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Stability and area growth of λ-hypersurfaces

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In this paper, We define a $\mathcal{F}$-functional and study $\mathcal{F}$-stability of λ-hypersurfaces, which extend a result of Colding-Minicozzi [6]. Lower bound growth and upper bound growth of area for complete and non-compact λ-hypersurfaces are studied.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth n-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. A family $X(\cdot, t)$ of smooth

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immersions:

$$X(\cdot,t): M \to \mathbb{R}^{n+1}$$

with $X(\cdot,0) = X(\cdot)$ is called a mean curvature flow if they satisfy

$$\frac{\partial X(p,t)}{\partial t} = \mathbf{H}(p,t),$$

where $\mathbf{H}(t) = \mathbf{H}(p,t)$ denotes the mean curvature vector of hypersurface $M_t = X(M^n,t)$ at point $X(p,t)$. Huisken [9] proved that the mean curvature flow M_t remains smooth and convex until it becomes extinct at a point in the finite time. If we rescale the flow about the point, the rescaling converges to the round sphere. An immersed hypersurface $X: M \to \mathbb{R}^{n+1}$ is called *a self-shrinker* if

$$H + \langle X, N\rangle = 0,$$

where H and N denote the mean curvature and the unit normal vector of $X: M \to \mathbb{R}^{n+1}$, respectively. $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow ups at a given singularity of the mean curvature flow.

Colding and Minicozzi [6] have introduced a notation of $\mathcal{F}$-functional and computed the first and the second variation formulas of the $\mathcal{F}$-functional. They have proved that an immersed hypersurface $X: M \to \mathbb{R}^{n+1}$ is a self-shrinker if and only if it is a critical point of the $\mathcal{F}$-functional. Furthermore, they have given a complete classification of the $\mathcal{F}$-stable complete self-shrinkers with polynomial area growth.

In [3], we consider a new type of mean curvature flow:

$$\frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \mathbf{H}(t), \tag{1.1}$$

with

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu},$$

where N is the unit normal vector of $X: M \to \mathbb{R}^{n+1}$. We define *a weighted volume* of M_t by

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu.$$

We can prove that the flow (1.1) preserves the weighted volume $V(t)$. Hence, we call the flow (1.1) *a weighted volume-preserving mean curvature flow.*

From a view of variations, self-shrinkers of mean curvature flow can be characterized as critical points of the weighted area functional. In [3], the authors give a definition of weighted volume and study the weighted area functional for variations preserving this volume. Critical points for the weighted area functional for variations preserving this volume are called λ-hypersurfaces by the authors in [3]. Precisely, an n-dimensional hypersurface $X : M \to \mathbb{R}^{n+1}$ in Euclidean space $\mathbb{R}^{n+1}$ is called *a λ-hypersurface* if

$$\langle X, N\rangle + H = \lambda, \tag{1.2}$$

where λ is a constant, H and N denote the mean curvature and unit normal vector of $X : M \to \mathbb{R}^{n+1}$, respectively.

Remark 1.1. *If $\lambda = 0$, $\langle X, N\rangle + H = \lambda = 0$, then $X : M \to \mathbb{R}^{n+1}$ is a self-shrinkers. Hence, the notation of λ-hypersurfaces is a natural generalization of the self-shrinkers of the mean curvature flow. The equation* (1.2) *also arises in the Gaussian isoperimetric problem.*

In this paper, we define $\mathcal{F}$-functional. The first and second variation formulas of $\mathcal{F}$-functional are given. Notation of $\mathcal{F}$-stability and $\mathcal{F}$-unstability of λ-hypersurfaces are introduced. We prove that spheres $S^n(r)$ with $r \le \sqrt{n}$ or $r > \sqrt{n+1}$ are $\mathcal{F}$-stable and spheres $S^n(r)$ with $\sqrt{n} < r \le \sqrt{n+1}$ are $\mathcal{F}$-unstable. In section 4, we study the weak stability of the weighted area functional for the weighted volume-preserving variations. In sections 5 and 6, the area growth of complete and non-compact λ-hypersurfaces are studied.

We should remark that this paper is the second part of our paper `arXiv:1403.3177`, which is divided into two parts. The first part has been published [4].

2. The first variation of $\mathcal{F}$-functional

In this section, we will give another variational characterization of λ-hypersurfaces.

The following lemmas can be found in [3].

Lemma 2.1. *If $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface, then we have*

$$\mathcal{L}\langle X, a\rangle = \lambda\langle N, a\rangle - \langle X, a\rangle, \tag{2.1}$$

$$\mathcal{L}\langle N, a\rangle = -S\langle N, a\rangle, \tag{2.2}$$

$$\frac{1}{2}\mathcal{L}(|X|^2) = n - |X|^2 + \lambda\langle X, N\rangle, \tag{2.3}$$

where $\mathcal{L}$ is an elliptic operator given by $\mathcal{L}f = \Delta f - \langle X, \nabla f\rangle$, Δ and ∇ denote the Laplacian and the gradient operator of the λ-hypersurface, respectively, $a \in \mathbb{R}^{n+1}$ is constant vector, S is the squared norm of the second fundamental form.

Lemma 2.2. *If $X : M \to \mathbb{R}^{n+1}$ is a hypersurface, u is a C^1-function with compact support and v is a C^2-function, then*

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle\nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}d\mu. \tag{2.4}$$

Corollary 2.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a complete hypersurface. If u, v are C^2 functions satisfying*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|X|^2}{2}}d\mu < +\infty, \tag{2.5}$$

then we have

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle\nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}d\mu. \tag{2.6}$$

Lemma 2.3. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with polynomial area growth, then*

$$\int_M (\langle X, a\rangle - \lambda\langle N, a\rangle)e^{-\frac{|X|^2}{2}}d\mu = 0, \tag{2.7}$$

$$\int_M \big(n - |X|^2 + \lambda\langle X, N\rangle\big)e^{-\frac{|X|^2}{2}}d\mu = 0, \tag{2.8}$$

$$\begin{aligned}\int_M \langle X, a\rangle|X|^2 e^{-\frac{|X|^2}{2}}d\mu \\ = \int_M \Big(2n\lambda\langle N, a\rangle + 2\lambda\langle X, a\rangle(\lambda - H) - \lambda\langle N, a\rangle|X|^2\Big)e^{-\frac{|X|^2}{2}}d\mu,\end{aligned} \tag{2.9}$$

$$\int_M \langle X, a\rangle^2 e^{-\frac{|X|^2}{2}}d\mu = \int_M \Big(|a^T|^2 + \lambda\langle N, a\rangle\langle X, a\rangle\Big)e^{-\frac{|X|^2}{2}}d\mu, \tag{2.10}$$

where $a^T = \sum_i < a, e_i > e_i$.

$$\begin{aligned}\int_M \Big(|X|^2 - n - \frac{\lambda(\lambda - H)}{2}\Big)^2 e^{-\frac{|X|^2}{2}}d\mu \\ = \int_M \Big\{(\frac{\lambda^2}{4} - 1)(\lambda - H)^2 + 2n - H^2 + \lambda^2\Big\}e^{-\frac{|X|^2}{2}}d\mu.\end{aligned} \tag{2.11}$$

Let $X(s): M \to \mathbb{R}^{n+1}$ a variation of X with $X(0) = X$ and $\frac{\partial}{\partial s}X(s)|_{s=0} = fN$. For $X_0 \in \mathbb{R}^{n+1}$ and a real number t_0, $\mathcal{F}$*-functional* is defined by

$$\begin{aligned}\mathcal{F}_{X_s,t_s}(s) &= \mathcal{F}_{X_s,t_s}(X(s))\\ &= (4\pi t_s)^{-\frac{n}{2}}\int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &\quad + \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\int_M \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,\end{aligned}$$

where X_s and t_s denote variations of X_0 and t_0. Let

$$\frac{\partial t_s}{\partial s} = h(s), \quad \frac{\partial X_s}{\partial s} = y(s), \quad \frac{\partial X(s)}{\partial s} = f(s)N(s),$$

one calls that $X: M \to \mathbb{R}^{n+1}$ is *a critical point of* $\mathcal{F}_{X_s,t_s}(s)$ if it is critical with respect to all normal variations and all variations in X_0 and t_0.

Lemma 2.4. *Let $X(s)$ be a variation of X with normal variation vector field $\frac{\partial X(s)}{\partial s}|_{s=0} = fN$. If X_s and t_s are variations of X_0 and t_0 with $\frac{\partial X_s}{\partial s}|_{s=0} = y$ and $\frac{\partial t_s}{\partial s}|_{s=0} = h$, then the first variation formula of $\mathcal{F}_{X_s,t_s}(s)$ is given by*

$$\begin{aligned}(2.12)\quad \mathcal{F}'_{X_0,t_0}(0) &= (4\pi t_0)^{-\frac{n}{2}}\int_M\Big(\lambda - (H + \langle\frac{X-X_0}{t_0}, N\rangle)\Big)fe^{-\frac{|X-X_0|^2}{2}}d\mu\\ &\quad + (4\pi t_0)^{-\frac{n}{2}}\int_M\Big(\langle\frac{X-X_0}{t_0}, y\rangle - \lambda\langle N, y\rangle\Big)e^{-\frac{|X-X_0|^2}{2}}d\mu\\ &\quad + (4\pi t_0)^{-\frac{n}{2}}\int_M\Big(\frac{|X-X_0|^2}{t_0} - n - \lambda\langle X-X_0, N\rangle\Big)\frac{h}{2t_0}e^{-\frac{|X-X_0|^2}{2}}d\mu.\end{aligned}$$

Proof. Defining

$$(2.13)\quad \mathbb{A}(s) = \int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s, \quad \mathbb{V}(s) = \int_M \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,$$

then

$$\begin{aligned}\mathcal{F}'_{X_s,t_s}(s) &= (4\pi t_s)^{-\frac{n}{2}}\mathbb{A}'(s) + \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\mathbb{V}'(s)\\ &\quad - (4\pi t_s)^{-\frac{n}{2}}\frac{n}{2t_s}h\mathbb{A}(s) - \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\frac{h}{2t_s}\mathbb{V}(s).\end{aligned}$$

Since

$$\mathbb{A}^{'}(s)=\int_M\bigg\{-\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle+\frac{|X(s)-X_s|^2}{2t_s^2}h \\ -H_s\langle\frac{\partial X(s)}{\partial s},N(s)\rangle\bigg\}e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s,$$

$$\mathbb{V}^{'}(s)=\int_M\langle\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s},N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,$$

we have

$$\begin{aligned}(2.14)\quad \mathcal{F}^{'}_{X_s,t_s}(s)=&(4\pi t_s)^{-\frac{n}{2}}\int_M-(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M\lambda f\langle N(s),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\ &+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M\lambda\langle -y,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\ &+(4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M-\frac{h\lambda}{2t_s}\langle X(s)-X_s,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu.\end{aligned}$$

If $s=0$, then $X(0)=X$, $X_s=X_0$, $t_s=t_0$ and

$$\begin{aligned}\mathcal{F}^{'}_{X_0,t_0}(0)=&(4\pi t_0)^{-\frac{n}{2}}\int_M\bigg(\lambda-(H+\langle\frac{X-X_0}{t_0},N\rangle)\bigg)fe^{-\frac{|X-X_0|^2}{2}}d\mu\\ &+(4\pi t_0)^{-\frac{n}{2}}\int_M\bigg(\langle\frac{X-X_0}{t_0},y\rangle-\lambda\langle N,y\rangle\bigg)e^{-\frac{|X-X_0|^2}{2}}d\mu\\ &+(4\pi t_0)^{-\frac{n}{2}}\int_M\bigg(\frac{|X-X_0|^2}{t_0}-n-\lambda\langle X-X_0,N\rangle\bigg)\frac{h}{2t_0}e^{-\frac{|X-X_0|^2}{2}}d\mu.\end{aligned}$$

□

From Lemma 2.4, we know that if $X:M\to\mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$, then

$$H+\langle\frac{X-X_0}{t_0},N\rangle=\lambda.$$

We next prove that if $H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda$, then $X : M \to \mathbb{R}^{n+1}$ must be a critical point of $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$. For simplicity, we only consider the case of $X_0 = 0$ and $t_0 = 1$. In this case, $H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda$ becomes

$$H + \langle X, N\rangle = \lambda. \tag{2.15}$$

Furthermore, we know that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$ if and only if $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}$-functional $\mathcal{F}_{X_0,t_0}(s)$ with respect to fixed X_0 and t_0.

Theorem 2.1. *$X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{X_s,t_s}(s)$ if and only if*

$$H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda.$$

Proof. We only prove the result for $X_0 = 0$ and $t_0 = 1$. In this case, the first variation formula (2.12) becomes

$$\begin{aligned} \mathcal{F}_{0,1}'(0) = {} & (4\pi)^{-\frac{n}{2}} \int_M \Big(\lambda - (H + \langle X, N\rangle)\Big) f e^{-\frac{|X|^2}{2}} d\mu \\ & + (4\pi)^{-\frac{n}{2}} \int_M \Big(\langle X, y\rangle - \lambda\langle N, y\rangle\Big) e^{-\frac{|X|^2}{2}} d\mu \\ & + (4\pi)^{-\frac{n}{2}} \int_M \Big(|X|^2 - n - \lambda\langle X, N\rangle\Big) \frac{h}{2} e^{-\frac{|X|^2}{2}} d\mu. \end{aligned} \tag{2.16}$$

If $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{0,1}$, then $X : M \to \mathbb{R}^{n+1}$ should satisfy $H + \langle X, N\rangle = \lambda$. Conversely, if $H + \langle X, N\rangle = \lambda$ is satisfied, then we know that $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface. Therefore, the last two terms in (2.16) vanish for any h and any y from (2.7) and (2.8) of Lemma 2.3. Therefore $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{0,1}$. □

Corollary 2.2. *$X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{X_s,t_s}(s)$ if and only if M is the critical point of $\mathcal{F}$-functional with respect to fixed X_0 and t_0.*

3. The second variation of $\mathcal{F}$-functional

In this section, we shall give the second variation formula of $\mathcal{F}$-functional.

Theorem 3.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a critical point of the functional $\mathcal{F}(s) = \mathcal{F}_{X_s,t_s}(s)$. The second variation formula of $\mathcal{F}(s)$ for $X_0 = 0$ and $t_0 = 1$ is given by*

$$
\begin{aligned}
(4\pi)^{\frac{n}{2}}\mathcal{F}''(0) = & -\int_M fLfe^{-\frac{|X|^2}{2}}d\mu + \int_M\big(-|y|^2+\langle X,y\rangle^2\big)e^{-\frac{|X|^2}{2}}d\mu \\
& + \int_M\Big\{2\langle N,y\rangle + (n+1-|X|^2)\lambda h - 2hH - 2\lambda\langle X,y\rangle\Big\}fe^{-\frac{|X|^2}{2}}d\mu \\
& + \int_M\Big\{(|X|^2-n-1)\langle X,y\rangle\Big\}he^{-\frac{|X|^2}{2}}d\mu \\
& + \int_M\Big\{\frac{n^2+2n}{4} - \frac{n+2}{2}|X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4}(\lambda-H)\Big\}h^2e^{-\frac{|X|^2}{2}}d\mu,
\end{aligned}
$$

where the operator L is defined by

$$L = \mathcal{L} + S + 1 - \lambda^2.$$

Proof. Let

$$
\begin{aligned}
I(s) = & (4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s + \langle\frac{X(s)-X_s}{t_s}, N(s)\rangle)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s \\
& + (4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M \lambda f\langle N(s),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu, \\
II(s) = & (4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s}, y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s \\
& + (4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M \lambda\langle -y,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu, \\
III(s) = & (4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2})he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s \\
& + (4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M -\frac{h\lambda}{2t_s}\langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,
\end{aligned}
$$

we have

$$
\begin{aligned}
& \mathcal{F}'(s) = I(s) + II(s) + III(s), \quad \mathcal{F}''(s) = I'(s) + II'(s) + III'(s), \\
& I'(s) = (4\pi t_s)^{-\frac{n}{2}}\int_M \frac{nh}{2t_s}(H_s + \langle\frac{X(s)-X_s}{t_s}, N(s)\rangle)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s \\
& \quad + (4\pi t_s)^{-\frac{n}{2}}\int_M -\Big(\frac{dH_s}{ds} + \langle\frac{\frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}}{t_s}, N(s)\rangle - \langle\frac{X(s)-X_s}{t_s^2}, N(s)\rangle h \\
& \qquad + \langle\frac{X(s)-X_s}{t_s}, \frac{dN(s)}{ds}\rangle\Big)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s
\end{aligned}
$$

$$
\begin{aligned}
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M -(H_s + \langle \frac{X(s)-X_s}{t_s}, N(s)\rangle) f^{'} e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (H_s + \langle \frac{X(s)-X_s}{t_s}, N(s)\rangle) \\
&\qquad \times (\langle \frac{X(s)-X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}\rangle + H_s f) f e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M -(H_s + \langle \frac{X(s)-X_s}{t_s}, N(s)\rangle) f \frac{|X(s)-X_s|^2}{2t_s^2} h \\
&\qquad \times e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h}{2t_s} \lambda \langle N(s), N\rangle f e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M \lambda f^{'} \langle N(s), N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M \lambda f \langle \frac{dN(s)}{ds}, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu,
\end{aligned}
$$

$$
\begin{aligned}
II^{'}(s) = {}& (4\pi t_s)^{-\frac{n}{2}} (-\frac{nh}{2t_s}) \int_M \langle \frac{X(s)-X_s}{t_s}, y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (\langle \frac{\frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}}{t_s}, y\rangle - \langle \frac{X(s)-X_s}{t_s^2}, y\rangle h) e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M \langle \frac{X(s)-X_s}{t_s}, y^{'}\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M \langle \frac{X(s)-X_s}{t_s}, y\rangle \Big(-\langle \frac{X(s)-X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}\rangle \\
&\qquad - H_s f \Big) e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M \langle \frac{X(s)-X_s}{t_s}, y\rangle \frac{|X(s)-X_s|^2}{2t_s^2} h e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} (-\frac{h}{2t_s}) \int_M -\lambda \langle N, y\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\lambda \langle N, y^{'}\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu,
\end{aligned}
$$

$$
\begin{aligned}
III^{'}(s) = {}& (4\pi t_s)^{-\frac{n}{2}} (-\frac{nh}{2t_s}) \int_M (-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2}) h e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (\frac{nh}{2t_s^2} - \frac{|X(s)-X_s|^2}{t_s^3} h + \frac{\langle X(s)-X_s, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}\rangle}{t_s^2}) \\
&\qquad \times h e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s
\end{aligned}
$$

$$
\begin{aligned}
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2}) h^{'} e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2}) h(-H_s f \\
&\qquad - \langle \frac{X(s)-X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}\rangle) e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2}) h \frac{|X(s)-X_s|^2}{2t_s^2} h e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} (-\frac{h}{2t_s}) \int_M -\frac{h}{2t_s} \lambda \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h^{'}\lambda}{2t_s} \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M (\frac{h}{2t_s^2} \langle X(s)-X_s, N\rangle \lambda h \\
&\qquad - \frac{1}{2t_s} \langle \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}, N\rangle \lambda h) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu.
\end{aligned}
$$

Since $X : M \to \mathbb{R}^{n+1}$ is a critical point, we get

$$
\begin{aligned}
H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda, \\
\int_M (n + \lambda\langle X-X_0, N\rangle - \frac{|X-X_0|^2}{t_0}) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0, \\
\int_M (\lambda\langle N, a\rangle - \langle \frac{X-X_0}{t_0}, a\rangle) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0.
\end{aligned}
$$

On the other hand,

$$
H^{'} = \Delta f + Sf, \quad N^{'} = -\nabla f.
$$

Using of the above equations and letting $s = 0$, we obtain

$$
\begin{aligned}
(4\pi t_0)^{\frac{n}{2}} \mathcal{F}^{''}(0) = &\int_M -fLf e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&+ \int_M \Big(\frac{2}{t_0}\langle N, y\rangle + \frac{2h}{t_0}\langle \frac{X-X_0}{t_0}, N\rangle + \frac{n-1}{t_0}\lambda h \\
&\qquad - \frac{|X-X_0|^2}{t_0^2}\lambda h - 2\lambda\langle \frac{X-X_0}{t_0}, y\rangle \Big) f e^{-\frac{|X-X_0|^2}{2t_0}} d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\int_M\Big(-\frac{n+2}{t_0}\langle\frac{X-X_0}{t_0},y\rangle+\frac{\lambda}{t_0}\langle N,y\rangle\\
&\qquad+\langle\frac{X-X_0}{t_0},y\rangle\frac{|X-X_0|^2}{t_0^2}\Big)he^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M\Big(\frac{n^2}{4t_0^2}+\frac{n}{2t_0^2}-\frac{n+2}{2t_0^3}|X-X_0|^2+\frac{|X-X_0|^4}{4t_0^4}\\
&\qquad+\frac{3\lambda}{4t_0}\langle\frac{X-X_0}{t_0},N\rangle\Big)h^2e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M\Big(-\frac{1}{t_0}\langle y,y\rangle+\langle\frac{X-X_0}{t_0},y\rangle^2\Big)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,
\end{aligned}
$$

where the operator L is defined by $L=\Delta+S+\frac{1}{t_0}-\langle\frac{X-X_0}{t_0},\nabla\rangle-\lambda^2$. When $t_0=1$, $X_0=0$, then $L=\mathcal{L}+S+1-\lambda^2$.

$$
\begin{aligned}
(4\pi)^{\frac{n}{2}}\mathcal{F}''(0)=&\int_M -fLfe^{-\frac{|X-|^2}{2}}d\mu\\
&+\int_M\Big(2\langle N,y\rangle+2\lambda h+(n-1)\lambda h-2hH\\
&\qquad-|X|^2\lambda h-2\lambda\langle X,y\rangle\Big)fe^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big(\lambda\langle N,y\rangle-(n+2)\langle X,y\rangle+\langle X,y\rangle|X|^2\Big)he^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big(\frac{n^2+2n}{4}-\frac{n+2}{2}|X|^2+\frac{|X|^4}{4}+\frac{3\lambda}{4}\langle X,N\rangle\Big)h^2e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M-(|y|^2-\langle X,y\rangle^2)e^{-\frac{|X|^2}{2}}d\mu\\
=&\int_M -fLfe^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big[2\langle N,y\rangle+(n+1-|X|^2)\lambda h-2hH-2\lambda\langle X,y\rangle\Big]fe^{-\frac{|X-|^2}{2}}d\mu\\
&+\int_M\Big\{(|X|^2-n-1)\langle X,y\rangle\Big\}he^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big(\frac{n^2+2n}{4}-\frac{n+2}{2}|X|^2+\frac{|X|^4}{4}+\frac{3\lambda}{4}(\lambda-H)\Big)h^2e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M(-|y|^2+\langle X,y\rangle^2)e^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}
$$

$\square$

Definition 3.1. *One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$ is $\mathcal{F}$-stable if, for every normal variation fN, there exist variations of X_0 and t_0 such that $\mathcal{F}''_{X_0,t_0}(0) \geq 0$;*

One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$ is $\mathcal{F}$-unstable if there exist a normal variation fN such that for all variations of X_0 and t_0, $\mathcal{F}''_{X_0,t_0}(0) < 0$.

Theorem 3.2. *If $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-stable; If $\sqrt{n} < r \leq \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-unstable.*

Proof. For the sphere $S^n(r)$, we have

$$X = -rN, \quad H = \frac{n}{r}, \quad S = \frac{H^2}{n} = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r$$

and

$$Lf = \mathcal{L}f + (S + 1 - \lambda^2)f = \Delta f + (\frac{n}{r^2} + 1 - \lambda^2)f. \tag{3.1}$$

Since we know that eigenvalues μ_k of Δ on the sphere $S^n(r)$ are given by

$$\mu_k = \frac{k^2 + (n-1)k}{r^2}, \tag{3.2}$$

and constant functions are eigenfunctions corresponding to eigenvalue $\mu_0 = 0$. For any constant vector $z \in \mathbb{R}^{n+1}$, we get

$$-\Delta\langle z, N\rangle = \Delta\langle z, \frac{X}{r}\rangle = \langle z, \frac{1}{r}HN\rangle = \frac{n}{r^2}\langle z, N\rangle, \tag{3.3}$$

that is, $\langle z, N\rangle$ is an eigenfunction of Δ corresponding to the first eigenvalue $\mu_1 = \frac{n}{r^2}$. Hence, for any normal variation with the variation vector field fN, we can choose a real number $a \in \mathbb{R}$ and a constant vector $z \in \mathbb{R}^{n+1}$ such that

$$f = f_0 + a + \langle z, N\rangle, \tag{3.4}$$

and f_0 is in the space spanned by all eigenfunctions corresponding to eigenvalues μ_k $(k \geq 2)$ of Δ on $S^n(r)$. Using Lemma 2.3, we get

(3.5)

$$\begin{aligned}
&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0)\\
= &\int_{S^n(r)} -(f_0 + a + \langle z, N\rangle) L(f_0 + a + \langle z, N\rangle) d\mu\\
&+ \int_{S^n(r)} [2\langle N, y\rangle + (n+1-r^2)\lambda h - 2\frac{n}{r}h + 2\lambda\langle rN, y\rangle](f_0 + a + \langle z, N\rangle) d\mu\\
&+ \int_{S^n(r)} \lambda\langle N, y\rangle (r^2 - n - 1) h d\mu\\
&+ \int_{S^n(r)} (\frac{n^2+2n}{4} - \frac{n+2}{2}r^2 + \frac{r^4}{4} + \frac{3}{4}r^2 - \frac{3}{4}n) h^2 d\mu\\
&+ \int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu\\
\geq &\int_{S^n(r)} \left\{ (\frac{n+2}{r^2} - 1 + \lambda^2) f_0^2 - (\frac{n}{r^2} + 1 - \lambda^2) a^2 + (\lambda^2 - 1)\langle z, N\rangle^2 \right\} d\mu\\
&+ \int_{S^n(r)} \left\{ 2(1+\lambda r)\langle N, y\rangle\langle N, z\rangle + [(n+1-r^2)\lambda - 2\frac{n}{r}] ah \right\} d\mu\\
&+ \int_{S^n(r)} \frac{1}{4}[r^4 - (2n+1)r^2 + n(n-1)] h^2 d\mu\\
&+ \int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu.
\end{aligned}$$

From Lemma 2.3, we have

$$\int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu = -\int_{S^n(r)} (1+\lambda r)\langle N, y\rangle^2 d\mu. \tag{3.6}$$

Putting (3.6) and $\lambda = \frac{n}{r} - r$ into (3.5), we obtain

$$\begin{aligned}
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) \geq &\int_{S^n(r)} \frac{1}{r^2} \left\{ (r^2 - n - \frac{1}{2})^2 + \frac{7}{4} \right\} f_0^2 d\mu\\
&+ \int_{S^n(r)} [r^4 - (2n+1)r^2 + n(n-1)](\frac{a}{r} + \frac{h}{2})^2 d\mu\\
&+ \int_{S^n(r)} \frac{1}{r^2}[r^4 - (2n+1)r^2 + n^2]\langle z, N\rangle^2 d\mu
\end{aligned} \tag{3.7}$$

$$+\int_{S^n(r)} 2(1+n-r^2)\langle N,y\rangle\langle N,z\rangle d\mu$$
$$+\int_{S^n(r)} -(1+n-r^2)\langle N,y\rangle^2 d\mu.$$

If we choose $h=-\frac{2a}{r}$ and $y=kz$, then we have

(3.8)
$$\begin{aligned}(4\pi)^{\frac{n}{2}}e^{\frac{r^2}{2}}\mathcal{F}''(0)\geq&\int_{S^n(r)}\frac{1}{r^2}\left\{(r^2-n-\frac{1}{2})^2+\frac{7}{4}\right\}f_0^2d\mu\\&+\int_{S^n(r)}\left\{\lambda^2-1+2(1+\lambda r)k-(1+\lambda r)k^2\right\}\langle z,N\rangle^2d\mu\\=&\int_{S^n(r)}\frac{1}{r^2}\left\{(r^2-n-\frac{1}{2})^2+\frac{7}{4}\right\}f_0^2d\mu\\&+\int_{S^n(r)}\left\{\lambda^2+\lambda r-(1+\lambda r)(1-k)^2\right\}\langle z,N\rangle^2d\mu.\end{aligned}$$

We next consider three cases:

Case 1: $r\leq\sqrt{n}$

In this case, $\lambda\geq 0$. Taking $k=1$, then we get

$$\mathcal{F}''(0)\geq 0.$$

Case 2: $r\geq\frac{1+\sqrt{1+4n}}{2}$.

In this case, $\lambda\leq -1$. Taking $k=2$, we can get

$$\mathcal{F}''(0)\geq 0.$$

Case 3: $\sqrt{n+1}<r<\frac{1+\sqrt{1+4n}}{2}$.

In this case, $-1<\lambda<0$, $1+\lambda r<0$, we can take k such that $(1-k)^2\geq\frac{\lambda(\lambda+r)}{1+\lambda r}$, then we have

$$\mathcal{F}''(0)\geq 0.$$

Thus, if $r\leq\sqrt{n}$ or $r>\sqrt{n+1}$, the n-dimensional round sphere $X:S^n(r)\to\mathbb{R}^{n+1}$ is $\mathcal{F}$-stable;

If $\sqrt{n} < r \leq \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-unstable. In fact, in this case, $-1 < \lambda < 0$, $1 + \lambda r \geq 0$. We can choose f such that $f_0 = 0$, then we have

$$\begin{aligned}
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) \leq & \int_{S^n(r)} (\lambda^2 - 1)\langle z, N\rangle^2 d\mu \\
& + \int_{S^n(r)} 2(1 + \lambda r)\langle N, y\rangle\langle N, z\rangle d\mu \\
& + \int_{S^n(r)} -(1 + \lambda r)\langle N, y\rangle^2 d\mu \\
= & (\lambda^2 + \lambda r) \int_{S^n(r)} \langle z, N\rangle^2 d\mu \\
& - (1 + \lambda r) \int_{S^n(r)} (\langle z, N\rangle - \langle y, N\rangle)^2 d\mu \\
< & \ 0.
\end{aligned} \tag{3.9}$$

This completes the proof of Theorem 3.2. □

According to Theorem 3.2, we would like to propose the following:

Problem 3.1. Is it possible to prove that spheres $S^n(r)$ with $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$ are the only $\mathcal{F}$-stable compact λ-hypersurfaces?

Remark 3.1. *Colding and Minicozzi [5] have proved that the sphere $S^n(\sqrt{n})$ is the only $\mathcal{F}$-stable compact self-shrinkers. In order to prove this result, the property that the mean curvature H is an eigenfunction of L-operator plays a very important role. But for λ-hypersurfaces, the mean curvature H is not an eigenfunction of L-operator in general.*

4. The weak stability of the weighted area functional for weighted volume-preserving variations

Define

$$\mathcal{T}(s) = (4\pi t_s)^{-\frac{n}{2}} \int_M e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s. \tag{4.1}$$

We compute the first and the second variation formulas of the general $\mathcal{T}$-functional for weighted volume-preserving variations with fixed X_0 and t_0.

By a direct calculation, we have

$$\begin{aligned}\mathcal{T}^{'}(s) &= (4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s,\\ \mathcal{T}^{''}(s) &= (4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)f^{'}e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M (H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)\\ &\qquad\times(\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}\rangle+H_sf)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\ &\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M -\left(\frac{dH_s}{ds}+\langle\frac{\frac{\partial X(s)}{\partial s}}{t_s},N(s)\rangle\right.\\ &\qquad\left.+\langle\frac{X(s)-X_s}{t_s},\frac{dN(s)}{ds}\rangle\right)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s.\end{aligned}$$

Lemma 4.1.

$$\int_M f^{'}(0)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0.$$

Proof. Since $V(t)=\int_M\langle X(t)-X_0,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=V(0)$ for any t, we have

$$\int_M f(t)\langle N(t),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0.$$

Hence, we get

$$\begin{aligned}0&=\frac{d}{dt}|_{t=0}\int_M f(t)\langle N(t),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\ &=\int_M f^{'}(0)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu.\end{aligned}$$

□

Since M is a critical point of $\mathcal{T}(s)$, we have

$$H+\langle\frac{X-X_0}{t_0},N\rangle=\lambda.$$

On the other hand, we have

$$H^{'}=\Delta f+Sf,\quad N^{'}=-\nabla f. \tag{4.2}$$

Then for $t_0 = 1$ and $X_0 = 0$, the second variation formula becomes

$$(4\pi)^{\frac{n}{2}}\mathcal{T}''(0) = \int_M -f\big(\mathcal{L}f + (S + 1 - \lambda^2)f\big)e^{-\frac{|X|^2}{2}}d\mu.$$

Theorem 4.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a critical point of the functional $\mathcal{T}(s)$ for the weighted volume-preserving variations with fixed $X_0 = 0$ and $t_0 = 1$. The second variation formula of $\mathcal{T}(s)$ is given by*

$$(4\pi)^{\frac{n}{2}}\mathcal{T}''(0) = \int_M -f\big(\mathcal{L}f + (S + 1 - \lambda^2)f\big)e^{-\frac{|X|^2}{2}}d\mu. \tag{4.3}$$

Definition 4.1. *A critical point $X : M \to \mathbb{R}^{n+1}$ of the functional $\mathcal{T}(s)$ is called weakly stable if, for any weighted volume-preserving normal variation, $\mathcal{T}''(0) \geq 0$;*

A critical point $X : M \to \mathbb{R}^{n+1}$ of the functional $\mathcal{T}(s)$ is called weakly unstable if there exists a weighted volume-preserving normal variation, such that $\mathcal{T}''(0) < 0$.

Theorem 4.2. *If $r \leq \frac{-1+\sqrt{1+4n}}{2}$ or $r \geq \frac{1+\sqrt{1+4n}}{2}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly stable; If $\frac{-1+\sqrt{1+4n}}{2} < r < \frac{1+\sqrt{1+4n}}{2}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly unstable.*

Proof. For the sphere $S^n(r)$, we have

$$X = -rN, \quad H = \frac{n}{r}, \quad S = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r$$

and

$$Lf = \mathcal{L}f + (S + 1 - \lambda^2)f = \Delta f + (\frac{n}{r^2} + 1 - \lambda^2)f. \tag{4.4}$$

Since we know that eigenvalues μ_k of Δ on the sphere $S^n(r)$ are given by

$$\mu_k = \frac{k^2 + (n-1)k}{r^2}, \tag{4.5}$$

and constant functions are eigenfunctions corresponding to eigenvalue $\mu_0 = 0$. For any constant vector $z \in \mathbb{R}^{n+1}$, we get

$$-\Delta\langle z, N\rangle = \frac{n}{r^2}\langle z, N\rangle, \tag{4.6}$$

that is, $\langle z, N\rangle$ is an eigenfunction of Δ corresponding to the first eigenvalue $\mu_1 = \frac{n}{r^2}$. Hence, for any weighted volume-preserving normal variation with the variation vector field fN satisfying

$$\int_{S^n(r)} fe^{-\frac{r^2}{2}} d\mu = 0,$$

we can choose a constant vector $z \in \mathbb{R}^{n+1}$ such that

$$f = f_0 + \langle z, N \rangle, \tag{4.7}$$

and f_0 is in the space spanned by all eigenfunctions corresponding to eigenvalues μ_k $(k \geq 2)$ of Δ on $S^n(r)$. By making use of Theorem 4.1, we have

$$\begin{aligned} (4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0) &= \int_{S^n(r)} -(f_0 + \langle z, N \rangle) L(f_0 + \langle z, N \rangle) d\mu \\ &\geq \int_{S^n(r)} \left\{ (\frac{n+2}{r^2} - 1 + \lambda^2) f_0^2 + (\lambda^2 - 1) \langle z, N \rangle^2 \right\} d\mu. \end{aligned} \tag{4.8}$$

According to $\lambda = \frac{n}{r} - r$, we obtain

$$\begin{aligned} (4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0) \geq & \int_{S^n(r)} \frac{1}{r^2} \{ (r^2 - n - \frac{1}{2})^2 + \frac{7}{4} \} f_0^2 d\mu \\ & + \int_{S^n(r)} (\frac{n}{r} - r - 1)(\frac{n}{r} - r + 1) \langle z, N \rangle^2 d\mu \geq 0 \end{aligned}$$

if

$$r \leq \frac{-1 + \sqrt{4n+1}}{2} \quad \text{or} \quad r \geq \frac{1 + \sqrt{4n+1}}{2}.$$

Thus, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly stable.

If

$$\frac{-1 + \sqrt{4n+1}}{2} < r < \frac{1 + \sqrt{4n+1}}{2},$$

choosing $f =< z, N >$, we have

$$\int_{S^n(r)} fe^{-\frac{r^2}{2}} d\mu = 0.$$

Hence, there exists a weighted volume-preserving normal variation with the variation vector filed fN such that

$$(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0) = \int_{S^n(r)} (\frac{n}{r} - r - 1)(\frac{n}{r} - r + 1) \langle z, N \rangle^2 d\mu < 0.$$

Thus, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly unstable. It finishes the proof. □

Remark 4.1. *From Theorem 3.2 and Theorem 4.2, we know the $\mathcal{F}$-stability and the weak stability are different. The $\mathcal{F}$-stability is a weaker notation than the weak stability.*

Remark 4.2. *Is it possible to prove that spheres $S^n(r)$ with $r \le \frac{-1+\sqrt{1+4n}}{2}$ or $r \ge \frac{1+\sqrt{1+4n}}{2}$ are the only weak stable compact λ-hypersurfaces?*

5. Properness and polynomial area growth for λ-hypersurfaces

For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Bishop and Gromov says that geodesic balls have at most polynomial area growth:

$$\mathrm{Area}(B_r(x_0)) \le Cr^n.$$

For n-dimensional complete and non-compact gradient shrinking Ricci soliton, Cao and Zhou [1] have proved geodesic balls have at most polynomial area growth. For self-shrinkers, Ding and Xin [7] proved that any complete non-compact properly immersed self-shrinker in the Euclidean space has polynomial area growth. X. Cheng and Zhou [5] showed that any complete immersed self-shrinker with polynomial area growth in the Euclidean space is proper. Hence any complete immersed self-shrinker is proper if and only if it has polynomial area growth.

It is our purposes in this section to study the area growth for λ-hypersurfaces. First of all, we study the equivalence of properness and polynomial area growth for λ-hypersurfaces. If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say M has polynomial area growth if there exist constant C and d such that for all $r \ge 1$,

$$\mathrm{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \le Cr^d, \tag{5.1}$$

where $B_r(0)$ is a round ball in $\mathbb{R}^{n+1}$ with radius r and centered at the origin.

Theorem 5.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then, there is a* *positive constant C such that for $r \geq 1$,*

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq C r^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}}, \tag{5.2}$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$.

Proof. Since $X : M \to \mathbb{R}^{n+1}$ is a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, we have

$$\langle X, N\rangle + H = \lambda.$$

Defining $f = \frac{|X|^2}{4}$, we have

$$f - |\nabla f|^2 = \frac{|X|^2}{4} - \frac{|X^T|^2}{4} = \frac{|X^\perp|^2}{4} = \frac{1}{4}(\lambda - H)^2, \tag{5.3}$$

$$\begin{aligned} \Delta f &= \frac{1}{2}(n + H\langle N, X\rangle) \\ &= \frac{1}{2}(n + \lambda\langle N, X\rangle - \langle N, X\rangle^2) \\ &= \frac{1}{2}n + \frac{\lambda^2}{4} - \frac{H^2}{4} - f + |\nabla f|^2. \end{aligned} \tag{5.4}$$

Hence, we obtain

$$|\nabla(f-\beta)|^2 \leq (f-\beta), \tag{5.5}$$

$$\Delta(f-\beta) - |\nabla(f-\beta)|^2 + (f-\beta) \leq (\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4}). \tag{5.6}$$

Since the immersion X is proper, we know that $\overline{f} = f - \beta$ is proper. Applying Theorem 2.1 of X. Cheng and Zhou [5] to $\overline{f} = f - \beta$ with $k = (\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4})$, we obtain

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq C r^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}}, \tag{5.7}$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$ and C is a constant. □

Remark 5.1. *The estimate in Theorem 5.1 is the best possible because the cylinders $S^k(r_0) \times \mathbb{R}^{n-k}$ satisfy the equality.*

Remark 5.2. *By making use of the same assertions as in X. Cheng and Zhou [5] for self-shrinkers, we can prove the weighted area of a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is bounded.*

By making use of to the same assertions as in X. Cheng and Zhou [5] for self-shrinkers, we can prove the following theorem. We will leave it for readers.

Theorem 5.2. *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete immersed λ-hypersurface with polynomial area growth, then $X : M \to \mathbb{R}^{n+1}$ is proper.*

6. A lower bound growth of the area for λ-hypersurfaces

For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Calabi and Yau says that geodesic balls have at least linear area growth:

$$\text{Area}(B_r(x_0)) \geq Cr.$$

Cao and Zhu [2] have proved that n-dimensional complete and non-compact gradient shrinking Ricci soliton must have infinite volume. Furthermore, Munteanu and Wang [12] have proved that areas of geodesic balls for n-dimensional complete and non-compact gradient shrinking Ricci soliton has at least linear growth. For self-shrinkers, Li and Wei [11] proved that any complete and non-compact proper self-shrinker has at least linear area growth.

In this section, we study the lower bound growth of the area for λ-hypersurfaces. The following lemmas play a very important role in order to prove our results.

Lemma 6.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete noncompact proper λ-hypersurface, then there exist constants $C_1(n, \lambda)$ and $c(n, \lambda)$ such that for all $t \geq C_1(n, \lambda)$,*

$$\begin{aligned} &\text{Area}(B_{t+1}(0) \cap X(M)) - \text{Area}(B_t(0) \cap X(M)) \\ &\quad \leq c(n, \lambda) \frac{\text{Area}(B_t(0) \cap X(M))}{t} \end{aligned} \tag{6.1}$$

and

$$\text{Area}(B_{t+1}(0) \cap X(M)) \leq 2\text{Area}(B_t(0) \cap X(M)). \tag{6.2}$$

Proof. Since $X : M \to \mathbb{R}^{n+1}$ is a complete λ-hypersurface, one has

$$\frac{1}{2}\Delta|X|^2 = n + H\langle N, X\rangle = n + H\lambda - H^2. \tag{6.3}$$

Integrating (6.3) over $B_r(0) \cap X(M)$, we obtain

$$\begin{aligned} & n\text{Area}(B_r(0) \cap X(M)) + \int_{B_r(0)\cap X(M)} H\lambda d\mu - \int_{B_r(0)\cap X(M)} H^2 d\mu \\ &= \frac{1}{2}\int_{B_r(0)\cap X(M)} \triangle|X|^2 d\mu \\ &= \frac{1}{2}\int_{\partial(B_r(0)\cap X(M))} \nabla|X|^2 \cdot \frac{\nabla\rho}{|\nabla\rho|} d\sigma \\ &= \int_{\partial(B_r(0)\cap X(M))} |X^T| d\sigma \\ &= \int_{\partial(B_r(0)\cap X(M))} \frac{|X|^2 - (\lambda - H)^2}{|X^T|} d\sigma \\ &= r(\text{Area}(B_r(0) \cap X(M)))' - \int_{\partial(B_r(0)\cap X(M))} \frac{(\lambda - H)^2}{|X^T|} d\sigma, \end{aligned} \tag{6.4}$$

where $\rho(x) := |X(x)|$, $\nabla\rho = \frac{X^T}{|X|}$. Here we used, from the co-area formula,

$$\big(\text{Area}(B_r(0) \cap X(M))\big)' = r\int_{\partial(B_r(0)\cap X(M))} \frac{1}{|X^T|} d\sigma. \tag{6.5}$$

Hence, we obtain

$$\begin{aligned} & (n + \frac{\lambda^2}{4})\text{Area}(B_r(0) \cap X(M)) - r(\text{Area}(B_r(0) \cap X(M)))' \\ &= \int_{B_r(0)\cap X(M)} (H - \frac{\lambda}{2})^2 d\mu - \int_{\partial(B_r(0)\cap X(M))} \frac{(\lambda - H)^2}{|X^T|} d\sigma, \end{aligned} \tag{6.6}$$

From (6.5), $(H - \lambda)^2 = \langle N, X\rangle^2 \le |X|^2 = r^2$ on $\partial(B_r(0) \cap X(M))$ and (6.6), we conclude

$$\int_{B_r(0)\cap X(M)} (H - \frac{\lambda}{2})^2 d\mu \le (n + \frac{\lambda^2}{4})\text{Area}(B_r(0) \cap X(M)). \tag{6.7}$$

Furthermore, we have

$$\int_{B_r(0)\cap X(M)} (H-\lambda)^2 d\mu \leq \int_{B_r(0)\cap X(M)} 2\big[(H-\frac{\lambda}{2})^2+\frac{\lambda^2}{4}\big] d\mu \leq (2n+\lambda^2)\mathrm{Area}(B_r(0)\cap X(M)), \tag{6.8}$$

$$\int_{B_r(0)\cap X(M)} H^2 d\mu \leq \int_{B_r(0)\cap X(M)} 2\big[(H-\frac{\lambda}{2})^2+\frac{\lambda^2}{4}\big] d\mu \leq (2n+\lambda^2)\mathrm{Area}(B_r(0)\cap X(M)). \tag{6.9}$$

(6.6) implies that

$$\begin{aligned}(6.10)\qquad &\big(r^{-n-\frac{\lambda^2}{4}}\mathrm{Area}(B_r(0)\cap X(M))\big)' \\ &= r^{-n-1-\frac{\lambda^2}{4}}\Big(r\big(\mathrm{Area}(B_r(0)\cap X(M))\big)' \\ &\qquad - (n+\frac{\lambda^2}{4})\mathrm{Area}(B_r(0)\cap X(M))\Big) \\ &= r^{-n-1-\frac{\lambda^2}{4}}\int_{\partial(B_r(0)\cap X(M))}\frac{(H-\lambda)^2}{|X^T|}d\sigma \\ &\quad - r^{-n-1-\frac{\lambda^2}{4}}\int_{B_r(0)\cap X(M)}(H-\frac{\lambda}{2})^2 d\mu.\end{aligned}$$

Integrating (6.10) from r_2 to r_1 ($r_1 > r_2$), one has

$$\begin{aligned}(6.11)\qquad &r_1^{-n-\frac{\lambda^2}{4}}\mathrm{Area}(B_{r_1}(0)\cap X(M)) - r_2^{-n-\frac{\lambda^2}{4}}\mathrm{Area}(B_{r_2}(0)\cap X(M)) \\ &= r_1^{-n-2-\frac{\lambda^2}{4}}\int_{B_{r_1}(0)\cap X(M)}(H-\lambda)^2 d\mu \\ &\quad - r_2^{-n-2-\frac{\lambda^2}{4}}\int_{B_{r_2}(0)\cap X(M)}(H-\lambda)^2 d\mu \\ &\quad + (n+2+\frac{\lambda^2}{4})\int_{r_2}^{r_1} s^{-n-3-\frac{\lambda^2}{4}}(\int_{B_s(0)\cap X(M)}(H-\lambda)^2 d\mu)ds \\ &\quad - \int_{r_2}^{r_1} s^{-n-1-\frac{\lambda^2}{4}}(\int_{B_s(0)\cap X(M)}(H-\frac{\lambda}{2})^2 d\mu)ds \\ &\leq \big(r_1^{-n-2-\frac{\lambda^2}{4}}+r_2^{-n-2-\frac{\lambda^2}{4}}\big)\int_{B_{r_1}(0)\cap X(M)}(H-\lambda)^2 d\mu.\end{aligned}$$

Here we used

$$\left(\int_{B_r(0)\cap X(M)} (H-\lambda)^2 d\mu\right)' = r\int_{\partial(B_r(0)\cap X(M))} \frac{(H-\lambda)^2}{|X^T|}d\sigma$$

and $\mathrm{Area}(B_r(0)\cap X(M))$ is non-decreasing in r from (6.5). Combining (6.11) with (6.8), we have

$$\begin{aligned}\frac{\mathrm{Area}(B_{r_1}(0)\cap X(M))}{r_1^{n+\frac{\lambda^2}{4}}} - \frac{\mathrm{Area}(B_{r_2}(0)\cap X(M))}{r_2^{n+\frac{\lambda^2}{4}}} \\ \leq (2n+\lambda^2)\Big(\frac{1}{r_1^{n+2+\frac{\lambda^2}{4}}} + \frac{1}{r_2^{n+2+\frac{\lambda^2}{4}}}\Big)\mathrm{Area}(B_{r_1}(0)\cap X(M)).\end{aligned} \tag{6.12}$$

Putting $r_1 = t+1$, $r_2 = t > 0$, we get

$$\begin{aligned}\left(1 - \frac{2(2n+\lambda^2)(t+1)^{n+\frac{\lambda^2}{4}}}{t^{n+2+\frac{\lambda^2}{4}}}\right)\mathrm{Area}(B_{t+1}(0)\cap X(M)) \\ \leq \mathrm{Area}(B_t(0)\cap X(M))\Big(\frac{t+1}{t}\Big)^{n+\frac{\lambda^2}{4}}.\end{aligned} \tag{6.13}$$

For t sufficiently large, one has, from (6.13),

$$\begin{aligned}&\mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_t(0)\cap X(M)) \\ &\leq \mathrm{Area}(B_t(0)\cap X(M))\left((1+\frac{1}{t})^n - 1 + \frac{C(t+1)^{2n+\lambda^2}4}{t^{2n+2+\lambda^2}}\right),\end{aligned} \tag{6.14}$$

where C is constant only depended on n, λ. Therefore, there exists some constant $C_1(n,\lambda)$ such that for all $t \geq C_1(n,\lambda)$,

$$\begin{aligned}&\mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_t(0)\cap X(M)) \\ &\leq c(n,\lambda)\frac{\mathrm{Area}(B_t(0)\cap X(M))}{t},\end{aligned} \tag{6.15}$$

$$\mathrm{Area}(B_{t+1}(0)\cap X(M)) \leq 2\mathrm{Area}(B_t(0)\cap X(M)), \tag{6.16}$$

where $c(n,\lambda)$ depends only on n and λ. This completes the proof of Lemma 6.1. □

The following Logarithmic Sobolev inequality for hypersurfaces in Euclidean space is due to Ecker [8],

Lemma 6.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface with measure $d\mu$. Then the following inequality*

$$\begin{aligned}(6.17)\qquad &\int_M f^2(\ln f^2)e^{-\frac{|X|^2}{2}}d\mu - \int_M f^2 e^{-\frac{|X|^2}{2}}d\mu \ln(\int_M f^2 e^{-\frac{|X|^2}{2}}d\mu)\\ &\leq 2\int_M |\nabla f|^2 e^{-\frac{|X|^2}{2}}d\mu + \frac{1}{2}\int_M |H + \langle X, N\rangle|^2 f^2 e^{-\frac{|X|^2}{2}}d\mu\\ &+ C_1(n)\int_M f^2 e^{-\frac{|X|^2}{2}}d\mu,\end{aligned}$$

$$\begin{aligned}(6.18)\qquad &\int_M f^2(\ln f^2)d\mu - \int_M f^2 d\mu \ln(\int_M f^2 d\mu)\\ &\leq 2\int_M |\nabla f|^2 d\mu + \frac{1}{2}\int_M |H|^2 f^2 d\mu + C_2(n)\int_M f^2 d\mu\end{aligned}$$

hold for any nonnegative function f for which all integrals are well-defined and finite, where $C_1(n)$ and $C_2(n)$ are positive constants depending on n.

Corollary 6.1. *For an n-dimensional λ-hypersurface $X : M \to \mathbb{R}^{n+1}$, we have the following inequality*

$$(6.19)\qquad \int_M f^2(\ln f)e^{-\frac{|X|^2}{2}}d\mu \leq \int_M |\nabla f|^2 e^{-\frac{|X|^2}{2}}d\mu + \frac{1}{2}C_1(n) + \frac{1}{4}\lambda^2$$

for any nonnegative function f which satisfies

$$(6.20)\qquad \int_M f^2 e^{-\frac{|X|^2}{2}}d\mu = 1.$$

Lemma 6.3. *([11]) Let $X : M \to \mathbb{R}^{n+1}$ be a complete properly immersed hypersurface. For any $x_0 \in M$, $r \leq 1$, if $|H| \leq \frac{C}{r}$ in $B_r(X(x_0)) \cap X(M)$ for some constant $C > 0$. Then*

$$(6.21)\qquad Area(B_r(X(x_0)) \cap X(M)) \geq \kappa r^n,$$

where $\kappa = \omega_n e^{-C}$.

Lemma 6.4. *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete and non-compact proper λ-hypersurface, then it has infinite area.*

Proof. Let

$$\Omega(k_1,k_2)=\{x\in M: 2^{k_1-\frac{1}{2}}\le \rho(x)\le 2^{k_2-\frac{1}{2}}\},$$

$$A(k_1,k_2)=\text{Area}(X(\Omega(k_1,k_2))),$$

where $\rho(x)=|X(x)|$. Since $X: M\to\mathbb{R}^{n+1}$ is a complete and non-compact proper immersion, $X(M)$ can not be contained in a compact Euclidean ball. Then, for k large enough, $\Omega(k,k+1)$ contains at least 2^{2k-1} disjoint balls

$$B_r(x_i)=\{x\in M:\rho_{x_i}(x)<2^{-\frac{1}{2}}r\},\quad x_i\in M,\ r=2^{-k}$$

where $\rho_{x_i}(x)=|X(x)-X(x_i)|$. Since, in $\Omega(k,k+1)$,

$$\begin{aligned}|H|&\le|H-\lambda|+|\lambda|=|\langle X,N\rangle|+|\lambda|\\&\le|X|+|\lambda|\le 2^k\sqrt{2}+|\lambda|\le\frac{\sqrt{2}+|\lambda|}{r},\end{aligned}\tag{6.22}$$

by using of Lemma 6.3, we get

$$A(k,k+1)\ge\kappa_1 2^{2k-1-kn},\tag{6.23}$$

with $\kappa_1=\omega_n e^{-(\sqrt{2}+|\lambda|)2^{-\frac{1}{2}}}2^{-\frac{n}{2}}$.

Claim: If A$rea(X(M))<\infty$, then, for every $\varepsilon>0$, there exists a large constant $k_0>0$ such that,

$$A(k_1,k_2)\le\varepsilon\quad\text{and}\quad A(k_1,k_2)\le 2^{4n}A(k_1+2,k_2-2),\qquad \text{if}\ \ k_2>k_1>k_0.\tag{6.24}$$

In fact, we may choose $K>0$ sufficiently large such that $k_1\approx\frac{K}{2}$, $k_2\approx\frac{3K}{2}$. Assume (6.24) does not hold, that is,

$$A(k_1,k_2)\ge 2^{4n}A(k_1+2,k_2-2).$$

If

$$A(k_1+2,k_2-2)\le 2^{4n}A(k_1+4,k_2-4),$$

then we complete the proof of the claim. Otherwise, we can repeat the procedure for j times, we have

$$A(k_1,k_2)\ge 2^{4nj}A(k_1+2j,k_2-2j).$$

When $j \approx \frac{K}{4}$, we have from (6.23)

$$\text{Area}(X(M)) \geq A(k_1, k_2) \geq 2^{nK} A(K, K+1) \geq \kappa_1 2^{2K-1}.$$

Thus, (6.24) must hold for some $k_2 > k_1$ because $\text{Area}(M) < \infty$. Hence for any $\varepsilon > 0$, we can choose k_1 and $k_2 \approx 3k_1$ such that (6.24) holds.

We define a smooth cut-off function $\psi(t)$ by

$$\psi(t) = \begin{cases} 1, & 2^{k_1+\frac{3}{2}} \leq t \leq 2^{k_2-\frac{5}{2}}, \\ 0, & \text{outside } [2^{k_1-\frac{1}{2}}, 2^{k_2-\frac{1}{2}}]. \end{cases} \quad 0 \leq \psi(t) \leq 1, \quad |\psi^{'}(t)| \leq 1. \tag{6.25}$$

Moreover, $\psi(t)$ can be defined in such a way that

$$0 \leq \psi'(t) \leq \frac{c_1}{2^{k_1-\frac{1}{2}}}, \quad t \in [2^{k_1-\frac{1}{2}}, 2^{k_1+\frac{3}{2}}], \tag{6.26}$$

$$-\frac{c_2}{2^{k_2-\frac{1}{2}}} \leq \psi'(t) \leq 0, \quad t \in [2^{k_2-\frac{5}{2}}, 2^{k_2-\frac{1}{2}}], \tag{6.27}$$

for some positive constants c_1 and c_2.

Letting

$$f(x) = e^{L+\frac{|X|^2}{4}} \psi(\rho(x)), \tag{6.28}$$

we choose the constant L satisfying

$$1 = \int_M f^2 e^{-\frac{|X|^2}{2}} d\mu = e^{2L} \int_{\Omega(k_1,k_2)} \psi^2(\rho(x)) d\mu. \tag{6.29}$$

We obtain from Corollary 6.1, $t \ln t \geq -\frac{1}{e}$ for $0 \leq t \leq 1$, $|\nabla \rho| \leq 1$ and $\psi^{'}(\rho(x)) \leq 0$ in $\Omega(k_1+2, k_2)$ that

$$\begin{aligned} \frac{1}{2}C_1(n) + \frac{1}{4}\lambda^2 \geq & \int_{\Omega(k_1,k_2)} e^{2L} \psi^2 (L + \frac{|X|^2}{4} + \ln \psi) d\mu \\ & - \int_{\Omega(k_1,k_2)} e^{2L} |\psi^{'} \nabla \rho + \psi \frac{X^T}{2}|^2 d\mu \end{aligned} \tag{6.30}$$

$$
\begin{aligned}
&\geq \int_{\Omega(k_1,k_2)} e^{2L}\psi^2(L+\frac{|X|^2}{4}+\ln\psi)d\mu \\
&\quad -\int_{\Omega(k_1,k_2)} e^{2L}|\psi^{'}|^2 d\mu - \frac{1}{4}\int_{\Omega(k_1,k_2)} e^{2L}\psi^2|X|^2 d\mu \\
&\quad -\frac{1}{2}\int_{\Omega(k_1,k_2)} e^{2L}\psi^{'}\psi\frac{|X^T|^2}{|X|}d\mu \\
&\geq L+\int_{\Omega(k_1,k_2)} e^{2L}\psi^2\ln\psi d\mu - \int_{\Omega(k_1,k_2)} e^{2L}|\psi^{'}|^2 d\mu \\
&\quad -\frac{1}{2}\int_{\Omega(k_1,k_1+2)} e^{2L}\psi^{'}\psi|X|d\mu \\
&\geq L-(\frac{1}{2e}+1)e^{2L}A(k_1,k_2)-2c_1e^{2L}A(k_1,k_1+2).
\end{aligned}
$$

Therefore, it follows from (6.24) that

$$
\begin{aligned}
\frac{1}{2}C_1(n)+\frac{1}{4}\lambda^2 &\geq L-(\frac{1}{2e}+1+2c_1)e^{2L}2^{4n}A(k_1+2,k_2-2) \\
&\geq L-(\frac{1}{2e}+1+2c_1)e^{2L}2^{4n}\int_{\Omega(k_1,k_2)}\psi^2(\rho(x))d\mu \\
&= L-(\frac{1}{2e}+1+2c_1)2^{4n}.
\end{aligned}
\tag{6.31}
$$

On the other hand, we have, from (6.24) and definition of $f(x)$,

$$
1 \leq e^{2L}\varepsilon. \tag{6.32}
$$

Letting $\varepsilon > 0$ sufficiently small, then L can be arbitrary large, which contradicts (6.31). Hence, M has infinite area. □

Theorem 6.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface. Then, for any $p \in M$, there exists a constant $C > 0$ such that*

$$
Area(B_r(X(x_0)) \cap X(M)) \geq Cr,
$$

for all $r > 1$.

Proof. We can choose $r_0 > 0$ such that $\text{Area}(B_r(0) \cap X(M)) > 0$ for $r \geq r_0$. It is sufficient to prove there exists a constant $C > 0$ such that

$$
\text{Area}(B_r(0) \cap X(M)) \geq Cr \tag{6.33}
$$

holds for all $r \geq r_0$. In fact, if (6.33) holds, then for any $x_0 \in M$ and $r > |X(x_0)|$,

$$B_r(X(x_0)) \supset B_{r-|X(x_0)|}(0), \tag{6.34}$$

and

$$\text{Area}(B_r(X(x_0)) \cap X(M)) \geq \text{Area}(B_{r-|X(x_0)|}(0) \cap X(M)) \geq \frac{C}{2} r, \tag{6.35}$$

for $r \geq 2|X(x_0)|$.

We next prove (6.33) by contradiction. Assume for any $\varepsilon > 0$, there exists $r \geq r_0$ such that

$$\text{Area}(B_r(0) \cap X(M)) \leq \varepsilon r. \tag{6.36}$$

Without loss of generality, we assume $r \in \mathbb{N}$ and consider a set:

$$D := \{k \in \mathbb{N} : \text{Area}(B_t(0) \cap X(M)) \leq 2\varepsilon t \quad \text{for any integer } t \text{ satisfying } r \leq t \leq k\}.$$

Next, we will show that $k \in D$ for any integer k satisfying $k \geq r$. For $t \geq r_0$, we define a function u by

$$u(x) = \begin{cases} t + 2 - \rho(x), & \text{in } B_{t+2}(0) \cap X(M) \setminus B_{t+1}(0) \cap X(M), \\ 1, & \text{in } B_{t+1}(0) \cap X(M) \setminus B_t(0) \cap X(M), \\ \rho(x) - (t-1), & \text{in } B_t(0) \cap X(M) \setminus B_{t-1}(0) \cap X(M), \\ 0, & \text{otherwise.} \end{cases} \tag{6.37}$$

Using Lemma 6.2, $|\nabla \rho| \leq 1$ and $t \ln t \geq -\frac{1}{e}$ for $0 \leq t \leq 1$, we have

$$\begin{aligned} &-\frac{1}{2}\int_M u^2 d\mu \ln\big\{\big(\text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M))\big)\big\} \\ &\leq C_0\Big(\text{Area}(B_{t+2}(0) \cap X(M)) - \text{Area}(B_{t-1}(0) \cap X(M))\Big) \\ &\quad + \frac{1}{4}\Big(\int_{B_{t+2}(0)\cap X(M)} H^2 d\mu - \int_{B_{t-1}(0)\cap X(M)} H^2 d\mu\Big), \end{aligned} \tag{6.38}$$

where $C_0 = 1 + \frac{1}{2e} + \frac{1}{2}C_2(n)$, $C_2(n)$ is the constant of Lemma 6.2.

For all $t \geq C_1(n,\lambda)+1$, we have from Lemma 6.1

$$
\begin{aligned}
(6.39)\quad & \mathrm{Area}(B_{t+2}(0)\cap X(M)) - \mathrm{Area}(B_{t-1}(0)\cap X(M)) \\
&\leq c(n,\lambda)\Big(\frac{\mathrm{Area}(B_{t+1}(0)\cap X(M))}{t+1} \\
&\quad + \frac{\mathrm{Area}(B_{t}(0)\cap X(M))}{t} + \frac{\mathrm{Area}(B_{t-1}(0)\cap X(M))}{t-1}\Big) \\
&\leq c(n,\lambda)\Big(\frac{2}{t+1} + \frac{1}{t} + \frac{1}{t}(1+\frac{1}{C_1(n,\lambda)})\Big)\mathrm{Area}(B_{t}(0)\cap X(M)) \\
&\leq C_2(n,\lambda)\frac{\mathrm{Area}(B_{t}(0)\cap X(M))}{t},
\end{aligned}
$$

where $C_2(n,\lambda)$ is constant depended only on n and λ. Note that we can assume $r \geq C_1(n,\lambda)+1$ for the r satisfying (6.36). In fact, if for any given $\varepsilon > 0$, all the r which satisfies (6.36) is bounded above by $C_1(n,\lambda)+1$, then $\mathrm{Area}(B_r(0)\cap X(M)) \geq Cr$ holds for any $r > C_1(n,\lambda)+1$. Thus, we know that M has at least linear area growth. Hence, for any $k \in D$ and any t satisfying $r \leq t \leq k$, we have

$$
(6.40)\quad \mathrm{Area}(B_{t+2}(0)\cap X(M)) - \mathrm{Area}(B_{t-1}(0)\cap X(M)) \leq 2C_2(n,\lambda)\varepsilon.
$$

Since

$$
(6.41)\quad \int_M u^2 d\mu \geq \mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_{t}(0)\cap X(M)),
$$

holds, if we choose ε such that $2C_2(n,\lambda)\varepsilon < 1$, from (6.38), we obtain

$$
\begin{aligned}
(6.42)\quad & (\mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_{t}(0)\cap X(M)))\ln(2C_2(n,\lambda)\varepsilon)^{-1} \\
&\leq 2C_0\Big(\mathrm{Area}(B_{t+2}(0)\cap X(M)) - \mathrm{Area}(B_{t-1}(0)\cap X(M))\Big) \\
&\quad + \frac{1}{2}\Big(\int_{B_{t+2}(0)\cap X(M)} H^2 d\mu - \int_{B_{t-1}(0)\cap X(M)} H^2 d\mu\Big).
\end{aligned}
$$

Iterating from $t = r$ to $t = k$ and taking summation on t, we infer, from Lemma 6.1 and the equation (6.9) that

$$\begin{aligned}(6.43)\quad &(\mathrm{Area}(B_{k+1}(0)\cap X(M)) - \mathrm{Area}(B_r(0)\cap X(M)))\ln(2C_2(n,\lambda)\varepsilon)^{-1}\\ &\le 6C_0\mathrm{Area}(B_{k+2}(0)\cap X(M)) + \frac{3}{2}\int_{B_{k+2}(0)\cap X(M)} H^2 d\mu\\ &\le \left[6C_0 + \frac{3}{2}(2n+\lambda^2)\right]\mathrm{Area}(B_{k+2}(0)\cap X(M))\\ &\le 2\left[6C_0 + \frac{3}{2}(2n+\lambda^2)\right]\mathrm{Area}(B_{k+1}(0)\cap X(M)).\end{aligned}$$

Hence, we get

$$\begin{aligned}(6.44)\quad &\mathrm{Area}(B_{k+1}(0)\cap X(M))\\ &\le \frac{\ln(2C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2C_2(n,\lambda)\varepsilon)^{-1} - 12C_0 - 3(2n+\lambda^2)}\mathrm{Area}(B_r(0)\cap X(M))\\ &\le \frac{\ln(2C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2C_2(n,\lambda)\varepsilon)^{-1} - 12C_0 - 3(2n+\lambda^2)}\varepsilon r.\end{aligned}$$

We can choose ε small enough such that

$$(6.45)\qquad \frac{\ln(2C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2C_2(n,\lambda)\varepsilon)^{-1} - 12C_0 - 3(2n+\lambda^2)} \le 2.$$

Therefore, it follows from (6.44) that

$$(6.46)\qquad \mathrm{Area}(B_{k+1}(0)\cap X(M)) \le 2\varepsilon r,$$

for any $k \in D$. Since $k+1 \ge r$, we have, from (6.46) and the definition of D, that $k+1 \in D$. Thus, by induction, we know that D contains all of integers $k \ge r$ and

$$(6.47)\qquad \mathrm{Area}(B_k(0)\cap X(M)) \le 2\varepsilon r,$$

for any integer $k \ge r$. This implies that M has finite volume, which contradicts with Lemma 6.4. Hence, there exist constants C and r_0 such that $\mathrm{Area}(B_r(0)\cap X(M)) \ge Cr$ for $r > r_0$. It completes the proof of Theorem 6.1. □

Remark 6.1. *The estimate in our theorem is the best possible because the cylinders* $S^{n-1}(r_0)\times\mathbb{R}$ *satisfy the equality.*

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