Macroscopic stability and simplicial norms of hypersurfaces

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We introduce a $\mathbb{Z}$–coefficient version of Guth's macroscopic stability inequality for almost-minimizing hypersurfaces. In manifolds with a lower bound on macroscopic scalar curvature, we use the inequality to prove a lower bound on areas of hypersurfaces in terms of the Gromov simplicial norm of their homology classes. We give examples to show that a very positive lower bound on macroscopic scalar curvature does not necessarily imply an upper bound on the areas of minimizing hypersurfaces.

1. Introduction

The main theorem of this paper is the following.

Theorem 1. *Let (M, g) be a closed Riemannian manifold of dimension n, with the property that in the universal cover of M every ball of radius 1 has volume less than or equal to some number V_0. We also assume that $\pi_1(M)$ is residually finite. Let Σ be a 2–sided embedded closed hypersurface in M, and let $i\colon \Sigma \hookrightarrow M$ denote the inclusion. Then we have the inequality*

$$\operatorname{Area} \Sigma \geq \operatorname{const}(n, V_0) \cdot \|i_*[\Sigma]\|_\Delta,$$

where the constant is positive and $\|i_*[\Sigma]\|_\Delta$ denotes the Gromov simplicial norm of the homology class $i_*[\Sigma] \in H_{n-1}(M; \mathbb{R})$ corresponding to Σ.

This theorem is a macroscopic analogue of theorems about minimal surfaces. The hypothesis about volumes of unit balls in the universal cover can be thought of as a lower bound on macroscopic scalar curvature; just as a lower bound on scalar curvature gives an upper bound on volumes of infinitesimal balls, here we have an upper bound on volumes of larger-scale balls. Gromov has suggested that theorems about scalar curvature should have analogues involving volumes of unit balls; see Guth's exposition [4] which includes a definition of the term "macroscopic scalar curvature".

The 3–dimensional version of Theorem 1 with minimal surfaces says that if the scalar curvature of M^3 is at least that of $\mathbb{H}^2 \times \mathbb{R}$, then every 2–sided stable minimal hypersurface has area at least that of the hyperbolic surface of the same genus; if the scalar curvature of M is at least that of $S^2 \times \mathbb{R}$, then every 2–sided stable minimal hypersurface is a sphere and has area at *most* that of S^2 [9–11]. This second statement, about positive scalar curvature, is false for macroscopic scalar curvature, as we discuss in Propositions 4 and 5 at the end of the paper.

In higher dimensions, comparing the scalar curvature to that of $\mathbb{H}^{n-1} \times \mathbb{R}$ gives a comparison between the $(n-1)$–dimensional area of a 2–sided stable minimal hypersurface and its Yamabe invariant [2, 8]. In Theorem 1, we compare the area of a hypersurface to its simplicial norm. Because this simplicial norm depends on the homology class of the embedding rather than on the intrinsic topology of the hypersurface, there is no need to include a hypothesis about the minimality of Σ.

The Gromov simplicial norm, introduced in [3], is a semi-norm on singular homology with coefficients in $\mathbb{R}$. Given a homology class h, the **simplicial norm** $\|h\|_\Delta$ is defined to be the infimum over all cycles c representing h of the sum of absolute values of coefficients in c. It is often zero. For closed manifolds the simplicial norm of the fundamental homology class is called the simplicial volume, and for closed hyperbolic manifolds the simplicial volume is a constant times the hyperbolic volume; the constant is the volume of a regular ideal simplex in n–dimensional hyperbolic space [3]. Thus, like the Yamabe invariant, the simplicial norm can be thought of as a topological generalization of hyperbolic volume. It has the nice property of being multiplicative under covering maps. For arbitrary continuous maps $f\colon X \to Y$ we have $\|f_* h\|_\Delta \le \|h\|_\Delta$ for all $h \in H_*(X; \mathbb{R})$.

Taking Theorem 1 in the case where M is diffeomorphic to the product of a hyperbolic $(n-1)$–manifold with S^1, and the hypersurface Σ is homologous to the hyperbolic factor, we get a comparison between the area of Σ and the hyperbolic area of the hyperbolic factor. Thus Theorem 1 is a generalization of the main theorem from the paper [1] of the present author with Kei Funano, under the additional hypothesis that Σ is 2–sided; for 1–sided Σ, the methods of that paper still prove something more.

In Section 2 we prove Lemma 2, the macroscopic stability inequality with $\mathbb{Z}$–coefficients, based on the stability lemma from Guth's paper [5]. We use it to prove Theorem 1 using a theorem from Guth's paper [6]. Then in Section 3 we discuss the case of positive lower bounds on macroscopic scalar curvature. An upper bound on the areas of minimizing hypersurfaces seems to be false, but in Proposition 6 we give an upper bound on the areas of a

2–dimensional surface in terms of areas of balls in the universal covers of balls in the surface.

2. Stability lemma and bounded negative curvature

We prove Theorem 1 using a macroscopic stability inequality. For stable minimal surfaces, the curvature of the ambient space and the curvature of the surface are related by the second variation formula and the Gauss equations. Here we measure curvature macroscopically in terms of volumes of balls, and instead of stable minimal hypersurfaces we use hypersurfaces that are almost area-minimizing in their homology class. The macroscopic stability inequality we use here is an adaptation of the one introduced by Guth in [5]. There he uses homology with coefficients in $\mathbb{Z}_2$, and here we use coefficients in $\mathbb{Z}$. As in [5] we define the **length** $L(\alpha)$ of any cohomology class α in degree 1 to be the infimal length of 1–cycles c with $\alpha(c) \neq 0$, and say that a hypersurface Σ is **minimizing up to** δ if it is within δ of the infimal area of hypersurfaces in its homology class.

Lemma 2 (Macroscopic stability inequality with $\mathbb{Z}$–coefficients).
Let (M, g) be a closed Riemannian manifold of dimension n. Let $\alpha \in H^1(M; \mathbb{Z})$. Suppose that $Z \subset M$ is a smooth embedded oriented hypersurface, Poincaré dual to α and minimizing up to δ (among embedded surfaces in its homology class). Suppose that $R < \frac{1}{2}L(\alpha)$. Then for any $p \in \Sigma$ we have

$$\text{Area } B_{(\Sigma, g|_\Sigma)}(p, R/2) \leq 2R^{-1} \text{Vol } B_{(M,g)}(p, R) + \delta.$$

We prove this lemma with $\mathbb{Z}$–coefficients because the Gromov simplicial norm does not work well with $\mathbb{Z}_2$–coefficients. We can count the nonzero coefficients in $\mathbb{Z}_2$–cycles, but the behavior under covering maps is not good. Therefore, in order to use the simplicial norm in Theorem 1 it is better to have the lemma with $\mathbb{Z}$–coefficients. I hope that the version with $\mathbb{Z}$–coefficients will be more useful in other situations as well.

In Guth's version, the left-hand side of the inequality measures the area, not of a ball taken inside Σ, but of the portion of Σ inside a ball in M. That version of the inequality may be false for $\mathbb{Z}$–coefficients. For instance, in a ball in M, the surface Σ may have several parallel sheets. Under $\mathbb{Z}_2$–coefficients, the sheets cancel and are homologous to something smaller, but

under $\mathbb{Z}$–coefficients they do not cancel, if they are oriented in the same direction. Thus, in the version with $\mathbb{Z}$–coefficients we show that each of these sheets inside a ball in M is not too large, which implies that the balls taken inside Σ are not too large.

Proof of Lemma 2. As in [5], we use the coarea inequality to select $t \in (\frac{1}{2}R, R)$ such that the sphere $S(p,t)$ in M has area at most

$$2R^{-1} \operatorname{Vol} B_{(M,g)}(p, R),$$

and such that $S(p,t)$ is smooth and has transverse intersection with Σ.

As in [5], the assumption $R < \frac{1}{2}L(\alpha)$ implies that every loop in the ball $B(p,t)$ has a zero signed intersection number with Σ. We group the components of $B(p,t) \setminus \Sigma$ into "levels" $L_1, \ldots, L_k$ and group the components of $\Sigma \cap B(p,t)$ into "dividers" $D_1, \ldots, D_{k-1}$, as shown in Figure 1, such that for any path from level L_i to level L_j, its signed intersection number with Σ is $j - i$, and each divider D_i is the boundary between levels L_i and L_{i+1}. We think of D_0 and D_k as empty dividers before level L_1 and after level L_k.

For each divider D_i, we consider all the ways to modify D_i by adding a $\mathbb{Z}$–linear combination of boundaries of levels, and let D_i' be the way with the least area. Then D_i' consists of some D_{j_i} (with $0 \le j_i \le k$) plus the portion of $S(p,t)$ lying between dividers D_i and D_{j_i}, as shown in Figure 1. Notice that the sequence $j_1, \ldots, j_{k-1}$ is (weakly) monotonic—the new surfaces $D_1', \ldots, D_{k-1}'$ may cover some dividers and pieces of $S(p,t)$ multiple times but they do not cross. Thus we can modify them very slightly so that replacing $D_1, \ldots, D_{k-1}$ by $D_1', \ldots, D_{k-1}'$ in Σ yields a new smooth embedded hypersurface homologous to Σ.

Because Σ is area-minimizing up to δ, we have

$$\sum_{i=1}^{k-1} \operatorname{Area} D_i' \ge \sum_{i=1}^{k-1} \operatorname{Area} D_i - \delta.$$

Thus, for each divider D_i we have

$$\operatorname{Area} D_i - \delta \le \operatorname{Area} D_i' \le \operatorname{Area} S(p,t) \le 2R^{-1} \operatorname{Vol} B(p,R).$$

The middle inequality is because D_i' was chosen to be of minimal area when compared with taking $j_i = 0$ or $j_i = k$, both of which result in surfaces contained in $S(p,t)$.
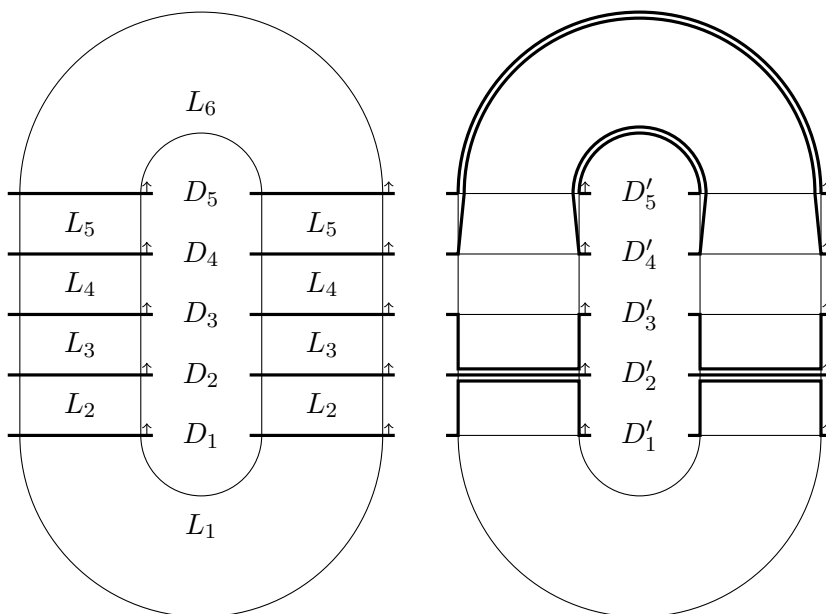
Figure 1: Given the intersection of hypersurface Σ (bold horizontal lines, with normal vectors to indicate orientation) with the ball $B(p, t)$ (annulus), we group the components of $B(p, t) \setminus \Sigma$ into "levels" L_i and group the components of $B(p, t) \cap \Sigma$ into "dividers" D_i. We replace each divider D_i by a homologous D'_i, minimizing area; the figure sketches only the topology, and not the metric information.

Applying these inequalities to the divider D_i containing the center p, we have

$$\text{Area } B_{(\Sigma, g|_\Sigma)}(p, R/2) \leq \text{Area } D_i \leq 2R^{-1} \text{Vol } B_{(M,g)}(p, R) + \delta.$$

$\square$

The proof of Theorem 1 is very similar to the proof of the main theorem in [1]; there, we prove a modified version of a theorem of Guth from [6], and the version we need for the present paper is as follows.

Theorem 3 ([1, 6]). *Let (Σ, g) be a closed oriented Riemannian manifold of dimension n, such that every ball of radius 1 in Σ has volume at most V_0. Let A be an aspherical topological space (i.e., its universal cover is contractible), and suppose that $f \colon \Sigma \to A$ is a continuous map such that $f(\gamma)$*

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is null-homotopic for every loop $\gamma \subset \Sigma$ of length at most 1. *Then we have*

$$\|f_*[\Sigma]\|_\Delta \le \mathrm{const}(n, V_0) \cdot \mathrm{Vol}\,\Sigma.$$

(Note that the theorem is vacuous in the case where the homology class $f_*[\Sigma]$ is zero.) The proof in [6] is quite technical and involved, but very roughly, the idea is to replace Σ by the nerve of a covering by balls such that the number of balls is at most proportional to the volume of Σ. The statement in [1], the "modified volume theorem", is not identical to the one given above—it is adapted for use in that paper—but we can follow the same proof. Specifically, in the language from [1], we take a good cover of Σ by balls, and let $\phi\colon \Sigma \to N$ be the corresponding map to the rectangular nerve. By [6, Lemma 9] we have $\|\phi_*[\Sigma]\|_\Delta \le \mathrm{const}(n, V_0) \cdot \mathrm{Vol}\,\Sigma$. By [1, Modified Lemma 7] there is a map $\psi\colon N \to A$ such that $\psi \circ \phi\colon \Sigma \to A$ is homotopic to f. Then, because Gromov simplicial norm is nonincreasing under mappings we have

$$\|f_*[\Sigma]\|_\Delta = \|\psi_*\phi_*[\Sigma]\|_\Delta \le \|\phi_*[\Sigma]\|_\Delta \le \mathrm{const}(n, V_0) \cdot \mathrm{Vol}\,\Sigma,$$

giving the desired inequality.

Using this theorem and Lemma 2 we can finish the proof of Theorem 1.

Proof of Theorem 1. We find a finite cover $\widehat{M}$ of M such that every loop in $\widehat{M}$ of length at most 2 is null-homotopic. This is possible because $\pi_1(M)$ is residually finite: only finitely many elements of $\pi_1(M)$ come from short loops, so we can choose $\pi_1(\widehat{M})$ to be a finite-index subgroup of $\pi_1(M)$ that excludes them all. Let $\widehat{Z}$ be one connected component of the preimage of Σ in $\widehat{M}$. We claim that it suffices to prove Theorem 1 for $\widehat{M}$ and $\widehat{\Sigma}$ instead of for M and Σ. Indeed, let $\pi\colon \widehat{M} \to M$ denote the covering map, let $\widehat{i}\colon \widehat{\Sigma} \hookrightarrow \widehat{M}$ denote the inclusion, and suppose that $\widehat{\Sigma}$ has k sheets over Σ. Because the orientation of Σ pulls back to $\widehat{\Sigma}$, all k sheets are oriented the same way. Thus, we know

$$k \cdot \mathrm{Area}\,Z = \mathrm{Area}\,\widehat{Z}$$

and

$$\|\widehat{i}_*[\widehat{Z}]\|_\Delta \ge \|\pi_*\widehat{i}_*[\widehat{Z}]\|_\Delta = \|k \cdot i_*[Z]\|_\Delta = k \cdot \|i_*[Z]\|_\Delta,$$

so if we can show

$$\mathrm{Area}\,\widehat{Z} \ge \mathrm{const}(n, V_0) \cdot \|\widehat{i}_*[\widehat{Z}]\|_\Delta,$$

then we can string together the inequalities to get

$$k \cdot \operatorname{Area} Z \geq \operatorname{const}(n, V_0) \cdot k \cdot \|i_*[Z]\|_\Delta,$$

which is k times the desired inequality.

Thus we may replace M by $\widehat{M}$ and assume that every loop in M of length at most 2 is null-homotopic. Then every unit ball in M has volume at most V_0. We may also assume that Σ is minimizing up to $\delta = 1$ among embedded hypersurfaces in its homology class. Applying Lemma 2 with $R = 1$ we find that every ball of radius $\frac{1}{2}$ in Σ has volume at most $2V_0 + 1$.

To apply Theorem 3 we need a map from Σ to an aspherical topological space. Let $A = K(\pi_1(M), 1)$ and let $f \colon \Sigma \to A$ be the composition of the inclusion $i \colon \Sigma \hookrightarrow M$ with the classifying map $M \to K(\pi_1(M), 1)$. This classifying map preserves simplicial norm (Mapping Theorem from Section 3.1 of [3], or see [7]), so we have

$$\|f_*[\Sigma]\|_\Delta = \|i_*[\Sigma]\|_\Delta.$$

Loops of length at most 2 are null-homotopic in M and hence also in Σ, so we may apply Theorem 3 (with Σ scaled by 2 to get a bound on unit balls rather than on balls of radius $\frac{1}{2}$) to get

$$\|f_*[\Sigma]\|_\Delta \leq \operatorname{const}(n, 2^{n-1}(2V_0 + 1)) \cdot \operatorname{Area} \Sigma,$$

and we obtain

$$\|i_*[\Sigma]\|_\Delta \leq \operatorname{const}(n, V_0) \cdot \operatorname{Area} \Sigma.$$

$\square$

3. Positive curvature

A *positive* lower bound on scalar curvature should give an *upper* bound on the areas of stable minimal hypersurfaces. In 3 dimensions, if the scalar curvature is at least that of $S^2 \times \mathbb{R}$, then every 2–sided stable minimal hypersurface is a sphere and has area at most that of S^2. We can ask whether the same is true for macroscopic scalar curvature: if every ball of radius 1 in the universal cover of a manifold M has very tiny volume, do the area-minimizing hypersurfaces have constrained topology and an upper bound on their areas? Proposition 4 shows that the topology is constrained in some sense, but Proposition 5 shows that we cannot expect an upper bound on the areas.

Proposition 4. *There exists a constant $V_0 > 0$ such that the following holds. Suppose that M is a 3–dimensional Riemannian manifold diffeomorphic to the product of a closed surface Σ with S^1, such that every ball of radius 1 in the universal cover of M has volume at most V_0. Then Σ must be the sphere or the projective plane.*

Proof. Guth proves in [6] that there is a constant $\delta(n)$ such that for every closed aspherical Riemannian n–manifold, the volumes of unit balls in the universal cover are at least $\delta(n)$. If Σ is a surface other than the sphere or the projective plane, then $\Sigma \times S^1$ is aspherical, so we may take $V_0 = \delta(3)$ to get the desired result. Alternatively, Guth proves in [5] a similar estimate for n–manifolds where there is a nonzero n–fold cup product. $\square$

Although having a very positive macroscopic scalar curvature constrains the topology of a manifold and its hypersurfaces, it does not necessarily result in an upper bound on the areas of stable minimal hypersurfaces.

Proposition 5. *For any $V_0 > 0$, any $K > 0$, and any n, there exists an n–dimensional closed Riemannian manifold M such that every ball of radius 1 in the universal cover of M has volume at most V_0, but simultaneously M contains an area-minimizing hypersurface of area at least K.*

Proof. Let Σ be a Riemannian $(n-1)$–manifold diffeomorphic to S^{n-1}, obtained by taking the boundary of a very small neighborhood of a very long segment in $\mathbb{R}^n$. Then let $M = \Sigma \times S^1$. By making the neighborhood very small we can guarantee that the unit balls in the universal cover of M have volume at most V_0, and by making the segment very long we can guarantee that the area of Σ is at least K. $\square$

For $n = 3$, the fact that these minimizing hypersurfaces have unbounded area implies that the ambient manifolds do not have a positive lower bound on scalar curvature. Indeed, along the length of the skinny sphere, the sphere looks like a cylinder, so the manifold looks locally flat. And yet, the unit balls in the universal cover do not look Euclidean. They look just like the unit balls downstairs and have very small volume. The reason why they do not look flat is that the universal cover is not a local construction. If we were to take the universal cover of only the cylindrical portion, then the balls would indeed be flat Euclidean balls.

This example suggests that maybe instead of taking the volumes of balls in the universal cover of the manifold M, we should take the volumes of balls in the universal covers of *balls*. The resulting number, like scalar curvature,

would depend only on local properties of M. However, it turns out that we can construct an example like the one in Proposition 5 with the additional property that the unit balls in the ambient manifold are simply connected. Thus, there does not seem to be an upper bound on volumes of minimizing hypersurfaces in terms of macroscopic scalar curvature information.

The modified example is constructed as follows. We start with a skinny sphere Σ as in Proposition 5, constructed as the boundary of the ε–neighborhood of a segment of length L, where ε is very small and L is very large. Let Σ' be the disjoint union of $10L$ copies of the boundary of the ε–neighborhood of a segment of length $\frac{1}{10}$. Then we take M to be the disjoint union of $\Sigma \times [-\frac{1}{10}, \frac{1}{10}]$ and $\Sigma' \times [\frac{3}{10}, 2 - \frac{3}{10}]$, connected by two short identical pieces that interpolate between them. Each of these connector pieces looks like a Morse function from $\frac{1}{10}$ to $\frac{3}{10}$, with $10L - 1$ critical points at $\frac{2}{10}$, as the level set Σ pinches off to become the level set Σ', like sausage links. Together, M looks like a bundle over the circle $\mathbb{R}/2\mathbb{Z}$ of length 2, with the fibers over the interval $(-\frac{2}{10}, \frac{2}{10})$ diffeomorphic to Σ, the fibers over the interval $(\frac{2}{10}, 2 - \frac{2}{10})$ diffeomorphic to Σ', and singular fibers at $\pm\frac{2}{10}$. We can adjust the metric to ensure that Σ is area-minimizing in its homology class, despite being of large area, and also ensure that every unit ball in M with center in Σ is simply connected with tiny volume.

In two dimensions, though, a condition about balls in the universal covers of balls does guarantee that the surface has small total area, as follows.

Proposition 6. *For every ε with $0 < \varepsilon < \frac{1}{8}$, the following holds. Let M be a closed surface with the property that for any point $p \in M$, the ball of radius 1 around any lift $\widetilde{p}$ of p in the universal cover $\widetilde{B(p,1)}$ of the unit ball at p has area at most ε. Then the area of M is at most 8ε.*

Proof. First we claim that the diameter of M is at most 2. Suppose to the contrary that there are two points in M with distance greater than 2, and let p be the midpoint of a length-minimizing geodesic between them. Using the coarea inequality we may select $t \in (\frac{1}{2}, 1)$ such that the total length of the sphere $S(\widetilde{p}, t)$ in $\widetilde{B(p,1)}$ is at most 2ε.

The sphere $S(\widetilde{p}, t)$ is a disjoint union of loops, and because $\widetilde{B(p,1)}$ is simply connected, each loop bounds a disk. Exactly one of the loops in $S(\widetilde{p}, t)$ encloses $\widetilde{p}$, and that loop has length at most 2ε. The geodesic through p lifts to a geodesic through $\widetilde{p}$, and the portion inside the loop has length at least $2t$. Thus, because $\varepsilon < \frac{1}{2}$, we have $2\varepsilon < 2t$ and we can homotope the geodesic to a shorter path by following the loop, giving a contradiction.

Thus, the diameter of M is at most 2. Next we describe how to cover M by at most eight unit balls. Then the area of M is at most 8ε, proving the proposition.

Start with $p_1 \in M$ arbitrary. Find $t_1 \in (\frac{1}{2}, 1)$ such that the sphere $S(\widetilde{p_1}, t_1)$ in $\widetilde{B(p_1, 1)}$ has total length at most 2ε. Find the loop L_1 in $S(\widetilde{p_1}, t_1)$ that encloses $\widetilde{p_1}$, and select $p_2 \in M$ to be any point on the image of L_1 in M.

Next we consider the universal cover of $B(p_1, 1) \cup B(p_2, 1)$. Each component of the preimage of $B(p_2, 1)$ in this covering space is covered by $\widetilde{B(p_2, 1)}$, so the ball of radius 1 around $\widetilde{p_2}$ in the universal cover of $B(p_1, 1) \cup B(p_2, 1)$ has area at most ε. Thus we may find $t_2 \in (\frac{1}{2}, 1)$ such that the sphere $S(\widetilde{p_2}, t_2)$ has length at most 2ε, and select $p_3 \in M$ to be on the image in M of the loop L_2 in $S(\widetilde{p_2}, t_2)$ enclosing $\widetilde{p_2}$.

Continue in this way: having selected p_k, take the universal cover of the union $B(p_1, 1) \cup \cdots \cup B(p_k, 1)$, select $t_k \in (\frac{1}{2}, 1)$ such that $S(\widetilde{p_k}, t_k)$ has length at most 2ε, and select $p_{k+1} \in M$ to be on the image of the loop L_k in $S(\widetilde{p_k}, t_k)$ that encloses $\widetilde{p_k}$. Once the union $B(p_1, 1) \cup \cdots \cup B(p_k, 1)$ is all of M, stop.

We claim that the union $B(p_1, 1) \cup \cdots \cup B(p_k, 1)$ covers the ball of radius $\sum_{i=1}^{k}(t_i - 2\varepsilon)$ around p_1. Notice that L_1 encloses at least at t_1–neighborhood of $\widetilde{p_1}$. Then, L_2 encloses at least a $(t_2 - 2\varepsilon)$–neighborhood around (the lift of the projection of) L_1, and in general each L_i encloses at least at $(t_i - 2\varepsilon)$–neighborhood around L_{i-1}. Thus, let q be any point outside $B(p_1, 1) \cup \cdots \cup B(p_k, 1)$, and find a length-minimizing geodesic γ from p_1 to q. Lifting γ to the universal cover of $B(p_1, 1)$, we can cut off a segment extending from $\widetilde{p_1}$ to L_1, with length at least t_1. Then, lifting γ to the universal cover of $B(p_1, 1) \cup B(p_2, 1)$, we can cut off a segment extending from L_1 to L_2, of length at least $t_2 - 2\varepsilon$, and so on. Reaching L_k we have cut off length at least $t_1 + (t_2 - 2\varepsilon) + \cdots + (t_k - 2\varepsilon)$ and still have not exhausted γ, so the distance from p_1 to q must be greater than $\sum_{i=1}^{k}(t_i - 2\varepsilon)$.

We know that the diameter of M is at most 2, and we have assumed that $\varepsilon < \frac{1}{8}$. Thus each $t_i - 2\varepsilon$ is at least $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $B(p_1, 1) \cup \cdots \cup B(p_8, 1)$ must cover all of M. $\qquad\square$

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