# On Gauduchon connections with Kähler-like curvature

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We study Hermitian metrics with a Gauduchon connection being "Kähler-like", namely, satisfying the same symmetries for curvature as the Levi-Civita and Chern connections. In particular, we investigate 6-dimensional solvmanifolds with invariant complex structures with trivial canonical bundle and with invariant Hermitian metrics. The results for this case give evidence for two conjectures that are expected to hold in more generality: first, if the Strominger-Bismut connection is Kähler-like, then the metric is pluriclosed; second, if another Gauduchon connection, different from Chern or Strominger-Bismut, is Kähler-like, then the metric is Kähler. As a further motivation, we show that the Kähler-like condition for the Levi-Civita connection assures that the Ricci flow preserves the Hermitian condition along analytic solutions.

## Introduction

We study Hermitian manifolds whose curvature tensor, with respect to a
"canonical" connection, satisfies further symmetries, as initiated by A. Gray
[Gra76], and by B. Yang and F. Zheng [YZ16]. Symmetries clearly include
the case of flat curvature [Boo58, WYZ16, GK06, DSV11, YZ17, VYZ18,
HLY18]. This is ultimately related to Yau's Problem 87 [Yau93] concerning
compact Hermitian manifolds with holonomy reduced to subgroups of U($n$).

   The setting is the following. Let $X$ be a complex manifold endowed with
a Hermitian metric. One may wonder which connection is more "natural"
to investigate the complex geometry of $X$.

   On the one side, the *Levi-Civita connection* $\nabla^{LC}$ is the only torsion-free
metric connection. In general, it does not preserve the complex structure,
this condition forcing the metric to be Kähler. The absence of torsion yields
that its curvature tensor $R^{LC}$ satisfies the symmetry called *first Bianchi
identity*:

$$\sum_{\sigma \in \mathfrak{G}} R^{LC}(\sigma x, \sigma y)\sigma z = 0.$$

   On the other side, the *Chern connection* $\nabla^{Ch}$ is another tool to investi-
gate the differential complex geometry of $X$. It is the only Hermitian con-
nection (namely, that preserves both the metric and the complex structure,)
being compatible with the Cauchy-Riemann operator on the holomorphic
tangent bundle. This yields that its curvature tensor $R^{Ch}$ satisfies the *type*

*condition* symmetry:

$$R^{Ch} \in \wedge^{1,1}(X; \mathrm{End}(T^{1,0}X)).$$

The condition $R^{Ch} = R^{LC}$ forces the metric to be Kähler, as proven in [YZ16, Theorem 1.1]. In the same spirit, in [LY17, Corollary 4.5], the authors prove that if the total scalar curvatures $s_{Ch} = s_{LC}$, then the metric is balanced. Notwithstanding, there are plenty of non-Kähler Hermitian metrics such that either $R^{LC}$ or $R^{Ch}$ satisfies both the symmetries of the Levi-Civita curvature and the symmetries of the Chern curvature. Such metrics are called *G-Kähler-like*, respectively *Kähler-like* by B. Yang and F. Zheng [YZ16].

Notice that no locally homogeneous examples of compact complex Hermitian surfaces with odd first Betti number exist such that the Levi-Civita connection is Kähler-like, thanks to a result by Muškarov [Mus01] that prevents $J$-invariant Ricci tensor.

We point out three remarks that motivate the (G-)Kähler-like conditions. First, they imply the cohomological property of the metric being *balanced* in the sense of Michelsohn [Mic82], as proven in [YZ16, Theorem 1.3]. This property plays a role in the Strominger system, and this is a first motivation in [YZ16]. Second, symmetries are clearly satisfied by zero curvature: classification results in case of certain specific connections are investigated in [Boo58, WYZ16, GK06, YZ17, VYZ18]. This is related to Yau's Problem 87 in [Yau93] on compact Hermitian manifolds with holonomy reduced to a subgroup of U($n$) (here the connection needs not to be the Levi-Civita connection). Third, we prove that the Kähler-like condition for the Levi-Civita connection assures that the Ricci flow preserves the Hermitian condition along (analytic, compare [Kot15]) solutions:

**Theorem 33.** *Let $g_0$ be a Hermitian metric on a compact complex manifold, and consider an analytic solution $(g(t))_{t \in (-\varepsilon, \varepsilon)}$ for $\varepsilon > 0$ of the Ricci flow*

$$\text{(RF)} \qquad \frac{d}{dt} g(t) = -\mathrm{Ric}(g(t)), \qquad g(0) = g_0.$$

*If the Levi-Civita connection of $g_0$ is Kähler-like, then $g(t)$ is Hermitian for any $t$.*

In fact, there are infinitely-many Hermitian connections, namely, preserving both the metric and the complex structure, but possibly having torsion. Prescribing components of the torsion selects a special family of Hermitian connections $\nabla^\varepsilon$ called *canonical connections in the sense of Gauduchon* [Gau97], varying $\varepsilon \in \mathbb{R}$. Such family includes and interpolates the Chern connection $\nabla^{Ch} = \nabla^0$ and the *Strominger-Bismut connection* $\nabla^+ = \nabla^{1/2}$, both of which have a role in complex geometry and heterotic string theory [Str86]. In this note, we extend the notion of *Kähler-like* (Definition 4) to the family $\{\nabla^\varepsilon\}_{\varepsilon \in \mathbb{R}}$ of Gauduchon connections.

Whereas the balanced condition is related to the Bott-Chern cohomology and to the Chern connection, the Strominger-Bismut connection is related to the Aeppli cohomology and to the notion of *pluriclosed* metrics (also known as SKT), see *e.g.* [Bis89]. Therefore, as a Strominger-Bismut counterpart of [YZ16, Theorem 1.3] (see Theorem 8), the following conjecture is suggested:

**Conjecture 1.** *Consider a compact complex manifold endowed with a Hermitian metric. If the Strominger-Bismut connection is Kähler-like, then the metric is pluriclosed.*

Notice that the particular case of Strominger-Bismut-flat metrics is studied in [WYZ16], where it is proven that a compact Hermitian manifold whose Strominger-Bismut connection is flat admits, as finite unbranched cover, a local Samelson space, given by the product of a compact semisimple Lie group and a torus, [WYZ16, Theorem 1].

Moreover, they prove in [WYZ16, Theorem 2] that a balanced Strominger-Bismut-flat metric is actually Kähler. It is known that Hermitian metrics that are both pluriclosed and balanced are in fact Kähler, see *e.g.* [AI01, Remark 1] or [FPS04, Proposition 1.4]. In fact, it is conjectured that no compact non-Kähler complex manifold can admit at the same time a balanced metric and a pluriclosed metric, see [FV15, Problem 3]. This conjecture has proven to be true, for example, for the twistor spaces of compact anti-self-dual Riemannian manifolds [Ver14], for compact complex manifolds in the Fujiki class $\mathcal{C}$ [Chi14], for $\sharp_k \mathbb{S}^3 \times \mathbb{S}^3$ for $k \geq 2$ endowed with the complex structures constructed from the conifold transition [FLY12], for nilmanifolds with invariant complex structures [FPS04, FV15, FV16]. So this last theorem of Wang, Yang, and Zheng is explained by the fact that the flatness of the Strominger-Bismut connection implies that the metric is pluriclosed, which gives a partial evidence for the above Conjecture 1 (see Section 1.5 for details).

The canonical connections by Gauduchon allow to tie the Chern and Strominger-Bismut connections. So, one may claim that a generic canonical connection shares properties with both of them. Again since no non-Kähler metric can be both balanced and pluriclosed, we conjecture the following, that extends [YZ17, Conjecture 1.3] on Gauduchon-flat connections.

**Conjecture 2.** *Consider a compact complex manifold endowed with a Hermitian metric. Consider a canonical connection in the Gauduchon family, different from the Strominger-Bismut connection and the Chern connection. If it is Kähler-like, then the metric is Kähler.*

We test these conjectures on a class of compact complex manifolds of complex dimension three given by quotients of Lie groups[1]. More precisely, we restrict to *Calabi-Yau solvmanifolds*, that is, compact quotients of solvable Lie groups by discrete subgroups, endowed with an invariant complex structure having a non-zero invariant closed (3,0)-form, which implies that the canonical bundle is holomorphically-trivial. We choose this class because of their role *e.g.* in constructing explicit invariant solutions to the Strominger system with respect to a Gauduchon connection [FY15, OUV17]. They will be equipped with an invariant Hermitian metric. When the group is nilpotent, we get *nilmanifolds* and they always satisfy the latter condition on triviality of the canonical bundle by [Sal01, BDV09]. Invariant complex structures on 6-dimensional nilmanifolds are ultimately classified by [COUV16], see also the references therein, and they never admit Kähler metrics, unless the nilmanifold be a torus, by topological obstructions [BG88, Has89]. Invariant complex structures on 6-dimensional solvmanifolds with trivial canonical bundle are classified by [FOU15].

The following theorem is a consequence of direct inspection performed with the help of the symbolic computation softwares, Sage [S+09] and Mathematica.

**Theorem 11.** *Consider the class of compact* 6*-dimensional solvmanifolds endowed with an invariant complex structure with holomorphically-trivial canonical bundle and with an invariant Hermitian metric.*

- *If the Strominger-Bismut connection is Kähler-like, then the metric is pluriclosed.*

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[1]After the appearence of our note, recent progress on the Conjectures 1 and 2 has been made by Zhao and Zheng [ZZ19] and by Fu and Zhou [FZ19], respectively. Moreover, at the time of the correction of the proofs of the article, we have also added the references [LS22] and [ZZ22] for further progress on the subject.

- *If a canonical connection in the Gauduchon family, different from the Strominger-Bismut connection and the Chern connection, is Kähler-like, then the metric is Kähler.*

*Therefore, Conjectures 1 and 2 are satisfied for this class of manifolds.*

A more detailed statement can be found in Theorem 13, and Proposition 34 finalizes the study showing that, on six-dimensional solvmanifolds with holomorphically-trivial canonical bundle endowed with an invariant Hermitian non-Kähler metric, the Levi-Civita connection is never Kähler-like.

In particular, we observe here some evidence following from the performed study.

Holomorphically-parallelizable complex structures are Chern-flat and then their Chern connection is clearly Kähler-like. Conversely, by [Boo58], compact Hermitian Chern-flat manifolds are given by quotients of complex Hermitian Lie groups. We have examples of non-Strominger-Bismut-flat metric whose Strominger-Bismut connection is Kähler-like.

The examples here confirm [YZ17, Conjecture 1.3], stating that compact Hermitian manifolds with a flat Gauduchon connection (different from Chern or Strominger-Bismut) must be Kähler. The analogous question for invariant Hermitian structures on Lie groups is asked and investigated in [VYZ18, Question 1.1], and our result gives further evidence for a positive answer, since our computations are also valid on Lie groups endowed with left-invariant structures (see Remark 14).

On a compact complex manifold, being balanced is a necessary condition for either the Levi-Civita or the Chern connections being Kähler-like, thanks to [YZ16, Theorem 1.3], but the converse does not hold true, as many examples below also show. It is expected that compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma admit balanced metrics. We ask *whether the $\partial\overline{\partial}$-Lemma property has any relation with the Kähler-like property.*

According to [FV15, Problem 3], no compact complex non-Kähler manifold is expected to bear both a balanced metric and a pluriclosed metric. Then, according to [YZ16, Theorem 1.3] and Conjecture 1, we expect that *a compact complex manifold admitting both a Hermitian metric with Kähler-like Chern connection and a (possibly different) Hermitian metric with Kähler-like Strominger-Bismut connection admits a Kähler metric.* By Theorem 11, this is obviously confirmed in the class of Calabi-Yau solvmanifolds of complex dimension 3. More in general, we ask *whether it is possible*

*to admit two different connections (with respect to possibly different metrics) being both Kähler-like.*

The Iwasawa manifold and its small deformations show that the property of being Kähler-like with respect to the Chern connection is not open under deformations of the complex structures. It is not even closed, *i.e.* the central limit of a holomorphic family of compact Hermitian manifolds with the Kähler-like property may not admit any Hermitian metric satisfying the Kähler-like property with respect to the Chern connection, see Corollary 12. We ask *whether the Kähler-like property with respect to the Strominger-Bismut connection (or, more generally, any other Gauduchon connection,) is open and/or closed by holomorphic deformations of the complex structure.*

## 1. Kähler-like connections

### 1.1. Curvatures

Let $X$ be a manifold endowed with a complex structure $J$ and a Hermitian structure $h = g - \sqrt{-1}\omega$.

Let $\nabla$ be any linear *metric* connection on $X$, that is, preserving the Riemannian metric $g$, namely, $\nabla g = 0$. (Denote with the same symbol its $\mathbb{C}$-bi-linear extension to $TX \otimes_{\mathbb{R}} \mathbb{C}$.) Consider its curvature operator:

$$R^{\nabla}(x, y) := \nabla^2(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]},$$

and define also the $(4, 0)$-tensor

$$R^{\nabla}(x, y, z, w) := g(R^{\nabla}(x, y)z, w).$$

Notice that, by the very definition, respectively by the condition $\nabla g = 0$,

$$R^{\nabla}(\_, \_) \in \wedge^2(X; \mathrm{End}(TX)) \qquad \text{and} \qquad R^{\nabla}(\_, \_, \_, \_) \in \wedge^2 X \otimes \wedge^2 X.$$

The *first Ricci curvature* and the *second Ricci curvature* are, respectively, the traces

$$\mathrm{Ric}^{(i)}(x, y) = \mathrm{tr}\, R^\nabla(x, y) \in \wedge^2 X,$$

$$\mathrm{Ric}^{(ii)}(z, w) = \mathrm{tr}_g R^\nabla(\_, \_, z, w) \in \wedge^2 X \subset \mathrm{End}(TX).$$

The *scalar curvature* is

$$\mathrm{Scal} = \mathrm{tr}_g \mathrm{Ric}^{(i)} = \mathrm{tr}\, \mathrm{Ric}^{(ii)} \in \mathcal{C}^\infty(X; \mathbb{R}).$$

## 1.2. Gauduchon connections

We focus on the *canonical connections* in the Gauduchon family as defined in [Gau97, Definition 2]: for $t \in \mathbb{R}$, the Hermitian connection $\nabla^{G_t} =: \nabla^{\frac{1-t}{4}, \frac{1+t}{4}}$ associated to $(J, g)$ is defined by

$$g(\nabla_x^{G_t} y, z) = g(\nabla_x^{LC} y, z) + \frac{1-t}{4}\, T(x, y, z) + \frac{1+t}{4}\, C(x, y, z),$$

where

$$T := Jd\omega := -d\omega(J\_, J\_, J\_), \qquad C := d\omega(J\_, \_, \_),$$

and $\nabla^{LC}$ denotes the Levi-Civita connection, that we can compute by

$$g(\nabla_x^{LC} y, z) = \frac{1}{2}\left(x\, g(y, z) + y\, g(x, z) - z\, g(x, y)\right.$$
$$\left. + g([x, y], z) - g([y, z], x) - g([x, z], y)\right).$$

Note that, in the space of metric connections $\nabla^{\varepsilon, \rho}$ as defined in [OUV17], Gauduchon connections sit on the line $\varepsilon + \rho = \frac{1}{2}$.

Other than

$$\nabla^{LC} = \nabla^{0,0},$$

special values are

$$\nabla^{Ch} = \nabla^{G_1} = \nabla^{0, \frac{1}{2}}, \qquad \nabla^+ = \nabla^{G_{-1}} = \nabla^{\frac{1}{2}, 0}, \qquad \nabla^- = \nabla^{-\frac{1}{2}, 0}$$

corresponding to the Chern, Strominger-Bismut, and anti-Strominger-Bismut connections, respectively. Moreover,

$$\nabla^{G_0} = \nabla^{\frac{1}{4}, \frac{1}{4}} =: \nabla^{lv}$$

is the *first canonical connection*, also called *associated connection* [Gau97, GK06], namely the projection onto the holomorphic tangent bundle of the

Levi-Civita connection. Another important connection is given by

$$\nabla^{G_{1/3}},$$

called the *minimal Gauduchon connection* [Gau97, YZ17] because it is distinguished by the property that it has the smallest total torsion among Gauduchon connections.

### 1.3. Symmetries of the curvature

If we focus on the *Levi-Civita connection* $\nabla^{LC}$, (namely, the unique metric torsion-free connection on $X$,) respectively on the *Chern connection* $\nabla^{Ch}$, (namely, the unique Hermitian connection being compatible with the Cauchy-Riemann operator $\bar{\partial}$ on the holomorphic tangent bundle,) we have further symmetries.

The Levi-Civita connection $\nabla := \nabla^{LC}$ satisfies the *first Bianchi identity*:

$$\text{(1Bnc)} \qquad \sum_{\sigma \in \mathfrak{G}} R^\nabla(\sigma x, \sigma y)\sigma z = 0.$$

(This follows by the Levi-Civita connection being torsion-free, where the torsion is defined as

$$T^\nabla(x, y) := \nabla_x y - \nabla_y x - [x, y],$$

and by the Jacobi identity for the Lie bracket $[\_, \_]$.) More in general, for a metric connection $\nabla$ with possibly non-zero torsion $T^\nabla$, we have (compare *e.g.* [Gau14, §1.16]):

$$\sum_{\sigma \in \mathfrak{G}} R^\nabla(\sigma x, \sigma y)\sigma z = d^\nabla T(x, y, z).$$

The Chern connection $\nabla := \nabla^{Ch}$ satisfies the *type condition*:

$$R^\nabla \in \wedge^{1,1}(X; \mathrm{End}(T^{1,0}X)),$$

namely,

$$\text{(Cplx)} \qquad R^\nabla(x, y, z, w) \;=\; R^\nabla(x, y, Jz, Jw) \;=\; R^\nabla(Jx, Jy, z, w)\,.$$

(The *J*-invariance in the third and fourth arguments, namely the property $R^\nabla \in \wedge^2(X; \mathrm{End}(T^{1,0}X))$, follows from $\nabla J = 0$ and $g(J\_, J\_) = g(\_, \_)$. The conclusion follows from $\nabla^{0,1} = \bar{\partial}$ yielding $(\nabla^{0,1})^2 = 0$ and by $\nabla$ being real.)

**Remark 3.** *Condition* (1Bnc) *for a metric connection implies*

(Symm) $$R^\nabla \in S^2 \wedge^2 X,$$

*namely,* $R^\nabla(x, y, z, w) = R^\nabla(z, w, x, y)$. *In this case, the first Bianchi identity holds for the* $(4, 0)$-*tensor* $R^\nabla$ *for permutations of any triple of indices. If moreover* $R^\nabla \in \wedge^2(X; \mathrm{End}(T^{1,0}X))$, *then actually* $R^\nabla \in \wedge^{1,1}(X; \mathrm{End}(\mathrm{T}^{1,0}\mathrm{X}))$.

### 1.4. Kähler-like symmetries

On a Kähler manifold, we have

$$\nabla^{LC} = \nabla^{Ch},$$

since $\nabla^{LC}J = 0$. In particular, $\nabla := \nabla^{LC}$ satisfies both (1Bnc) and (Cplx).

**Definition 4.** *Let* $X$ *be a complex manifold endowed with a Hermitian structure. Let* $\nabla$ *be a metric connection on it. We say that* $\nabla$ *is* Kähler-like *if it satisfies* (1Bnc) *and* (Cplx).

**Remark 5 (comparison with Yang and Zheng's [YZ16]).** *We claim that the above definition specializes to Yang and Zheng's definitions in [YZ16] in case of Chern connection and Levi-Civita connection. Recall that, in [YZ16, page 2], B. Yang and F. Zheng call a Hermitian structure:*

- Kähler-like (in the sense of [YZ16]) *if, for any* $(1, 0)$-*tangent vector* $X$, $Y$, $Z$, *and* $W$, *it holds*

$$R^{Ch}(X, \bar{Y}, Z, \bar{W}) = R^{Ch}(Z, \bar{Y}, X, \bar{W}) ;$$

- G-Kähler-like (in the sense of [YZ16]), *in honour of Gray, if, for any* $(1, 0)$-*tangent vector* $X$, $Y$, $Z$, *and* $W$, *it holds*

$$R^{LC}(X, Y, \bar{Z}, \bar{W}) = R^{LC}(X, Y, Z, \bar{W}) = 0 .$$

*Then:*

1) *a Hermitian structure is Kähler-like in the sense of [YZ16] if and only if its Chern connection is Kähler-like in the sense of Definition 4;*

2) *a Hermitian structure is G-Kähler-like in the sense of [YZ16] if and only if its Levi-Civita connection is Kähler-like in the sense of Definition 4.*

*Proof.* *(1)* Assume that the Chern connection $\nabla^{Ch}$ is Kähler-like in the sense of Definition 4. In particular, by (1Bnc), we get:

$$R^{Ch}(X, \bar{Y}, Z, \bar{W}) = -R^{Ch}(\bar{Y}, Z, X, \bar{W}) - R^{Ch}(Z, X, \bar{Y}, \bar{W})$$
$$= R^{Ch}(Z, \bar{Y}, X, \bar{W}) + g(R^{Ch}(X, Z)\bar{Y}, \bar{W})$$
$$= R^{Ch}(Z, \bar{Y}, X, \bar{W}),$$

since $g(R^{Ch}(X, Z)\bar{Y}, \bar{W}) = 0$.

Conversely, assume that the Hermitian structure is Kähler-like in the sense of [YZ16]. We already noticed that the Chern connection satisfies (Cplx). We prove (1Bnc). It suffices to prove it for

$$(x, y, z, w) \in \{(\bar{X}, Y, Z, \bar{W}),\ (X, \bar{Y}, Z, \bar{W}),\ (X, Y, \bar{Z}, \bar{W})\},$$

where $X$, $Y$, $Z$, and $W$ are $(1, 0)$-tangent vector fields. For example, in the first case, we have:

$$R^{\nabla}(\bar{X}, Y, Z, \bar{W}) + R^{\nabla}(Y, Z, \bar{X}, \bar{W}) + R^{\nabla}(Z, \bar{X}, Y, \bar{W})$$
$$= -R^{\nabla}(Y, \bar{X}, Z, \bar{W}) + R^{\nabla}(Z, \bar{X}, Y, \bar{W}) = 0,$$

where we used the skew-symmetries of $R^{\nabla}$, the type properties of $R^{\nabla}$, and the condition of Kähler-like in the sense of [YZ16].

*(2)* Assume that the Levi-Civita connection $\nabla^{LC}$ is Kähler-like in the sense of Definition 4. In particular, by (Cplx), we get: for any $(1, 0)$-tangent vector fields $X$, $Y$, $Z$, and $W$,

$$R^{LC}(X, Y, \bar{Z}, \bar{W}) = R^{LC}(X, Y, J\bar{Z}, J\bar{W}) = -R^{LC}(X, Y, \bar{Z}, \bar{W}),$$

whence

$$R^{LC}(X, Y, \bar{Z}, \bar{W}) = 0\ ;$$

moreover, by using (Symm), we get:

$$R^{LC}(X, Y, Z, \bar{W}) = R^{LC}(Z, \bar{W}, X, Y) = R^{LC}(Z, \bar{W}, JX, JY)$$
$$= -R^{LC}(Z, \bar{W}, X, Y) = -R^{LC}(X, Y, Z, \bar{W}),$$

whence

$$R^{LC}(X, Y, Z, \bar{W}) = 0.$$

Conversely, assume that the Hermitian structure is G-Kähler-like in the sense of [YZ16]. By [Gra76], we have also: for any $(1, 0)$-tangent vector fields

$X$, $Y$, $Z$, and $W$,

(Gray)                    $R^{LC}(X, Y, Z, W) = 0.$

We already noticed that the Levi-Civita connection satisfies (1Bnc). We prove (Cplx). By (Symm), it suffices to show $R(x, y, z, w) = R(z, y, Jz, Jw)$. By the definition of G-Kähler-like and by (Gray), and up to conjugation and (skew-)symmetries, we are reduced to the terms of curvatures:

$$R(\bar{X}, Y, Z, \bar{W}) = -R(Y, \bar{X}, Z, \bar{W}), \qquad R(X, \bar{Y}, Z, \bar{W}),$$
$$R(X, Y, Z, \bar{W}) = 0, \qquad R(X, Y, \bar{Z}, \bar{W}) = 0,$$
$$R(\bar{X}, Y, \bar{Z}, \bar{W}) = 0, \qquad R(X, \bar{Y}, \bar{Z}, \bar{W}) = 0,$$
$$R(\bar{X}, \bar{Y}, Z, \bar{W}) = 0, \qquad R(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = 0,$$

whence to the case: $(x, y, z, w) = (X, \bar{Y}, Z, \bar{W})$. Of course, in this case (Cplx) holds.                                                                            □

We will use also the following characterization of the Kähler-like condition.

**Remark 6.** *A metric linear connection $\nabla$ on a manifold endowed with a Hermitian structure is Kähler-like in the sense of Definition 4 if and only if, for any $(1,0)$-tangent vector fields $X$, $Y$, $Z$, and $W$, for any tangent vector fields $x$, $y$, $z$, and $w$,*

(1Bnc')        $B(X, \bar{Y}, Z, \bar{W}) := R^{\nabla}(X, \bar{Y}, Z, \bar{W}) - R^{\nabla}(Z, \bar{Y}, X, \bar{W}) = 0,$

(Cplx')                    $R^{\nabla}(x, y, Z, W) = R^{\nabla}(X, Y, z, w) = 0.$

*Proof.* Assume that $\nabla$ is Kähler-like in the sense of Definition 4. Moreover, we get also (Symm). Then:

$$R^{\nabla}(x, y, Z, W) = R^{\nabla}(x, y, JZ, JW) = -R^{\nabla}(x, y, Z, W),$$
$$R^{\nabla}(X, Y, z, w) = R^{\nabla}(z, w, X, Y) = R^{\nabla}(z, w, JX, JY)$$
$$= -R^{\nabla}(z, w, X, Y) = -R^{\nabla}(X, Y, z, w),$$

whence we get (Cplx'). Now, we have:

$$R^\nabla(X, \bar{Y}, Z, \bar{W}) = -R^\nabla(\bar{Y}, Z, X, \bar{W}) - R^\nabla(Z, X, \bar{Y}, \bar{W})$$
$$= R^\nabla(Z, \bar{Y}, X, \bar{W}) \,,$$

that is, (1Bnc').

On the other hand, up to anti-symmetries and conjugation, the only non-zero terms of the curvature are

$$R^\nabla(X, \bar{Y}, Z, \bar{W}) \,.$$

Therefore, (Cplx) is satisfied. Also, it follows that (1Bnc') yields (1Bnc).  □

**Remark 7.** *We notice that the possible several traces become equal as soon as we assume further symmetries of the curvature tensor. In this direction, compare the results by Liu and Yang in [LY17], comparing scalar curvatures. In particular, [LY17, Corollaries 4.4 and 4.5] state conditions concerning equalities between total scalar curvatures that force the metric to be Kähler, respectively balanced.*

### 1.5. Relations to special Hermitian metrics

A Hermitian metric $\omega$ on a complex manifold is called *balanced* [Mic82] if $d_\omega^* \omega = 0$, where $d_\omega^*$ denotes the adjoint operator of $d$ with respect to the $L^2$-pairing induced by the metric associated to $\omega$; in other words, $d\omega^{n-1} = 0$, where $n$ is the complex dimension of the manifold.

We recall the following obstruction to being Kähler-like, respectively G-Kähler-like in the sense of [YZ16].

**Theorem 8 ([YZ16, Theorem 1.3]).** *If a Hermitian metric on a compact complex manifold has either Levi-Civita or Chern connection being Kähler-like, then it is balanced in the sense of Michelsohn [Mic82].*

A Hermitian metric $\omega$ on a complex manifold is called *pluriclosed* (also known as *SKT*) if $\partial\bar{\partial}\omega = 0$ [Bis89]. Using a result in [WYZ16], and inspired by the argument in [YZ17], we show the following result for Strominger-Bismut-flat Hermitian manifolds, which can be seen as an evidence for Conjecture 1 in the special case of flat connection.

**Theorem 9.** *Let $X$ be a complex manifold of complex dimension $n$ endowed with a Hermitian metric. If the Strominger-Bismut connection is flat, then the metric is pluriclosed.*

*Proof.* Denote by $\omega$ the Strominger-Bismut-flat Hermitian metric on the manifold $X$. The pluriclosed condition $\partial\bar\partial\omega = 0$ is a local property. The Strominger-Bismut connection $\nabla^+$ being flat, we can choose an unitary parallel frame $(e_j, \bar e_j)_j$ in a neighbourhood of any point $p$ of $X$, that is, $\nabla^+ e_j = \nabla^+ \bar e_j = 0$. Furthermore, by [WYZ16, Proof of Theorem 1, page 14], it has (local) structure constants

$$[e_i, e_j] = c_{ij}^k e_k, \qquad [e_i, \bar e_j] = \overline{c_{jk}^i} e_k - c_{ik}^j \bar e_k,$$

where $c_{ij}^k \in \mathbb{C}$ are constants satisfying the required identities, and $i, j, k \in \{1, \ldots, n\}$. In addition, the metric satisfies

$$\langle [X, Y], Z \rangle = -\langle [X, Z], Y \rangle,$$

for any $X, Y, Z$ in $(e_j, \bar e_j)_{1 \le j \le n}$. Hereafter, we shorten *e.g.* $e^{i\bar j} := e^i \wedge \bar e^j$, where $(e^j, \bar e^j)_j$ is the dual coframe of $(e_j, \bar e_j)_j$, and we use Einstein notation on summation over repeated indices. In this notation, we have that with respect to the unitary parallel frame $(e_j, \bar e_j)_j$ the metric $\omega$ is expressed locally as

$$\omega = \frac{\sqrt{-1}}{2} e^{k\bar k}.$$

In other words, we are locally reduced to the Lie algebra case where the complex structure and the metric are both left invariant, and moreover, the metric is actually bi-invariant. In these conditions one has $\partial\bar\partial\omega = 0$ [SSTV88] (see also [Enr13, page 568]), so the metric $\omega$ is pluriclosed. $\qquad\square$

## 2. Hermitian geometry on six-dimensional Calabi-Yau solvmanifolds

We consider 6-dimensional *solvmanifolds* $X = \Gamma\backslash G$, namely, compact quotients of connected simply-connected solvable Lie groups by co-compact discrete subgroups. In particular, we consider solvmanifolds endowed with invariant Hermitian structures $(J, g)$, that is, structures whose lift to the universal cover is invariant under left-translations. Such structures are encoded in the associated Lie algebra $\mathfrak{g}$. As a matter of notation, we will denote Lie algebras in the concise form in [Sal01]: *e.g.* $\mathfrak{rh}_3 = (0, 0, -12, 0)$ means that

the four-dimensional Lie algebra $\mathfrak{rh}_3$ admits a basis $(e_1, e_2, e_3, e_4)$ such that $[e_1, e_2] = e_3$, the other brackets being trivial. The notation actually refers to the dual $\mathfrak{rh}_3^*$: the dual basis $(e^1, e^2, e^3, e^4)$ satisfies $de^1 = de^2 = de^4 = 0$ and $de^3 = -e^1 \wedge e^2$, where we will shorten $e^{12} := e^1 \wedge e^2$. Here, we use the following formula to relate the differential $d\colon \mathfrak{g}^\vee \to \wedge^2 \mathfrak{g}^\vee$ and the bracket $[\_, \_]\colon \wedge^2 \mathfrak{g} \to \mathfrak{g}$:

$$d\alpha(x, y) = -\alpha([x, y]).$$

### 2.1. Metric formulas

Let

$$\left(\varphi^1, \varphi^2, \varphi^3\right)$$

be an invariant co-frame of $(1, 0)$-forms with respect to $J$. With respect to the frame $(\varphi_1, \varphi_2, \varphi_3, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ dual to $\left(\varphi^1, \varphi^2, \varphi^3, \bar{\varphi}^1, \bar{\varphi}^2, \bar{\varphi}^3\right)$, we first set the structure constants

$$[\varphi_I, \varphi_H] =: c_{IH}^K \varphi_K.$$

(Capital letters here vary in $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ and refer to the corresponding component; the Einstein summation is assumed.)

The generic invariant Hermitian structure $\omega = g(\_, J\_)$ is given by

$$(2.1) \qquad 2\,\omega = \sqrt{-1}r^2\,\varphi^{1\bar{1}} + \sqrt{-1}s^2\,\varphi^{2\bar{2}} + \sqrt{-1}t^2\,\varphi^{3\bar{3}}$$
$$+ u\,\varphi^{1\bar{2}} - \bar{u}\,\varphi^{2\bar{1}} + v\,\varphi^{2\bar{3}} - \bar{v}\,\varphi^{3\bar{2}} + z\,\varphi^{1\bar{3}} - \bar{z}\,\varphi^{3\bar{1}}$$

where the coefficients satisfy [Uga07, page 189]

$$r^2 > 0, \qquad s^2 > 0, \qquad t^2 > 0,$$

$$r^2 s^2 > |u|^2, \qquad r^2 t^2 > |z|^2, \qquad s^2 t^2 > |v|^2,$$

$$r^2 s^2 t^2 + 2\mathrm{Re}\left(\sqrt{-1}\,\bar{u}z\bar{v}\right) > t^2|u|^2 + r^2|v|^2 + s^2|z|^2.$$

(Hereafter, we shorten *e.g.* $\varphi^{3\bar{1}} := \varphi^3 \wedge \bar{\varphi}^1$.) That is to say, the matrix $(g_{KL})_{K,L}$ is positive-definite. Its inverse is denoted by

$$(g^{KL})_{K,L} := (g_{KL})_{K,L}^{-1}.$$

Let $\Omega$ be the matrix associated to the Hermitian metric, i.e.

$$\Omega = \begin{pmatrix} \sqrt{-1}\,\frac{r^2}{2} & \frac{u}{2} & \frac{z}{2} \\ -\frac{\bar{u}}{2} & \sqrt{-1}\,\frac{s^2}{2} & \frac{v}{2} \\ -\frac{\bar{z}}{2} & -\frac{\bar{v}}{2} & \sqrt{-1}\,\frac{t^2}{2} \end{pmatrix}$$

Then,

$$8\sqrt{-1}\det\Omega = r^2 s^2 t^2 - r^2|v|^2 - s^2|z|^2 - t^2|u|^2 \tag{2.2}$$
$$+ 2\,\mathsf{Re}\,(\sqrt{-1}\,\bar{u}\bar{v}z) > 0.$$

We can express the Christoffel symbols of $\nabla^{LC}$ as

$$(\Gamma^{LC})_{IH}^K = \frac{1}{2}\,g^{KL}\,(g([\varphi_I,\varphi_H],\varphi_L) - g([\varphi_H,\varphi_L],\varphi_I) - g([\varphi_I,\varphi_L],\varphi_H))$$

$$= \frac{1}{2}\,c_{IH}^K - \frac{1}{2}\,g^{KA}g_{BI}c_{HA}^B - \frac{1}{2}\,g^{KA}g_{BH}c_{IA}^B.$$

Therefore, for $(\varepsilon,\rho)$, we compute the Christoffel symbols of the $\nabla^{\varepsilon,\rho}$ connection:

$$(\Gamma^{\varepsilon,\rho})_{IH}^K = (\Gamma^{LC})_{IH}^K + \varepsilon g^{KL}T_{IHL} + \rho g^{KL}C_{IHL},$$

where

$$T_{IHL} = -d\omega(J\varphi_I, J\varphi_H, J\varphi_L), \qquad C_{IHL} = d\omega(J\varphi_I, \varphi_H, \varphi_L).$$

We can then express the $(4,0)$-Riemannian curvature of $\nabla^{\varepsilon,\rho}$ as

$$(R^{\varepsilon,\rho})_{IHKL} = g_{AL}(\Gamma^{\varepsilon,\rho})_{HK}^B(\Gamma^{\varepsilon,\rho})_{IB}^A - g_{AL}(\Gamma^{\varepsilon,\rho})_{IK}^B(\Gamma^{\varepsilon,\rho})_{HB}^A$$
$$- g_{AL}c_{IH}^B(\Gamma^{\varepsilon,\rho})_{BK}^A.$$

### 2.2. Complex structures

We consider 6-dimensional *Calabi-Yau solvmanifolds*, meaning that they are endowed with an invariant complex structure having a non-zero invariant closed (3,0)-form, thus their canonical bundle is holomorphically-trivial. This includes nilmanifolds [Sal01, Uga07, COUV16] and the solvmanifolds in [FOU15]. We recall here their classification.

Six-dimensional nilpotent Lie algebras have been classified by Morozov in 34 different classes, 18 of which admit invariant complex structures by Salamon [Sal01]: they are $\mathfrak{h}_1, \ldots, \mathfrak{h}_{16}, \mathfrak{h}_{19}^-, \mathfrak{h}_{26}^+$. The complex structures on them are classified into four families in [COUV16]. We recall the complex structure equations and the underlying Lie algebras in Table 1.

We also consider solvmanifolds other than nilmanifolds. Classification of invariant complex structures in dimension 6 such that the canonical bundle is holomorphically-trivial is obtained in [Ota14, FOU15]. We recall the complex structure equations and the underlying Lie algebras in Table 2.

| Name | Complex structure | Lie algebra |
|------|-------------------|-------------|
| (Np) | $d\varphi^1 = d\varphi^2 = 0,\; d\varphi^3 = \rho\,\varphi^{12},\;$ where $\rho \in \{0,1\}$ | $\rho = 0 : \mathfrak{h}_1 = (0,0,0,0,0,0)$ |
|      |                                                                                          | $\rho = 1 : \mathfrak{h}_5 = (0,0,0,0,13+42,14+23)$ |
| (Ni) | $d\varphi^1 = d\varphi^2 = 0,$ $d\varphi^3 = \rho\,\varphi^{12} + \varphi^{1\bar{1}} + \lambda\,\varphi^{1\bar{2}} + D\,\varphi^{2\bar{2}},$ where $\rho \in \{0,1\}, \lambda \in \mathbb{R}^{\geq 0}, D \in \mathbb{C}$ with $\mathsf{Im}\,D \geq 0$ | $\mathfrak{h}_2 = (0,0,0,0,12,34)$ |
|      |  | $\mathfrak{h}_3 = (0,0,0,0,0,12+34)$ |
|      |  | $\mathfrak{h}_4 = (0,0,0,0,12,14+23)$ |
|      |  | $\mathfrak{h}_5 = (0,0,0,0,13+42,14+23)$ |
|      |  | $\mathfrak{h}_6 = (0,0,0,0,12,13)$ |
|      |  | $\mathfrak{h}_8 = (0,0,0,0,0,12)$ |
| (Nii) | $d\varphi^1 = 0,\quad d\varphi^2 = \varphi^{1\bar{1}},$ $d\varphi^3 = \rho\varphi^{12} + B\,\varphi^{1\bar{2}} + c\,\varphi^{2\bar{1}},$ where $\rho \in \{0,1\}, B \in \mathbb{C}, c \in \mathbb{R}^{\geq 0},$ with $(\rho, B, c) \neq (0,0,0)$ | $\mathfrak{h}_7 = (0,0,0,12,13,23)$ |
|      |  | $\mathfrak{h}_9 = (0,0,0,0,12,14+25)$ |
|      |  | $\mathfrak{h}_{10} = (0,0,0,12,13,14)$ |
|      |  | $\mathfrak{h}_{11} = (0,0,0,12,13,14+23)$ |
|      |  | $\mathfrak{h}_{12} = (0,0,0,12,13,24)$ |
|      |  | $\mathfrak{h}_{13} = (0,0,0,12,13+14,24)$ |
|      |  | $\mathfrak{h}_{14} = (0,0,0,12,14,13+42)$ |
|      |  | $\mathfrak{h}_{15} = (0,0,0,12,13+42,14+23)$ |
|      |  | $\mathfrak{h}_{16} = (0,0,0,12,14,24)$ |
| (Niii) | $d\varphi^1 = 0,\quad d\varphi^2 = \varphi^{13} + \varphi^{1\bar{3}},$ $d\varphi^3 = \sqrt{-1}\rho\,\varphi^{1\bar{1}} \pm \sqrt{-1}(\varphi^{1\bar{2}} - \varphi^{2\bar{1}}),\quad$ where $\rho \in \{0,1\}$ | $\mathfrak{h}_{19}^- = (0,0,0,12,23,14-35)$ |
|      |  | $\mathfrak{h}_{26}^+ = (0,0,12,13,23,14+25)$ |

Table 1: Invariant complex structures on six-dimensional nilmanifolds up to linear equivalence, see [ABD11, UV14, COUV16].

In particular, notice that the family (Np) and the family (Siv1) consist of *holomorphically-parallelizable* structures, namely, such that the holomorphic tangent bundle is holomorphically-trivial. The case (Np) with $\rho = 0$ corresponds to the complex torus, the case (Np) with $\rho = 1$ corresponds to the holomorphically-parallelizable Iwasawa manifold, and the case (Siv1) corresponds to the holomorphically-parallelizable Nakamura manifold.

We recall that, by [Wan54, Theorem 1], holomorphically-parallelizable manifolds can be regarded, up to a holomorphic homeomorphism, as quotients of a connected simply-connected complex Lie group by a discrete subgroup. Clearly, any invariant metric on a holomorphically-parallelizable manifold is Chern-flat. (On the other side, by [Boo58], a compact complex

| Name | Complex structure | Lie algebra |
|------|-------------------|-------------|
| (Si) | $d\varphi^1 = A\varphi^{13} + A\varphi^{1\bar{3}},$ $d\varphi^2 = -A\varphi^{23} - A\varphi^{2\bar{3}}, \quad d\varphi^3 = 0,$ where $A = \cos\theta + \sqrt{-1}\sin\theta, \theta \in [0, \pi)$ | $\mathfrak{g}_1 = (15, -25, -35, 45, 0, 0)$ when $\theta = 0$ $\mathfrak{g}_2^\alpha = (\alpha \times 15 + 25, -15 + \alpha \times 25, -\alpha \times 35 + 45, -35 - \alpha \times 45, 0, 0)$ with $\alpha = \frac{\cos\theta}{\sin\theta} \geq 0$, when $\theta \neq 0$ |
| (Sii) | $d\varphi^1 = 0, \quad d\varphi^2 = -\frac{1}{2}\varphi^{13} - \left(\frac{1}{2} + \sqrt{-1}x\right)\varphi^{1\bar{3}} + \sqrt{-1}x\,\varphi^{3\bar{1}},$ $d\varphi^3 = \frac{1}{2}\varphi^{12} + \left(\frac{1}{2} - \frac{\sqrt{-1}}{4x}\right)\varphi^{1\bar{2}} + \frac{\sqrt{-1}}{4x}\varphi^{2\bar{1}},$ where $x \in \mathbb{R}^{>0}$ | $\mathfrak{g}_3 = (0, -13, 12, 0, -46, -45)$ |
| (Siii1) | $d\varphi^1 = \sqrt{-1}\varphi^{13} + \sqrt{-1}\varphi^{1\bar{3}}$ $d\varphi^2 = -\sqrt{-1}\varphi^{23} - \sqrt{-1}\varphi^{2\bar{3}}$ $d\varphi^3 = \pm\varphi^{1\bar{1}}$ | $\mathfrak{g}_4 = (23, -36, 26, -56, 46, 0)$ |
| (Siii2) | $d\varphi^1 = \varphi^{13} + \varphi^{1\bar{3}}$ $d\varphi^2 = -\varphi^{23} - \varphi^{2\bar{3}}$ $d\varphi^3 = \varphi^{12} + \varphi^{2\bar{1}}$ | $\mathfrak{g}_5 = (24 + 35, 26, 36, -46, -56, 0)$ |
| (Siii3) | $d\varphi^1 = \sqrt{-1}\varphi^{13} + \sqrt{-1}\varphi^{1\bar{3}}$ $d\varphi^2 = -\sqrt{-1}\varphi^{23} - \sqrt{-1}\varphi^{2\bar{3}}$ $d\varphi^3 = \varphi^{1\bar{1}} + \varphi^{2\bar{2}}$ | $\mathfrak{g}_6 = (24 + 35, -36, 26, -56, 46, 0)$ |
| (Siii4) | $d\varphi^1 = \sqrt{-1}\varphi^{13} + \sqrt{-1}\varphi^{1\bar{3}}$ $d\varphi^2 = -\sqrt{-1}\varphi^{23} - \sqrt{-1}\varphi^{2\bar{3}}$ $d\varphi^3 = \pm(\varphi^{1\bar{1}} - \varphi^{2\bar{2}})$ | $\mathfrak{g}_7 = (24 + 35, 46, 56, -26, -36, 0)$ |
| (Siv1) | $d\varphi^1 = -\varphi^{13}, \quad d\varphi^2 = \varphi^{23}, \quad d\varphi^3 = 0$ | |
| (Siv2) | $d\varphi^1 = 2\sqrt{-1}\varphi^{13} + \varphi^{3\bar{3}}, \quad x \in \{0, 1\}$ $d\varphi^2 = -2\sqrt{-1}\varphi^{23} + x\,\varphi^{3\bar{3}}, \quad d\varphi^3 = 0$ | $\mathfrak{g}_8 = (16 - 25, 15 + 26, -36 + 45, -35 - 46, 0, 0)$ |
| (Siv3) | $d\varphi^1 = A\,\varphi^{13} - \varphi^{1\bar{3}}$ $d\varphi^2 = -A\,\varphi^{23} + \varphi^{2\bar{3}}, \quad d\varphi^3 = 0$ $A \in \mathbb{C}$ with $|A| \neq 1$ | |
| (Sv) | $d\varphi^1 = -\varphi^{3\bar{3}}$ $d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^{12} + \frac{1}{2}\varphi^{13} - \frac{\sqrt{-1}}{2}\varphi^{2\bar{1}}$ $d\varphi^3 = -\frac{\sqrt{-1}}{2}\varphi^{13} + \frac{\sqrt{-1}}{2}\varphi^{3\bar{1}}$ | $\mathfrak{g}_9 = (45, 15 + 36, 14 - 26 + 56, -56, 46, 0)$ |

Table 2: Invariant complex structures on six-dimensional solvmanifolds non-nilmanifolds with holomorphically-trivial canonical bundle up to linear equivalence, see [Ota14, FOU15].

Hermitian manifold is Chern-flat if and only if its universal cover is holomorphically isometric to a complex Lie group endowed with an invariant Hermitian metric.) In [YZ16, Theorem 1.2], the authors prove that, on a compact complex manifold of complex dimension $n \geq 3$, a metric such that both its Levi-Civita and its Chern connection are Kähler-like is actually Kähler. Recall that, by [Wan54, Corollary 2], holomorphically-parallelizable

compact Kähler manifolds are complex torus. Then, on holomorphically-parallelizable nilmanifolds and solvmanifolds different than tori, the Chern connection associated to an invariant Hermitian structure is always Kähler-like, while the Levi-Civita connection is never.

Note that this case also includes the case of the *special Lie algebra* $\mathfrak{sl}(2; \mathbb{C})$, with structure equations

$$\text{(sl2C)} \qquad d\varphi^1 = \varphi^{23}, \quad d\varphi^2 = -\varphi^{13}, \quad d\varphi^3 = \varphi^{12},$$

other than the already mentioned *Iwasawa manifold* (Np) with $\rho = 1$, and *Nakamura manifold* (Siv1).

## 2.3. Special Hermitian metrics

In view of [YZ16, Theorem 1.3] and Conjectures 1 and 2, we are particularly interested on nilmanifolds and solvmanifolds admitting invariant complex structures with holomorphically-trivial canonical bundle and (invariant) balanced or pluriclosed metrics.

As for balanced, according to [Uga07], and [COUV16, UV14, UV15], they are:

- either $\mathfrak{h}_1$ with a holomorphically-parallelizable complex structure in Family (Np) with $\rho = 0$; any Hermitian metric is in fact Kähler;

- or $\mathfrak{h}_2$, $\ldots$, $\mathfrak{h}_6$ with a nilpotent complex structure in Family (Ni); a generic Hermitian metric as in (2.1) is balanced if and only if $r^2 = 1$, $v = z = 0$, and $s^2 + D = \sqrt{-1}\,\bar{u}\,\lambda$;

- or $\mathfrak{h}_5$ with a holomorphically-parallelizable complex structure in Family (Np) with $\rho = 1$, corresponding to the Iwasawa manifold; in this case, any invariant Hermitian metric is balanced;

- or $\mathfrak{h}_{19}^-$ with a non-nilpotent complex structure in Family (Niii); balanced metrics in (2.1) are characterized by $u = z = 0$, and either $t^2 = 1$ and $v = 0$, or $t^2 > 0$ and $v = 1$;

- or $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ with a splitting-type complex structure in Family (Si); a generic metric as in (2.1) is balanced if and only if $v = z = 0$; they are non-Kähler except for $\mathfrak{g}_2^0$ with $u = v = z = 0$;

- or $\mathfrak{g}_3$ (Sii) or $\mathfrak{g}_5$ (Siii2) or $\mathfrak{g}_7$ (Siii4); balanced metrics in (2.1) are characterized by $v = z = 0$, and moreover in case $\mathfrak{g}_5$ take $u \in \mathbb{R}$, respectively in case $\mathfrak{g}_7$ take $r^2 = s^2$;

- or $\mathfrak{g}_8$ with a holomorphically-parallelizable splitting-type complex structure in Family (Siv1): any metric is balanced; or with a splitting-type complex structure in Family (Siv3): balanced metrics as in (2.1) are characterized by $v = z = 0$.

As for pluriclosed metrics, they exist only on:

- $\mathfrak{h}_1$ with a holomorphically-parallelizable complex structure in Family (Np) with $\rho = 0$; any Hermitian metric is in fact Kähler;

- $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5, \mathfrak{h}_8$ in Family (Ni), and in this case any Hermitian metric is pluriclosed, see [FPS04, Theorems 1.2, 3.2];

- $\mathfrak{g}_2^0$ in Family (Si) and $\mathfrak{g}_4$ (Siii1); pluriclosed metrics are given by $u = 0$ in the form (2.1); see [FOU15, Theorem 4.1].

Finally, we review the existence of Kähler metrics. By [BG88] and, more in general, [Has89], Kähler metrics do not exist on non-tori nilmanifolds (with invariant or non-invariant complex structures). On the other side, Kähler metrics exist on:

- $\mathfrak{h}_1$, namely, the complex torus;

- $\mathfrak{g}_2^0$ in Family (Si), and the Kähler metrics are given by the diagonal ones, $u = v = z = 0$, see [Ota14, Theorem 5.1.3].

**Remark 10.** *Observe that the Lie algebras* $\mathfrak{h}_7, \ldots, \mathfrak{h}_{16}, \mathfrak{h}_{26}^+$ *and* $\mathfrak{g}_9$ *do not admit neither balanced nor pluriclosed metrics.*

## 3. Kähler-like Gauduchon connections on six-dimensional Calabi-Yau solvmanifolds

In this section we study the existence of Kähler-like connections on the class of 6-dimensional Calabi-Yau solvmanifolds.

Let us denote the Gauduchon connections by $\nabla^\varepsilon := \nabla^{G_{1-4\varepsilon}}$ and by $R^\varepsilon$ their corresponding curvature. We recall that with respect to this notation, $\nabla^{\varepsilon=0} = \nabla^{Ch}$ and $\nabla^{\varepsilon=1/2} = \nabla^+$.

**Theorem 11.** *Let* $X = \Gamma \backslash G$ *be a six-dimensional solvmanifold endowed with an invariant complex structure* $J$ *such that the canonical bundle is holomorphically-trivial. Denote by* $\mathfrak{g}$ *the Lie algebra associated to* $G$ *and let* $\omega$ *be the* $(1,1)$*-form associated to any invariant* $J$*-Hermitian metric on* $X$. *We have the following.*

- If the Chern connection $\nabla^{Ch}$ is Kähler-like, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1$, $\mathfrak{h}_5$, $\mathfrak{g}_1$, $\mathfrak{g}_2^{\alpha \geq 0}$, or $\mathfrak{g}_8$, and the Hermitian metric $\omega$ is balanced.

- If the Strominger-Bismut connection $\nabla^+$ is Kähler-like, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_8$, $\mathfrak{g}_2^0$, or $\mathfrak{g}_4$, and the Hermitian metric $\omega$ is pluriclosed.

- If a Gauduchon connection $\nabla^\varepsilon$ is Kähler-like for some $\varepsilon \in \mathbb{R} \setminus \{0, \frac{1}{2}\}$, then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_1$ or $\mathfrak{g}_2^0$, and the Hermitian metric $\omega$ is Kähler.

Notice that Theorem 11 yields that Conjectures 1 and 2 are satisfied for such class of six-dimensional Calabi-Yau solvmanifolds. It also shows that the property of being Kähler-like for the Chern connection is neither open nor closed under holomorphic deformations of the complex structure: as for non-closedness, it follows by

**Corollary 12.** *The Lie algebra $\mathfrak{g}_8$ admits Chern-flat Hermitian metrics for complex structures in Family (Siv3), which admit limits in Family (Siv1) which are not even balanced [FOU15, Theorem 5.2].*

The following result makes Theorem 11 more precise, in specifying the complex structures $J$ and the $J$-Hermitian metrics $\omega$ in each case when the connections are Kähler-like.

**Theorem 13.** *In the conditions of Theorem 11, we have the following.*

- *The Chern connection $\nabla^{Ch}$ is Kähler-like precisely in the following cases:*
  - $\mathfrak{h}_1$, *with $J$ in Family (Np) with $\rho = 0$, with any $\omega$ given by (2.1), which is in fact Kähler and Chern-flat;*
  - $\mathfrak{h}_5$, *with $J$ in the holomorphically-parallelizable Family (Np) with $\rho = 1$, with any $\omega$ given by (2.1), which is in fact Chern-flat;*
  - $\mathfrak{g}_1$, *with $J$ in Family (Si) with $\theta = 0$, with $\omega$ given by (2.1) with $u = v = z = 0$, which is in fact Chern-flat;*
  - $\mathfrak{g}_2^0$, *with $J$ in Family (Si) with $\theta = \frac{\pi}{2}$, with $\omega$ given by (2.1) with $u = v = z = 0$, which is in fact Kähler and Chern-flat;*
  - $\mathfrak{g}_2^\alpha$ *with $\alpha > 0$, with $J$ in Family (Si) with $\theta \notin \{0, \frac{\pi}{2}\}$, with $\omega$ given by (2.1) with $u = v = z = 0$, which is in fact Chern-flat;*
  - $\mathfrak{g}_8$, *with $J$ in the holomorphically-parallelizable Family (Siv1), with any $\omega$ given by (2.1), which is in fact Chern-flat;*
  - $\mathfrak{g}_8$, *with $J$ in Family (Siv3), with $\omega$ given by (2.1) with $u = v = z = 0$, which is in fact Chern-flat.*

*In any case, the metric is balanced by [YZ16, Theorem 1.3].*

- *The Strominger-Bismut connection $\nabla^+$ is Kähler-like precisely in the previous Kähler cases (see $\mathfrak{h}_1$ and $\mathfrak{g}_2^0$) and in the following cases:*
  - *$\mathfrak{h}_2$, with $J$ in Family (Ni) with $\rho = \lambda = 0$ and $D = \sqrt{-1}$, with $\omega$ given by (2.1) with $r^2 = 1$, $u = v = z = 0$ (not Strominger-Bismut-flat);*
  - *$\mathfrak{h}_8$, with $J$ in Family (Ni) with $\rho = \lambda = D = 0$, with any $\omega$ given by (2.1) (not Strominger-Bismut-flat);*
  - *$\mathfrak{g}_4$, with $J$ in Family (Siii1), with $\omega$ given by (2.1) with $u = v = z = 0$ (not Strominger-Bismut-flat).*
  
  *In any case, the metric is pluriclosed.*

- *The Gauduchon connection $\nabla^\varepsilon$ for some $\varepsilon \in \mathbb{R} \setminus \{0, \frac{1}{2}\}$ is Kähler-like precisely in the previous Kähler cases (see $\mathfrak{h}_1$ and $\mathfrak{g}_2^0$).*

The rest of the section is devoted to the proofs of Theorems 11 and 13. Recall that by Remark 6 we know that $\nabla^\varepsilon$ is Kähler-like if and only if the identities (1Bnc') and (Cplx') hold; more concretely,

$$R^\varepsilon_{ij\bullet\bullet} = R^\varepsilon_{\bullet\bullet k\ell} = 0, \text{ and } B^\varepsilon_{i\bar{j}k\bar{\ell}} := R^\varepsilon_{i\bar{j}k\bar{\ell}} - R^\varepsilon_{k\bar{j}i\bar{\ell}} = 0,$$

for any $i, j, k, \ell \in \{1, 2, 3\}$, where for instance $B^\varepsilon_{1\bar{2}3\bar{1}}$ denotes $B^\varepsilon(X_1, \bar{X}_2, X_3, \bar{X}_1)$. Throughout this section, and also Section 4.2, we make use of this notation.

**Remark 14.** *We remark that we are doing calculations at the level of the Lie algebra, equivalently, on invariant objects on the Lie group. In particular, this means that our results hold true also at the level of the* non-compact *Lie group. Compare also [VYZ18, Theorem 1.3], where no compactness assumptions is supposed, but the extra assumption on invariant parallel frame, to study flat Gauduchon connections on Lie groups.*

### 3.1. Nilmanifolds

### 3.1.1. Holomorphically-parallelizable nilmanifolds in Family (Np).
Consider the complex structure equations

$$d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = \varphi^{12},$$

and a generic (balanced) Hermitian metric given by (2.1). Let $\{X_1, X_2, X_3\}$ be the (1,0)-basis of vectors dual to $\{\varphi^1, \varphi^2, \varphi^3\}$, that is, $[X_1, X_2] = -X_3$.

**Proposition 15.** *The Gauduchon connection $\nabla^\varepsilon$ is never Kähler-like unless $\varepsilon = 0$.*

*Proof.* A direct calculation shows that $R^\varepsilon_{ij\bullet\bullet} = R^\varepsilon_{\bullet\bullet k\ell} = 0$ for any $i, j, k, \ell \in \{1, 2, 3\}$. On the other hand, what symmetries of Bianchi-type concerns, we can observe that:

$$B^\varepsilon_{1\bar{1}3\bar{3}} = \frac{2\varepsilon^2 t^4 (r^2 t^2 - |z|^2)}{8\sqrt{-1}\det\Omega} = 0 \Longleftrightarrow \varepsilon = 0.$$

Moreover, $B^0 \equiv 0$, *i.e.* $\nabla^{Ch}$ is Kähler-like; in fact, this is true for any holomorphically-parallelizable manifold. $\square$

**3.1.2. Nilmanifolds in Family (Ni).** According to [UV15, equations (2.4)–(2.5)], any Hermitian structure can be expressed as:

$$(3.1) \qquad d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = \rho\,\varphi^{12} + \varphi^{1\bar{1}} + \lambda\,\varphi^{1\bar{2}} + D\,\varphi^{2\bar{2}},$$
$$2\omega = \sqrt{-1}(\varphi^{1\bar{1}} + s^2\,\varphi^{2\bar{2}} + t^2\,\varphi^{3\bar{3}}) + u\,\varphi^{1\bar{2}} - \bar{u}\,\varphi^{2\bar{1}},$$

where $\rho \in \{0, 1\}$, $\lambda \geq 0$, $\mathsf{Im}\, D \geq 0$, and $s^2 > |u|^2$, $t^2 > 0$; *i.e.* we can take $v = z = 0$ and $r^2 = 1$ in the generic expression (2.1).

**Lemma 16.** *In the notation as above, if $\nabla^\varepsilon$ is Kähler-like, then $\varepsilon = \frac{1}{2}$, $\rho = 0$ and $\omega$ is pluriclosed.*

*Proof.* Let us take the element $R^\varepsilon_{231\bar{3}} = \frac{2\varepsilon^2 \rho s^2 t^6}{8\sqrt{-1}\det\Omega}$. It vanishes if and only if $\varepsilon\rho = 0$. If $\varepsilon = 0$, then $B^0_{1\bar{1}3\bar{3}} = -\frac{s^2 t^4}{2(s^2 - |u|^2)} \neq 0$, hence $\rho = 0$.
 Now, substituting $\rho = 0$ we find:

$$B^\varepsilon_{1\bar{1}2\bar{1}} = \frac{-\lambda(1 - 2\varepsilon)^2 t^2}{2} = 0 \Longleftrightarrow \varepsilon = \frac{1}{2} \text{ or } \lambda = 0.$$

Finally, if $\lambda = 0$, then $B^\varepsilon_{1\bar{3}3\bar{1}} = \frac{s^2 t^4 \varepsilon(2\varepsilon - 1)}{s^2 - |u|^2} \neq 0$, hence $\varepsilon = \frac{1}{2}$.
 For the last statement, according to [FPS04, equation (3)], the Hermitian metric $\omega$ is pluriclosed if and only if the parameters in the complex structure (Ni) satisfy

$$\rho + \lambda^2 - (D + \bar{D}) = 0.$$

Now, in the case $\rho = 0$, we have that:

$$B^{\frac{1}{2}}_{1\bar{1}2\bar{2}} = \frac{-t^2}{2}(\lambda^2 - (D + \bar{D})),$$

whence the statement. □

**Proposition 17.** *In the notation as above, $\nabla^\varepsilon$ is Kähler-like if and only if $\varepsilon = \frac{1}{2}$ and*

- *either the Lie algebra is $\mathfrak{h}_2$ and the Hermitian structure is given by $(\rho, \lambda, D) = (0, 0, \sqrt{-1})$ and $u = 0$,*

- *or the Lie algebra is $\mathfrak{h}_8$ and the Hermitian structure is anyone defined on the complex structure $(\rho, \lambda, D) = (0, 0, 0)$.*

*Proof.* In [Uga07], it is shown that the only Lie algebras underlying equations (3.1) with $\rho = 0$ admitting a Hermitian pluriclosed metric are $\mathfrak{h}_2$ and $\mathfrak{h}_8$. Moreover, $\mathsf{Re}\, D = \frac{\lambda^2}{2}$. Taking into account the classification of complex structures up to equivalence [COUV16], $\lambda$ can take only the value 0, so we are forced to:

- $(\rho, \lambda, D) = (0, 0, \sqrt{-1})$, namely, the Lie algebra $\mathfrak{h}_2$;

- $(\rho, \lambda, D) = (0, 0, 0)$, namely, the Lie algebra $\mathfrak{h}_8$.

We are going to study the two cases above. First, the computation of the curvature elements for $\rho = 0$ and $\varepsilon = 1/2$ yields that, for any $i, j, k, \ell \in \{1, 2, 3\}$,

$$R^{\frac{1}{2}}_{ij\bullet\bullet} = R^{\frac{1}{2}}_{\bullet\bullet k\ell} = 0.$$

With respect to the relations coming from the Bianchi identity, we summarize the results of the computations in Appendix I. In particular, $\varepsilon = \frac{1}{2}$, $\rho = 0$, $\lambda = 0$, and $\mathsf{Re}\, D = 0$ yield that

$$B^{\frac{1}{2}} \equiv 0 \iff B^{\frac{1}{2}}_{1\bar{3}3\bar{2}} = \frac{t^4 u (D - \bar{D})}{16 \det \Omega} = 0 \iff u \, \mathsf{Im}\, D = 0.$$

□

**Remark 18.** *In the cases above, the Strominger-Bismut connection is not flat. In fact, $R^+_{1\bar{1}1\bar{1}} = R^+_{2\bar{2}2\bar{2}} = t^2 \neq 0$ for $\mathfrak{h}_2$, and $R^+_{1\bar{1}1\bar{1}} = t^2 \neq 0$ for $\mathfrak{h}_8$, the other components of the curvature being zero.*

### 3.1.3. Nilmanifolds in Family (Nii). 
Consider the complex structure equations

$$d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{1\bar{1}}, \quad d\varphi^3 = \rho\, \varphi^{12} + B\varphi^{1\bar{2}} + c\, \varphi^{2\bar{1}},$$

where $\rho \in \{0, 1\}$, $c \geq 0$, $B \in \mathbb{C}$ satisfying $(\rho, B, c) \neq (0, 0, 0)$.

**Proposition 19.** *For nilmanifolds in Family (Nii), the Gauduchon connection $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* For the Chern connection, i.e. $\varepsilon = 0$, the result follows directly from the fact that $\omega$ is never balanced and by [YZ16, Theorem 1.3].

As a consequence, we consider $\varepsilon \neq 0$ and the following elements:

$$B^\varepsilon_{3\bar{2}2\bar{3}} = -\frac{t^4(s^2t^2 - |v|^2)}{4\sqrt{-1}\det\Omega}\left(\varepsilon^2\rho^2 - 2|B|^2\left(\varepsilon - \frac{1}{2}\right)\left(\varepsilon - \frac{1}{4}\right)\right),$$

$$B^\varepsilon_{3\bar{3}2\bar{2}} = \frac{\varepsilon\, t^4(s^2t^2 - |v|^2)}{4\sqrt{-1}\det\Omega}\left(\varepsilon c^2 - 2|B|^2\left(\varepsilon - \frac{1}{4}\right)\right),$$

where recall the expression for $\det\Omega$ as in (2.2).

Observe first that $R^\varepsilon_{232\bar{3}} = \frac{2\varepsilon^2\rho\bar{B}(s^2t^2 - |v|^2)t^4}{8\sqrt{-1}\det\Omega} = 0$ if and only if $\rho B = 0$. If $B = 0$, then

$$(B^\varepsilon_{3\bar{2}2\bar{3}}, B^\varepsilon_{3\bar{3}2\bar{2}}) = (0,0) \Longleftrightarrow (\rho, c) = (0,0)$$

which is a contradiction. Therefore, we may assume $B \neq 0$ and $\rho = 0$. Now,

$$B^\varepsilon_{3\bar{2}2\bar{3}} = 0 \Longleftrightarrow \varepsilon = \frac{1}{2} \text{ or } \varepsilon = \frac{1}{4}.$$

If $\varepsilon = \frac{1}{4}$, then

$$B^{\frac{1}{4}}_{3\bar{3}2\bar{2}} = 0 \Longleftrightarrow c = 0$$

but in this case one can check that $B^{\frac{1}{4}}_{2\bar{1}1\bar{2}} = \frac{|B|^2t^2}{4} \neq 0$.

If $\varepsilon = \frac{1}{2}$, then $R^{\frac{1}{2}}_{231\bar{2}} = \frac{c(s^2t^2 - |v|^2)^2}{16\sqrt{-1}\det\Omega}$ and $R^{\frac{1}{2}}_{232\bar{1}} = \frac{-\bar{B}(s^2t^2 - |v|^2)^2}{16\sqrt{-1}\det\Omega}$, therefore

$$(R^{\frac{1}{2}}_{231\bar{2}}, R^{\frac{1}{2}}_{232\bar{1}}) = (0,0) \Longleftrightarrow (B, c) = (0,0),$$

providing a contradiction. $\qquad\square$

**3.1.4. Nilmanifolds in Family (Niii).** Consider the complex structure equations

$$d\varphi^1 = 0, \quad d\varphi^2 = \varphi^{13} + \varphi^{1\bar{3}}, \quad d\varphi^3 = \sqrt{-1}\nu\,\varphi^{1\bar{1}} + \sqrt{-1}\,\delta\,(\varphi^{1\bar{2}} - \varphi^{2\bar{1}}),$$

where $\nu = \{0, 1\}$ and $\delta = \pm 1$.

**Proposition 20.** *For nilmanifolds in Family (Niii), the Gauduchon connection $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* Consider the Bianchi identity relation $B^\varepsilon_{3\bar{3}2\bar{3}}$. Now,

$$B^\varepsilon_{3\bar{3}2\bar{3}} = \frac{(4\varepsilon^2 + 4\varepsilon - 1)(-\sqrt{-1}s^2 + \delta t^2)(s^2 t^2 - |v|^2)v}{16\sqrt{-1}\det\Omega} = 0$$

$$\Longleftrightarrow \begin{cases} v = 0, \\ \varepsilon = \frac{-1\pm\sqrt{2}}{2}. \end{cases}$$

We start supposing $v = 0$. Now, $B^\varepsilon_{3\bar{2}1\bar{2}} = \frac{2\varepsilon^2\bar{z}(\sqrt{-1}s^2 + \delta t^2)s^4}{8\sqrt{-1}\det\Omega} = 0$ if and only if $z\varepsilon = 0$. In the case $\varepsilon = 0$, one can check that $B^0_{3\bar{3}2\bar{2}} = \frac{-s^4 t^2(s^2 + \delta\sqrt{-1}t^2)}{16\sqrt{-1}\det\Omega}$ is always a non-zero element. On the other hand, if $z = 0$, just take

$$B^\varepsilon_{2\bar{2}3\bar{3}} = \frac{s^2 t^2(s^2 + \sqrt{-1}\delta t^2)\left(2\varepsilon s^2 + \sqrt{-1}(1 - 2\varepsilon)\delta t^2\right)(4\varepsilon - 1)}{16\sqrt{-1}\det\Omega}.$$

One has $B^\varepsilon_{2\bar{2}3\bar{3}} = 0 \Longleftrightarrow \varepsilon = \frac{1}{4}$. But in this case, $B^{\frac{1}{4}}_{1\bar{1}3\bar{3}} = \frac{-s^2}{2} \neq 0$.

Finally, if $\varepsilon_0 = \frac{-1\pm\sqrt{2}}{2}$ is a root of the polynomial $4\varepsilon^2 + 4\varepsilon - 1$, then:

$$B^{\varepsilon_0}_{3\bar{3}2\bar{2}} = 0$$
$$\Longleftrightarrow (s^2 t^2 - |v|^2)\left((s^2 + \sqrt{-1}\,\delta t^2)\left[(\mp 7 + 5\sqrt{2})s^2 \pm \sqrt{-1}t^2(4 \pm 3\sqrt{2})\right]\right.$$
$$\left. -\sqrt{-1}\,\delta|v|^2(\mp 3 + 2\sqrt{2})\right) = 0,$$

but the last expression is always non-zero. □

### 3.2. Solvmanifolds

**3.2.1. Solvmanifolds in Family (Si).** By Section 2.2, for this family of complex structures the underlying Lie algebras are $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ with $\alpha \geq 0$.

**Proposition 21.** *The Chern connection $\nabla^{Ch}$ is Kähler-like if and only $\omega$ given by (2.1) satisfies $u = v = z = 0$. Moreover, in these cases, the Chern curvature vanishes identically.*

*Proof.* Using [YZ16, Theorem 1.3], a necessary condition for $\nabla^{Ch}$ to be Kähler-like is that $\omega$ must be a balanced metric. According to [FOU15, Theorem 4.5], $\omega$ is balanced if and only if $v = z = 0$. Now, with this restriction, it is immediate to see that $B^0 \equiv 0$ if and only if $u = 0$ (see Appendix II). The last statement follows by direct inspection. □

**Remark 22.** *The previous result gives an example of a non-Kähler metric with Kähler-like Chern connection on a solvmanifold endowed with a left-invariant complex structure which does not admit any basis of (left-invariant) holomorphic vector fields.*

*Note indeed that the above diagonal metrics are Kähler if and only if the parameter satisfies $A = \sqrt{-1}$, corresponding to the Lie algebra $\mathfrak{g}_2^0$. Clearly, by [Boo58], the complex solvmanifold is in fact biholomorphic to a holomorphically-parallelizable manifold. But it is presented in a different way as quotient of a Lie group, and as such it is not parallelizable by left-invariant holomorphic vector fields.*

**Proposition 23.** *For $\varepsilon \neq 0$, the Gauduchon connection $\nabla^\varepsilon$ is Kähler-like if and only if $\mathfrak{g} = \mathfrak{g}_2^0$ and $\omega$ is Kähler.*

*Proof.* We prove firstly that the underlying Lie algebra must be $\mathfrak{g}_2^0$, corresponding to $A = \sqrt{-1}$ in family (Si), as:

$$B^\varepsilon_{1\bar{1}2\bar{2}} = \frac{8\,\varepsilon^2(r^2s^2 - |u|^2)}{8\sqrt{-1}\,\det\Omega}\left[(\mathsf{Re}\,A)^2(r^2s^2 - |u|^2) + |A|^2|u|^2\right] = 0,$$

implies $A = \sqrt{-1}$ and $u = 0$.

Now, as it is shown in Appendix III, $B^\varepsilon_{1\bar{1}3\bar{3}} = B^\varepsilon_{2\bar{2}3\bar{3}} = 0$ if and only if $(2\varepsilon - 1)v = (2\varepsilon - 1)z = 0$.

If $v = z = 0$, in particular the metric $\omega$ is Kähler and one has that $B^\varepsilon \equiv 0$ and $R^\varepsilon \equiv 0$.

On the other hand, consider now the Strominger-Bismut connection, *i.e.* $\varepsilon = \frac{1}{2}$. Then, $B^{\frac{1}{2}} \equiv 0$ if and only if $(B^{\frac{1}{2}}_{1\bar{3}3\bar{3}}, B^{\frac{1}{2}}_{2\bar{3}3\bar{3}}) = (\frac{\sqrt{-1}z}{2}, \frac{\sqrt{-1}v}{2}) = (0,0)$, which implies $v = z = 0$. $\qquad\square$

### 3.2.2. Solvmanifolds in Family (Sii).

By Section 2.2, the Lie algebra underlying this family of complex structures is $\mathfrak{g}_3$.

**Proposition 24.** *For any solvmanifold in Family (Sii), the Gauduchon connection $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* First let us suppose $\varepsilon = 0$. Using [YZ16, Theorem 1.3], a necessary condition for $\nabla^{Ch}$ to be Kähler-like is that $\omega$ must be a balanced metric. According to [FOU15, Theorem 4.5], $\omega$ is balanced if and only if $u = z = 0$.

Now, with this restriction,

$$B^0_{1\bar{2}2\bar{1}} = \frac{-(t^2 - 2\sqrt{-1}s^2 x)(1 + 4x^2)}{32x^2} \neq 0,$$

whence the statement.

In what follows, we will suppose that $\varepsilon \neq 0$ and a generic Hermitian metric $\omega$. Consider the elements $R^\varepsilon_{2\bar{3}2\bar{3}}$ and $R^\varepsilon_{2\bar{3}3\bar{3}}$:

$$\begin{cases} R^\varepsilon_{2\bar{3}2\bar{3}} = \frac{\varepsilon(\sqrt{-1}+2x)(s^2t^2-|v|^2)(-v^2+2\sqrt{-1}s^2t^2x-4s^4x^2+2\varepsilon(t^4+4s^4x^2)-2\sqrt{-1}|v|^2x)}{64x\sqrt{-1}\det\Omega}, \\ R^\varepsilon_{2\bar{3}3\bar{3}} = \frac{\varepsilon(\sqrt{-1}+2x)(s^2t^2-|v|^2)(v(\sqrt{-1}(-1+2\varepsilon)t^2+4\varepsilon s^2x)+4\sqrt{-1}x(\sqrt{-1}\varepsilon t^2-s^2x+2\varepsilon s^2x)\bar{v})}{64x\sqrt{-1}\det\Omega}. \end{cases}$$

Now:

$$\begin{cases} R^\varepsilon_{2\bar{3}2\bar{3}} = 0 \iff 2\varepsilon t^4 - v^2 + 2\sqrt{-1}x(s^2t^2 - |v|^2) + 4s^4x^2(2\varepsilon - 1) = 0, \\ R^\varepsilon_{2\bar{3}3\bar{3}} = 0 \iff v\left(4\sqrt{-1}\,\varepsilon s^2 x - (2\varepsilon - 1)t^2\right) \\ \qquad\qquad -4x\bar{v}\left(\sqrt{-1}\,\varepsilon t^2 + (2\varepsilon - 1)s^2 x\right) = 0. \end{cases}$$

Adding and subtracting the two expressions above, we obtain the equivalent system:

$$\begin{cases} (2\sqrt{-1}\,s^2 x + 2\varepsilon(t^2 - 2\sqrt{-1}\,s^2 x) - v)(t^2 + 2\sqrt{-1}\,s^2 x + v + 2\sqrt{-1}\,x\bar{v}) = 0, \\ (2\sqrt{-1}\,s^2 x + 2\varepsilon(t^2 - 2\sqrt{-1}\,s^2 x) + v)(t^2 + 2\sqrt{-1}\,s^2 x - v - 2\sqrt{-1}\,x\bar{v}) = 0. \end{cases}$$

As a matter of notation, let us write the first equation as $AB = 0$ and the second as $CD = 0$.

Let us focus our attention in the first equation and consider the case $A = 0$. We can express $v$ as:

$$v = 2\sqrt{-1}\,s^2 x + 2\varepsilon(t^2 - 2\sqrt{-1}\,s^2 x).$$

Substituting this value in the second equation we obtain that:

$$\begin{cases} C = 4(\varepsilon t^2 - \sqrt{-1}s^2 x(2\varepsilon - 1)) \neq 0, \\ \\ D = (1 - 2\varepsilon)(t^2 - 4s^2 x^2) + 4\sqrt{-1}\,\varepsilon x(s^2 - t^2) = 0 \iff \begin{cases} \operatorname{Re} D = 0, \\ \operatorname{Im} D = 0, \end{cases} \end{cases}$$

Now,

$$\begin{cases} \mathrm{Re}\,D = 0, \\ \mathrm{Im}\,D = 0, \end{cases} \iff \begin{cases} (1 - 2\varepsilon)(t^2 - 4s^2x^2) = 0, \\ s^2 - t^2 = 0, \end{cases}$$

$$\iff \begin{cases} (1 - 2\varepsilon)(1 - 4x^2) = 0, \\ s^2 - t^2 = 0, \end{cases}$$

so, we obtain two different possibilities:

$$s^2 = t^2, \quad \varepsilon = \frac{1}{2}, \quad \text{or} \quad s^2 = t^2, \quad x = \frac{1}{2}.$$

If we consider $A \neq 0$, then $B = 0$ and therefore $v + 2\sqrt{-1}\,x\bar{v} = -t^2 - 2\sqrt{-1}\,s^2x$. Substituting this expression in $D$ we get that

$$D = 2(t^2 + 2\sqrt{-1}x^2x) \neq 0,$$

which implies that $C = 0$. Observe that system $B = C = 0$ is the same as $A = D = 0$ just changing the sign of $v$. One get the same solutions as before.

To finish the proof, we study first the case $s^2 = t^2$ and $\varepsilon = \frac{1}{2}$. Directly,

$$R^{\frac{1}{2}}_{121\bar{2}} = \frac{t^2(2x - \sqrt{-1})}{16x} \neq 0.$$

On the other hand, if $s^2 = t^2$ and $x = \frac{1}{2}$, $B^{\varepsilon}_{2\bar{2}3\bar{1}} = \frac{\sqrt{-1}\varepsilon(2\varepsilon - 1)t^4}{\bar{u} + \sqrt{-1}\bar{z}} \neq 0.$ $\qquad \square$

### 3.2.3. Solvmanifolds in Families (Siii1), (Siii3), (Siii4).

Recall that from Section 2.2, the Lie algebras underlying (Siii1), (Siii3), and (Siii4) are, respectively, $\mathfrak{g}_4$, $\mathfrak{g}_6$, and $\mathfrak{g}_7$. In order to give an unified argument, we will gather the complex equations as follows:

$$d\varphi^1 = \sqrt{-1}\,(\varphi^{13} + \varphi^{1\bar{3}}), \quad d\varphi^2 = -\sqrt{-1}\,(\varphi^{23} + \varphi^{2\bar{3}}),$$

$$d\varphi^3 = x\,\varphi^{1\bar{1}} + y\,\varphi^{2\bar{2}},$$

where $(x, y) = (\pm 1, 0)$ for $\mathfrak{g}_4$, $(x, y) = (1, 1)$ for $\mathfrak{g}_6$ and $(x, y = -x) = (\pm 1, \mp 1)$ for $\mathfrak{g}_7$. In particular, $x \neq 0$.

**Proposition 25.** *The Chern connection $\nabla^{Ch}$ is never Kähler-like.*

*Proof.* Using [YZ16, Theorem 1.3], a necessary condition for $\nabla^{Ch}$ to be Kähler-like is that $\omega$ must be a balanced metric. A direct computation shows

that $\omega$ is balanced if and only if $v = z = 0$ and $s^2x + r^2y = 0$. Observe that this last equation has no solutions on $\mathfrak{g}_4$ and $\mathfrak{g}_6$, which implies that $\nabla^{Ch}$ is never Kähler-like for these algebras. On $\mathfrak{g}_7$, since $x = -y$, the only option is $r^2 = s^2$. Now, if we compute the Bianchi identity-symmetries $B^0$, we get that $B^0_{1\bar{1}2\bar{2}} = \frac{t^2}{2} \neq 0$. $\qquad\square$

**Proposition 26.** *In the notation as above, consider $\varepsilon \neq 0$. Then, $\nabla^\varepsilon$ is Kähler-like if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_4$, $\varepsilon = \frac{1}{2}$, and $u = v = z = 0$ in (2.1). In this case, the Strominger-Bismut connection is non-flat.*

*Proof.* From now on, consider $\varepsilon \neq 0$. Let us focus on Chern-symmetries $R^\varepsilon_{121\bar{3}}$ and $R^\varepsilon_{122\bar{3}}$:

$$R^\varepsilon_{121\bar{3}} = \frac{-\varepsilon\,z}{8\sqrt{-1}\,\det\Omega}\left(v\left[\sqrt{-1}\,(2\varepsilon+1)uvx + z\left(s^2x(2\varepsilon+1) - r^2y(2\varepsilon-1)\right)\right] + \sqrt{-1}\,(2\varepsilon-1)y\bar{u}z^2\right),$$

$$R^\varepsilon_{122\bar{3}} = \frac{\varepsilon\,v}{8\sqrt{-1}\,\det\Omega}\left(v\left[\sqrt{-1}\,(2\varepsilon-1)uvx + z\left(s^2x(2\varepsilon-1) - r^2y(2\varepsilon+1)\right)\right] + \sqrt{-1}\,(2\varepsilon+1)y\bar{u}z^2\right).$$

Trivially, this two elements vanishes if $v = z = 0$. Suppose that $vz \neq 0$. Then, the system $R^\varepsilon_{121\bar{3}} = R^\varepsilon_{122\bar{3}} = 0$ is equivalent to

$$\begin{cases} A := v\left[\sqrt{-1}\,(2\varepsilon+1)uvx + z\left(s^2x(2\varepsilon+1) - r^2y(2\varepsilon-1)\right)\right] \\ \qquad + \sqrt{-1}\,(2\varepsilon-1)y\bar{u}z^2 = 0, \\ B := v\left[\sqrt{-1}\,(2\varepsilon-1)uvx + z\left(s^2x(2\varepsilon-1) - r^2y(2\varepsilon+1)\right)\right] \\ \qquad + \sqrt{-1}\,(2\varepsilon+1)y\bar{u}z^2 = 0. \end{cases}$$

Consider the following homogeneous system, whose solutions are precisely $A = B = 0$:

$$\begin{cases} A(1+2\varepsilon) + B(1-2\varepsilon) = 0, \\ A(1-2\varepsilon) + B(1+2\varepsilon) = 0. \end{cases}$$

This last system reduces to

$$\begin{cases} 8\,\varepsilon\,v\,x\,(\sqrt{-1}uv + s^2z) = 0, \\ 8\varepsilon\,\sqrt{-1}\,z\,y\,(\sqrt{-1}r^2v + \bar{u}z) = 0. \end{cases} \iff \begin{cases} \sqrt{-1}uv + s^2z = 0, \\ y\,(\sqrt{-1}r^2v + \bar{u}z) = 0. \end{cases}$$

$$\iff \begin{cases} u = \dfrac{\sqrt{-1}\,s^2\,z}{v}, \\ \dfrac{\sqrt{-1}\,y}{\bar{v}}\,(r^2|v|^2 - s^2|z|^2) = 0. \end{cases}$$

Moreover, substituting $u = \dfrac{\sqrt{-1}\,s^2\,z}{v}$ in (2.2), we get:

$$8\sqrt{-1}\,\det\Omega = \frac{(s^2t^2 - |v|^2)(r^2|v|^2 - s^2|z|^2)}{|v|^2} \neq 0.$$

So, we are forced to consider $y = 0$. However, in this particular case, $R^{\varepsilon}_{123\bar{2}} = \frac{4\sqrt{-1}\,\varepsilon^2 s^4 z}{s^2t^2 - |v|^2} \neq 0$.

Let us study now the case when $vz = 0$. Suppose first $z = 0$ and $v \neq 0$. Now, $R^{\varepsilon}_{121\bar{2}} = \frac{-\varepsilon^2 u^2 v^2 x}{2\det\Omega} = 0$ if and only if $u = 0$. But imposing this condition, we get that $R^{\varepsilon}_{231\bar{1}} = -\varepsilon vx \neq 0$.

On the other hand, if $v = 0$ and $z \neq 0$, symmetry $R^{\varepsilon}_{121\bar{3}} = \frac{-\varepsilon(2\varepsilon-1)yz^3\bar{u}}{8\det\Omega} = 0$ is equivalent to $(2\varepsilon - 1)uy = 0$. If $\varepsilon = \frac{1}{2}$ (corresponding to the Strominger-Bismut connection), then

$$\begin{cases} R^{\frac{1}{2}}_{132\bar{2}} = \frac{yz}{2} = 0, \\ R^{\frac{1}{2}}_{121\bar{1}} = \frac{s^2z^2x\bar{u}}{8\sqrt{-1}\det\Omega} = 0, \end{cases} \iff \quad u = y = 0.$$

But now, $R^{\frac{1}{2}}_{133\bar{3}} = \frac{\sqrt{-1}z}{2} \neq 0$.

Suppose now $\varepsilon \neq \frac{1}{2}$. If $u = 0$, directly from $R^{\varepsilon}_{132\bar{2}} = \varepsilon yz$ one gets $y = 0$. Analogously, if we suppose $y = 0$, then $R^{\varepsilon}_{121\bar{1}} = \frac{-\sqrt{-1}\varepsilon s^2 z^2\bar{u}x}{4\det\Omega}$ implies $u = 0$. So, if $\varepsilon \neq \frac{1}{2}$ we may assume $u = y = 0$. At this point, observe that $R^{\varepsilon}_{131\bar{3}} = \frac{\varepsilon(2\varepsilon-1)(r^2 - \sqrt{-1}xt^2)z^2}{8\sqrt{-1}\det\Omega} = 0$ if and only if $\varepsilon = \frac{1}{2}$.

Finally, if $v = z = 0$, then

$$R^{\varepsilon}_{133\bar{1}} = \frac{\varepsilon((-1+2\varepsilon)s^2t^4x^2 + (8\varepsilon r^2u + 2\sqrt{-1}t^2ux)\bar{u})}{8\sqrt{-1}t^2\det\Omega} = 0$$

if and only if the complex number

$$(2\varepsilon - 1)s^2t^4x^2 + 8\varepsilon r^2|u|^2 + \sqrt{-1}(2t^2x|u|^2) = 0.$$

Observe that the imaginary part vanishes if and only if $u = 0$. Furthermore, one can check that if $u = 0$, then all the Chern-symmetries are zero if and only if $\varepsilon = \frac{1}{2}$. As for Bianchi identity-symmetries, in the case $u = v = z = 0$ and $\varepsilon = \frac{1}{2}$, we have $R^{\frac{1}{2}}_{ij\bullet\bullet} = R^{\frac{1}{2}}_{\bullet\bullet k\ell} = 0$ for any $i, j, k, \ell \in \{1, 2, 3\}$ if and only if $y = 0$, so the Lie algebra is isomorphic to $\mathfrak{g}_4$.

Finally, note that $R^{\frac{1}{2}}_{1\bar{1}1\bar{1}} = t^2$, whence the last statement. $\qquad\square$

**Remark 27.** *We notice that any diagonal metric on $\mathfrak{g}_4$ gives examples of a non-flat Kähler-like Strominger-Bismut connection on a non-Kähler manifold. (For the existence of lattices, see [FOU15, Proposition 2.10].)*

**3.2.4. Solvmanifolds in Family (Siii2).** By Section 2.2, the underlying Lie algebra is $\mathfrak{g}_5$.

**Proposition 28.** *The connection $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* Let us suppose $\varepsilon = 0$. Using [YZ16, Theorem 1.3], a necessary condition for $\nabla^{Ch}$ to be Kähler-like is that $\omega$ must be a balanced metric. According to [FOU15, Theorem 4.5], $\omega$ is balanced if and only if $v = z = 0$ and $u \in \mathbb{R}$. Now, with these restrictions,

$$B^0_{1\bar{1}2\bar{1}} = -r^2 \neq 0,$$

whence the statement.

From now on, consider $\varepsilon \neq 0$ and a generic Hermitian metric $\omega$. Let us focus on Chern-symmetries $R^\varepsilon_{121\bar{3}}$ and $R^\varepsilon_{122\bar{3}}$ and develop a similar argument as that in Proposition 26:

$$
\begin{aligned}
R^\varepsilon_{121\bar{3}} &= \frac{\varepsilon\, z}{8\sqrt{-1}\,\det\Omega} \left(\sqrt{-1}\left[(2\varepsilon + 1)r^2v^2 - z(2\varepsilon - 1)(\sqrt{-1}uv + s^2 z)\right]\right. \\
&\quad \left. + (2\varepsilon + 1)\bar{u}vz\right), \\
R^\varepsilon_{122\bar{3}} &= \frac{-\varepsilon\, v}{8\sqrt{-1}\,\det\Omega} \left(\sqrt{-1}\left[(2\varepsilon - 1)r^2v^2 - z(2\varepsilon + 1)(\sqrt{-1}uv + s^2 z)\right]\right. \\
&\quad \left. + (2\varepsilon - 1)\bar{u}vz\right).
\end{aligned}
$$

Clearly, this two elements vanishes if $v = z = 0$. Suppose that $vz \neq 0$. Then, the system $R^\varepsilon_{121\bar{3}} = R^\varepsilon_{122\bar{3}} = 0$ is equivalent to

$$
\begin{cases}
A := \sqrt{-1}\left[(2\varepsilon + 1)r^2v^2 - z(2\varepsilon - 1)(\sqrt{-1}uv + s^2 z)\right] + (2\varepsilon + 1)\bar{u}vz = 0, \\
B := \sqrt{-1}\left[(2\varepsilon - 1)r^2v^2 - z(2\varepsilon + 1)(\sqrt{-1}uv + s^2 z)\right] + (2\varepsilon - 1)\bar{u}vz = 0.
\end{cases}
$$

Consider the following homogeneous system, whose solutions are precisely $A = B = 0$:

$$\begin{cases} A(1 + 2\varepsilon) + B(1 - 2\varepsilon) = 0, \\ A(1 - 2\varepsilon) + B(1 + 2\varepsilon) = 0. \end{cases}$$

This last system is equivalent to

$$\begin{cases} \sqrt{-1}uv + s^2 z = 0, \\ \sqrt{-1}r^2 v + \bar{u}z = 0. \end{cases} \iff v = z = 0,$$

which contradicts our initial hypothesis.

We can suppose now that $vz = 0$. In fact, in this case $v = z = 0$ (use $R^\varepsilon_{122\bar{2}} = \frac{\sqrt{-1}\varepsilon^2 s^4 z^2}{2 \det \Omega} = 0$ when $v = 0$ and $R^\varepsilon_{121\bar{1}} = \frac{\sqrt{-1}\varepsilon^2 r^4 v^2}{2 \det \Omega}$ if $z = 0$). Now, $B^\varepsilon_{1\bar{1}2\bar{1}} = (4\varepsilon - 1)r^2 = 0$ if and only if $\varepsilon = \frac{1}{4}$ but now $B^{\frac{1}{4}}_{1\bar{1}3\bar{3}} = -r^2 \neq 0$. $\square$

### 3.2.5. Solvmanifolds in Families (Siv1), (Siv2), and (Siv3).

Recall that by Section 2.2, the underlying Lie algebra to these complex structures is $\mathfrak{g}_8$.

**Proposition 29.** *For the holomorphically-parallelizable structure in Family (Siv1), we have that the Chern connection $\nabla^{Ch}$ is flat and $\nabla^{\varepsilon \neq 0}$ is never Kähler-like.*

*Proof.* The result about Chern connection comes from the fact that the structure is complex-parallelizable. When $\varepsilon \neq 0$, one can see that $R^\varepsilon_{ij\bullet\bullet} = R^\varepsilon_{\bullet\bullet k\ell} \equiv 0$ for any $i, j, k, \ell \in \{1, 2, 3\}$ if and only if $\varepsilon = \frac{1}{2}$ (see for example $R^\varepsilon_{133\bar{1}} = -(2\varepsilon - 1)\varepsilon r^2$). But now, $B^{\frac{1}{2}}_{1\bar{1}2\bar{2}} = \frac{r^4 s^4 - |u|^4}{16\sqrt{-1}\det \Omega}$ if and only if $r^4 s^4 - |u|^4 = 0$ which is a contradiction. $\square$

**Proposition 30.** *For the complex structures in Family (Siv2), $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* The result for the Chern connection (namely $\varepsilon = 0$) follows directly observing that complex structures in Family (Siv2) do not admit balanced metrics and by [YZ16, Theorem 1.3].

If $\varepsilon \neq 0$, then

$$R^\varepsilon_{123\bar{1}} = -\frac{\varepsilon^2(r^2s^2 - |u|^2)(xr^2s^2 + x|u|^2 + 2\sqrt{-1}r^2\bar{u})}{2\det\Omega},$$

and

$$R^\varepsilon_{123\bar{2}} = -\frac{\varepsilon^2(r^2s^2 - |u|^2)(r^2s^2 - 2\sqrt{-1}xs^2u + |u|^2)}{2\det\Omega}.$$

Hence, $R^\varepsilon_{123\bar{1}} = R^\varepsilon_{123\bar{2}} = 0$ if and only if

$$xr^2s^2 + x|u|^2 + 2\sqrt{-1}r^2\bar{u} = 0, \quad r^2s^2 + |u|^2 - 2\sqrt{-1}xs^2u = 0.$$

If $x = 1$, these equations imply $\mathsf{Re}\, u = 0$, $\mathsf{Im}\, u = -r^2$, and $r^2 = s^2$, which is a contradiction to the positive definiteness of the metric. Hence, $x = 0$ and $R^\varepsilon_{123\bar{2}}$ is always different from zero. $\qquad\square$

**Proposition 31.** *For the complex structures in Family (Siv3), $\nabla^\varepsilon$ is Kähler-like if and only if $\varepsilon = 0$ and $\omega$ is diagonal. Moreover, in these cases, $\nabla^{Ch}$ is flat.*

*Proof.* Let us start with the case $\varepsilon = 0$ and imposing $\omega$ to be balanced. In this situation, $v = z = 0$. Moreover, $B^0 \equiv 0$ if and only if $u = 0$ (see for example $B^0_{13\bar{3}\bar{1}} = \frac{2r^2|u|^2}{r^2s^2 - |u|^2}$). Observe that we do not get restriction on the parameter $A$.

Finally, if $\varepsilon \neq 0$, we focus on Bianchi identity-symmetry $B^\varepsilon_{1\bar{1}2\bar{2}}$. Observe that

$$B^\varepsilon_{1\bar{1}2\bar{2}} = \frac{2\varepsilon^2(r^2s^2 - |u|^2)(r^2s^2|A - 1|^2 + |u|^2|A + 1|^2)}{8\sqrt{-1}\det\Omega} \neq 0,$$

whence the statement. $\qquad\square$

**3.2.6. Solvmanifolds in Family (Sv).** By Section 2.2, the underlying Lie algebra is $\mathfrak{g}_9$.

**Proposition 32.** *The connection $\nabla^\varepsilon$ is never Kähler-like.*

*Proof.* If $\varepsilon = 0$, then the result follows directly by observing that $\mathfrak{g}_9$ does not admit balanced metrics and by [YZ16, Theorem 1.3].

From now on, consider $\varepsilon \neq 0$. Let us focus on the Chern-symmetry $R^\varepsilon_{121\bar{2}}$:

$$R^\varepsilon_{121\bar{2}} = \frac{\bar{v}\left[(2r^2s^2 - |u|^2)v - (\sqrt{-1}s^2\bar{u})z\right]}{32\sqrt{-1}\det\Omega}.$$

Hence,

$$R^\varepsilon_{121\bar{2}} = 0 \iff \bar{v}\left[(2r^2s^2 - |u|^2)v - (\sqrt{-1}s^2\bar{u})z\right] = 0.$$

Suppose first that $v \neq 0$. Then, we can express

$$v = \frac{\sqrt{-1}s^2\bar{u}z}{2r^2s^2 - |u|^2}.$$

Observe that, since $v \neq 0$, also $uz \neq 0$. Using this particular value for $v$, $R^\varepsilon_{122\bar{1}} = \frac{\varepsilon(1-2\varepsilon)s^2\bar{u}^2}{4(rs^2-|u|^2)} = 0$ if and only if $\varepsilon = \frac{1}{2}$, but now $R^{\frac{1}{2}}_{131\bar{2}} = \frac{s^2u\bar{z}}{4(r^2s^2-|u|^2)} \neq 0$.

On the other hand, if $v = 0$, then $R^\varepsilon_{122\bar{3}} = \frac{\varepsilon(1-2\varepsilon)s^2z\bar{u}^2}{16\sqrt{-1}\det\Omega} = 0$ holds if and only if $(2\varepsilon - 1)uz = 0$. None of the three cases leads to Kähler-like connections: if $\varepsilon = \frac{1}{2}$, then $R^{\frac{1}{2}}_{233\bar{3}} = \frac{s^4|z|^2}{32\sqrt{-1}\det\Omega} = 0$ to conclude that $z = 0$ but then $R^{\frac{1}{2}}_{123\bar{1}} = \frac{-(r^2s^2-|u|^2)}{4t^2} \neq 0$; if $z = 0$, then $R^\varepsilon_{131\bar{3}} = \frac{\varepsilon(2\varepsilon-1)r^4}{t^2}$ vanishes if and only if $\varepsilon = \frac{1}{2}$, that has already been studied; finally, if $u = 0$, then take $R^\varepsilon_{121\bar{1}} = \frac{-\varepsilon r^2s^2z}{4(r^2s^2-|z|^2)}$. This element is zero if and only if $z = 0$, which has already been analysed. $\qquad\square$

## 4. Kähler-like Levi-Civita connection

### 4.1. Hermitian condition along the Ricci-flow

In this section, we prove that the Kähler-like condition for the Levi-Civita connection assures that the Ricci flow preserves the Hermitian condition of the initial metric along analytic solutions. We wonder whether, as an application, one may prove that non-toral nilmanifolds do not admit invariant Hermitian metrics whose Levi-Civita connection is Kähler-like, thanks to the results by Lauret [Lau11] on the Ricci flow for nilpotent Lie groups. In fact, we will prove that this is true in dimension six, see Proposition 34.

**Theorem 33.** *Let $g_0$ be a Hermitian metric on a compact complex manifold, and consider an analytic solution $(g(t))_{t\in(-\varepsilon,\varepsilon)}$ for $\varepsilon > 0$ of the Ricci*

*flow*

(RF) $$\frac{d}{dt}g(t) = -\mathrm{Ric}(g(t)), \qquad g(0) = g_0.$$

*If the Levi-Civita connection of $g_0$ is Kähler-like, then $g(t)$ is Hermitian for any $t$.*

*Proof.* The proof follows by the evolution equation for the curvatures along the Ricci flow in [Ham82, Corollary 7.3 and Theorem 7.1]:

$$\frac{d}{dt}\mathrm{Ric}(t)_{JK} = \Delta_t \mathrm{Ric}(t)_{JK} + 2g(t)^{PQ}g(t)^{RS}R(t)_{PJKR}\mathrm{Ric}(t)_{QS}$$
$$- 2g(t)^{PQ}\mathrm{Ric}(t)_{JP}\mathrm{Ric}(t)_{QK},$$

$$\frac{d}{dt}R(t)_{IJKL} = \Delta_t R(t)_{IJKL}$$
$$+ 2\left(g(t)^{PR}g(t)^{QS}R(t)_{IPJQ}R(t)_{KRLS}\right.$$
$$-g(t)^{PR}g(t)^{QS}R(t)_{IPJQ}R(t)_{LRKS}$$
$$-g(t)^{PR}g(t)^{QS}R(t)_{IPLQ}R(t)_{JRKS}$$
$$\left.+g(t)^{PR}g(t)^{QS}R(t)_{IPKQ}R(t)_{JRLS}\right)$$
$$- g(t)^{PQ}\left(R(t)_{PJKL}\mathrm{Ric}(t)_{QI} + R(t)_{IPKL}\mathrm{Ric}(t)_{QJ}\right.$$
$$\left.+R(t)_{IJPL}\mathrm{Ric}(t)_{QK} + R(t)_{IJKP}\mathrm{Ric}(t)_{QL}\right).$$

Here, capital letters $I, J, \ldots$ vary in $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ while small letters $i, j, \ldots$ vary in $\{1, \ldots, n\}$, and for simplicity we just write the dependence in $t$ meaning to refer to the quantities for $g(t)$. The dot denotes the derivative at zero, *e.g.* $\dot{\mathrm{Ric}}_{jk} = \frac{d}{dt}\big|_{t=0}\mathrm{Ric}_{jk}(t)$.

The Kähler-like condition assures that

$$R_{ijKL}(0) = 0, \qquad \mathrm{Ric}_{ij}(0) = 0.$$

This, together with the formula for the evolution of the Ricci curvature tensor, whence
$$\dot{\mathrm{Ric}}_{jk} = 0,$$
yields that
$$\dot{g}_{jk} = \ddot{g}_{jk} = 0.$$

Moreover, notice that the Kähler-like assumption forces $R_{IJKL}(0)$ to be zero whenever three or more of the indices in $(I, J, K, L)$ have the same type. The same holds for the tensor $g(0)^{PR}g(0)^{QS}R(0)_{PIQJ}R(0)_{RKSL}$, that

appears in the right-hand-side for $\dot{R}$. From the evolution equation of the Riemann curvature tensor, we get the same property also for $\dot{R}_{IJKL}$. Then we get

$$\ddot{g}_{jk} = 0$$

by straightforward computations.

Further derivatives behave similarly. Indeed, the only remark to be noticed is that $\dot{R}_{IJKL}$ involves the tensors $R_{\bullet\bullet\bullet}(0)$, $\dot{R}_{\bullet\bullet\bullet}$, $\mathrm{Ric}_{\bullet\bullet}(0)$, $\dot{\mathrm{Ric}}_{\bullet\bullet}$, other than the Hermitian metric $g_{\bullet\bullet}(0)$ and its time-derivative $\dot{g}_{\bullet\bullet}$, whence it still satisfies the property to vanish whenever three out of $(I, J, K, L)$ have the same type.

So we get that the solution $g(t)_{jk}$ has vanishing time-derivatives of any order at 0. Being analytic, it is constant, so $g(t)_{jk} = g(0)_{jk} = 0$, that is, $g(t)$ is still Hermitian. $\qquad\square$

### 4.2. Kähler-like Levi-Civita connection on six-dimensional Calabi-Yau solvmanifolds

In this section, we prove the following:

**Proposition 34.** *Let $(X, J, g)$ be a complex six-dimensional solvmanifold with holomorphically-trivial canonical bundle endowed with an invariant Hermitian metric. The Levi-Civita connection is Kähler-like if and only if the metric is Kähler.*

*Proof.* First, consider the nilpotent case. We provide explicit computations. Without loss of generality, we can reduce to the case when the Hermitian metric is balanced, thanks to [YZ16, Theorem 1.3], and we can exclude the holomorphically-parallelizable case (Np) thanks to the fact that holomorphically-parallelizable manifolds are Chern-flat and to [YZ16, Theorem 1.2]. So we only remain with Family (Ni) with the exception of Lie algebra $\mathfrak{h}_8$, and the Family (Niii) in fact only for $\mathfrak{h}_{19}^-$. We distinguish complex structures of nilpotent, respectively non-nilpotent type, see [UV15].

For nilpotent complex structures on nilpotent Lie algebras, argue as follows. First of all, the generic Hermitian metric as in (2.1) is balanced if and only if it is equivalent to one with parameters $r^2 = 1$, $v = z = 0$, and also $s^2 + D = \sqrt{-1}\bar{u}\lambda$, see [UV15]. We compute, for example, $R_{1\bar{1}12}^{LC} = -\frac{3}{8}\rho\,t^2$. So the Levi-Civita connection of a balanced metric on a nilmanifold with complex structure in Family (Ni) with $\rho = 1$ is never Kähler-like. This forces $\rho = 0$, and the Lie algebra to be either $\mathfrak{h}_3$ (with $\lambda = 0$ and $D = -1$) or $\mathfrak{h}_5$

(with $\lambda = 1$ and $D \in [0, \frac{1}{4})$). Moreover, since $R^{LC}_{3\bar{2}13} = \frac{1}{8} \frac{\sqrt{-1}\, t^4\, u\, \bar{D}}{s^2 - |u|^2}$, then either $u = 0$ (case $\mathfrak{h}_3$) or $D = 0$ (case $\mathfrak{h}_5$).

In the first case, that is, when $\rho = \lambda = 0$ and $D = -1$, we compute $B^{LC}_{1\bar{1}3\bar{3}} = \frac{(\lambda^2 + \rho^2 + s^2 + \sqrt{-1}\,\lambda u - \sqrt{-1}\,\lambda \bar{u})t^4}{8\,(s^2 - u\bar{u})}$ that reduces to $B^{LC}_{1\bar{1}3\bar{3}} = \frac{s^2 t^4}{8\,(s^2 - u\bar{u})} \neq 0$, in fact, $B^{LC}_{1\bar{1}3\bar{3}} = \frac{t^4}{8} \neq 0$, by using $s^2 = 1$ and $u = 0$. Therefore, in this case, the Levi-Civita connection is not Kähler-like.

In the second case, that is, when $\rho = D = 0$ and $\lambda = 1$, we compute $B^{LC}_{1\bar{1}2\bar{2}} = -\frac{1}{8}\left(\lambda^2 + 2\rho^2 - D - \bar{D}\right)t^2$ that reduces to $B^{LC}_{1\bar{1}2\bar{2}} = -\frac{1}{8}\,t^2 \neq 0$. Therefore, also in this second case, the Levi-Civita connection is not Kähler-like.

Finally, for non-nilpotent complex structure on nilpotent Lie algebras, argue as follows. The generic Hermitian balanced metric as in (2.1) forces $\nu = 0$ (case $\mathfrak{h}_{19}^-$) and can be reduced to parameters $u = z = 0$, (and moreover either $v = 1$, or $v = 0$ and $t^2 = 1$,) see [UV15]. We compute, for example, $R^{LC}_{3\bar{2}2\bar{3}} = -\frac{s^4 + t^4}{8r^2} \neq 0$ and $R^{LC}_{2\bar{2}3\bar{3}} = 0$. Therefore the Levi-Civita connection is not Kähler-like.

We restrict now our attention to the solvable non-nilpotent case.

We remind that by [YZ16, Theorem 1.3], the Levi-Civita connection be Kähler-like implies that the metric is balanced. Hence and by [FOU15] we restrict our attention to the Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2^\alpha$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$, and $\mathfrak{g}_8$.

If the Lie algebra is isomorphic to $\mathfrak{g}_1$ or $\mathfrak{g}_2^\alpha$ (case (Si)) then:

$$
\begin{aligned}
R^{\mathrm{LC}}_{12\bar{1}\bar{2}} &= \frac{(A^2 + \bar{A}^2)(r^2 s^2 - |u|^2) + 2|A|^2(r^2 s^2 + |u|^2)}{4t^2} = 0 \\
&\Leftrightarrow (\mathsf{Re}\,A)^2 r^2 s^2 + (\mathsf{Im}\,A)^2 |u|^2 = 0, \\
&\Leftrightarrow \mathsf{Re}\,A = 0,\ u = 0, \\
&\Leftrightarrow A = \sqrt{-1},\ u = 0.
\end{aligned}
$$

That is, the Lie algebra is $\mathfrak{g}_2^0$ and $\omega$ is Kähler.

If the Lie algebra is isomorphic to $\mathfrak{g}_3$ (case (Sii)) then:

$$
R^{\mathrm{LC}}_{23\bar{2}\bar{3}} = \frac{(1 + 4x^2)(t^4 + 4s^4 x^2)}{128 r^2 x^2} \neq 0.
$$

If the Lie algebra is isomorphic to $\mathfrak{g}_5$ (case (Siii2)) then:

$$
R^{\mathrm{LC}}_{13\bar{2}\bar{3}} = \frac{-\sqrt{-1}(4r^2 s^2 + t^4)u}{8(r^2 s^2 - u^2)} = 0 \quad \Leftrightarrow \quad \sqrt{-1}(4r^2 s^2 + t^4)u = 0 \quad \Leftrightarrow \quad u = 0.
$$

But $u = 0$ implies $R^{\mathrm{LC}}_{23\bar{3}\bar{1}} = \frac{-t^2}{2} \neq 0$.

If the Lie algebra is isomorphic to $\mathfrak{g}_7$ (case (Siii4)) then:

$$R^{\mathrm{LC}}_{23\bar{2}\bar{3}} = \frac{s^2(t^4 + 4|u|^2)}{8(s^4 - |u|^2)} \neq 0.$$

If the Lie algebra is isomorphic to $\mathfrak{g}_8$, then the complex structure belongs to Families (Siv1) or (Siv3). In the first case we directly find that $R^{\mathrm{LC}}_{13\bar{3}\bar{1}} = \frac{r^2}{8} \neq 0$. In the second case the Kähler-like condition implies the following system:

$$R^{\mathrm{LC}}_{13\bar{3}\bar{1}} = \frac{r^2 \left[ (A-1)^2 r^2 s^2 - (A^2 - 6A + 1)|u|^2 \right]}{8(r^2 s^2 - |u|^2)} = 0,$$

$$R^{\mathrm{LC}}_{13\bar{3}\bar{2}} = \frac{-\sqrt{-1}u \left[ (A^2 + 6A + 1)r^2 s^2 - (A+1)^2 |u|^2 \right]}{8(r^2 s^2 - |u|^2)} = 0.$$

If $u = 0$ then the first equation implies $(A-1)^2 r^2 s^2 = 0$ which is not possible. On the other hand if $u \neq 0$ then the vanishing of the brackets corresponds to a homogeneous linear system in $r^2 s^2$ and $|u|^2$. The equations are linear dependent if and only if $A = 0$, but in this case the second equation reduces to $r^2 s^2 - |u|^2 = 0$. $\qquad\square$

## Appendix I. Bianchi and Chern symmetries for the Gauduchon connections on nilmanifolds in Family (Ni)

$$R^{\varepsilon}_{121\bar{1}} = 2\, t^2\, \varepsilon (1 - \varepsilon)\rho,$$

$$R^{\varepsilon}_{122\bar{1}} = \lambda\, R^{\varepsilon}_{121\bar{1}},$$

$$R^{\varepsilon}_{122\bar{2}} = \bar{D}\, R^{\varepsilon}_{121\bar{1}},$$

$$R^{\varepsilon}_{123\bar{3}} = \frac{t^4\, \varepsilon}{s^2 - |u|^2}\rho(2\varepsilon - 1)(s^2 + \sqrt{-1}\lambda u + \bar{D}),$$

$$R^{\varepsilon}_{131\bar{3}} = \frac{-2\sqrt{-1}\, t^4\, \varepsilon^2}{s^2 - |u|^2}\rho\, u,$$

$$R^{\varepsilon}_{132\bar{3}} = \frac{-2\, t^4\, \varepsilon^2}{s^2 - |u|^2}\rho\, (\sqrt{-1}\lambda u + \bar{D}),$$

$$R^{\varepsilon}_{133\bar{1}} = \frac{-t^4\, \varepsilon}{s^2 - |u|^2}(2\varepsilon - 1)(s^2 + \sqrt{-1}\lambda u),$$

$$R^{\varepsilon}_{133\bar{2}} = \frac{-\sqrt{-1}\,t^4\,\varepsilon}{s^2 - |u|^2}(2\varepsilon - 1)\,u\,\bar{D},$$

$$R^{\varepsilon}_{231\bar{3}} = \frac{2\,s^2\,t^4\,\varepsilon^2}{s^2 - |u|^2}\rho,$$

$$R^{\varepsilon}_{232\bar{3}} = \frac{2\,t^4\,\varepsilon^2}{s^2 - |u|^2}\rho\,(\lambda s^2 - \sqrt{-1}\bar{u}\bar{D}),$$

$$R^{\varepsilon}_{233\bar{1}} = \frac{-t^4\,\varepsilon}{s^2 - |u|^2}(2\varepsilon - 1)(\lambda(s^2 + \sqrt{-1}\lambda u + \bar{D}) - \sqrt{-1}\bar{u}\bar{D}),$$

$$R^{\varepsilon}_{233\bar{2}} = \frac{-t^4\,\varepsilon}{s^2 - |u|^2}(2\varepsilon - 1)\bar{D}(\sqrt{-1}\lambda u + \bar{D}).$$

$$B^{\varepsilon}_{1\bar{1}2\bar{1}} = \frac{-t^2}{2}(2\varepsilon - 1)^2\,\lambda,$$

$$B^{\varepsilon}_{1\bar{1}2\bar{2}} = \frac{-t^2}{2}\left[2\varepsilon(\lambda^2 - D + 4\rho\varepsilon) + \bar{D}(4\varepsilon^2 - 6\varepsilon + 1)\right],$$

$$B^{\varepsilon}_{1\bar{1}3\bar{3}} = \frac{t^4}{2(s^2 - |u|^2)}\left[4\,\rho\varepsilon^2 - s^2(2\varepsilon - 1)(4\varepsilon - 1)\right],$$

$$B^{\varepsilon}_{1\bar{2}2\bar{1}} = \frac{t^2}{2}\left[2\varepsilon(4\rho\varepsilon - \bar{D}) + (D - \lambda^2)(4\varepsilon^2 - 6\varepsilon + 1)\right],$$

$$B^{\varepsilon}_{1\bar{2}2\bar{2}} = \bar{D}B^{\varepsilon}_{1\bar{1}2\bar{1}},$$

$$B^{\varepsilon}_{1\bar{2}3\bar{3}} = \frac{-t^4}{2(s^2 - |u|^2)}\left[4\,\sqrt{-1}\,\rho u\varepsilon^2 + (2\varepsilon - 1)(4\varepsilon - 1)(\lambda s^2 + \sqrt{-1}uD)\right],$$

$$B^{\varepsilon}_{1\bar{3}2\bar{3}} = \frac{2\,t^4\,\varepsilon^2}{s^2 - |u|^2}\rho(D + s^2 - \sqrt{-1}\lambda\bar{u}),$$

$$B^{\varepsilon}_{1\bar{3}3\bar{1}} = \frac{-t^4\,\varepsilon}{s^2 - |u|^2}\left[s^2 + 2\varepsilon(\lambda^2 - s^2 - 2\lambda\mathsf{Im}\,u)\right],$$

$$B^{\varepsilon}_{1\bar{3}3\bar{2}} = \frac{t^4\,\varepsilon}{s^2 - |u|^2}\left[(4\varepsilon - 1)(\lambda s^2 + \sqrt{-1}uD) - 2\varepsilon\bar{D}(\lambda + \sqrt{-1}u)\right],$$

$$B^{\varepsilon}_{2\bar{1}3\bar{3}} = \overline{B^{\varepsilon}_{1\bar{2}3\bar{3}}},$$

$$B^{\varepsilon}_{2\bar{2}3\bar{3}} = \frac{-t^4}{2(s^2 - |u|^2)}\left[(2\varepsilon - 1)(4\varepsilon - 1)(\lambda^2 s^2 - 2\lambda\mathsf{Im}\,(uD) + |D|^2) - 4s^2\rho\varepsilon^2\right],$$

$$B^{\varepsilon}_{2\bar{3}3\bar{1}} = \overline{B^{\varepsilon}_{1\bar{3}3\bar{2}}},$$

$$B^{\varepsilon}_{2\bar{3}3\bar{2}} = \frac{t^4\,\varepsilon}{s^2 - |u|^2}\left[(2\varepsilon - 1)|D|^2 + (4\varepsilon - 1)\lambda(\lambda s^2 - 2\mathsf{Im}\,(uD))\right].$$

## Appendix II. Bianchi symmetries for the Levi-Civita connection on solvmanifolds $\mathfrak{g}_1$ and $\mathfrak{g}_2^\alpha$ in Family (Si)

$$B^0_{1\bar{3}3\bar{1}} = \frac{2\,r^2|u|^2}{r^2 s^2 - |u|^2}, \qquad B^0_{1\bar{3}3\bar{2}} = \frac{-2\,\sqrt{-1}\,r^2 s^2 u}{r^2 s^2 - |u|^2},$$

$$B^0_{2\bar{3}3\bar{1}} = \overline{B^0_{1\bar{3}3\bar{2}}}, \qquad B^0_{2\bar{3}3\bar{2}} = \frac{2\,s^2|u|^2}{r^2 s^2 - |u|^2}.$$

## Appendix III. Bianchi and Chern symmetries for the Gauduchon connections on solvmanifolds $\mathfrak{g}_2^0$ in Family (Si)

$$R^{\varepsilon}_{131\bar{3}} = \frac{\varepsilon(2\varepsilon - 1)r^2 s^2 z^2}{8\sqrt{-1}\,\det\Omega},$$

$$R^{\varepsilon}_{132\bar{3}} = \frac{-\varepsilon(2\varepsilon - 1)r^2 s^2 vz}{8\sqrt{-1}\,\det\Omega},$$

$$R^{\varepsilon}_{133\bar{3}} = \frac{\varepsilon\,z}{8\det\Omega}\left[r^2 s^2 t^2 - 2\varepsilon r^2|v|^2 + 2(\varepsilon - 1)s^2|z|^2\right],$$

$$R^{\varepsilon}_{231\bar{3}} = -R^{\varepsilon}_{132\bar{3}},$$

$$R^{\varepsilon}_{232\bar{3}} = \frac{\varepsilon(2\varepsilon - 1)r^2 s^2 v^2}{8\sqrt{-1}\,\det\Omega},$$

$$R^{\varepsilon}_{233\bar{3}} = \frac{\varepsilon\,v}{8\det\Omega}\left[r^2 s^2 t^2 + 2(\varepsilon - 1)r^2|v|^2 - 2\varepsilon s^2|z|^2\right].$$

$$B^{\varepsilon}_{1\bar{1}3\bar{3}} = \frac{-\varepsilon(2\varepsilon - 1)r^2 s^2|z|^2}{8\sqrt{-1}\,\det\Omega},$$

$$B^{\varepsilon}_{1\bar{2}3\bar{3}} = \frac{\varepsilon(2\varepsilon - 1)r^2 s^2\bar{v}z}{8\sqrt{-1}\,\det\Omega},$$

$$B^{\varepsilon}_{1\bar{3}3\bar{1}} = \frac{(2\varepsilon - 1)(4\varepsilon - 1)r^2 s^2|z|^2}{16\sqrt{-1}\,\det\Omega},$$

$$B^\varepsilon_{1\bar{3}3\bar{2}} = \frac{-(2\varepsilon - 1)(4\varepsilon - 1)r^2 s^2 \bar{v} z}{16\sqrt{-1}\,\det\Omega},$$

$$B^\varepsilon_{1\bar{3}3\bar{3}} = \frac{z}{16\det\Omega}\left[(4\varepsilon - 1)r^2 s^2 t^2 + 2(2\varepsilon^2 - 4\varepsilon + 1)r^2|v|^2 - 4\varepsilon^2 s^2|z|^2\right],$$

$$B^\varepsilon_{2\bar{1}3\bar{3}} = \overline{B^\varepsilon_{1\bar{2}3\bar{3}}},$$

$$B^\varepsilon_{2\bar{2}3\bar{3}} = \frac{-\varepsilon(2\varepsilon - 1)r^2 s^2|v|^2}{8\sqrt{-1}\,\det\Omega},$$

$$B^\varepsilon_{2\bar{3}3\bar{1}} = \overline{B^\varepsilon_{1\bar{3}3\bar{2}}},$$

$$B^\varepsilon_{2\bar{3}3\bar{2}} = \frac{(2\varepsilon - 1)(4\varepsilon - 1)r^2 s^2|v|^2}{16\sqrt{-1}\,\det\Omega},$$

$$B^\varepsilon_{2\bar{3}3\bar{3}} = \frac{v}{16\det\Omega}\left[(4\varepsilon - 1)r^2 s^2 t^2 - 4\varepsilon^2 r^2|v|^2 + 2(2\varepsilon^2 - 4\varepsilon + 1)s^2|z|^2\right].$$

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