/(n-2)}$, where u is constructed in Lemma 3.2.

Proposition 2.2. *Let Z_∞ be an end-periodic manifold whose end is modeled on the infinite cyclic cover of X . Then, for any metric $g' = \sigma^2 g_\infty$ as above,*

$$(3) \quad \tilde{\xi}_\alpha(X, \mathcal{D}^+) = k \cdot \text{ind } \mathcal{D}^+(Z_\infty, g') - \text{ind } \mathcal{D}_\alpha^+(Z_\infty, g').$$

Proof. First note that the metric g' has uniformly positive scalar curvature, therefore, the operators $\mathcal{D}^+(Z_\infty, g')$ and $\mathcal{D}_\alpha^+(Z_\infty, g')$ are uniformly invertible at infinity so their L^2 closures are Fredholm. The Dirac operators on Z_∞ corresponding to the metrics g_∞ and $g' = \sigma^2 g_\infty$ are related by the formulas

$$\mathcal{D}^+(Z_\infty, g') = \sigma^{-(n+1)/2} \circ \mathcal{D}^+(Z_\infty, g_\infty) \circ \sigma^{(n-1)/2},$$

$$\mathcal{D}_\alpha^+(Z_\infty, g') = \sigma^{-(n+1)/2} \circ \mathcal{D}_\alpha^+(Z_\infty, g_\infty) \circ \sigma^{(n-1)/2}.$$

It follows that, for any L^2 spinor φ in the kernel of $\mathcal{D}^+(Z_\infty, g_\infty)$, the spinor $\varphi' = \sigma^{-(n-1)/2}\varphi$ is in the kernel of $\mathcal{D}^+(Z_\infty, g')$ and, moreover,

$$\|\varphi'\|_{L^2(Z_\infty, g')}^2 = \int_{Z_\infty} |\varphi'|^2 d \text{vol}_{g'} = \int_{Z_\infty} \sigma |\varphi|^2 d \text{vol}_{g_\infty} \leq C \|\varphi\|_{L^2(Z_\infty, g_\infty)}^2$$

for some constant $C > 0$. Since this construction is reversible, it establishes an isomorphism between the kernels of $\mathcal{D}^+(Z_\infty, g_\infty)$ and $\mathcal{D}^+(Z_\infty, g')$. A similar argument applied to the cokernel of $\mathcal{D}^+(Z_\infty, g_\infty)$, and then to the kernel and cokernel of $\mathcal{D}_\alpha^+(Z_\infty, g_\infty)$, establishes the equalities $\text{ind } \mathcal{D}^+(Z_\infty, g') = \text{ind } \mathcal{D}^+(Z_\infty, g_\infty)$ and $\text{ind } \mathcal{D}_\alpha^+(Z_\infty, g') = \text{ind } \mathcal{D}_\alpha^+(Z_\infty, g_\infty)$. The statement now follows from Proposition 2.1. \square

3. Cobordisms of positive scalar curvature

We begin by recalling the definition of the bordism group $\Omega_n^{\text{spin},+}(BG)$; see [4, 5]. Given a discrete group G , consider the triples (X, g, f) consisting of a closed oriented spin manifold X of dimension n , a positive scalar curvature metric g on X , and a continuous map $f : X \rightarrow BG$. Two triples (X_0, g_0, f_0) and (X_1, g_1, f_1) represent the same class in $\Omega_n^{\text{spin},+}(BG)$ if there is a spin cobordism Z between X_0 and X_1 which admits a positive scalar curvature metric g_Z such that $g_Z = g_i + dt^2$ in collar neighborhoods of $X_i, i = 0, 1$, and a continuous map $Z \rightarrow BG$ extending the maps $X_0 \rightarrow BG$ and $X_1 \rightarrow BG$.

To get the statement about the fundamental group in Theorem 1, we need the following lemma, well-known to experts.

Lemma 3.1. *Let $n \geq 4$ and let G be a finite group. Then any class in $\Omega_n^{\text{spin},+}(BG)$ is represented by a triple (X, g, f) where $f_* : \pi_1(X) \rightarrow \pi_1(BG) = G$ is an isomorphism.*

Proof. First note that we may assume that f_* is surjective. To see this, choose a finite generating set $\{\gamma_1, \dots, \gamma_k\}$ for Γ . By [10], the spin cobordism W obtained by adding k 1-handles to $X \times I$ has a psc metric extending the one on X . Evidently, the map $f : X \rightarrow BG$ extends over W , inducing a map on $\pi_1(W) \cong \pi_1(X) * F\langle g_1, \dots, g_k \rangle \rightarrow G$ that sends the generators g_j to γ_j . Let (X_1, g_1, f_1) denote the new boundary component of W ; then $\pi_1(X_1) \rightarrow \pi_1(W)$ is an isomorphism and so $(f_1)_*$ is a surjection.

Let K denote the kernel of $(f_1)_*$. Since G is finite, K is of finite index in $\pi_1(X_1)$ and hence is finitely generated. Hence one can do a further surgery on circles representing these elements, preserving the spin structure, and changing the fundamental group to G . Since $n \geq 4$, the circles along which the surgeries are done have codimension at least 3. Using [10] once more, the cobordism gotten by adding 2-handles to $X_1 \times I$ is a spin cobordism with a psc metric and with a map to $B\Gamma$ inducing an isomorphism on the fundamental group. The new boundary component of this cobordism is the desired representative of the original class in $\Omega_n^{\text{spin},+}(B\Gamma)$. \square

From now on, we will assume that $3 \leq n \leq 6$. Let (X_0, g_0) and (X_1, g_1) be n -dimensional closed Riemannian spin manifolds of positive scalar curvature, and fix a choice of primitive cohomology classes $\gamma_i \in H^1(X_i; \mathbb{Z})$, $i = 0, 1$. According to [22], there exist smoothly embedded volume minimizing hypersurfaces $Y_i \subset X_i$ which are Poincaré dual to γ_i and which admit positive scalar curvature metrics g_{Y_i} in the same conformal class as the metrics $g_i|_{Y_i}$ obtained by restriction.

Cutting X_i open along Y_i , where $i = 0, 1$, we obtain cobordisms W_i with boundary $\partial W_i = Y_i^- \cup Y_i^+$ as in Section 2. Note for future use that the infinite cyclic covers $\tilde{X}_i \rightarrow X_i$ can be written as the infinite unions

$$\tilde{X}_i = \bigcup_{j \in \mathbb{Z}} W_i$$

with appropriate identifications of the boundary components, and denote the respective ‘half-infinite’ unions by

$$\tilde{X}_i^- = \bigcup_{j \leq 0} W_i \quad \text{and} \quad \tilde{X}_i^+ = \bigcup_{j \geq 0} W_i.$$

Let us now assume that (X_0, g_0, γ_0) and (X_1, g_1, γ_1) represent the same class in the psc-bordism group $\Omega_n^{\text{spin},+}(S^1)$. Then, according to [3, Theorem 5], there exists a spin cobordism Z from Y_0 to Y_1 with a positive scalar curvature metric g_Z which is a product metric $g_Z = dt^2 + g_{Y_i}$ near Y_i , $i = 0, 1$. The metric g_Z does not extend in any obvious way to a positive scalar curvature metric on the end-periodic manifold

$$(4) \quad Z_\infty = \tilde{X}_0^- \cup Z \cup \tilde{X}_1^+.$$

To construct such a metric, we proceed as follows. Note that Z_∞ has two ends, one of which, \tilde{X}_1^+ , is attached to the component Y_1 of the boundary of Z , and the other, \tilde{X}_0^- , to the component Y_0 with reversed orientation. Correspondingly, we will write Y_0^- and Y_1^- for the boundaries of \tilde{X}_0^- and \tilde{X}_1^+ . We will use a similar convention for the boundary components of W_i so the ‘-’ component will be the one closer to the compact piece. Using [1], replace the metric g_i on W_i , $i = 0, 1$, with a new metric g'_i satisfying the following conditions:

- 1) $g'_i = g_i$ away from a neighborhood of Y_i^- ,
- 2) $g'_i = dt^2 + g_{Y_i}$ near Y_i^- , and
- 3) the conformal class $(W_i, [g'_i])$ is Yamabe positive; cf. Section 4.1.

Note that the scalar curvature of g'_i may be negative. Using the product structure of g'_i , equip Z_∞ with the smooth metric

$$g_\infty = \begin{cases} g_0 & \text{on} & \bigcup_{j \leq -1} W_0 \\ g'_0 & \text{on} & \bigcup_{j=0} W_0 \\ g_Z & \text{on} & Z \\ g'_1 & \text{on} & \bigcup_{j=0} W_1 \\ g_1 & \text{on} & \bigcup_{j \geq 1} W_1. \end{cases}$$

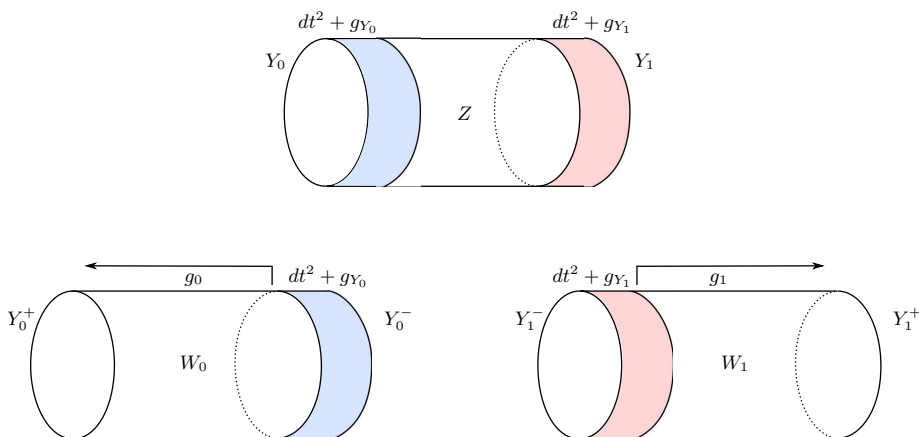


Figure 1: Initial metric g_∞ .

The gluing near Z can be visualized in Figure 1. In what follows, we will use the notations

$$X = X_0 \sqcup X_1, \quad \tilde{X} = \tilde{X}_0 \sqcup \tilde{X}_1, \quad W = W_0 \sqcup W_1, \quad Y = Y_0 \sqcup Y_1$$

and, for the two metrics on W ,

$$g_W = g_0 \sqcup g_1 \quad \text{and} \quad g'_W = g'_0 \sqcup g'_1.$$

We will also fix a smooth function $f : Z_\infty \rightarrow \mathbb{R}$ such that $f(\tau(x)) = x + 1$, where τ is the covering translation on either end, f is constant on each copy of Y , $f^{-1}([-1, 0]) = Z$, and $f^{-1}([j, j + 1])$ is the j th copy of W . The following is our main technical result, which will be proved in the next section. It roughly states that, in spite of the possible negative scalar curvature that g'_i introduces, the non-compact manifold (Z_∞, g_∞) admits a well-controlled conformal change to a uniformly positive scalar curvature metric.

Lemma 3.2. *There exists a positive function $u : Z_\infty \rightarrow \mathbb{R}$ such that*

- 1) $|u - 1| \leq Ce^{-Bf}$ for some constants $B > 0$ and $C > 0$, and
- 2) the scalar curvature of the metric $u^{4/(n-2)}g_\infty$ is bounded from below by a positive constant.

With Lemma 3.2 in place, we can complete the proof of Theorem 2. Assume that (X_0, g_0, γ_0) and (X_1, g_1, γ_1) represent the same class in the group $\Omega_n^{\text{spin},+}(S^1)$, and that the representations $\alpha_0 : \pi_1(X_0) \rightarrow U(k)$ and

$\alpha_1 : \pi_1(X_1) \rightarrow U(k)$ extend to a representation of the fundamental group of the psc cobordism. Then the end-periodic manifold (4) admits a uniformly positive scalar curvature metric g' to which Proposition 2.2 applies. This results in the formula

$$\tilde{\xi}_\alpha(X_1, \mathcal{D}^+) - \tilde{\xi}_\alpha(X_0, \mathcal{D}^+) = k \cdot \text{ind } \mathcal{D}^+(Z_\infty, g') - \text{ind } \mathcal{D}_\alpha^+(Z_\infty, g').$$

Since the metric g' has positive scalar curvature, both indices on the right hand side of this formula must vanish, leading us to the conclusion that

$$\tilde{\xi}_\alpha(X_0, \mathcal{D}^+) = \tilde{\xi}_\alpha(X_1, \mathcal{D}^+).$$

4. Proof of the main lemma

Our proof of Lemma 3.2 will be a modification of the argument from [6, Proposition 4.6] dealing with a cylindrical Yamabe problem. We begin by recalling some basic facts about the conformal Laplacian on compact manifolds with boundary which will be essential in our proof.

4.1. Conformal Laplacian on compact manifolds with boundary

Let (M, g) be a compact oriented n -dimensional Riemannian manifold with non-empty boundary ∂M . Denote by ν the outward normal vector to ∂M , by R_g the scalar curvature of M , and by H_g the mean curvature of ∂M . Consider the following pair of operators acting on $C^\infty(M)$:

$$(5) \quad \begin{cases} L_g = -\Delta_g + c_n R_g & \text{in } M \\ B_g = \partial_\nu + 2c_n H_g & \text{on } \partial M, \end{cases}$$

where $c_n = (n - 2)/(4(n - 1))$. These operators describe the change in the scalar and boundary mean curvatures under a conformal change of metric: a standard computation shows that, for a positive function $\varphi \in C^\infty(M)$, the scalar and boundary mean curvatures of the metric $\tilde{g} = \varphi^{4/(n-2)}g$ are given by the formulas

$$(6) \quad \begin{cases} R_{\tilde{g}} &= c_n^{-1} \varphi^{-(n+2)/(n-2)} L_g \varphi & \text{in } M \\ H_{\tilde{g}} &= \frac{1}{2} c_n^{-1} \varphi^{-n/(n-2)} B_g \varphi & \text{on } \partial M. \end{cases}$$

Associated with the operators (5) and every real-valued function $0 \neq \varphi \in C^\infty(M)$ is the Rayleigh quotient

$$\frac{Q_g(\varphi)}{\int_M \varphi^2 d\mu}$$

where

$$(7) \quad Q_g(\varphi) = \int_M (|\nabla\varphi|^2 + c_n R_g \varphi^2) d\mu + 2c_n \int_{\partial M} H_g \varphi^2 d\sigma$$

and $d\mu$ and $d\sigma$ denote the volume forms associated to g and $g|_{\partial M}$, respectively. According to the standard elliptic theory,

$$(8) \quad \lambda = \inf_{0 \neq \varphi \in H^1(M)} \frac{Q_g(\varphi)}{\int_M \varphi^2 d\mu}$$

is the principal eigenvalue of the boundary value problem (L_g, B_g) and there is a positive function $\varphi \in C^\infty(M)$, unique up to scalar multiplication, such that

$$(9) \quad \begin{cases} L_g \varphi = \lambda \varphi & \text{in } M \\ B_g \varphi = 0 & \text{on } \partial M. \end{cases}$$

This eigenvalue problem was first studied by Escobar [8] in the context of the Yamabe problem on manifolds with boundary.

Let φ be a positive solution of (9) and consider the metric $\tilde{g} = \varphi^{4/(n-1)}g$. It follows from (6) that the boundary mean curvature of the metric \tilde{g} vanishes, and that the scalar curvature of \tilde{g} has a constant sign agreeing with the sign of λ . In particular, the sign of λ is an invariant of the conformal class $[g]$ of metric g . A conformal manifold $(M, [g])$ is called *Yamabe positive, negative, or null* if λ is respectively positive, negative, or zero; see Escobar [7, 8].

4.2. Preliminary eigenvalue estimates

For each positive integer k , denote by Z_k the compact submanifold of Z_∞ obtained by attaching only the first k copies of W to Z . Let g_k be the metric g_∞ restricted to Z_k , and $\lambda(k)$ the principal eigenvalue of the boundary value problem (9) for the operators (L_{g_k}, B_{g_k}) . Since the hypersurfaces Y_0 and Y_1

were chosen to be minimal, each Z_k has vanishing boundary mean curvature, and the Rayleigh quotient (8) for $\lambda(k)$ takes the form

$$(10) \quad \lambda(k) = \inf_{0 \neq \varphi \in H^1(Z_k)} \frac{\int_{Z_k} (|\nabla \varphi|^2 + c_n R_{g_k} \varphi^2) d\mu}{\int_{Z_k} \varphi^2 d\mu}$$

Proposition 4.1. *There are positive constants C_1 and C_2 depending only on the metrics g_Z , g_W , and g'_W such that, for all positive k ,*

$$C_1 \leq \lambda(k) \leq C_2.$$

Proof. The conformal manifolds $(Z, [g_Z])$, $(W, [g_W])$, and $(W, [g'_W])$ are all Yamabe positive by construction. As mentioned in Section 4.1, this is equivalent to the positivity of the principal eigenvalue of the boundary value problem (5). Since the boundaries of (Z, g_Z) , (W, g_W) , and (W, g'_W) are all minimal, their mean curvatures vanish and the boundary conditions in (5) reduce to the Neumann boundary conditions, $\partial_\nu \varphi = 0$. This implies that the principal Neumann eigenvalues $\lambda(L_{g_Z})$, $\lambda(L_{g_W})$, and $\lambda(L_{g'_W})$ are all positive.

Let us consider the principal Neumann eigenvalue $\lambda(1) = \lambda(L_{g_{W_1}})$ of the operator $L_{g_{W_1}}$. Since W_1 is split into (Z, g_Z) and (W, g'_W) ,

$$\lambda(L_{g_{W_1}}) = \inf \{ Q_{g_{W_1}}(\varphi) \mid 0 \neq \varphi \in H^1(W_1), \|\varphi\|_{L^2}^2 = 1 \}$$

can be estimated from below by

$$\begin{aligned} & \inf \left\{ Q_{g_Z}(\varphi_1) + Q_{g'_W}(\varphi_2) \mid \varphi_1 \in H^1(Z), \varphi_2 \in H^1(W), \right. \\ & \qquad \qquad \qquad \left. \|\varphi_1\|_{L^2}^2 + \|\varphi_2\|_{L^2}^2 = 1 \right\} \\ &= \inf_{a \in [0,1]} \inf \left\{ a \cdot \frac{Q_{g_Z}(\varphi_1)}{\|\varphi_1\|_{L^2}^2} + (1-a) \cdot \frac{Q_{g'_W}(\varphi_2)}{\|\varphi_2\|_{L^2}^2} \mid \varphi_1 \in H^1(Z), \right. \\ & \qquad \qquad \qquad \left. \varphi_2 \in H^1(W), \|\varphi_1\|_{L^2}^2 = a, \|\varphi_2\|_{L^2}^2 = 1-a \right\}, \end{aligned}$$

which is in turn estimated from below by

$$\inf_{a \in [0,1]} \left\{ a \cdot \lambda(L_{g_Z}) + (1-a) \cdot \lambda(L_{g'_W}) \right\} = \min \left\{ \lambda(L_{g_Z}), \lambda(L_{g'_W}) \right\} > 0.$$

By splitting Z_k into (Z_{k-1}, g_{k-1}) and (W, g_W) and proceeding inductively, one can use the above argument to show that

$$\lambda(k) \geq \min \left\{ \lambda(L_{g_Z}), \lambda(L_{g_W}), \lambda(L_{g'_W}) \right\} > 0$$

for all positive k . This gives the desired lower bound on the eigenvalues $\lambda(k)$. To obtain the upper bound, choose the constant test function $\varphi = 1$ in the Rayleigh quotient to obtain

$$\lambda(k) \leq \frac{1}{\text{vol}(Z_k)} \cdot \int_{Z_k} c_n R_{g_k} d\mu \leq c_n \cdot \sup_{Z_k} (R_{g_k}).$$

Since the scalar curvature of g_∞ is uniformly bounded from above, this gives the desired upper bound on $\lambda(k)$. □

4.3. Strategy of the proof

The function $u : Z_\infty \rightarrow \mathbb{R}$ whose existence is claimed in Lemma 3.2 will be obtained as a solution of the equation $L_{g_\infty}(u) = h$, where $h : Z_\infty \rightarrow \mathbb{R}$ is a smooth positive function such that

- 1) $h = c_n R_{g_\infty}$ on $Z_\infty \setminus Z_3$ and
- 2) $h \geq |c_n R_{g_\infty}|$ on Z_3 .

To solve this equation, we will first solve the equation $L_{g_\infty}(v) = \tilde{h}$ for the function $\tilde{h} = h - c_n R_{g_\infty}$, which is positive and compactly supported in Z_3 , and then let $u = 1 + v$. The equation $L_{g_\infty}(v) = \tilde{h}$ will be solved by the barrier method.

4.4. The barrier method

Let $C_1 > 0$ be the constant from Proposition 4.1 and introduce the constant

$$\bar{\lambda} = \frac{1}{2} \min \left\{ C_1, \inf_{Z_\infty \setminus Z_3} c_n R_{g_\infty} \right\}.$$

Note that $\bar{\lambda}$ is positive, and also that the function $c_n R_{g_\infty} - \bar{\lambda}$ is positive on $Z_\infty \setminus Z_3$. The two propositions that follow provide two ingredients for the barrier construction.

Proposition 4.2. *For any point $x_0 \in Z$ there exists a smooth positive function $w : Z_\infty \rightarrow \mathbb{R}$ such that*

$$\begin{cases} (L_{g_\infty} - \bar{\lambda})(w) = 0 & \text{in } Z_\infty \\ w(x_0) = 1. \end{cases}$$

Proof. It follows from Proposition 4.1 that $0 < \bar{\lambda} < \lambda(k)$ for all positive k , hence the operators $L_{g_k} - \bar{\lambda}$ with the Neumann boundary conditions are all invertible modulo constant functions. Since the principal eigenvalue of an elliptic operator with the Dirichlet boundary condition is greater than or equal to the principal eigenvalue of the same operator with the Neumann boundary condition, it follows that $L_{g_k} - \bar{\lambda}$ with the Dirichlet boundary condition is also invertible. The rest of the proof follows the argument of [26, Theorem 2.1]. For each positive k , consider a smooth solution w'_k of the problem

$$\begin{cases} (L_{g_\infty} - \bar{\lambda})(w'_k) = 0 & \text{in } Z_k \\ w'_k = 0 & \text{on } \partial Z_k \end{cases}$$

which is positive in the interior of Z_k . Define $w_k = w'_k/w'_k(x_0)$. Then a standard argument using the Harnack inequality and Schauder estimates shows that there is subsequence of w_k which converges to a positive function $w \in C^\infty(Z_\infty)$ on compact subsets of Z_∞ in the $C^{k,\alpha}$ topology. Proposition 4.2 follows. \square

Proposition 4.3. *There are positive constants B and C_3 such that the function $\varphi = e^{-Bf} : Z_\infty \rightarrow \mathbb{R}$ satisfies the inequality*

$$L_{g_\infty}(\varphi) \geq C_3 \varphi \quad \text{on } Z_\infty \setminus Z_3.$$

Proof. Let $B > 0$ be an arbitrary constant, to be determined later, and $\varphi = e^{-Bf}$. Then

$$\begin{aligned} \Delta_{g_\infty} \varphi &= B^2 |\nabla f|^2 e^{-Bf} - (\Delta_{g_\infty} f) B e^{-Bf} \\ &\leq B^2 \max(|\nabla f|^2) e^{-Bf} + \max(|\Delta_{g_\infty} f|) B e^{-Bf} \\ &= B(B \max(|\nabla f|^2) + \max(|\Delta_{g_\infty} f|)) \varphi. \end{aligned}$$

This allows us to make the estimate

$$\begin{aligned} L_{g_\infty} \varphi &= -\Delta_{g_\infty} \varphi + c_n R_{g_\infty} \varphi \\ (11) \quad &\geq (-B(B \max(|\nabla f|^2) + \max(|\Delta_{g_\infty} f|)) + c_n \min(R_{g_\infty})) \varphi. \end{aligned}$$

From the construction of f , we know that $\max(|\nabla f|^2)$ and $\max(\Delta_{g_\infty} f)$ are both finite. Moreover, the minimum of R_{g_∞} on $Z_\infty \setminus Z_3$ is positive. Combining these facts with inequality (11), we conclude that $B > 0$ can be chosen small enough so that

$$L_{g_\infty} \varphi \geq C_3 \varphi \quad \text{on } Z_\infty \setminus Z_3,$$

where $C_3 = \frac{1}{2} c_n \min(R_{g_\infty})$. □

The barrier for the equation $L_{g_\infty}(v) = \tilde{h}$ is now constructed by piecing together the functions w and φ .

Proposition 4.4. *There are continuous functions $\bar{\varphi}, \underline{\varphi} : Z_\infty \rightarrow \mathbb{R}$ which satisfy the inequalities*

$$L_{g_\infty}(\underline{\varphi}) \leq \tilde{h} \quad \text{and} \quad L_{g_\infty}(\bar{\varphi}) \geq \tilde{h}$$

weakly. Moreover, there is a positive constant C_4 such that $|\underline{\varphi}|, |\bar{\varphi}| \leq C_4 \varphi$ on $Z_\infty \setminus Z_3$.

Proof. We start by choosing constants $\alpha, \beta > 0$ such that $\alpha w < \beta \varphi$ on ∂Z_3 and both $L_{g_\infty}(\alpha w) \geq \tilde{h}$ on Z_∞ and $L_{g_\infty}(\beta \varphi) \geq \tilde{h}$ on $Z_\infty \setminus Z_3$.

Case 1: There exists a positive integer $k_0 \geq 4$ such that $\beta \varphi \leq \alpha w$ on $Z_{k_0+1} \setminus Z_{k_0}$. In this case, we define

$$\bar{\varphi}(x) := \begin{cases} \alpha w(x) & \text{if } x \in Z_3 \\ \min(\alpha w(x), \beta \varphi(x)) & \text{if } x \in Z_{k_0} \setminus Z_3 \\ \beta \varphi(x) & \text{if } x \in Z_\infty \setminus Z_{k_0} \end{cases}$$

and let $\underline{\varphi} = -\bar{\varphi}$. The functions $\bar{\varphi}$ and $\underline{\varphi}$ are continuous super- and sub-solutions of the equation $L_{g_\infty}(\cdot) = \tilde{h}$ on Z_∞ . This follows from the fact that the minimum (resp. maximum) of two super-solutions (resp. sub-solutions) is again a super-solution (resp. sub-solution); see for instance [11, Section 2.8].

Case 2: The assumption of Case 1 does not hold. In other words, for every integer $k \geq 4$, there is a point $x_k \in Z_{k+1} \setminus Z_k$ such that $\alpha w(x_k) < \beta \varphi(x_k)$ and, in particular, $\alpha w(x_k) < \beta e^{-Bk}$.

Since the the distance between x_k and the boundary components ∂Z_k and ∂Z_{k+1} is less than the diameter of W for all k , the Harnack inequality

for w provides us with a constant $D > 0$ such that

$$\alpha w(x) \leq D\alpha w(x_k) < D\beta e^{-Bk}$$

for all $x \in Z_{k+1} \setminus Z_k$. It now follows that there is a constant $C_4 > 0$ such that $\alpha w \leq C_4\varphi$ on $Z_\infty \setminus Z_4$. In this case, we simply define $\bar{\varphi} = \alpha w$ and $\underline{\varphi} = -\bar{\varphi}$. □

Having constructed the barrier, we can put it to use proving the following existence result.

Proposition 4.5. *There exists a smooth function $v : Z_\infty \rightarrow \mathbb{R}$ such that*

- 1) $L_{g_\infty}(v) = \tilde{h}$ on Z_∞ , and
- 2) $\underline{\varphi} \leq v \leq \bar{\varphi}$ on Z_∞ .

Proof. For each $k \geq 3$, we find a solution $v_k : Z_k \rightarrow \mathbb{R}$ to the problem

$$\begin{cases} L_{g_\infty}(v_k) = \tilde{h} & \text{in } W_k \\ v_k = \underline{\varphi} & \text{on } \partial W_k. \end{cases}$$

Since $\underline{\varphi}$ is a subsolution, one can apply the maximum principle to $v_k - \underline{\varphi}|_{Z_k}$ to conclude that $v_k \geq \underline{\varphi}|_{Z_k}$. Likewise, one concludes that $v_k \leq \bar{\varphi}|_{Z_k}$. A standard argument now shows that v_k converges in the $C^{2,\alpha}$ topology to a smooth function v on Z_∞ which satisfies $L_{g_\infty}(v) = \tilde{h}$. □

4.5. Finishing the proof

We are now ready to finish the proof of Lemma 3.2. The following proposition ensures that the function $u = 1 + v$, where v is the function of Proposition 4.5, is positive and hence can serve as a conformal factor.

Proposition 4.6. *The function $u = 1 + v$ solves the equation $L_{g_\infty}(u) = h$. Moreover, $u > 0$ on Z_∞ .*

Proof. The following argument is reproduced from the proof of [6, Proposition 4.6]. From the exponential decay of the function v in Proposition 4.5 it is clear that $u > 0$ on $Z_\infty \setminus Z_k$ for some sufficiently large $k > 3$. Let $\varphi_0 > 0$ be a positive eigenfunction corresponding to the principal eigenvalue $\lambda^D(k)$

of the operator L_{g_k} on Z_k with the Dirichlet boundary condition, that is,

$$\begin{cases} L_{g_k}(\varphi_0) = \lambda^D(k) \varphi_0 & \text{in } Z_k \\ \varphi_0 = 0 & \text{on } \partial Z_k. \end{cases}$$

Then one may consider the function u/φ_0 and calculate as in [6, Proposition 4.6] that

$$\Delta_{g_k} \left(\frac{u}{\varphi_0} \right) + \frac{2}{\varphi_0} \left\langle \nabla \varphi_0, \nabla \left(\frac{u}{\varphi_0} \right) \right\rangle - \lambda^D(k) \left(\frac{u}{\varphi_0} \right) = -\frac{h}{\varphi_0} < 0.$$

It follows that at an interior minimum in Z_k , the function u/φ_0 is positive; the minimum of u/φ_0 must lie in the interior because $\varphi_0 = 0$ on ∂Z_k . \square

It follows from formula (6) that the metric $u^{4/(n-2)}g_\infty$ has positive scalar curvature. Since $u - 1 = v$ decays exponentially by Proposition 4.5, the proof of Lemma 3.2 is complete.

5. Proof of Theorem 1

We will prove Theorem 1 by constructing connected psc manifolds M with fundamental group Γ and representations α so that the invariants $\tilde{\xi}_\alpha$ take on infinitely many distinct values. Note that the connectedness of M is an essential part of Theorem 1; disconnected examples could be constructed much more easily.

In dimension six, it follows from [5, Theorem 0.1] that, for each finite group G with $r_5(G) > 0$, there are closed connected spin manifolds Y^5 which admit infinitely many psc metrics g_i distinguished up to bordism by invariants $\tilde{\xi}_\alpha(Y, g_i)$, where α is a unitary representation of G . By Lemma 3.1 we may assume that $\pi_1(Y) \cong G$. The pull back via projection $\mathbb{Z} \times G \rightarrow G$ gives rise to a unitary representation of $\mathbb{Z} \times G$ called again α . Since the periodic η -invariants of [19] satisfy $\eta(S^1 \times Y, \mathcal{D}_\alpha^+) = \eta_\alpha(Y)$ when $S^1 \times Y$ is given a product metric, Theorem 2 implies that the periodic invariants $\tilde{\xi}_\alpha(S^1 \times Y, dt^2 + g_i)$ are all distinct up to bordism. Note that this argument gives the *a priori* stronger conclusion that all of these non-bordant metrics live on the same manifold.

This argument does not work in dimension 4, because 3-dimensional space forms support a unique psc metric, up to isotopy [18]. One issue is that the standard technique for pushing a psc metric across a cobordism, which is crucial to the constructions in [4, 5], does not work for pushing a metric across a 4-dimensional cobordism. In general, one is not able to push

a psc metric across a 5-dimensional cobordism either. However, Section 9.3 in [19] shows how to create psc-cobordisms between 4-manifolds, at the expense of taking connected sums with some unknown number of copies of $S^2 \times S^2$.

For a non-simply connected space form S^3/G , the proof of [19, Theorem 9.5], sketched below, shows that for an appropriate α and any $N \geq 1$, there is a non-negative number m_N such that

$$\tilde{\xi}_\alpha((S^1 \times (S^3/G)) \# m_N \cdot (S^2 \times S^2))$$

takes on N different values. By letting N go to infinity, we see that $\Omega_4^{\text{spin},+}(S^1 \times BG)$ must be infinite. We note that, in contrast to the 6-dimensional result, it is not clear if infinitely many non-bordant metrics could be supported on the same manifold. \square

Remark 5.1. The hypothesis in the 6-dimensional case that Y admit a spin structure can presumably be omitted. This would involve extending the results of [19] to include ‘twisted’ spin structures as discussed in the introduction of [5].

Remark 5.2. Because the metrics in the 6-dimensional case are product metrics, we do not really need the end-periodic index theorem to prove Theorem 1 in this case; the Atiyah–Patodi–Singer theorem [2] (as extended in Proposition 2.2) would suffice. On the other hand, it does not seem possible to prove the 4-dimensional case of Theorem 1 without the use of [19].

For the sake of completeness, we provide here a sketch of the construction of the metrics on the 4-manifolds described in the proof above. Details can be found in [19, Section 9]

Write $Y = S^3/G$, and let g be a psc metric descending from the round metric on S^3 . Lemma 9.7 of [19] provides a representation $\alpha : \pi_1(Y) \rightarrow U(k)$ for which the invariant $\tilde{\xi}_\alpha(Y, g)$ is non-zero. For any n , the finiteness of the spin cobordism group $\Omega_3^{\text{spin}}(B\pi_1(Y))$ and some additional topological arguments are used construct a spin cobordism V_n with $\pi_1(V_n) \cong \pi_1(Y)$. There is an extension $\tilde{\alpha}$ of the representation α so that

$$\partial(V_n, \tilde{\alpha}) = (Y, \alpha) - (nd + 1) \cdot (Y, \alpha)$$

Next, cross V_n with a circle to obtain a cobordism W_n from $S^1 \times ((nd + 1) \cdot Y)$ (the lower part of the boundary) to $S^1 \times Y$. It can be assumed that

this cobordism has a handlebody decomposition with handles of index only 2 and 3. Write k_n for the number of 3-handles.

The argument of [4, 5] would at this point be to push the psc metric from the bottom end of W_n to $S^1 \times Y$, but the presence of the 3-handles prohibits this, since they are attached along spheres of codimension 2. However, the psc metric can be pushed across

$$S^1 \times ((nd + 1) \cdot Y) \times I$$

plus the 2-handles. The upper boundary of this manifold is shown to be $(S^1 \times Y) \# k_n \cdot (S^2 \times S^2)$, which therefore acquires a psc metric. By construction, all of these manifolds have fundamental group $\mathbb{Z} \times G$, and their $\tilde{\xi}_\alpha$ invariants grow linearly with n and hence by Theorem 2, their bordism classes are distinct. By adding additional copies of $S^2 \times S^2$ to these manifolds, one can obtain diffeomorphic manifolds carrying an arbitrary number of these distinct bordism classes. \square

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