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# Level curves of minimal graphs

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We consider minimal graphs $u = u(x, y) > 0$ over domains $D \subset R^2$ bounded by an unbounded Jordan arc $\gamma$ on which $u = 0$. We prove an inequality on the curvature of the level curves of $u$, and prove that if $D$ is concave, then the sets $u(x, y) > C$ $(C > 0)$ are all concave. A consequence of this is that solutions, in the case where $D$ is concave, are also superharmonic.

## 1. Introduction

Let $D$ be a plane domain bounded by an unbounded Jordan arc $\gamma$. In this paper we consider the boundary value problem for the minimal surface equation

$$\begin{cases} \operatorname{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \quad \text{and } u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \gamma \end{cases} \tag{1.1}$$

We shall study the curvature $\kappa = \pm|d\varphi/ds|$ for level curves $u = C$ $(C > 0)$ where $\varphi$ is the angle of the tangent vector to the curve, and the sign will be taken to be + when the curve bends away from the set where $u > C$.

**Theorem 1.** *There exists a constant $K$ depending on $u$ such that, if $u$ as in (1.1) and $C > 0$, the curvature $\kappa = \kappa(C)$ of the level curve $u = C$ satisfies the inequality*

$$|\kappa| \le \frac{K}{C}. \tag{1.2}$$

Further comments regarding the constant $K$ are given in §6.

Our next result concerns solutions whose domains are concave. There is a literature (see [3] and references cited there) regarding the propogation of convexity for level curves of solutions to partial differential equations over convex domains.

However, regarding the possible geometry of $D$ in (1.1), it follows from a theorem of Nitsche [6, p.256] that $D$ cannot be convex unless $D$ is a halfplane since (1.1) cannot have nontrivial solutions over domains contained in a sector of opening less than $\pi$. On the other hand, amongst the examples given in [5], there is a continuum of graphs which do have concave domains; specifically those given parametrically in the right half plane $\mathbf{H}$ by

$$z(\zeta) = (\zeta+1)^{\gamma} - \frac{1}{\gamma(2-\gamma)}(\bar{\zeta}+1)^{2-\gamma} \quad (\zeta \in \mathbf{H}, \quad 1 < \gamma < 2) \tag{1.3}$$

together with the height function $2\Re e\,\zeta$. A concave domain $D$ is taken to be one whose complement is an unbounded convex domain. The boundary of $D$ is then a curve which bends away from the domain.

In §6 we will verify that the domains for the graphs of (1.3) are concave. In this note we shall prove the following

**Theorem 2.** *If $u$ is a solution to (1.1) with $D$ concave and bounded by a $C^2$ curve $\gamma$, then the sets where $u > C$ are concave for each $C > 0$.*

This has the curious consequence

**Corollary.** *If $u$ is as in Theorem 2 above, then $u$ is also superharmonic in $D$.*

## 2. Preliminaries

For a solution $u$ to the minimal surface equation over a simply connected domain $D$ we shall slightly abuse notation by using $u$ to also denote the solution to (1.1) when given in parametric form. We shall make use of the parametrization of the surface given by $u$ in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), u(\zeta))$ with $\zeta$ in the right half plane $\mathbf{H}$. Our notation will then be given by

$$f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta = \sigma + i\tau \in \mathbf{H}. \tag{2.1}$$

Then $f(\zeta)$ is univalent and harmonic, and since $D$ is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \quad \zeta = \sigma + i\tau \in \mathbf{H} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in $\mathbf{H}$,

$$|h'(\zeta)| > |g'(\zeta)|, \tag{2.3}$$

and

$$u(\zeta) = 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.4}$$

(cf. [2, §10.2]).

Now, $u(\zeta)$ is harmonic and positive in $\mathbf{H}$ and vanishes on $\partial\mathbf{H}$. Thus, (cf. [7, p. 151]),

$$u(\zeta) = k_0\, \Re e\, \zeta, \tag{2.5}$$

where $k_0$ is a positive constant. This with (2.4) gives

$$g'(\zeta) = -\frac{k}{h'(\zeta)} \qquad (k = k_0^2/4). \tag{2.6}$$

Then from (2.3) we have, in particular, that

$$|h'(\zeta)| \geq \sqrt{k}. \tag{2.7}$$

It follows from (2.5) that the level curves of $u$ can be parametrized by $f(\sigma_0 + i\tau)$ for $-\infty < \tau < \infty$ and fixed values $\sigma_0$. Then the curvature $\kappa$ corresponding to height $\sigma_0$ with the sign convention given at the begining for

$$\varphi = \arctan(y_\tau / x_\tau)$$

is given by

$$\kappa = \kappa(\sigma_0, \tau) = \frac{d\varphi}{ds} = \frac{1}{(x_\tau^2 + y_\tau^2)^{3/2}}(x_\tau y_{\tau\tau} - y_\tau x_{\tau\tau}). \tag{2.8}$$

To compute (2.8) we use (2.1) and (2.6) to write

$$\begin{aligned}
(2.9)\qquad x_\tau &= \frac{\partial}{\partial\tau}\Re e(h+\overline{g}) = \Re e\, i(h'-k/h') \\
&= -\Im m(h'-k/h') = -(|h'|^2+k)\Im m\frac{1}{\overline{h}'} \\
(2.10)\qquad x_{\tau\tau} &= -\frac{\partial}{\partial\tau}\Im m(h'-k/h') = -\Re e(h''+kh''/h'^2) \\
(2.11)\qquad y_\tau &= \frac{\partial}{\partial\tau}\Im m(h+\overline{g}) = \Im m\, i(h'+k/h') \\
&= \Re e(h'+k/h') = (|h'|^2+k)\Re e\frac{1}{\overline{h}'} \\
(2.12)\qquad y_{\tau\tau} &= \frac{\partial}{\partial\tau}\Re e(h'+k/h') = -\Im m(h''-kh''/h'^2)
\end{aligned}$$

Substituting (2.9)-(2.12) into (2.8) we get

$$\begin{aligned}
\kappa = \frac{|h'|^3}{4(|h'|^2+k)^2}\Bigg(&-(\frac{1}{\overline{h}'}-\frac{1}{h'})(h''-k\frac{h''}{h'^2}-\overline{h}''+k\frac{\overline{h}''}{\overline{h}'^2}) \\
&+(\frac{1}{\overline{h}'}+\frac{1}{h'})(h''+k\frac{h''}{h'^2}+\overline{h}''+k\frac{\overline{h}''}{\overline{h}'^2})\Bigg)
\end{aligned}$$

which simplifies down to

$$\kappa = \frac{|h'|}{|h'|^2+k}\,\Re e\,\frac{h''}{h'}. \tag{2.13}$$

Summarizing this, we have

**Lemma 1.** *With $u$ as in (1.1) and $k_0$ as in (2.5), then the locus of $u=C$ is the set $\zeta=\sigma_0+i\tau$, where $\sigma_0=C/k_0$ and $-\infty<\tau<\infty$. The curvature $\kappa$ at each point of this level set satisfies (2.13).*

The proof of Theorem 2 uses the comparison of $\kappa$ in (2.13) with the corresponding curvature $\kappa_1$ of the image of the line $\sigma_0+i\tau \quad (-\infty<\tau<\infty)$ under $h$. Since $\arg h' = \Im m\,\log h'$, the formula (2.8) gives

$$\kappa_1 = \frac{1}{|h'|}\Re e\frac{h''}{h'}. \tag{2.14}$$

## 3. Proof of Theorem 1

Since $f$ in (2.2) is a univalent harmonic mapping, we may convert the estimate from [1, Lemma 1] (cf. also ( [2, p. 153])) for a univalent harmonic mapping $F = H + \overline{G}$ in the unit disk $\mathbf{U}$ to a mapping of the half plane $\mathbf{H}$.

**Lemma 2.** *Let $u$ be as in (1.1) and $f = h + \overline{g}$ as in (2.2). Then*

$$\left|\frac{h''(\zeta)}{h'(\zeta)}\right| \leq A/\sigma$$

*for some absolute constant $A$.*

*Proof of Lemma 2.* For the univalent harmonic mapping $F = H + \overline{G}$ of $\mathbf{U}$, the estimate of [1] is

$$\left|\frac{H''(w)}{H'(w))}\right| \leq \frac{A_1}{1-|w|}, \qquad w \in \mathbf{U}$$

for some absolute constant $A_1$. Now, for $f(\zeta) = h(\zeta) + \overline{g(\zeta)}$, let

$$F(w) = f\left(\frac{1+w}{1-w}\right), \quad w \in \mathbf{U}.$$

Then,

$$h(\zeta) = H\left(\frac{\zeta-1}{\zeta+1}\right),$$

$$h'(\zeta) = H'\left(\frac{\zeta-1}{\zeta+1}\right)\frac{2}{(\zeta+1)^2},$$

and

$$h''(\zeta) = H''\left(\frac{\zeta-1}{\zeta+1}\right)\frac{4}{(\zeta+1)^4} - H'\left(\frac{\zeta-1}{\zeta+1}\right)\frac{4}{(\zeta+1)^3}.$$

Thus,

$$\begin{aligned}
\left|\frac{h''(\zeta)}{h'(\zeta)}\right| &\leq \frac{2}{|\zeta+1|}\left(\frac{1}{|\zeta+1|}\frac{A_1}{1-\left|\frac{\zeta-1}{\zeta+1}\right|}+1\right)\\
&\leq \frac{2}{|\zeta+1|}\left(\frac{A_1}{|\zeta+1|-|\zeta-1|}+1\right)\\
&\leq \frac{2}{|\zeta+1|}\left(\frac{A_2(|\zeta+1|+|\zeta-1|)}{4\sigma}+1\right)\\
&\leq A/\sigma
\end{aligned}$$

for some absolute constant $A$. □

*Proof of Theorem 1.* From Lemma 1, Lemma 2, and (2.7) it follows that, on the level set $u = C$,

$$|\kappa| \leq \frac{A}{\sqrt{k}C}. \tag{3.1}$$

□

## 4. Proof of Theorem 2

For convenience, we dismiss the trivial case where $u$ is planar, and hence we may assume that $h'$ is nonconstant.

From the given hypothesis, it follows that $\gamma$ must have asymptotic angles in both directions as $z \to \infty$. By a rotation we may assume that the asymptotic tangent vectors have directions $\pm\alpha$ for some $0 \leq \alpha \leq \pi/2$.

From the concavity of $D$ and the assumption that the asymptotic tangents to $\gamma$ have angles $\pm\alpha$, it follows that $y_\tau \geq 0$ for $\sigma = 0$. Thus, from (2.11) it follows that for $\sigma = 0$, $\Re e\, 1/\overline{h}' \geq 0$, and hence $\Re e\, 1/h' \geq 0$. Since, by (2.7) $1/h'$ is bounded in $\mathbf{H}$, this means that $\Re e\, 1/h' > 0$ thoughout $\mathbf{H}$. This in turn gives

$$\Re e\, h'(\zeta) > 0 \qquad \zeta \in \mathbf{H}. \tag{4.1}$$

Let $\psi(\tau) = \arg h'(i\tau)$. It follows from (2.13) and (2.14) that $0 \leq \kappa_1 \not\equiv 0$ on $\partial\mathbf{H}$ so that

$$\frac{d\psi}{d\tau} = \frac{\partial}{\partial\tau}\Im m(\log h') = \Re e \frac{h''}{h'} \geq 0 \quad \text{when } \tau = 0. \tag{4.2}$$

By (4.1)

$$-\pi/2 \leq \psi(\tau) \leq \pi/2. \tag{4.3}$$

Now, $-\pi/2 < \Im m(\log h') < \pi/2$ in $\mathbf{H}$, and in particular is a bounded harmonic function in $\mathbf{H}$. So for $\zeta = \sigma + i\tau \in \mathbf{H}$,

$$\Im m\, \log h'(\zeta) = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{\psi(t)dt}{\sigma^2 + (t-\tau)^2}.$$

Then

$$\begin{aligned}\Re e\frac{h''(\zeta)}{h'(\zeta)} &= \frac{\partial}{\partial\tau}\Im m\,\log h'(\zeta)\\ &= \frac{\partial}{\partial\tau}\left(\frac{\sigma}{\pi}\int_{-\infty}^{\infty}\frac{\psi(t)dt}{\sigma^2+(t-\tau)^2}\right)\\ &= \frac{2\sigma}{\pi}\int_{-\infty}^{\infty}\frac{(t-\tau)\psi(t)dt}{(\sigma^2+(t-\tau)^2)^2}.\end{aligned}$$

An integration by parts yields

$$\Re e\frac{h''}{h'} = \frac{\sigma}{\pi}\left(\frac{-\psi(t)}{\sigma^2+(t-\tau)^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\psi'(t)dt}{\sigma^2+(t-\tau)^2}\right).$$

By (4.3) it follows that the first term on the right vanishes, and by (4.2) the second term is positive. Thus $\kappa_1$ in (2.14) and hence $\kappa$ in (2.13) are positive in **H**. □

## 5. Proof of the corollary

We may write the minimal surface equation for $u$ as

$$\frac{\Delta u + F}{|\nabla u|^3} = 0$$

where $F = F(u, x, y) = u_y^2 u_{xx} + u_x^2 u_{yy} - 2u_x u_y u_{xy}$.

Now, for a given function $v(x, y) > 0$ the curvature of the level set $v(x, y) = 0$ is given by $F(v, x, y)/|\nabla v|^3$ [4, p. 72] which is positive when the curve bends away from the interior of the domain. Since Theorem 2 shows that the level sets $u = c$ which bound the sets $u > c$ each have positive curvature, then applying this to $F(u - c, x, y)$ we find that $\Delta u < 0$ and hence $u$ is superharmonic in $D$ □

## 6. Concluding remarks

For the examples (1.3) of §1,

$$\Re e\frac{h''}{h'} = \Re e\,\frac{\gamma-1}{\zeta+1} > 0.$$

for $1 < \gamma < 2$ so that by (2.13) these have concave domains.

Furthermore, using (2.13), this shows that Theorem 1 is sharp. Regarding the constant $K$ in Theorem 1, the scaling factor $k$ in (3.1) is consistent with the fact that $\kappa$ would be rescaled by replacing $u(x,y)$ by $cu(x/c, y/c)$ for $0 < c < \infty$.

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