A softer connectivity principle

Luis Guijarro[†] and Frederick Wilhelm[‡]

We give soft, quantitatively optimal extensions of the classical
Sphere Theorem, Wilking's connectivity principle and Frankel's
Theorem to the context of Ric_k curvature. The hypotheses are soft
in the sense that they are satisfied on sets of metrics that are open
in the C^2–topology.

Introduction

A Riemannian manifold M has k^{th}–intermediate Ricci curvature $\geq \ell$ if for
any orthonormal $(k+1)$–frame $\{v, w_1, w_2, \ldots, w_k\}$, the sectional curvature
sum, $\Sigma_{i=1}^{k} \sec(v, w_i)$, is $\geq \ell$ ([28], [25]). For brevity we write $Ric_k M \geq \ell$. In
this article, we consider some of the topological implications that positive
Ric_k curvature imposes on a manifold.

We start by combining Berger's proof of the diameter sphere theorem,
[22], with the Jacobi field comparison lemma from [13] to obtain the following
generalization of the classical $\frac{1}{4}$–pinched sphere theorem.

Theorem A. *Let M be a complete Riemannian n–manifold with $Ric_k \geq k$.
If there is a point $p \in M$ with $\mathrm{conj}_p > \frac{\pi}{2}$, then the universal cover of M is
$(n-k)$–connected. In particular, if $k \leq \frac{n}{2}$, then the universal cover of M is
homeomorphic to the sphere.*

Since manifolds with $1 \leq \sec < 4$ have $Ric_1 \geq 1$ and conjugate radii $> \frac{\pi}{2}$,
Theorem A generalizes the classical sphere theorem.

The assumption $\mathrm{conj}_p > \frac{\pi}{2}$ cannot be weakened. Indeed, $Ric_{n-2}(\mathbb{C}P^n) \geq
n-2$, while $\mathrm{conj}(\mathbb{C}P^n) = \frac{\pi}{2}$, yet $\mathbb{C}P^n$ is not 2–connected. In Section 6, we
give examples of various versions of Theorem A, including examples that
show the connectivity estimate cannot be improved.

For our second result, let N be a smoothly embedded submanifold of a Riemannian manifold M, and denote by $S_v : T_p N \to T_p N$ the shape operator of N corresponding to a unit vector v normal to N. For brevity we write $S_v|_W$ for the composition of S_v restricted to some subspace W of $T_p N$ with orthogonal projection $T_p N \longrightarrow W$. We also denote by foc_N the focal radius of N.

Theorem B. *Let M be a simply connected, complete Riemannian n–manifold with $\mathrm{Ric}_k M \geq k$, and let $N \subset M$ be a compact, connected, embedded, ℓ–dimensional submanifold.*

If for some $r \in \left[0, \frac{\pi}{2} \right)$,

$$(1) \qquad\qquad\qquad \mathrm{foc}_N > r,$$

and for all unit vectors v normal to N and all k–dimensional subspaces $W \subset TN$,

$$(2) \qquad\qquad \left| \mathrm{Trace}\left(S_v|_W \right) \right| \leq k \cot \left(\frac{\pi}{2} - r \right),$$

then the inclusion $N \hookrightarrow M$ is $(2\ell - n - k + 2)$–connected.

Recall that an inclusion $\iota : N \hookrightarrow M$ is called q–connected if the relative homotopy groups $\pi_k(M, N) = 0$ for all $k \leq q$. Equivalently, ι induces an isomorphism on homotopy groups up to dimension $(q - 1)$ and a surjection of qth–homotopy groups.

In the event that $\pi_1(M) \neq 0$, Theorem B implies $\pi_k(M, N) = 0$ for all integers k in the range $2 \leq k \leq 2\ell - n - k + 2$. To see this, apply Theorem B the the universal cover of M and use the fact that covering maps induce isomorphisms on higher relative homotopy groups (Exercise 6, p. 358, [15]).

The Clifford torus in $\mathbb{S}^3$ shows that Theorem B is optimal in the sense that the conclusion is false if (2) is replaced with a weak inequality (see Example 6.3).

The proofs of both theorems also work in the non-simply connected case if the conjugate and focal radius hypotheses are replaced with the corresponding hypotheses on injectivity and normal injectivity radius. If, in addition, we assume that $r = 0$ and $k = 1$, then Theorem B recovers Wilking's connectivity principle (Theorem 1, [27]). Theorem B also generalizes the various versions of the connectivity principle in [7]. In the case $r = 0$ and $k = n - 1$, the proof of Theorem B recovers Frankel's theorem that the inclusion of a minimal hypersurface in a manifold with positive Ricci curvature induces a surjection of fundamental groups ([9]). Similarly, the proof

of Theorem A recovers Petersen's result that a manifold with $\mathrm{Ric} \geq n - 1$ is simply connected if it has a point whose injectivity radius is $> \frac{\pi}{2}$ ([22], page 181).

We also include a generalization of Frankel's Theorem on intersections of totally geodesic submanifolds in positive sectional curvature. We allow at the same time for positive Ric_k curvature and non-totally geodesic submanifolds. The result is quantitatively optimal and appears as Theorem 5.1.

The proofs of Theorems A and B use a version of the Long Homotopy Lemma that applies to submanifolds. We have not been able to find the required formulation in the literature, so we include it here (see, e.g., page 235 in [4] for the Long Homotopy Lemma for points. Also see [2] or [3]).

Lemma C. *(Long Homotopy Lemma for Submanifolds) Let M be a Riemannian manifold and N a closed, embedded submanifold. Let $\gamma : [0, b] \longrightarrow M$ be a unit speed geodesic with $b < 2 \, \mathrm{foc}_N$ and $\gamma(0), \gamma(b) \in N$ with $\gamma'(0), \gamma'(b) \perp N$. Suppose that H is a homotopy so that*

$$H_1(t) = \gamma(t),$$
$$H_s(0), H_s(b) \in N \quad \text{for all } s \in [0, 1], \text{ and}$$
$$H_0(t) \in N \quad \text{for all } t \in [0, b].$$

Then for some t, the length of the curve $H_t \geq 2 \, \mathrm{foc}_N$.

The paper is structured as follows: We start with a section reviewing the comparison theorem for Jacobi fields that we need in the rest of the paper and use it to derive Corollary 1.8 about the index of Lagrangians in k-th Ricci curvature. Section 2 recalls the definition of the index of a geodesic both for the endpoint and the endmanifold cases. The main result in this section is Theorem 2.2, which is due to N. Hingston and D. Kalish. The proof of Theorem A appears in Section 3, and the proof of the quantitative connectivity principle (Theorem B) is in Section 4. Section 5 contains the proof of our extension of Frankel's Theorem, and Section 6 gives some examples showing the sharpness of our theorems. The proof of the Long Homotopy Lemma for submanifolds has been postponed to an appendix.

Remark. An advantage of Theorem B and Theorem 5.1 is that they apply to an open set of metrics. The main theorem of [21] shows that the set of metrics to which either the original connectivity principle or Frankel's theorem can be applied is residual.

Remark. Our techniques yield versions of Theorems A, B, and 5.1 for radial intermediate Ricci curvature. We leave the details to the reader.

1. Transverse Jacobi field comparison

In this section, we review Wilking's transverse Jacobi equation and some comparison results developed for it in [13]. To simplify the writing, any vectors or vector fields along a geodesic will always be normal to it, and we assume throughout that $\dim M = n$.

1.1. Lagrangian subspaces of Jacobi fields

Let $\gamma : I \to M$ be a unit speed geodesic, and denote by $\mathcal{J}$ the vector space of normal Jacobi fields along it. $\mathcal{J}$ is a vector space of dimension $2n - 2$. Using symmetries of the curvature tensor, we see that for any pair $J_1, J_2 \in \mathcal{J}$,

$$\omega\left(J_1, J_2\right) = \left\langle J_1', J_2 \right\rangle - \left\langle J_1, J_2' \right\rangle,$$

is constant along γ and hence defines a symplectic form ω on $\mathcal{J}$.

Definition 1.1. *We will say that an $(n-1)$–dimensional subspace $\mathcal{L}$ of $\mathcal{J}$ is Lagrangian when ω vanishes on $\mathcal{L}$.*

Definition 1.2. *Let $\mathcal{V}$ be a subspace of $\mathcal{L}$. We will say that $\mathcal{V}$ has full index at $\bar{t}$ if any $J \in \mathcal{L}$ with $J(\bar{t}) = 0$ belongs to $\mathcal{V}$. We will also say that $\mathcal{V}$ has full index on an interval I if it has full index at each point of I.*

For a fixed interval I, we denote by $\mathcal{K}$ the minimal subspace of $\mathcal{L}$ that has full index on I. Thus

(3) $\mathcal{K} \equiv \operatorname{span}\left\{ J \in \mathcal{L} \mid J(t) = 0 \text{ for some } t \in I \right\}.$

1.2. The Riccati operator related to a Lagrangian

The set of times t where

$$(4) \qquad \{J(t) \mid J \in \mathcal{L}\} = \dot{\gamma}(t)^{\perp}$$

is open and dense in I. Observe that these are the times where no nontrivial field in $\mathcal{L}$ vanishes. For these t we get a well-defined Riccati operator

$$(5) \qquad S_t : \dot{\gamma}(t)^{\perp} \longrightarrow \dot{\gamma}(t)^{\perp}, \qquad S_t(v) = J'_v(t),$$

where J_v is the unique $J_v \in \mathcal{L}$ so that $J_v(t) = v$. The Jacobi equation then decomposes into the two first order equations

$$S_t(J) = J', \qquad S'_t + S_t^2 + R = 0,$$

where S'_t is the covariant derivative of S_t along γ, and R is the curvature along γ, that is $R(\cdot) = R(\cdot, \dot{\gamma})\dot{\gamma}$ (see Equation 1.7.1 in [11]). Although S_t depends on the choice of $\mathcal{L}$, we will never discuss more than one family $\mathcal{L}$, so we omit this dependence from our notation.

Given $t \in I$, and a subspace $\mathcal{V} \subset \mathcal{L}$, we get a subspace of $\gamma'^{\perp}$ by setting

$$(6) \qquad \mathcal{V}(t) = \{J(t) \mid J \in \mathcal{V}\} \oplus \{J'(t) \mid J \in \mathcal{V} \text{ and } J(t) = 0\}.$$

The second summand vanishes for almost every t (cf Lemma 1.7 of [11]), even at such times S_t is well defined on a subspace of $\gamma'^{\perp}$. Indeed,

Lemma 1.3. *([13, Remark 3.3]) Let $\mathcal{K}$ the subspace of $\mathcal{L}$ defined in (3). Then equation (5) defines S_t on $\mathcal{K}(t)^{\perp}$ for every $t \in I$.*

When $\mathcal{L}$ has fields vanishing at $t \in I$ and $v \in \mathcal{K}(t)^{\perp}$, there are many J_v in $\mathcal{L}$ with $J_v(t) = v$; Lemma 1.3 says that $J'_v(t)$ is independent of these choices.

Example 1.4. An especially useful Lagrangian appears when we take a geodesic $\gamma : \mathbb{R} \to M$ normal to a submanifold N at time zero and consider those Jacobi fields obtained from geodesic variations that leave N orthogonally. More precisely, let $p = \gamma(0)$, and denote by $S_{\gamma'(0)} : T_pN \to T_pN$ the shape operator of N determined by $\gamma'(0)$. The Lagrangian $\mathcal{L}_N$ is the set of

Jacobi fields along γ such that

$$(7) \qquad\qquad J(0) \in T_p N \quad \text{and} \quad J'^T = S_{\gamma'(0)} J(0),$$

where J'^T is the component of $J'(0)$ tangent to S (details can be found at [4, page 227]).

For this Lagrangian, the Riccati operator S_0 agrees with $S_{\gamma'(0)}$. This follows immediately from equation (7) above.

1.3. Comparison lemma for the transverse Jacobi equation

For a subspace $W_t \subset \gamma'(t)^{\perp}$ we will use

$$P_{W,t} : \gamma'^{\perp} \longrightarrow W_t$$

to denote the orthogonal projection onto W_t. For simplicity of notation we will write

$$\text{Trace}\, S_t|_W \text{ for Trace}\left(P_{W,t} \circ S_t|_W\right).$$

In the case of positive curvature, Lemma 4.23 in [13] gives us the following.

Lemma 1.5 (Intermediate Ricci Comparison). *Let* $\gamma : [0, t_{\max}] \longrightarrow M$ *be a unit speed geodesic in a complete Riemannian n–manifold M with* $\text{Ric}_k(\dot\gamma, \cdot) \geq k$. *Let $\mathcal{L}$ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S, and let $W_0 \perp \gamma'(0)$ be a k–dimensional subspace so that*

$$(8) \qquad\qquad \text{Trace}\, S_0|_{W_0} \leq k \cdot \cot(s_0)$$

for some $s_0 \in (0, \pi)$. Denote by $\mathcal{V}$ the subspace of $\mathcal{L}$ formed by those Jacobi fields that are orthogonal to W_0 at 0, and by $H(t)$ the subspace of $\gamma'(t)^{\perp}$ that is orthogonal to $\mathcal{V}(t)$ at each $t \in [0, t_{\max}]$. Assume that $\mathcal{V}$ is of full index in the interval $[0, t_{\max}]$.

Then for all $t \in [0, t_{\max}]$,

$$(9) \qquad\qquad \text{Trace}\, S_t|_{H(t)} \leq k \cdot \cot(t + s_0).$$

By Remark 4.20 of [13] we also have

Lemma 1.6. *Let $\gamma : [0, t_{\max}] \longrightarrow M$ be a unit speed geodesic in a complete Riemannian n–manifold M with $\mathrm{Ric}_k(\dot{\gamma}, \cdot) \geq k$. Let $\mathcal{L}$ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S. Let $\mathcal{V} \subset \mathcal{L}$ be a subspace of full index, and for $t \in [0, t_{\max}]$, let $H(t)$ be a k–dimensional subspace of $\gamma'(t)^{\perp}$ that is orthogonal to $\mathcal{V}(t)$.*

Then

$$\mathrm{Trace}\, S_t|_{H(t)} \leq k \cdot \cot(t).$$

In particular, if

(10) $\dim \mathcal{V} \leq n - 1 - k, \;\; \text{then}\;\; t_{\max} < \pi.$

Remark 1.7. Like Lemma 1.5, Lemma 1.6, follows by combining classical Riccati comparison ([5],[6]) with the transverse the Jacobi equation from [26].

The analog in Lemma 1.6 of the initial trace Hypothesis (8) of Lemma 1.5 is

(11) $\displaystyle\liminf_{t \to 0} \left(k \cot(t) - \mathrm{Trace}\, S_t|_{H(t)} \right) \geq 0.$

Since (11) holds for all k–dimensional subspaces of Jacobi fields along all geodesics in all Riemannian manifolds, it is not included in the statement of Lemma 1.6.

There is an improved version of (10) when $\mathcal{L}$ is nonsingular at $t = 0$. We give this in the next corollary which is related to [10, Proposition 3.4], [13, Theorem 1.1], and [12, Theorem G].

Corollary 1.8. *For $r > 0$, let $\gamma : \left[0, \frac{\pi}{2} + r \right] \longrightarrow M$ be a unit speed geodesic in a complete Riemannian n–manifold M with $\mathrm{Ric}_k(\dot{\gamma}, \cdot) \geq k$. Let $\mathcal{L}$ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S. Set*

$$\mathcal{K} \equiv \left\{ J \in \mathcal{L} \;\middle|\; J(t) = 0 \text{ for some } t \in \left(0, \frac{\pi}{2} + r \right] \right\}.$$

If there is an ℓ–dimensional subspace $U_0 \subset \gamma'^{\perp}$ so that for all k–dimensional subspaces $W \subset U_0$,

(12) $\mathrm{Trace}\, S_0|_W \leq k \cot\left(\dfrac{\pi}{2} - r \right),$

then $\dim \mathcal{K} \geq \ell - k + 1$.

Proof. Suppose that $\dim \mathcal{K} < \ell - k + 1$. Since all of the following are subspaces of the $(n-1)$–dimensional space $\gamma'(0)^{\perp}$, we get that

$$
\begin{aligned}
\dim(U_0 \cap \mathcal{K}(0)^{\perp}) &= \dim U_0 + \dim \mathcal{K}(0)^{\perp} - \dim(U_0 + \mathcal{K}(0)^{\perp}) \\
&\geq \dim U_0 + n - 1 - \dim \mathcal{K}(0) - (n-1) \\
&\geq \dim U_0 - \dim \mathcal{K} \\
&> \ell - (\ell - k + 1) \\
&= k - 1.
\end{aligned}
$$

Since $\dim(U_0 \cap \mathcal{K}(0)^{\perp}) \geq k$, inequality (12) together with Lemma 1.5 gives us that for all values of $t \in \left(0, \frac{\pi}{2} + r\right]$, there is a nonzero $v \in \mathcal{K}(t)^{\perp}$ so that

$$
\langle S_t v, v \rangle \leq \cot\left(\frac{\pi}{2} - r + t\right) |v|^2 .
$$

But $\lim_{s \to \pi^-} \cot(s) = -\infty$; thus there is some $t_1 \in \left(0, \frac{\pi}{2} + r\right]$ so that S_t is not well defined on $\mathcal{K}(t)^{\perp}$. Since this contradicts Lemma 1.3, $\dim \mathcal{K} \geq \ell - k + 1$. $\qquad\square$

2. The index of a geodesic

Some of the results of this section are well known and can be found in [19]; the exception is probably the Hingston-Kalish Theorem, which can be found in [16].

2.1. The endpoints case

For a pair of points $p, q \in M$, let $\Omega_{p,q}$ denote the space of piecewise smooth paths connecting p to q, that are parameterized on $[0,1]$, and let

$$
E : \Omega_{p,q} \to [0, \infty), \qquad E(\alpha) = \frac{1}{2} \int_0^1 |\alpha'|^2 \, dt
$$

denote the energy function. E is related to arc length via

$$
\text{length}(\alpha) \leq \sqrt{2E(\alpha)}, \tag{13}
$$

with equality if and only if α has constant speed ([22], pages 182–183).

Following [19], we will work with a finite dimensional submanifold $P \subset \Omega_{p,q}$ that is homotopy equivalent to $\Omega_{p,q}$, and we will continue using E for the restriction of the energy function to P. The critical points of E are the unit geodesics connecting p to q.

Definition 2.1. *Let $\gamma : [0, 1] \to M$ be a geodesic between p and q. The index of γ is the index of γ as a critical point of E.*

When p and q are not conjugate along γ, γ is a nondegenerate critical point of E. When they are conjugate, a kernel appears agreeing with the set of Jacobi fields vanishing at p and q simultaneously. In both cases, the Morse Index Lemma affirms that the index of γ agrees with the number of conjugate points to $\gamma(0)$ along γ in the interval $(0, 1)$.

2.2. The endmanifold case

Given two smooth embedded submanifolds N and $\tilde{N}$ of a Riemannian manifold M, we let $\Omega(N, \tilde{N})$ be the space of all piecewise smooth curves from N to $\tilde{N}$ parameterized on $[0, b]$. It is well known that the critical points of the energy functional $E : \Omega(N, \tilde{N}) \longrightarrow \mathbb{R}$ are precisely the geodesics $\gamma : [0, b] \longrightarrow M$ from N to $\tilde{N}$ that are normal to N and $\tilde{N}$ at both endpoints.

In what follows, we only consider continuous variations $f : [0, b] \times (-\varepsilon, \varepsilon) \longrightarrow M$ of such geodesics γ with

$$(14) \qquad f(0, s) \in N, \quad f(b, s) \in \tilde{N}, \quad \text{and} \quad V(t) := \frac{\partial f}{\partial s}(t, 0).$$

In addition, assume that there is a partition $0 = t_0 < t_1 < t_2 < \cdots < t_{k+1} = b$ so that

$$(15) \qquad f|_{[t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)} \text{ is smooth for all } i \in \{1, 2, \ldots, k\}.$$

The second variation formula gives us

$$E''(0) = -\int_0^b \left\langle V'' + R(V, \dot{\gamma}) \dot{\gamma}, V \right\rangle dt + \left\langle V' - S_{\gamma'} V, V \right\rangle \big|_0^b$$
$$+ \sum_i \left\langle V'(t_i^-) - V'(t_i^+), V(t_i) \right\rangle,$$

where $S_{\gamma'}$ is the shape operator of N or $\tilde{N}$ for the normal γ',

$$V'(t_i^-) = \lim_{t \to t_i^-} V'(t), \quad \text{and} \quad V'(t_i^+) = \lim_{t \to t_i^+} V'(t).$$

(See, for instance, [4, pages 198-199])

As usual the index of the critical point γ of $E : \Omega(N, \tilde{N}) \longrightarrow \mathbb{R}$ is defined to be the maximal dimension of the space of variation fields V of variations f that satisfy (14) and (15) for which $E''(0)$ is negative.

Let $\mathcal{L}_N$ be as in Example 1.4 and set

$$\mathcal{K} \equiv \operatorname{span}\{J \in \mathcal{L}_N \mid J(t) = 0 \text{ for some } t \in (0, b]\} \text{ and}$$
$$\mathcal{K}_b \equiv \{J \in \mathcal{L}_N \mid J(b) = 0\}.$$

Denote by $\mathcal{L}_{N,\tilde{N}}$ the subspace of $\mathcal{L}_N$ whose fields are tangent to $\tilde{N}$ at $\gamma(b)$, and let A be the symmetric bilinear form

$$A : \mathcal{L}_{N,\tilde{N}} \times \mathcal{L}_{N,\tilde{N}} \longrightarrow \mathbb{R},$$
$$(16) \qquad A(J_1, J_2) = \left\langle J_1'(b) - S_{\dot{\gamma}(b)} J_1, J_2(b) \right\rangle.$$

Thus A is the difference in $T_{\gamma(b)}M$ of the Riccati operator of $\mathcal{L}$ and the shape operator of $\tilde{N}$ at $\gamma(b)$ in the direction of $\gamma'(1)$.

The following theorem was proven in [16] (cf also [17]).

Theorem 2.2 (Hingston-Kalish Theorem). *The index of γ is equal to*

$$(17) \qquad \operatorname{index} A + \text{number of focal pts in } (0, b] - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp)$$

where the focal points are counted with their multiplicities.

There is a small difference in the formulation of Theorem 2.2 and the statement in [16]. Hingston and Kalish express the index of γ as

$$(18) \qquad \operatorname{index} A + \text{number of focal pts in } (0, b) + m_T$$

where

$$m_T = \dim \operatorname{Proj}_{\tilde{N}} \mathcal{K}_b(b),$$

and $\operatorname{Proj}_{\tilde{N}}$ is orthogonal projection onto $\tilde{N}$. Since $\dim \mathcal{K}_b(b)$ is the multiplicity of b as a N–focal point,

$$m_T = \dim \mathcal{K}_b(b) - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp)$$
$$= \text{the multiplicity of } b \text{ as a } N\text{-focalpoint} - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp),$$

and (17) and (18) agree.

3. Conjugate radius and positive curvature

By taking the contrapositive of (10) in Lemma 1.6, we get the following.

Lemma 3.1. *Let M be a complete Riemannian manifold with $\mathrm{Ric}_k \geq k$. Any geodesic in M of length $> \pi$ has index $\geq n - k$.*

To complete the proof of Theorem A, we will need the following result from Petersen's textbook.

Theorem 3.2 (Theorem 6.5.2 [22]). *Let $f : N \longrightarrow \mathbb{R}$ be a smooth proper function defined on a differentiable manifold N. If b is a regular value of f and all critical points in f in $f^{-1}[a, b]$ have index $\geq m$, then the inclusion*

$$f^{-1}(-\infty, a] \subset f^{-1}(-\infty, b]$$

is $(m-1)$–connected.

The above theorem will be applied to finite dimensional approximations of the space of closed paths based at some $p \in M$. A standard reference for this is [19, Chapter 16].

Proof of Theorem A. Without loss of generality, assume that M is simply connected. There is nothing to prove if $k = n - 1$, so we will assume that $k \leq n - 2$.

Choose some point $p \in M$ with $\mathrm{conj}_p > \frac{\pi}{2}$, and some number b with $\pi < b < 2\,\mathrm{conj}_p$. Let Ω_p be the loops based at p parameterized on $[0, 1]$.

If $E : \Omega_p \longrightarrow \mathbb{R}$ denotes the energy functional and σ is a critical point of E with $\mathrm{length}\,(\sigma) > b$, then by Lemma 3.1,

$$(19) \hspace{4cm} \mathrm{Index}\,(\sigma) \geq n - k.$$

Recall from [19, Theorem 16.2] that there is a finite dimensional approximation, N, of Ω_p so that for any $a \geq 0$, the sublevel set $N \cap E^{-1}[0, a]$ is a deformation retract of $E^{-1}[0, a]$. Thus we can apply Theorem 3.2 to N and obtain similar statements for Ω_p. We will do that in what follows without any further comment.

Since σ has constant speed, it follows from (13) that

$$E\,(\sigma) = \frac{1}{2}\,\mathrm{length}\,(\sigma)^2.$$

Since $n - k \geq 2$, the combination of Theorem 3.2 and Inequality (19) gives that the inclusion $E^{-1}\left[0, b^2/2\right] \hookrightarrow \Omega_p$ is 1–connected, i.e.,

$$\pi_1(\Omega_p, E^{-1}[0, b^2/2]) = 0.$$

From $\pi_1(M) = 0$, we get that $\pi_0(\Omega_p) = 0$, and the long exact sequence of the pair $(\Omega_p, E^{-1}[0, b^2/2])$ reads

$$0 = \pi_1(\Omega_p, E^{-1}[0, b^2/2]) \longrightarrow \pi_0(E^{-1}[0, b^2/2]) \longrightarrow \pi_0(\Omega_p) = 0.$$

Thus $\pi_0\left(E^{-1}\left[0, b^2/2\right]\right) = 0$, i.e., $E^{-1}\left[0, b^2/2\right]$ is connected.

Next we show that there is no geodesic loop $\gamma : [0, 1] \longrightarrow M$ at p of length ℓ with $\ell \leq b$. Since $E^{-1}[0, b^2/2]$ is connected, if there were such a loop, then γ would be homotopic to a constant loop through loops γ_t of energy $\leq b^2/2$ and hence of length

$$\text{length}(\gamma_t) \leq \sqrt{2E(\gamma_t)} \leq \sqrt{2b^2/2} = b$$

by (13).

On the other hand, by the original Long Homotopy Lemma (see [2] and [3]), any homotopy of γ to a constant loop must pass through a curve whose length is at least $2\,\text{conj}_p > b$, yielding a contradiction. Thus there can be no geodesic loop at p with length $\leq b$. Combining this with (19), it follows that all critical points of $E : \Omega_p \longrightarrow \mathbb{R}$ have index $\geq n - k$. Thus by Theorem 3.2, Ω_p is $(n - k - 1)$–connected, and M is $(n - k)$–connected. □

4. Quantitative connectivity principle

In this section we prove Theorem B. Our first result is a simple linear algebra fact that will allow us to turn the trace estimates of (8) and (9) into estimates on the values of S_t.

Proposition 4.1. *Let $A : U \longrightarrow U$ be a self adjoint endomorphism of an ℓ– dimensional inner product space. Suppose that there is a $k \in \{1, 2, \ldots, \ell - 1\}$ so that for all k–dimensional subspaces $W \subset U$,*

$$\text{Trace}\, A|_W \leq k \cdot \lambda.$$

Then there is an $(\ell - k + 1)$–dimensional subspace $V \subset U$ so that for all unit $v \in V$,

$$\langle Av, v \rangle \leq \lambda.$$

Proof. Let $\{u_1, \ldots u_m, v_1, \ldots, v_{\ell-m}\}$ be an orthonormal set of eigenvectors for A, for which the first m-vectors $\{u_1, \ldots u_m\}$ is the maximal subset that

satisfies

$$\langle Au_i, u_i \rangle > \lambda.$$

Then $m \le k - 1$. Thus

$$\ell - m \ge \ell - (k - 1),$$

and the subspace V spanned by $v_1, \ldots, v_{\ell-k+1}$ satisfies the conclusion of the proposition. $\qquad\square$

Next we state the main lemma for the proof of Theorem B. It will gives us estimates on the index of a geodesic orthogonal at its endpoints to a submanifold of sufficiently big dimension.

Lemma 4.2. *Let M be a complete Riemannian n–manifold with $\mathrm{Ric}_k M \ge k$, and let $N \subset M$ be an ℓ–dimensional submanifold for some $\ell \ge k$ that is compact, connected and embedded.*

Suppose that $\gamma : [0, b] \longrightarrow M$ is a unit speed geodesic perpendicular to N at both endpoints such that for all k–dimensional subspaces W tangent to N at the endpoints of γ, we have

$$(20) \qquad \left| \mathrm{Trace}\, S_{\gamma'(0)}|_W \right|, \left| \mathrm{Trace}\, S_{\gamma'(b)}|_W \right| \le k \cot \left(\frac{\pi}{2} - r \right)$$

for some $0 < r < \frac{\pi}{2}$. Then if the length of γ is larger than $2r$, we have

$$\mathrm{index}(\gamma) \ge 2\ell - n - k + 2.$$

Proof. In what follows, all subspaces of Jacobi fields are understood to be perpendicular to γ. Recall that $\mathcal{L}_N$ denotes those Jacobi fields corresponding to variations of γ by geodesics that leave N orthogonally at time zero. Let

$$\mathcal{K} := \mathrm{span}\left\{ J \in \mathcal{L}_N(0) \mid J(t) = 0 \text{ for some } t \in (0, b] \right\}.$$

These are precisely the Jacobi fields that create focal points on $(0, b]$, but a given field can vanish multiple times, so we only have the inequality

$$\text{number of focal pts in } (0, b] \ge \dim(\mathcal{K}).$$

We divide the proof into two cases, depending on the length of γ:

- *Case 1:* If $b \geq \frac{\pi}{2} + r$, then by Corollary 1.8,

$$\text{(21)} \qquad \dim\left(\mathcal{K}\right) \geq \ell - k + 1.$$

On the other hand, Theorem 2.2 allows us to estimate the index of γ as

$$\text{(22)} \qquad \text{index}(\gamma) \geq \dim\left(\mathcal{K}\right) - \dim(\mathcal{K}_b\left(b\right) \cap T_{\gamma(b)} N^\perp).$$

Since $\mathcal{K}_b \perp \gamma'(b)$, we have

$$\mathcal{K}_b\left(b\right) \cap T_{\gamma(b)} N^\perp \subset \gamma'\left(b\right)^\perp \cap T_{\gamma(b)} N^\perp.$$

Together with (21) and (22), this gives us

$$\begin{aligned} \text{index}\left(\gamma\right) &\geq \ell - k + 1 - (n - 1 - \ell) \\ &= 2\ell - n - k + 2, \end{aligned}$$

as desired.

- *Case 2:* Suppose $b < \frac{\pi}{2} + r$. If $\dim\mathcal{K} \geq \ell - k + 1$, then we can proceed as in Case 1 to conclude that

$$\text{Index}\left(\gamma\right) \geq \ell - n - k + 2.$$

If $\dim\mathcal{K} \leq \ell - k$, set

$$U_0 := \mathcal{K}\left(0\right)^\perp \cap T_{\gamma(0)} N,$$

and note that, since N is ℓ-dimensional,

$$\dim U_0 \geq \ell - \dim\mathcal{K} \geq k.$$

Next we apply Lemma 1.5 to the k–dimensional subspaces W_0 of U_0. To justify this, observe that since $U_0 \perp \mathcal{K}(0)$, the space $\mathcal{V}$ in Lemma 1.5 is of full index. Thus hypothesis (20) implies there is a subspace U_b of $\mathcal{K}(b)^\perp$ with the same dimension as U_0 so that for all k–dimensional subspaces $W \subset U_b$,

$$\text{(23)} \qquad \text{Trace } S_b|_W \leq k \cot\left(\frac{\pi}{2} - r + b\right).$$

On the other hand, using the formula for A in (16), for any k-dimensional subspace W in $U_b \cap T_{\gamma(b)} N$,

$$\text{Trace } A|_W = \text{Trace } S_b|_W - \text{Trace } S_{\gamma'(b)}|_W \leq \text{Trace } S_b|_W + |\text{Trace } S_{\gamma'(b)}|_W|$$

Combining this with (20) and (23), for any such W,

$$(24) \qquad \text{Trace } A|_W \leq k \left(\cot \left(\frac{\pi}{2} - r + b \right) + \cot \left(\frac{\pi}{2} - r \right) \right).$$

Together with Proposition 4.1, this implies there is a subspace V of $U_b \cap T_{\gamma(b)} N$ with

$$(25) \qquad \dim V \geq \dim(U_b \cap T_{\gamma(b)} N) - k + 1$$

so that for all unit $v \in V$,

$$\langle Av, v \rangle \leq \cot \left(\frac{\pi}{2} - r + b \right) + \cot \left(\frac{\pi}{2} - r \right).$$

Since $2r < b < \frac{\pi}{2} + r$,

$$(26) \qquad\qquad\qquad \langle Av, v \rangle < 0,$$

for all $v \in V$. Thus,

$$\text{index } A \geq \dim V$$
$$(27) \qquad\qquad\qquad \geq \dim(U_b \cap T_{\gamma(b)} N) - k + 1.$$

Observe that (27) also holds when $\dim(U_b \cap T_{\gamma(b)} N) < k$, since in that case (27) just reads index $A \geq 0$, which is obviously true. Therefore, the rest of the argument is independent of the dimension of $U_b \cap T_{\gamma(b)} N$.

The next goal is to estimate $\dim(U_b \cap T_{\gamma(b)} N)$. To this aim, observe that $U_b \perp \mathcal{K}(b)$, thus

$$U_b + T_{\gamma(b)} N \subset \left(\mathcal{K}(b) \cap T_{\gamma(b)} N^\perp \right)^\perp.$$

Since $\mathcal{K}_b \subset \mathcal{K}$, we have

$$\dim(U_b + T_{\gamma(b)} N) \leq n - 1 - \dim(\mathcal{K}_b(b) \cap T_{\gamma(b)} N^\perp).$$

Using also $\dim U_b = \dim U_0 \geq \ell - \dim \mathcal{K}$,

$$\dim(U_b \cap T_{\gamma(b)} N) = \dim U_b + \dim T_{\gamma(b)} N - \dim(U_b + T_{\gamma(b)} N)$$
$$\geq \ell - \dim \mathcal{K} + \ell - \left(n - 1 - \dim \left(\mathcal{K}_b(b) \cap T_{\gamma(b)} N^\perp \right) \right)$$
$$= 2\ell - n + 1 - \dim \mathcal{K} + \dim(\mathcal{K}_b(b) \cap T_{\gamma(b)} N^\perp).$$

Combining this with Theorem 2.2 and Inequality (27), we see that

$$
\begin{aligned}
\text{index}\,\gamma &\geq \text{index}\,A + \dim\mathcal{K} - \dim(\mathcal{K}_b\,(b) \cap T_{\gamma(b)}N^\perp)\\
&\geq \dim(U_b \cap T_{\gamma(b)}N) - k + 1 + \dim\mathcal{K} - \dim(\mathcal{K}_b\,(b) \cap T_{\gamma(b)}N^\perp)\\
&\geq 2\ell - n - k + 2,
\end{aligned}
$$

as claimed. $\qquad\qquad\square$

Proof of Theorem B. The result follows from Lemma 4.2, provided we can show that there is no geodesic $\gamma : [0,1] \longrightarrow M$ with $\text{len}(\gamma) = b \leq 2r$ and $\gamma(0), \gamma(1) \in N$ with $\gamma'(0), \gamma'(1) \perp N$.

Since N is connected and $\pi_1(M) = 0$, the long exact homotopy sequence of (M, N) gives $\pi_1(M, N) = 0$. So without loss of generality we may assume that $2\ell - n - k + 2 \geq 2$.

Let Ω_N be the space of piecewise smooth paths in M that start and end in N and are parameterized on $[0,1]$. Let $E : \Omega_N \longrightarrow \mathbb{R}$ be the energy functional. By Lemma 4.2, for all $\tilde{r} > r$, if σ is a critical point of E with $\sqrt{2E(\sigma)} = \text{length}(\sigma) \geq 2\tilde{r}$, then

$$
\begin{aligned}
\text{index}(\sigma) &\geq 2\ell - n - k + 2\\
&\geq 2.
\end{aligned}
$$

Combining this with Theorem 3.2 it follows that $\pi_1\left(\Omega_N, E^{-1}\left[0, 2\tilde{r}^2\right]\right) = 0$. Since $\pi_1(M, N) = 0$ and $\pi_0(\Omega_N) = 0$, the long exact sequence,

$$
\pi_1\left(\Omega_N, E^{-1}\left[0, 2\tilde{r}^2\right]\right) \longrightarrow \pi_0\left(E^{-1}\left[0, 2\tilde{r}^2\right]\right) \longrightarrow \pi_0\left(\Omega_N\right),
$$

yields $\pi_0\left(E^{-1}\left[0, 2\tilde{r}^2\right]\right) = 0$.

Now suppose that we have a geodesic $\gamma : [0,1] \longrightarrow M$ with $\gamma(0), \gamma(1) \in N$, $\text{len}(\gamma) = b \leq 2r$, and $\gamma'(0), \gamma'(1) \perp N$. Choose $\tilde{r} \in (r, \text{focal radius}(N))$. Since $\pi_0(E^{-1}[0, 2\tilde{r}^2]) = 0$, there is a homotopy of γ to a path in N through paths in Ω_N of energy $\leq 2\tilde{r}^2$ and hence of length

$$
\text{length}(\gamma_t) \leq \sqrt{2E(\gamma_t)} \leq \sqrt{2 \cdot 2\tilde{r}^2} = 2\tilde{r}.
$$

On the other hand, by the Long Homotopy Lemma (C), any homotopy of γ to a path in N must pass through a curve whose length is $\geq 2\,\text{foc}_N > 2\tilde{r}$, yielding a contradiction. Thus there can be no geodesic $\gamma : [0,1] \longrightarrow M$ with $\gamma(0), \gamma(b) \in N$, $\text{len}(\gamma) = b \leq 2r < 2\tilde{r}$ and $\gamma'(0), \gamma'(b) \perp N$. Thus the inclusion $N \hookrightarrow M$ is $(2l - n - k + 2)$–connected as claimed. $\qquad\square$

5. Quantitative version of Frankel's theorem

Frankel's Theorem [8], a classical result in Riemannian geometry, asserts that in the presence of positive sectional curvature, two closed totally geodesic submanifolds whose dimensions add to at least the dimension of the ambient manifold necessarily intersect. The theorem has a simple proof using the second variation formula. Since it appeared, the theorem has been generalized in at least two different directions:

- The positive sectional curvature has been changed to positive k-th Ricci curvature (see for instance [18] or [24]).

- The totally geodesic restriction on the submanifolds has been changed to restrictions on their second fundamental forms; in this case, the conclusion is usually relaxed to an estimate of the relative distance of the submanifolds (as for instance in [20] and [24]).

- Frankel showed in [9] that two minimal surfaces in a manifold of positive Ricci curvature always intersect. This was generalized to integrally positive Ricci curvature in [23]: if the integral of the portion of the Ricci curvature that is $< n - 1$ is small, then the distance between minimal submanifolds is small.

There are more references dealing with Frankel's Theorem, but we include only the above as an illustration.

Our next result gives the quantitatively optimal generalization of Frankel's Theorem to intermediate Ricci curvature. In particular, it assumes a weaker condition on the second fundamental form of the submanifolds than the one appearing in [24].

Theorem 5.1. *Let M be a complete Riemannian manifold with $\mathrm{Ric}_k M \geq k$. Let N and $\tilde{N}$ be two compact embedded submanifolds of M so that*

$$\dim N + \dim \tilde{N} \geq \dim M + k - 1.$$

Suppose that for some $r, \tilde{r} \in \left(0, \frac{\pi}{2}\right)$ and all unit vectors v normal to N, all k-dimensional subspaces $W \subset TN$, all unit vectors $\tilde{v}$ normal to $\tilde{N}$, and all k-dimensional subspaces $\tilde{W} \subset T\tilde{N}$,

$$(28) \quad |\operatorname{Trace} S_v|_W| \leq k \cot\left(\frac{\pi}{2} - r\right) \quad and \quad |\operatorname{Trace} S_{\tilde{v}}|_{\tilde{W}}| \leq k \cot\left(\frac{\pi}{2} - \tilde{r}\right).$$

Then

(29) $\text{dist}(N, \tilde{N}) \leq r + \tilde{r}.$

Proof. If N and $\tilde{N}$ intersect, then Inequality (29) is obvious, so assume that $N \cap \tilde{N} = \varnothing$. It follows that there is a minimal geodesic segment $\gamma : [0, b] \longrightarrow M$ between N and $\tilde{N}$.

Observe first that the length of γ cannot exceed $\frac{\pi}{2} + r$; otherwise, Inequality (28) and Corollary 1.8 give a field in $\mathcal{L}_N$ that vanishes before time b. This contradicts the minimality of γ, thus $b \leq \frac{\pi}{2} + r$.

As before we set

$$\mathcal{K} \equiv \text{span} \left\{ J \in \mathcal{L}_N \mid J(t) = 0 \text{ for some } t \in (0, b] \right\}, \text{ and}$$
$$\mathcal{K}_b \equiv \left\{ J \in \mathcal{L}_N \mid J(b) = 0 \right\}.$$

Since γ is minimal, no field in $\mathcal{K}$ vanishes before time b, in particular

(30) $\mathcal{K} = \mathcal{K}_b.$

Further, by Theorem 2.2,

$$0 = \text{index}\,\gamma$$
$$= \text{index}\,A + \text{number of focal pts in } (0, b] - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp)$$
$$\geq \text{index}\,A + \dim\mathcal{K} - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp).$$

But $\dim\mathcal{K} - \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp) \geq 0$, so

(31) $\text{index}\,A = 0 \text{ and } \dim\mathcal{K} = \dim(\mathcal{K}_b(b) \cap T\tilde{N}^\perp).$

Combined with $\mathcal{K} = \mathcal{K}_b$, we have

(32) $\mathcal{K}_b(b) \subset T\tilde{N}^\perp.$

To complete the proof, we show that index $A = 0$ implies that $b \leq r + \tilde{r}$. To do this, use Inequality (28) and apply Lemma 1.5 with $W_0 \equiv \mathcal{K}(0)^\perp \cap T_{\gamma(0)}N$. It follows that there is a $(\dim W_0)$–dimensional subspace H_b of $T_{\gamma(b)}M$ that is perpendicular to $\gamma'(b)$ and $\mathcal{K}(b)$ so that for all k–dimensional

subspaces $W \subset H_b$,

(33) $$\text{Trace}\,(S|_W) \leq k \cot\left(\frac{\pi}{2} - r + b\right).$$

Suppose for the moment that one of these k–dimensional subspaces is also contained in $T_{\gamma(b)}\tilde{N}$. Then Inequalities (28) and (33) together with the fact that $\tilde{r} \in \left(0, \frac{\pi}{2}\right)$ give

$$k\left(\cot\left(\frac{\pi}{2} - r + b\right) + \cot\left(\frac{\pi}{2} - \tilde{r}\right)\right) \geq \text{Trace}\,(S|_W - S_{\dot{\gamma}}|_W)$$
$$= \text{Trace}\,(A|_W).$$

Since index $A = 0$,

(34) $$\cot\left(\frac{\pi}{2} - r + b\right) \geq -\cot\left(\frac{\pi}{2} - \tilde{r}\right).$$

Since $b - r \leq \frac{\pi}{2}$, $r \in \left(0, \frac{\pi}{2}\right)$, and $b > 0$,

$$0 < \frac{\pi}{2} - r + b \leq \pi.$$

Combining this with $\tilde{r} \in \left[0, \frac{\pi}{2}\right)$ and (34), we can conclude that

$$\frac{\pi}{2} - r + b \leq \frac{\pi}{2} + \tilde{r}.$$

This will give us

$$\text{dist}(N, \tilde{N}) = b$$
$$\leq r + \tilde{r},$$

once we prove there is a k–dimensional subspace $U \subset H_b \cap T_{\gamma(b)}\tilde{N}$.

To do this we set

$$m = \dim M, \quad n = \dim N, \quad \text{and} \quad \tilde{n} = \dim \tilde{N},$$

and note that

$$\dim H_b = \dim W_0$$
$$= \dim(T_{\gamma(0)}N \cap \mathcal{K}(0)^{\perp})$$
$$\geq n - \dim \mathcal{K}.$$

Since both H_b and $T_{\gamma(b)}\tilde{N}$ are perpendicular to $\mathcal{K}_b(b)$ and $\gamma'(b)$,

$$
\begin{aligned}
\dim(H_b + T_{\gamma(b)}\tilde{N}) &\leq m - 1 - \dim \mathcal{K}_b(b) \\
&= m - 1 - \dim \mathcal{K}, \text{ by (30)}.
\end{aligned}
$$

The previous two inequalities give us

$$
\begin{aligned}
\dim(H_b \cap T_{\gamma(b)}\tilde{N}) &= \dim H_b + \dim T_{\gamma(b)}\tilde{N} - \dim(H_b + T_{\gamma(b)}\tilde{N}) \\
&\geq n - \dim(\mathcal{K}) + \tilde{n} - (m - 1 - \dim(\mathcal{K})) \\
&\geq n + \tilde{n} - m + 1 \\
&\geq k,
\end{aligned}
$$

as desired. $\square$

6. Examples

Our first example shows that the Ricci curvature version of Theorem A is optimal in the sense that the conclusion cannot be strengthened to say that M is 2–connected.

Example 6.1. Let S_3^2 be the 2–sphere with constant curvature 3. Then

$$
\mathrm{Ric}\left(S_3^2 \times S_3^2\right) \equiv 3 \equiv \mathrm{Ric}\left(\mathbb{S}^4\right),
$$

and for all $p \in S_3^2 \times S_3^2$,

$$
\mathrm{conj}_p = \mathrm{inj}_p = \pi\sqrt{\frac{1}{3}} > \frac{\pi}{2}.
$$

As predicted by Theorem A, $S_3^2 \times S_3^2$ is 1–connected, and since $S_3^2 \times S_3^2$ is not 2–connected, the conclusion of the Ricci curvature version of Theorem A cannot be strengthened.

The Octonionic projective plane provides an example of Theorem A that is truly about intermediate Ricci curvature, as well as an example showing that the conclusion about connectivity cannot be strengthened.

Example 6.2. $\mathbb{O}P^2$ with the canonical metric whose curvatures are in $[1, 4]$ is a 16–manifold with $Ric_9\mathbb{O}P^2 \geq 12$ in which each point has injectivity

radius $\frac{\pi}{2}$. If we multiply this metric by $\frac{12}{9}$ we get

$$Ric_9 \left(\sqrt{\frac{12}{9}} \mathbb{O}P^2 \right) \geq 9$$

and

$$\text{inj} \left(\sqrt{\frac{12}{9}} \mathbb{O}P^2 \right) = \frac{\pi}{2} \frac{\sqrt{12}}{\sqrt{9}} = \frac{\sqrt{3}}{3} \pi > \frac{\pi}{2}.$$

As predicted by Theorem A, $\mathbb{O}P^2$ is 7–connected; however, as it is not 8–connected, we see that the conclusion of Theorem A is optimal for 16–manifolds with $Ric_9 \geq 9$.

The next example shows that Theorem B is optimal in the sense that the estimate (2) cannot be replaced with a weak inequality.

Example 6.3. View $\mathbb{S}^3$ as the unit sphere in $\mathbb{C} \oplus \mathbb{C}$. The Clifford torus

$$N = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = \frac{1}{\sqrt{2}} \right\}$$

has

$$|\text{II}_N| = 1 = \cot \left(\frac{\pi}{2} - \frac{\pi}{4} \right) \text{ and } \text{foc}_N = \frac{\pi}{4}.$$

As predicted by Theorem B, the inclusion $N \hookrightarrow \mathbb{S}^3$ is 1–connected, but as it is not 2–connected, Inequality (2) cannot be be weakened to $\text{foc}_N \geq r$.

Our final example shows that Theorem 5.1 is optimal in the sense that the estimate (29) cannot be replaced with a strict inequality.

Example 6.4. As before take

$$N = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = \frac{1}{\sqrt{2}} \right\},$$

to be the Clifford Torus. Take

$$\tilde{N} \equiv \left\{ (z_1, 0) \in \mathbb{C}^2 \mid |z_1| = 1 \right\}$$

to be a coordinate circle. Then

$$|\text{II}_N| = 1 = \cot \left(\frac{\pi}{2} - \frac{\pi}{4} \right), \quad |\text{II}_{\tilde{N}}| = 0 = \cot \left(\frac{\pi}{2} - 0 \right),$$
$$\text{and} \quad \text{dist}(N, \tilde{N}) \leq \frac{\pi}{4} + 0,$$

as predicted by Theorem 5.1; however, as $\mathrm{dist}(N, \tilde{N}) = \pi/4$, we see that (29) cannot be replaced with a strict inequality.

7. Appendix: Proof of the Long Homotopy Lemma

In this appendix, we will provide a proof of the Long Homotopy Lemma for Submanifolds, as stated in Lemma C in the introduction. The proof has some points in contact with [1, Lemma 4.3]. To facilitate the reading, we will recall some of the notation used in Lemma C.

N is a closed, embedded submanifold in a Riemannian manifold M, and $\gamma : [0, b] \to M$ is a unit speed geodesic orthogonal to N at its endpoints $\gamma(0)$, $\gamma(b) \in N$. We denote by $\nu(N)$ the normal bundle of N in M, and by N_0 the zero section of $\nu(N)$; finally, let

$$\exp_N^\perp : \nu(N) \longrightarrow M$$

be the normal exponential map.

Let $D(N, \mathrm{foc}_N) \subset \nu(N)$ be the set of vectors normal to N with length less than foc_N. Since $\exp_N^\perp : D(N, \mathrm{foc}_N) \longrightarrow M$ is a local diffeomorphism, $g^* := \exp_N^{\perp *} g$ is a Riemannian metric on $D(N, \mathrm{foc}_N)$. In this metric, $D(N, \mathrm{foc}_N)$ agrees with the foc_N-neighborhood of N_0, the zero section of $\nu(N)$. We will denote such g^*-neighborhoods by $B(N_0, r)$ for $r \geq 0$.

Lemma 7.1. *Let $\alpha : [0, b] \to M$ be a curve with $\alpha(0) \in N$ and $\mathrm{length}(\alpha) = \ell < \mathrm{foc}_N$. Given a point $\bar{q} \in N_0$ with $\pi(\bar{q}) = \alpha(0)$, there is a unique lift $\bar{\alpha} : [0, b] \to \nu(N)$ with $\bar{\alpha}(0) = \bar{q}$; moreover, $\bar{\alpha}$ is entirely contained in $\overline{B(N_0, \ell)}$.*

Proof. There is an open cover $\{U_\alpha\}_\alpha$ of $\overline{B(N_0, \ell)}$ so that each map

$$\exp_N^\perp \Big|_{U_\alpha} : (U_\alpha, g^*) \longrightarrow (\exp_N^\perp(U_\alpha), g)$$

is an isometry for all α. The argument then follows the standard proof for lifts of curves in covering spaces. $\qquad \square$

The next lemma considers homotopies

$$H : [0, b] \times [0, 1] \to M, \quad H = H(t, s),$$

sending the three sides

$$\{0\} \times [0, 1], \quad [0, b] \times \{0\}, \quad \{b\} \times [0, 1]$$

of the rectangle $[0, b] \times [0, 1]$ into N. To simplify the writing, we will use $\sqcup$ to denote the union of the above three sides. We will also use H_s to denote the maps

$$H_s : [0, b] \to M, \quad t \to H(t, s)$$

for $s \in [0, 1]$. Finally, define the function $h(s) = \text{length}(H_s)/2$. If H has enough regularity (e.g, smooth), $h : [0, 1] \to \mathbb{R}$ is continuous.

Lemma 7.2. *Let $H : [0, b] \times [0, 1] \to M$ be a homotopy such that*

- *$H(\sqcup) \subset N$ and*

- *$\text{length}(H_s) < 2\,\text{foc}_N$ for every $0 \leq s \leq 1$.*

Then H has a lift

$$\tilde{H} : [0, b] \times [0, 1] \longrightarrow B(N_0, \text{foc}_N) \subset \nu(N)$$

with $\tilde{H}(\sqcup)$ contained in the zero section N_0.

Proof. We can assume, without loss of generality, that H is smooth. Thus, as observed above, $h(s)$ is a continuous function. The condition on the length of the curves H_s implies that there is a number r with $0 < r < \text{foc}_N$ so that the length of any H_s does not exceed $2r$.

Consider the curves

$$L_s : [0, h(s)] \to M, \qquad L_s(t) = H(t, s),$$

and

$$R_s : [h(s), b] \to M, \qquad R_s(t) = H(t, s).$$

By the choice of $h(s)$, the lengths of L_s and R_s do not exceed r. Lemma 7.1 gives us lifts

$$\bar{L}_s : [0, h(s)] \to \nu(S), \qquad \bar{R}_s : [h(s), b] \to \nu(S)$$

sending $\sqcup$ continuously into N_0. Moreover, the uniqueness of the lifts together with their being constructed using local inverses of $\exp_N^\perp |_{U_\alpha}$ shows that $\bar{L}(t, s) := \bar{L}_s(t)$ and $\bar{R}(t, s) := \bar{R}_s(t)$ are continuous in their respective domains.

We claim that the map $\bar{H}(t, s)$ defined by

$$\bar{H}(t, s) := \begin{cases} \bar{L}(t, s), & \text{for } 0 \leq t \leq h(s), \\ \bar{R}(t, s), & \text{for } h(s) \leq t \leq 1, \end{cases}$$

obtained by pasting together $\bar{L}$ and $\bar{R}$ is continuous. By the Pasting Lemma we just need to show that $\bar{L}(h(s), s) = \bar{R}(h(s), s)$ for all $s \in [0, 1]$. This can be done using a clopen argument; define

$$A := \{s \in [0, 1] \ : \ \bar{L}(h(s'), s') = \bar{R}(h(s'), s') \text{ for all } s' \leq s\},$$

and observe that $0 \in A$, and A is clearly closed. It remains to see that it is also open. This is a consequence of the way lifts are constructed. Indeed, suppose $s_0 \in A$, let $p_0 = H(h(s_0), s_0)$, and choose some U_α containing $\bar{L}(h(s_0), s_0) = \bar{R}(h(s_0), s_0)$ where $\exp_N^\perp$ is a diffeomorphism. Then p_0 lies in $\exp_N^\perp(U_\alpha)$, so

$$\bar{L}(h(s), s) = \exp_N^\perp |_{U_\alpha}^{-1} \circ H(h(s), s),$$
$$\text{and} \quad \bar{R}(h(s), s) = \exp_N^\perp |_{U_\alpha}^{-1} \circ H(h(s), s),$$

for s close to s_0. Thus A is open, and by connectedness, $A = [0, 1]$. $\qquad\square$

Lemma 7.3. *Let $\gamma : [0, b] \to M$ be a unit speed geodesic with $\gamma(0), \gamma(b) \in N$, $b < 2\,\mathrm{foc}_N$, and $\gamma'(0), \gamma'(b) \perp N$. Then there is no lift of γ to a curve $\hat{\gamma} : [0, b] \longrightarrow B(N_0, \mathrm{foc}_N)$ with $\hat{\gamma}(0), \hat{\gamma}(b) \in N_0$.*

Proof. Let $\tilde{\gamma}$ be a lift of γ to $\nu(N)$ with $\tilde{\gamma}(0) \in N_0$. Since $\exp^\perp$ is a local diffeomorphism in $B(N_0, \mathrm{foc}_N)$, we have

$$\tilde{\gamma}(t) = t\gamma'(0)$$

in any interval $[0, c]$ with $b/2 < c < \mathrm{foc}_N$. Thus,

$$|\tilde{\gamma}(c)| = \mathrm{dist}\,(\tilde{\gamma}(c), N_0) = c > \frac{b}{2}.$$

Since $\mathrm{dist}(\tilde{\gamma}(c), \tilde{\gamma}(b)) < b - c < b/2$, it follows that $\tilde{\gamma}(b)$ cannot lie in N_0. $\qquad\square$

Proof of Lemma C (The long homotopy Lemma). By Lemma 7.3, H has no lift to a homotopy with values in $B(N_0, \mathrm{foc}_N)$ whose curve end points are

in N_0. So by Lemma 7.2, for some $s \in [0,1]$ the curve

$$t \longmapsto H_s(t)$$

must be longer than $2 \operatorname{foc}_N$. $\qquad\square$

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Department of Mathematics, Universidad Autónoma de Madrid
and ICMAT CSIC-UAM-UCM-UC3M
28049 Madrid, Spain
E-mail address: luis.guijarro@uam.es

Department of Mathematics, University of California
Riverside, CA 92521, USA
E-mail address: fred@math.ucr.edu