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Effective operators for changing sign Robin Laplacian in thin two- and three-dimensional curved waveguides

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We study the Laplacian in some thin curved domains, in the plane and space, with particular types of Robin boundary conditions and cross-sections. We derive, when the diameters of the cross sections tend to zero, nontrivial effective Schrödinger operators on the reference curve by means of norm resolvent convergences. Besides the changing sign in the Robin parameter, for which no renormalization is necessary, another novelty is that the torsion (in the spatial case) plays no role to effective operators.

1. Introduction

We study the Laplace operator in some planar strips and three-dimensional curved tubes, subject to certain Robin boundary conditions; such regions Ω_ϵ are built over a reference curve $\Gamma(s)$ by appropriately moving a bounded cross section along Γ. In this paper we investigate effective self-adjoint operators as the cross-section S (a square in the spatial case) of the region tends (uniformly) to zero as a parameter ϵ vanishes.

Related studies, with more general cross-sections, as the behavior of the essential spectrum and eigenvalues expansions in terms of the small diameter of the model, have also been discussed in the literature, mainly with Neumann and Dirichlet boundary conditions; see [1, 6, 9, 12, 17, 21] and references therein. There are few works that consider Robin boundary conditions, as [3, 14, 15, 19] with positive coupling parameters in the plane, with ϵ-scaled and positive parameters in space [2], and combination of Robin with other types of conditions [11].

Here we examine (particular) Robin type conditions, i.e., we investigate effective operators for the Laplace operator on thin domains whose Robin parameter $\tilde{\gamma}$ is not constant and changes sign, a situation that has not been considered in the literature of thin regions (to our best knowledge). See ahead for detailed descriptions.

Since the domains Ω_ϵ and Γ have different dimensions, suitable identifications are required, and here we will approach effective operators in the norm resolvent sense. As in other works, their actions can be characterized by one-dimensional operators that depend on geometric characteristics of the thin domain, and here new classes of effective Schrödinger operators are obtained.

With our choice of boundary conditions, the effective potentials (see the actions of the effective operators for tubes in (3) and for planar strips in (5)) may be attractive or repulsive and, with some surprize, in the three-dimensional case the torsion of Γ plays no role in such singular limits!

In what follows we are going to be more specific in the description of our setting. We formally use

$$\frac{\partial u}{\partial \vec{\nu}} + \tilde{\gamma} u = 0, \quad \text{on } \partial\Omega_\epsilon, \tag{1}$$

to introduce our boundary condition, where $\epsilon > 0$ is small and $\partial\Omega_\epsilon$ denotes the boundary of Ω_ϵ. In this context, equation (1) should be understood in the sense of traces. As already mentioned, the function $\tilde{\gamma} : \partial\Omega_\epsilon \to \mathbb{R}$ is bounded and changes sign. In fact, in the study of the three-dimensional case we put

$\tilde{\gamma} := \gamma \circ \mathcal{L}_\epsilon^{-1}$ (the natural change of coordinates $\mathcal{L}_\epsilon$ for tubes is presented in (12)), where the bounded function $\gamma : \partial\Omega \to \mathbb{R}$ is defined on the border of the straight region $\Omega = S \times \mathbb{R}$, $S = (0,1) \times (0,1)$ (a square). The considered possibilities of boundary conditions for tubes are the following.

Let a boundary "parameter function" $\alpha : \mathbb{R} \to \mathbb{R}$ be given and we take the function $\gamma : \partial\Omega \to \mathbb{R}$ on the border of Ω given by

$$(2) \quad \gamma(y,s) = \begin{cases} -\alpha(s), & (y_1,y_2) \in \{(0,1] \times \{0\} \cup \{0\} \times [0,1)\},\ s \in \mathbb{R} \\ \alpha(s), & (y_1,y_2) \in \{\{1\} \times (0,1] \cup [0,1) \times \{1\}\},\ s \in \mathbb{R} \end{cases} .$$

Some hypotheses on Γ and α will be imposed in Section 2.

For unbounded tubes with Dirichlet condition, it was obtained in [5], via Γ- convergence, that the effective operator (through strong resolvent convergence) is given by

$$Tw = -w'' + \left(C(S)\tau^2(s) - \frac{k^2(s)}{4} \right) w, \qquad \operatorname{dom} T = H^2(\mathbb{R}),$$

with $C(S) > 0$, where $k(s)$ and $\tau(s)$ are, respectively, the curvature and the torsion of the reference curve; see Theorem 5 in [5], which was based on studies of bounded tubes in [1]. By applying the technique of [12, 13] combined with an additional change of variables, in [8] a norm resolvent convergence was obtained also for unbounded tubes; related results were also obtained in [18] and with minimal regularity assumptions (e.g., some noncontinuous curvatures of the reference curves are allowed).

Although in the studies of Robin boundary condition in [2] the Γ-convergence was employed, here the method of [12] will be our main tool. We will then establish a type of norm resolvent convergence to effective operators in space whose actions was found to be

$$(3) \qquad Tw = -w'' + \left(-2\alpha^2(s) - \alpha(s)k(s)\right) w, \qquad w \in \operatorname{dom} T = H^2(\mathbb{R}),$$

see Theorem 2.2. Particularly, note the absence of torsion $\tau(s)$ in the effective potentials (compare with the Dirichlet case [1, 5] and Robin condition with positive and scaled parameter [2]) and that, depending on the values of the functions α and k, the potential may be attractive or repulsive.

In the two-dimensional case, i.e., when we deal with unbounded curved strips over a reference curve Γ, a similar analysis is performed. In this case we study a Robin Laplacian on $\Omega_\epsilon \subset \mathbb{R}^2$, under the boundary condition (1) with $\tilde{\gamma} = \gamma \circ f_\epsilon^{-1}$. It is natural to express the Laplacian in the coordinates (s,u) determined by the inverse of f_ϵ described in (7). Now we consider the same notation as before, that is, let a boundary parameter $\alpha : \mathbb{R} \to \mathbb{R}$ be given; then the function $\gamma : \partial\Omega \to \mathbb{R}$ on the border of $\Omega := \mathbb{R} \times (0,1)$ (straight strip) is proposed to be

$$\gamma(s,u) = \begin{cases} -\alpha(s), & (s,u) \in \mathbb{R} \times \{0\} \\ \alpha(s), & (s,u) \in \mathbb{R} \times \{1\} \end{cases} . \tag{4}$$

We have found that, as $\epsilon \to 0$, the effective operator for strips with this boundary condition may be identified with

$$Tw = -w'' + \big(-\alpha^2(s) - \alpha(s)k(s)\big)\, w, \qquad w \in \operatorname{dom} T = H^2(\mathbb{R}). \tag{5}$$

The paper is organized as follows. In Section 2 we present the planar and spatial models and state our main results in Theorems 2.1 and 2.2 (see also Remark 2.3). We introduce appropriate quadratic forms in Sections 3-4, which are used in the proofs of the main results. Furthermore, we discuss some information about the Robin Laplacian on the respective cross sections, and explain how effective operators are obtained. In Section 5 we introduce some intermediate, but fundamental, convergences. The proofs of the main theorems are concluded in Section 6; however, in order to improve readability of the core of the work, we leave the proofs of some technical steps to three appendices.

Some notation used in the text. The symbol $A \sqsubseteq B$ indicates that A is a dense subset of B. The curvature and torsion of curves will be denoted, respectively, by $k(s)$ and $\tau(s)$. The norms on $\mathrm{L}^2, \mathrm{L}^\infty$ are respectively denoted by $\|\cdot\|_2, \|\cdot\|_\infty$. We denote by $\operatorname{dom} g$ the domain of the operator or quadratic form g. The norm on the Sobolev space $H^1(\Omega)$ of order 1 is denoted by $\|\cdot\|_{1,2}$. The outward pointing unit normal is denoted by $\vec{\nu}$, so that $\frac{\partial u}{\partial \vec{\nu}}$ is the outward normal derivative of u (this was already used in this Introduction).

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2. Preliminaries and Main Results

2.1. Strips

In what follows we precise the regions Ω_ϵ, which are modelled by infinitely long curved waveguides in $\mathbb{R}^2$, and state our main results. The general idea is to consider the curved regions Ω_ϵ when its cross-section ϵS diminishes to a point as $\epsilon \downarrow 0$, and study the behavior of the family of operators associated with the corresponding quadratic forms. Through appropriate identifications, we will be able to approximate such a family of operators, by means of a norm resolvent convergence, by a one-dimensional effective operator.

Let $\Gamma : \mathbb{R} \to \mathbb{R}^2 : \{s \mapsto (\Gamma_1(s), \Gamma_2(s))\}$ be an infinite planar curve of class $C^3(\mathbb{R})$ and with unit speed, i.e., $\|\dot{\Gamma}(s)\| = 1$ for all $s \in \mathbb{R}$. We assume that Γ is an embedding. The vectors $N := (-\dot{\Gamma}_2, \dot{\Gamma}_1)$ defines a unit *normal* vector field and the pair $(\dot{\Gamma}, N)$ gives a distinguished orthonormal frame. The *curvature* of Γ is the scalar function defined by $k = \det(\dot{\Gamma}, \ddot{\Gamma})$. We note that k is a function of class $C^1(\mathbb{R})$. Furthermore, we assume that $k \in W^{1,\infty}(\mathbb{R})$. Let the curved strip, which is the configuration space $\Omega_\epsilon \subset \mathbb{R}^2$, be defined by

$$\Omega_\epsilon = \left\{(x, y) \in \mathbb{R}^2; (x, y) = \Gamma(s) + \epsilon u N(s), s \in \mathbb{R}, u \in (0, 1)\right\}. \tag{6}$$

We consider, for small $\epsilon > 0$, that the strips Ω_ϵ are not self-intersecting. In fact, we are introducing the mapping f_ϵ from the straight strip $\overline{\Omega}$, where $\Omega = \mathbb{R} \times (0, 1)$, to $\mathbb{R}^2$ defined by

$$f_\epsilon(s, u) = \Gamma(s) + \epsilon u N(s), \tag{7}$$

and make the hypothesis that f_ϵ is injective, and since $k \in L^\infty(\mathbb{R})$ the mapping f_ϵ is a C^2-diffeomorphism, whose image $\Omega_\epsilon = f_\epsilon(\mathbb{R} \times (0, 1))$ has the geometrical meaning of an open nonself-intersecting curved strip along Γ.

The Robin Laplacian we consider, $-\Delta_R^{\Omega_\epsilon}$ in Ω_ϵ, is the unique self-adjoint operator on $L^2(\Omega_\epsilon)$ associated with the quadratic form $b_\epsilon^{\Omega_\epsilon}$ given by

$$b_\epsilon^{\Omega_\epsilon}(\phi) = \int_{\Omega_\epsilon} |\nabla\phi|^2 \,\mathrm{d}x\mathrm{d}y + \int_{\partial\Omega_\epsilon} \tilde{\gamma}|\mathrm{tr}_\epsilon(\phi)|^2 \,\mathrm{d}\sigma_\epsilon\,, \quad \mathrm{dom}\, b_\epsilon^{\Omega_\epsilon} = H^1(\Omega_\epsilon), \tag{8}$$

where the function $\mathrm{tr}_\epsilon(\phi)$ denotes the trace of $\phi \in \mathrm{dom}\, b_\epsilon^{\Omega_\epsilon}$ and $\mathrm{d}\sigma_\epsilon$ the one-dimensional surface measure on $\partial\Omega_\epsilon$. In terms of natural coordinates $(x, y) = f_\epsilon(s, u)$, with $(s, u) \in \partial\Omega$, we have, by definition, that $\tilde{\gamma} : \partial\Omega_\epsilon \to \mathbb{R}$ is bounded (recall (4)).

At the end of this subsection we present our main result (Theorem 2.1) for curved strips. For this, let us begin with the introduction the closed subspace E of $\mathcal{H} = \mathrm{L}^2(\Omega)$, which consists of functions independent of the longitudinal variable u, i.e., let $E \subset \mathcal{H}$ be the subspace given by

$$E = \{w(s)1; w \in \mathrm{L}^2(\mathbb{R})\}.$$

Since the functions in E depend only on s, E can be identified with $\mathrm{L}^2(\mathbb{R})$. Hence, we may identify an operator on $\mathrm{L}^2(\mathbb{R})$ with an operator acting on E and vice versa.

For each $\phi \in \mathcal{H} = E \oplus E^\perp$ the following holds,

$$\phi = P(\phi) + P_{E^\perp}(\phi), \quad \text{with} \quad P(\phi)(s,u) = \int_0^1 \phi(s,r)\,\mathrm{d}r, \quad \text{a.e.} \quad s \in \mathbb{R}, \tag{9}$$

where $P = P_E$ and $P_{E^\perp}$ stand for the orthogonal projections from $\mathrm{L}^2(\Omega)$ onto the subspaces E and $E^\perp$, respectively. For future reference, introduce the linear surjective isometry

$$\pi_0 : E \to \mathrm{L}^2(\mathbb{R}) : \{w1 \mapsto w\}.$$

We are now in position to formulate our first main result; in Section 3 we describe the operator T_ϵ precisely. For technical reasons, the proof of our Theorem 2.1 requires that $\alpha \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$.

Theorem 2.1. *Consider the self-adjoint operator* T_ϵ *in* $\mathrm{L}^2(\Omega)$ *unitarily equivalent to the Robin Laplacian operator* $-\Delta_R^{\Omega_\epsilon}$ *in* $\mathrm{L}^2(\Omega_\epsilon)$*. If* T *denotes the self-adjoint operator in* $\mathrm{L}^2(\mathbb{R})$ *given by* (5)*, then for some* $c_1 > 0$ *the uniform resolvent convergence*

$$\left\|(\mathrm{T}_\epsilon + c_1)^{-1} - \left[\pi_0^{-1} \circ (T + c_1)^{-1} \circ \pi_0 \oplus 0_{E^\perp}\right]\right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \longrightarrow 0, \quad \epsilon \to 0,$$

holds true, where $0_{E^\perp}$ *is the null operator on the subspace* $E^\perp$*. The choice of* c_1 *is done in Lemma 3.4.*

The fact that a "big" subspace is discarded in the limit process is what allows us to identify operators in $\mathrm{L}^2(\Omega)$ with operators in $\mathrm{L}^2(\mathbb{R})$. Moreover, this identification occurs via a type of norm convergence of resolvents.

2.2. Tubes

We consider a special ϵ-tubular neighborhood Ω_ϵ of some curves in $\mathbb{R}^3$. It follows the ideas of the planar model. Let $\Gamma : \mathbb{R} \to \mathbb{R}^3$ be a simple curve of class $\mathrm{C}^3(\mathbb{R})$ with $\|\dot{\Gamma}(s)\| = 1$ for all $s \in \mathbb{R}$. The *curvature* k of the reference curve Γ is defined by $k(s) = \|\ddot{\Gamma}(s)\|$, for all $s \in \mathbb{R}$. We choose the orthonormal basis of vector fields (T, N, B) of its *tangent*, *normal* and *binormal*, respectively, and assume that the (distinguished) Frenet frame is globally defined. The *curvature* and *torsion* functions associated with Γ, denoted by k and τ, respectively, are supposed to satisfy the *Frenet equations*

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}; \tag{10}$$

the torsion τ is defined by (10).

In order to guarantee that the distinguished Frenet frame exists one may impose the condition $k(s) \neq 0$ everywhere, but this is not strictly necessary; in case $k(s) \neq 0$ in a compact interval I, for instance, it is possible to extend the distinguished Frenet frame to all s by using suitable constant frames outside I (see [10]).

Consider the set

$$\Omega_\epsilon = \{x \in \mathbb{R}^3;\, x = \Gamma(s) + \epsilon y_1 N(s) + \epsilon y_2 B(s), s \in \mathbb{R}, (y_1, y_2) \in S\},$$

which is obtained by properly translating the region ϵS along the curve Γ; recall that here $S = (0,1) \times (0,1)$.

Introduce the Robin Laplacian $-\Delta_R^{\Omega_\epsilon}$ in Ω_ϵ as the unique self-adjoint operator in $\mathrm{L}^2(\Omega_\epsilon)$ associated with the closed and lower bounded quadratic form F_ϵ given by

$$\mathrm{F}_\epsilon(\psi) = \int_{\Omega_\epsilon} |\nabla\psi|^2 \,\mathrm{d}x + \int_{\partial\Omega_\epsilon} \tilde{\gamma} |\mathrm{tr}_\epsilon(\psi)|^2 \,\mathrm{d}\sigma_\epsilon\,, \quad \mathrm{dom}\,\mathrm{F}_\epsilon = H^1(\Omega_\epsilon), \tag{11}$$

where $\mathrm{d}\sigma_\epsilon$ denotes the bidimensional surface measure on the boundary $\partial\Omega_\epsilon$; the bounded function $\tilde{\gamma} : \partial\Omega_\epsilon \to \mathbb{R}$ is given by (2).

The standard strategy is similar to the case of planar strips, i.e., a natural change of coordinates given by $\mathcal{L}_\epsilon^{-1}$ is performed, so that the region in (11) becomes the straight tube $\Omega := S \times \mathbb{R}$, which is independent of $\epsilon > 0$.

Consider the mapping $\mathcal{L}_\epsilon : \overline{\Omega} \to \overline{\Omega}_\epsilon$, for each $\epsilon > 0$,

$$\mathcal{L}_\epsilon(y, s) := \Gamma(s) + \epsilon y_1 N(s) + \epsilon y_2 B(s)\,. \tag{12}$$

Denote $\beta_\epsilon = \beta_\epsilon(y, s) = \beta_\epsilon(y_1, \epsilon) := (1 - \epsilon k(s) y_1)$; a unitary transformation will identify the Hilbert space $\mathrm{L}^2(\Omega_\epsilon)$ with $\mathrm{L}^2(\Omega)$, the latter with the inner product

$$(\psi, \phi)_\epsilon = \int_\Omega \bar{\psi}(y, s)\phi(y, s)\epsilon^2 \beta_\epsilon(y, s)\, \mathrm{d}y\mathrm{d}s, \quad \forall\ \psi, \phi \in \mathrm{L}^2(\Omega)\,. \tag{13}$$

In order to ensure that this identification is meaningful, we will assume that $\|k\|_\infty, \|\tau\|_\infty < \infty$, so that $\mathcal{L}_\epsilon$ is a diffeomorphism. For technical reasons, the results of our main Theorem 2.2 require that $\tau \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$ and $k, \alpha \in W^{2,\infty}(\mathbb{R}) \cap \mathrm{C}^2(\mathbb{R})$; so our hypothesis that Γ is of class C^3 (on top of the assumed existence of a distinguished Frenet frame).

The Jacobian determinant of $\mathcal{L}_\epsilon$ is found to be $\det \nabla \mathcal{L}_\epsilon = \epsilon^2 \beta_\epsilon$ where $\beta_\epsilon = (1 - \epsilon k(s) y_1)$. Indeed, after some calculations, the Jacobian $\nabla \mathcal{L}_\epsilon$ matrix is given by

$$\nabla \mathcal{L}_\epsilon(y, s) := \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \\ \beta_\epsilon & -\tau\epsilon y_2 & \tau\epsilon y_1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{14}$$

where we put

$$e_1 = \frac{\partial \mathcal{L}_\epsilon}{\partial y_1} \quad e_2 = \frac{\partial \mathcal{L}_\epsilon}{\partial y_2}, \quad e_3 = \frac{\partial \mathcal{L}_\epsilon}{\partial s}.$$

The inverse of the matrix in (14) is given by

$$\begin{bmatrix} \frac{\tau(s) y_2}{\beta_\epsilon(y,s)} & -\frac{\tau(s) y_1}{\beta_\epsilon(y,s)} & \frac{1}{\beta_\epsilon(y,s)} \\ \frac{1}{\epsilon} & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \end{bmatrix}.$$

Since k is bounded, for ϵ small enough, we obtain that $\beta_\epsilon > 0$ on $\Omega = S \times \mathbb{R}$ and then it follows that $\mathcal{L}_\epsilon$ is a local diffeomorphism. By requiring that $\mathcal{L}_\epsilon$ is injective (it is usual to assume the tube Ω_ϵ is nonself-intersecting), a global diffeomorphism is obtained.

Finally, after a suitable identification and by (37), we take into account the family as follows $\{(b_\epsilon + c_1)\}_{\epsilon > 0}$ of quadratic forms in $\mathcal{H} = \mathrm{L}^2(\Omega)$. Since space $E = \{w(s)1; w \in \mathrm{L}^2(\mathbb{R})\} \subset \mathcal{H}$ is closed and $\pi_0 : E \to \mathrm{L}^2(\mathbb{R})$ $\{w1 \mapsto w\}$ identifies these spaces, we are able to state our main result in the three-dimensional case.

Theorem 2.2. *Let B_ϵ be the self-adjoint operator associated with b_ϵ. Then, for some $c_1 > 0$, the uniform resolvent convergence*

$$\left\| (B_\epsilon + c_1)^{-1} - \left[\pi_0^{-1} \circ (T + c_1)^{-1} \circ \pi_0 \oplus 0_{E^\perp} \right] \right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \longrightarrow 0, \quad \epsilon \to 0,$$

holds true, with T defined in (3) and $0_{E^\perp}$ denotes the null operator on $E^\perp$. The choice of c_1 is done in Lemma 4.10.

Remark 2.3. *In the proofs of Theorems 2.1 and 2.2, an intermediate step will be necessary, and a relevant family of closed subspaces of $\mathcal{H}$ will be considered; this takes into account the first eigenfunction of the Robin Laplacian on the respective cross sections; see Sections 3.3 and 4.3 for more details.*

3. Two-dimensional forms

In the two-dimensional case, the dimensional reduction will be produced by means of Proposition 3.1 in [12] together with a uniform convergence of quadratic forms. In fact, note that the determinant Jacobian matrix of the transformation f_ϵ is equal to $\epsilon\beta_\epsilon$ where $\beta_\epsilon = 1 - \epsilon u k(s) > 0$ for $\epsilon > 0$ small enough.

Initially, we employ the unitary transformation (15) to simplify the strip region so that we may work in the Hilbert space $\mathrm{L}^2(\Omega, \epsilon\beta_\epsilon \mathrm{d}s\mathrm{d}u)$, where $\Omega = f_\epsilon(\Omega_\epsilon)$ (straight strip), but the price to pay is a more complicated action of the Robin Laplacian $-\Delta_R^{\Omega_\epsilon}$.

Next, using the unitary transformation V_ϵ below, we can study the asymptotic behavior of quadratic forms in the Hilbert space $\mathrm{L}^2(\Omega, \mathrm{d}s\mathrm{d}u)$.

Our first unitary transformation is given by

$$U_\epsilon : \mathrm{L}^2(\Omega_\epsilon) \;\to\; \mathrm{L}^2(\Omega, \epsilon\beta_\epsilon \mathrm{d}s\mathrm{d}u), \qquad \psi \;\mapsto\; \phi = \psi \circ f_\epsilon \,. \tag{15}$$

This leads to the operator $\mathrm{J}_\epsilon = U_\epsilon(-\Delta_R^{\Omega_\epsilon})U_\epsilon^{-1}$ in $\mathrm{L}^2(\Omega, \epsilon\beta_\epsilon \mathrm{d}s\mathrm{d}u)$, which is associated with the quadratic form $b_\epsilon^\Omega(\phi) = b_\epsilon^{\Omega_\epsilon}(U_\epsilon^{-1}(\phi))$, and a direct calculation leads to $\operatorname{dom} b_\epsilon^\Omega = H^1(\Omega, \epsilon\beta_\epsilon \mathrm{d}s\mathrm{d}u)$ and

$$\begin{aligned} b_\epsilon^\Omega(\phi) = {}& \epsilon \int_\Omega \frac{|\partial_s \phi|^2}{\beta_\epsilon} \,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon} \int_\Omega |\partial_u \phi|^2 \beta_\epsilon \,\mathrm{d}s\,\mathrm{d}u \\ & + \int_{\mathbb{R}} \alpha(s) \left(|\mathrm{tr}(\phi)(s,1)|^2 \beta_\epsilon(s,1) - |\mathrm{tr}(\phi)(s,0)|^2 \right) \mathrm{d}s. \end{aligned} \tag{16}$$

By means of the unitary mapping

$$V_\epsilon : \mathrm{L}^2(\Omega, \epsilon\beta_\epsilon \,\mathrm{d}s\,\mathrm{d}u) \to \mathrm{L}^2(\Omega) = \mathrm{L}^2(\Omega,\, \mathrm{d}s\,\mathrm{d}u) : \{\phi \longmapsto (\sqrt{\epsilon\beta_\epsilon})\phi\}$$

we identify $\mathrm{L}^2(\Omega, \epsilon\beta_\epsilon)$ with $\mathrm{L}^2(\Omega)$. Note that $V_\epsilon(H^1(\Omega, \epsilon\beta_\epsilon \,\mathrm{d}s\mathrm{d}u)) = H^1(\Omega)$; since the derivative $k' \in \mathrm{L}^\infty(\mathbb{R})$, we have that

$$t_\epsilon(\phi) := b_\epsilon^\Omega(V_\epsilon^{-1}(\phi)), \quad \text{with} \quad \phi \in H^1(\Omega),$$

is well defined. Furthermore, $\mathrm{T}_\epsilon = V_\epsilon(\mathrm{J}_\epsilon)V_\epsilon^{-1}$ is the corresponding associated operator. After some maths, the quadratic form t_ϵ is explicitly given by

$$\begin{aligned} t_\epsilon(\phi) = &\int_\Omega \frac{|\partial_s\phi|^2}{\beta_\epsilon^2}\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u \\ &+ \frac{1}{\epsilon}\int_{\partial\Omega} \alpha|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma + \frac{1}{4}\int_\Omega \frac{k^2}{\beta_\epsilon^2}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u \\ &+ \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar\phi\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u + \epsilon\int_\Omega u\frac{k'}{\beta_\epsilon^3}\mathrm{Re}(\bar\phi\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u \\ &+ \epsilon^2\int_\Omega \frac{u^2}{4}\frac{|k'|^2}{\beta_\epsilon^4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u. \end{aligned}$$

We now introduce the quadratic form $\tilde t_\epsilon$, with $\mathrm{dom}\,\tilde t_\epsilon = H^1(\Omega)$,

$$\begin{aligned} \tilde t_\epsilon(\phi) := &\int_\Omega |\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma \\ &+ \int_\Omega \frac{k^2}{4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar\phi\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u \end{aligned}$$

which was obtained from t_ϵ by omitting the last two terms and replacing β_ϵ^2 by the constant 1 in the first and fourth integrals.

3.1. Estimates for straight strips

For technical reasons, we will deal with (strictly) positive quadratic forms, for both strips and tubes. Hence we choose appropriate positive constants c_1, c_2 so that the family of quadratic forms $\tilde a_\epsilon = \tilde t_\epsilon + c_1$ satisfies $\tilde a_\epsilon \geq c_2$, for ϵ small enough. Lemma 3.4 and Proposition 3.5 provide the main properties of such quantities.

Lemma 3.4. *Under the regularity assumptions $k, \alpha \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, there exist positive constants c_1, c_2 so that the quadratic form $\tilde a_\epsilon (= \tilde t_\epsilon + c_1) >$*

c_2 is closed and bounded from below by c_2; moreover, $\tilde{a}_\epsilon(\phi) \geq (2\epsilon)^{-2}\|\partial_u\phi\|_2^2$ for all $\phi \in \operatorname{dom}\tilde{a}_\epsilon$.

Proof. We begin by recalling the family $\{t_\epsilon\}_{\epsilon>0}$, $\operatorname{dom} t_\epsilon = H^1(\Omega)$,

$$
\begin{aligned}
t_\epsilon(\phi) = &\int_\Omega \frac{|\partial_s\phi|^2}{\beta_\epsilon^2}\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma\\
&+ \frac{1}{4}\int_\Omega \frac{k^2}{\beta_\epsilon^2}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{\phi}\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u\\
&+ \epsilon\int_\Omega u\frac{k'}{\beta_\epsilon^3}\mathrm{Re}(\bar{\phi}\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u + \epsilon^2\int_\Omega \frac{u^2}{4}\frac{|k'|^2}{\beta_\epsilon^4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u.
\end{aligned}
$$

First, the inequality holds

$$
\begin{aligned}
&\int_\Omega \frac{|\partial_s\phi|^2}{\beta_\epsilon^2}\,\mathrm{d}s\,\mathrm{d}u + \epsilon\int_\Omega u\frac{k'}{\beta_\epsilon^3}\mathrm{Re}(\bar{\phi}\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u\\
&\qquad + \epsilon^2\int_\Omega \frac{u^2}{4}\frac{|k'|^2}{\beta_\epsilon^4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u \geq -4\|k'\|_\infty\int_\Omega |\phi|^2\,\mathrm{d}s\,\mathrm{d}u
\end{aligned}
$$

for every ϵ sufficiently small.

We proceed as follows to limit the remaining terms. Let $Q_\epsilon(\phi)$ denote

$$
\begin{aligned}
Q_\epsilon(\phi) = &\frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma\\
&+ \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{\phi}\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u.
\end{aligned}
\tag{17}
$$

By using integration by parts we obtain

$$
\begin{aligned}
&\frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{\phi}\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u\\
&\qquad = -\int_\Omega \frac{k^2}{2\beta_\epsilon^2}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \frac{k}{2\beta_\epsilon}|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma
\end{aligned}
\tag{18}
$$

so that (17) becomes, where $\alpha_k = \alpha + \frac{k}{2}$,

$$
\begin{aligned}
Q_\epsilon(\phi) = &\frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha_k|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma\\
&+ \frac{1}{\epsilon}\int_{\partial\Omega} \frac{\epsilon u k^2}{2\beta_\epsilon}|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma - \int_\Omega \frac{k^2}{2\beta_\epsilon^2}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u
\end{aligned}
\tag{19}
$$

and we have used that $\frac{k}{2}\left(\frac{1}{\beta_\epsilon}-1\right)=\frac{\epsilon u k^2}{2\beta_\epsilon}$. By symmetry, we can verify that $\int_{\partial\Omega}\frac{uk^2}{2\beta_\epsilon}|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma$ is positive. Since $\beta_\epsilon\to 1, \epsilon\to 0$, we find that

$$-\int_\Omega \frac{k^2}{2\beta_\epsilon^2}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u \geq -\|k^2\|_\infty\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u$$

and it then follows that

$$Q_\epsilon(\phi)\geq -\|\alpha_k\|_\infty^2\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u-\|k\|_\infty^2\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u. \tag{20}$$

Finally, we may choose $c_2=\|\alpha_k\|_\infty^2+\|k\|_\infty^2+4\|k'\|_\infty+4\|k'\|_\infty^2$ so that

$$t_\epsilon(\phi)\geq -c_2\int_\Omega|\phi|^2\,\mathrm{d}s\mathrm{d}u,\quad \phi\in\operatorname{dom}t_\epsilon.$$

To apply Friedlander-Solomyak technique, we consider positive quadratic forms. Then we can choose $c_1=2c_2$, to obtain

$$t_\epsilon(\phi)+c_1\|\phi\|_2^2\geq c_2\|\phi\|_2^2.$$

In view of inequality (21) and the fact that $\beta_\epsilon\to 1$ uniformly as $\epsilon\to 0$, one can choose ϵ_0 small enough such that, for $0<\epsilon<\epsilon_0$, there exists $L>0$ (independent of ϵ) such that

$$\begin{aligned}\epsilon\int_\Omega\frac{uk'}{\beta_\epsilon^3}\mathrm{Re}(\bar\phi\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u&\\ \geq -4\|k'\|_\infty\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u&-4\epsilon\|k'\|_\infty\int_\Omega|\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u\end{aligned} \tag{21}$$

and

$$\begin{aligned}\int_\Omega\frac{|\partial_s\phi|^2}{\beta_\epsilon^2}\,\mathrm{d}s\,\mathrm{d}u&-4\epsilon\|k'\|_\infty\int_\Omega|\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u\\ &\geq L\int_\Omega|\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u\geq 0.\end{aligned} \tag{22}$$

Furthermore,

$$t_\epsilon(\phi)+c_1\|\phi\|_2^2\geq(2\epsilon)^{-2}\|\partial_u\phi\|_2^2\quad\text{and}\quad t_\epsilon(\phi)+c_1\|\phi\|_2^2\geq L\|\partial_s\phi\|_2^2$$

with the latter inequality obtained thanks to $c_1>0$ and (22). Note that the above proof allows us to obtain a constant $\tilde c>0$, for ϵ small enough, such that $\|\phi\|_{1,2}^2 \le \tilde{c}(t_\epsilon + c_1)(\phi)$, for each $\phi \in H^1(\Omega)$. Thus, the quadratic forms in the family $\{t_\epsilon + c_1\}_{\epsilon>0}$ are closed.

Similarly, one may check that the lemma holds true for the sequence $\{\tilde{t}_\epsilon\}_\epsilon$ defined by

$$\begin{aligned}\tilde{t}_\epsilon(\phi) = &\int_\Omega |\partial_s \phi|^2 \,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon^2}\int_\Omega |\partial_u \phi|^2 \,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha |\mathrm{tr}(\phi)|^2 \nu_2 \,\mathrm{d}\sigma \\ &+ \int_\Omega \frac{k^2}{4}|\phi|^2 \,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{\phi}\partial_u \phi)\,\mathrm{d}s\,\mathrm{d}u\end{aligned}$$

i.e, one gets $\tilde{t}_\epsilon(\phi) \ge -c_2\|\phi\|^2$ for all $\phi \in H^1(\Omega)$. To verify that $\tilde{a}_\epsilon = \tilde{t}_\epsilon + c_1$ is closed, just note that there is a constant $\mathrm{d} > 0$ such that $\|\phi\|_{1,2}^2 \le \mathrm{d}\,\tilde{a}_\epsilon(\phi)$, for ϵ small enough. □

Proposition 3.5 justifies the option of the family $\{\tilde{a}_\epsilon\}_{\epsilon>0}$ instead of original $\{t_\epsilon + c_1\}_{\epsilon>0}$. The choice of the constant c_1 is from Theorem 1 in [7] and the uniform convergence $\beta_\epsilon \to 1$. Let $\tilde{A}_\epsilon$ denote the self-adjoint operator associated with $\tilde{a}_\epsilon$.

Proposition 3.5. *Let c_1, c_2 be the constants obtained in Lemma 3.4. Then, for ϵ small enough, there exist $\delta, \tilde{\delta} > 0$ so that*

$$\Big|(t_\epsilon + c_1)(\phi) - \tilde{a}_\epsilon(\phi)\Big| \le (\epsilon\delta)\,\tilde{a}_\epsilon(\phi), \qquad \phi \in \mathrm{dom}\, t_\epsilon,$$

$$\Big\|(\mathrm{T}_\epsilon + c_1)^{-1} - \tilde{A}_\epsilon^{-1}\Big\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \le \tilde{\delta}\epsilon\,.$$

Proof. It is enough to verify the hypotheses of Theorem 1 in [7]. For ϵ small enough, we have the inequality $\|\beta_\epsilon^{-2} - 1\|_\infty \le \epsilon\mathrm{E}$ with $\mathrm{E} > 0$ depending only on $\|k\|_\infty$. Then

$$\begin{aligned}\Big|(t_\epsilon + c_1)(\phi) - \tilde{a}_\epsilon(\phi)\Big| \le\, &(\epsilon\mathrm{E})\int_\Omega |\partial_s\phi|^2 \,\mathrm{d}s\,\mathrm{d}u + (\epsilon\mathrm{E})\int_\Omega \frac{k^2}{4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u \\ &+ \left|\epsilon\int_\Omega u\frac{k'}{\beta_\epsilon^3}\mathrm{Re}(\bar{\phi}\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u + \epsilon^2\int_\Omega \frac{u^2}{4}\frac{|k'|^2}{\beta_\epsilon^4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u\right|.\end{aligned}$$

Now we estimate

$$\begin{aligned}\left|\epsilon\int_\Omega u\frac{k'}{\beta_\epsilon^3}\mathrm{Re}(\bar\phi\partial_s\phi)\,\mathrm{d}s\,\mathrm{d}u+\epsilon^2\int_\Omega\frac{u^2}{4}\frac{|k'|^2}{\beta_\epsilon^4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u\right|\\ \le 4\epsilon\|k'\|_\infty\left[\int_\Omega|\phi|^2+|\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u\right]+4\epsilon\|k'\|_\infty^2\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u\end{aligned}$$

and for $\delta=1+\mathrm{E}+4\|k'\|_\infty+4\|k'\|_\infty^2$, we may produce

$$\begin{aligned}\Big|(t_\epsilon+c_1)(\phi)-\tilde a_\epsilon(\phi)\Big|\le(\delta\epsilon)\int_\Omega|\partial_s\phi|^2\,\mathrm{d}s\,\mathrm{d}u+(\delta\epsilon)\int_\Omega\frac{k^2}{4}|\phi|^2\,\mathrm{d}s\,\mathrm{d}u\\ +(\delta\epsilon)\Big[2(4\|k'\|_\infty^2+4\|k'\|_\infty)\Big]\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

In the proof of Lemma 3.4 it was found that

$$\begin{aligned}\frac{1}{\epsilon^2}\int_\Omega|\partial_u\phi|^2\,\mathrm{d}s\,\mathrm{d}u+\frac{1}{\epsilon}\int_{\partial\Omega}\alpha|\mathrm{tr}(\phi)|^2\nu_2\,\mathrm{d}\sigma\\ +\int_\Omega\frac{k}{\epsilon\beta_\epsilon}\mathrm{Re}(\bar\phi\partial_u\phi)\,\mathrm{d}s\,\mathrm{d}u+2(2\|\alpha_k\|_\infty^2+2\|k\|_\infty^2)\int_\Omega|\phi|^2\,\mathrm{d}s\,\mathrm{d}u\ge 0.\end{aligned}$$

Thus, $\Big|(t_\epsilon+c_1)(\phi)-\tilde a_\epsilon(\phi)\Big|\le(\delta\epsilon)[\tilde{t_\epsilon}+c_1](\phi)$, and an application of Theorem 1 in [7] completes the proof. □

3.2. Robin Laplacian on the interval

Some results for Robin Laplacian $-\Delta_\alpha^I$ on the cross-section $I=(0,1)$ are presented for a constant parameter $\alpha\in\mathbb{R}$; in particular, we briefly discuss its self-adjointness. Assume that

$$-\psi'(0)-\alpha\psi(0)=0\quad\text{and}\quad\psi'(1)+\alpha\psi(1)=0\tag{23}$$

and let $\mathrm{dom}\,(-\Delta_\alpha^I)=\{\psi\in H^2(0,1);\psi\text{ satisfies (23)}\}$, where $-\Delta_\alpha^I$ has the usual action of (weak) second derivative in $\mathrm{L}^2(I)$. This operator is associated with the sesquilinear form $b_\alpha\ge-|\alpha|^2$ in the Hilbert space $\mathrm{L}^2(I)$ given by

$$\begin{aligned}&b_\alpha(\phi,\psi):=\int_0^1\overline{\phi'(y)}\,\psi'(y)\,\mathrm{d}y+\alpha\Big(\overline{\phi(1)}\psi(1)-\overline{\phi(0)}\psi(0)\Big),\\ &\phi,\psi\in\mathrm{dom}\,b_\alpha=H^1(I).\end{aligned}$$

By following an idea in [14] and [16], Example VI. 2.16, a proof of Theorem 3.6 is obtained (since it is standard, it will be omitted here).

Theorem 3.6. *Let $\alpha \in \mathbb{R}$. Then, the above Laplacian $-\Delta_\alpha^I$ is the unique self-adjoint operator associated with b_α.*

Now we present a short discussion about the eigenfunctions and eigenvalues of $-\Delta_\alpha^I$; we only discuss the case $\alpha \neq 0$ and this will be very important ahead (see the next section and Section 4.3). Denote by $-\Delta_D^I$ and $-\Delta_N^I$ the usual Laplacian in $\mathrm{L}^2(I)$ with Dirichlet and Neumann boundary conditions, respectively. The eigenvalues of $-\Delta_\alpha^I$ are given by

$$\lambda_0^I = -\alpha^2, \quad \lambda_n^I = n^2\pi^2, \quad n \geq 1,$$

with corresponding normalized eigenfunctions

$$\phi_0(y) = c\, e^{-y\alpha}, \quad \text{with} \quad c = \left(\frac{2\alpha}{1-e^{-2\alpha}}\right)^{1/2}, \tag{24}$$

$$\phi_n(y) = \frac{n\pi}{(n^2\pi^2+\alpha^2)^{1/2}}\left(\psi_n^N(y) - \frac{\alpha}{n\pi}\psi_n^D(y)\right). \tag{25}$$

Here $\psi_n^D(y) := \sqrt{2}\sin(n\pi y)$ and $\psi_n^N(y) := \sqrt{2}\cos(n\pi y)$ for $n \geq 1$, are eigenfunctions of $-\Delta_D^I$ and $-\Delta_N^I$, respectively. The collection $\{\phi_n\}_{n=1}^\infty \cup \{\phi_0\}$ is an orthonormal basis of $\mathrm{L}^2(I)$.

3.3. Effective potential and operators: interval cross-section

As already mentioned, there is an intermediate step in the proof of Theorem 2.1. It consists of an application of the technique of [12], and the choice of a secondary closed subspace $\mathcal{H}_\epsilon$ of $\mathcal{H}$, along with the orthogonal decomposition $\mathcal{H} = \mathcal{H}_\epsilon \oplus \mathcal{H}_\epsilon^\perp$.

In what follows the subspace $\mathcal{H}_\epsilon$ will consist of the functions $w(s)\phi_0^\epsilon$ with $w \in \mathrm{L}^2(\mathbb{R})$; we have denoted by $\phi_0^\epsilon(s,\cdot)$ the positive normalized eigenfunction corresponding to the lowest eigenvalue $\lambda_{0,\epsilon}^I(s) < 0$ of Robin Laplacian $-\Delta_{\epsilon\alpha_k(s)}^I$ in $\mathrm{L}^2(I)$, with $\alpha_k = \alpha + \frac{k}{2}$, that is,

$$\mathcal{H}_\epsilon = \{w\phi_0^\epsilon; w \in \mathrm{L}^2(\mathbb{R})\} \quad \text{with} \quad \phi_0^\epsilon(s,u) = \frac{e^{-\epsilon\alpha_k(s)u}}{\left(\int_0^1 |e^{-\epsilon\alpha_k(s)u}|^2\,\mathrm{d}u\right)^{1/2}}.$$

Of course, we may consider a linear surjective isometry π_ϵ from $\mathcal{H}_\epsilon$ into $\mathrm{L}^2(\mathbb{R})$, defined by

$$\pi_\epsilon : \mathcal{H}_\epsilon \to \mathrm{L}^2(\mathbb{R}) : \{w\phi_0^\epsilon \mapsto w\}. \tag{26}$$

In order to explicit an effective potential, let $\phi \in \mathcal{H} = \mathrm{L}^2(\Omega)$, so that we have the decomposition

$$\phi = w(s)\phi_0^\epsilon + \phi_\perp \quad \text{with} \quad w\phi_0^\epsilon \in \mathcal{H}_\epsilon, \quad \phi_\perp \in \mathcal{H}_\epsilon^\perp.$$

From such a decomposition, we can conclude that $w(s) = \int_0^1 \phi\phi_0^\epsilon \, \mathrm{d}u$; moreover, $w\phi_0^\epsilon \in H^1(\Omega)$ whenever $\phi \in H^1(\Omega)$. The hypothesis $\phi_\perp \in \mathcal{H}_\epsilon^\perp$ implies

$$\int_0^1 \phi_0^\epsilon \phi_\perp(s,u) \, \mathrm{d}u = 0 \quad \text{a.e.} \quad s \in \mathbb{R}.$$

Assuming, in addition, that $\phi_\perp \in H^1(\Omega)$, then one can differentiate such identity to get

$$\int_0^1 \phi_0^\epsilon(s,u)\partial_s\phi_\perp(s,u) \, \mathrm{d}u = -\int_0^1 \partial_s\phi_0^\epsilon(s,u)\phi_\perp(s,u) \, \mathrm{d}u \quad \text{a.e.} \quad s \in \mathbb{R}.$$

Next, the restriction $\tilde{a}_\epsilon|_{d_\epsilon}$, with $d_\epsilon = \{w\phi_0^\epsilon; w \in H^1(\mathbb{R})\} \sqsubseteq \mathcal{H}_\epsilon$, implies

$$\tilde{a}_\epsilon(w\phi_0^\epsilon) = \int_{\mathbb{R}} \Big(|w'|^2 + |w|^2[V_\epsilon^{\mathrm{eff}} + c_1]\Big) \, \mathrm{d}s$$

with effective potential $V_\epsilon^{\mathrm{eff}}(s)$ satisfying the uniform convergence

$$V_\epsilon^{\mathrm{eff}} \to V^{\mathrm{eff}}(s) = -\alpha^2(s) - \alpha(s)k(s), \quad \epsilon \to 0.$$

Indeed, let $\phi = w\phi_0^\epsilon$, with $w \in H^1(\mathbb{R})$; then by integrating by parts and Theorem 3.6,

$$\int_\Omega |\partial_s\phi|^2 \, \mathrm{d}s \, \mathrm{d}u = \int_{\mathbb{R}} \left(|w'|^2 + |w|^2 \left[\int_I |\partial_s\phi_0^\epsilon|^2 \, \mathrm{d}u\right]\right) \mathrm{d}s, \tag{27}$$

$$\frac{1}{\epsilon^2}\int_\Omega |\partial_u\phi|^2 \, \mathrm{d}s \, \mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega} \alpha_k |\mathrm{tr}(\phi)|^2 \nu_2 \, \mathrm{d}\sigma = -\int_{\mathbb{R}} |w|^2(\alpha_k)^2 \, \mathrm{d}s, \tag{28}$$

$$\begin{aligned} &\frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{\phi}\partial_u\phi) \, \mathrm{d}s \, \mathrm{d}u - \frac{1}{\epsilon}\int_{\partial\Omega} \frac{k}{2}|\mathrm{tr}(\phi)|^2\nu_2 \, \mathrm{d}\sigma \\ &\qquad = \int_{\mathbb{R}} k\alpha_k|w|^2 \left[1 - \int_I \frac{|\phi_0^\epsilon|^2}{\beta_\epsilon} \, \mathrm{d}u\right] \mathrm{d}s. \end{aligned} \tag{29}$$

Under the assumptions on the curvature function k and since β_ϵ converges uniformly to 1, we have uniform limits

$$\left[1 - \int_I \frac{|\phi_0^\epsilon|^2}{\beta_\epsilon} \, \mathrm{d}u\right] \to 0.$$

Therefore,

$$\tilde{a}_\epsilon(\phi) = \int_{\mathbb{R}} \left(|w'|^2 + |w|^2 [V_\epsilon^{\text{eff}} + c_1] \right) \mathrm{d}s,$$

where

$$V_\epsilon^{\text{eff}}(s) = \int_I |\partial_s \phi_0^\epsilon|^2 \,\mathrm{d}u - \alpha_k^2(s) + k(s)\alpha_k(s) \left[1 - \int_I \frac{|\phi_0^\epsilon|^2}{\beta_\epsilon} \,\mathrm{d}u \right] + \frac{k^2(s)}{4}$$

which is obtained from (27)-(29), with $V_\epsilon^{\text{eff}} \to V^{\text{eff}}$ uniformly.

In what follows, let us denote by q_ϵ the quadratic form identified with $\tilde{a}_\epsilon|_{d_\epsilon}$, defined in $\operatorname{dom} q_\epsilon = H^1(\mathbb{R})$, and let T_{q_ϵ} be the associated operator. Explicitly,

$$\begin{aligned} q_\epsilon(w) &= \int_{\mathbb{R}} \left(|w'|^2 + [V_\epsilon^{\text{eff}} + c_1] |w|^2 \right) \mathrm{d}s, \\ T_{q_\epsilon}(w) &= -w'' + [V_\epsilon^{\text{eff}} + c_1] w\,, \qquad w \in H^2(\mathbb{R}). \end{aligned}$$

Let q be the quadratic form, bounded from below by c_2,

$$q(w) = \int_{\mathbb{R}} \left(|w'|^2 + [V^{\text{eff}} + c_1] |w|^2 \right) \mathrm{d}s, \quad \operatorname{dom} q = H^1(\mathbb{R}),$$

whose associated *effective* operator is given by $[T + c_1]$, with

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V^{\text{eff}}, \qquad \operatorname{dom} T = H^2(\mathbb{R}). \tag{30}$$

Now we are in a position to prove the following results.

Theorem 3.7. *Suppose $k, \alpha \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$, then the following convergence holds true*

$$\left\| (T_{q_\epsilon})^{-1} - (T + c_1)^{-1} \right\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}))} \longrightarrow 0\,, \quad \epsilon \to 0.$$

Proof. If $u, v \in H^1(\mathbb{R})$, then

$$\begin{aligned} &\left| \left\langle (T_{q_\epsilon})^{1/2} u, (T_{q_\epsilon})^{1/2} v \right\rangle - \left\langle (T + c_1)^{1/2} u, (T + c_1)^{1/2} v \right\rangle \right| \\ &\qquad \le \| V_\epsilon^{\text{eff}} - V^{\text{eff}} \|_\infty \|u\|_2 \|v\|_2 \end{aligned}$$

since

$$\left| \left\langle (T_{q_\epsilon})^{1/2} u, (T_{q_\epsilon})^{1/2} v \right\rangle - \left\langle (T + c_1)^{1/2} u, (T + c_1)^{1/2} v \right\rangle \right| = |q_\epsilon(u,v) - q(u,v)|$$

and

$$|q_\epsilon(u,v) - q(u,v)| \le \|V_\epsilon^{\mathrm{eff}} - V^{\mathrm{eff}}\|_\infty \|u\|_2 \|v\|_2 \,.$$

Taking $u = (T + c_1)^{-1} g$ and $v = (T_{q_\epsilon})^{-1} h$, with $g, h \in \mathrm{L}^2(\mathbb{R})$, we have

$$\begin{aligned} &\left| \left\langle (T + c_1)^{-1} g, h \right\rangle - \left\langle g, (T_{q_\epsilon})^{-1} h \right\rangle \right| \\ &\qquad \le \Big(\|(T + c_1)^{-1}\| \, \|V_\epsilon^{\mathrm{eff}} - V^{\mathrm{eff}}\|_\infty \, \|(T_{q_\epsilon})^{-1}\| \Big) \|h\|_2 \|g\|_2. \end{aligned}$$

Therefore, by letting $\epsilon \to 0$,

$$\left\| (T_{q_\epsilon})^{-1} - (T + c_1)^{-1} \right\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}))} \le c_2^{-2} \|V_\epsilon^{\mathrm{eff}} - V^{\mathrm{eff}}\|_\infty \to 0 \,,$$

and the proof is complete. □

Corollary 3.8. *Consider the restriction $q_\epsilon = \tilde{a}_\epsilon|_{d_\epsilon}, d_\epsilon \sqsubseteq \mathcal{H}_\epsilon$, with associated self-adjoint operator $Q_\epsilon \ge c_2$. Then, $Q_\epsilon = \pi_\epsilon^{-1} \circ (T_{q_\epsilon}) \circ \pi_\epsilon$, and*

$$\left\| \left[Q_\epsilon^{-1} \oplus 0 \right] - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0 \right] \right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \longrightarrow 0, \quad \epsilon \to 0.$$

Proof. It is enough to note that

$$\begin{aligned} &\left\| \left[Q_\epsilon^{-1} \oplus 0 \right] - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0 \right] \right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \\ &\qquad \le \left\| (T_{q_\epsilon})^{-1} - (T + c_1)^{-1} \right\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}))}. \end{aligned}$$

□

Lemma 3.9. *Let $(T + c_1)^{-1} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ be as in Theorem 3.7. Then,*

$$\left\| \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0 \right] - \left[\pi_0^{-1} \circ (T + c_1)^{-1} \circ \pi_0 \oplus 0_{E^\perp} \right] \right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \to 0,$$
$\epsilon \to 0,$

where 0 and $0_{E^\perp}$ are the null operators on the subspaces $\mathcal{H}_\epsilon^\perp$ and $E^\perp$, respectively.

Proof. Let $\phi = P(\phi) + P_{E^\perp}(\phi)$ and $\phi = w\phi_0^\epsilon + \phi_\perp$, with $\|\phi\| = 1$ (recall that $P = P_E$ and $P_{E^\perp}$ denote the orthogonal projections onto E and $E^\perp$, respectively). Then,

$$\begin{aligned}\Big\|\Big[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\Big](\phi) - \Big[\pi_0^{-1} \circ (T + c_1)^{-1} \circ \pi_0 \oplus 0_{E^\perp}\Big](\phi)\Big\|_{\mathrm{L}^2(\Omega)} \\ = \Big\|\phi_0^\epsilon (T + c_1)^{-1} w - (T + c_1)^{-1} P(\phi)\Big\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}&\Big\|\phi_0^\epsilon (T + c_1)^{-1} w - (T + c_1)^{-1} P(\phi)\Big\|_{\mathrm{L}^2(\Omega)} \\ &\quad \le \Big\|\phi_0^\epsilon (T + c_1)^{-1} w - (T + c_1)^{-1} w\Big\|_{\mathrm{L}^2(\Omega)} \\ &\qquad + \Big\|(T + c_1)^{-1} w - (T + c_1)^{-1} P(\phi)\Big\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

The first term on the r.h.s. above vanishes as $\epsilon \to 0$. Indeed, given $\delta > 0$, by means of uniform convergence, there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ such that $|\phi_0^\epsilon - 1|^2 < \frac{\delta^2}{\|(T+c_1)^{-1}\|^2}$, whenever $0 < \epsilon < \epsilon_0$. Therefore,

$$\begin{aligned}&\|\phi_0^\epsilon (T + c_1)^{-1} w - (T + c_1)^{-1} w\|_{\mathrm{L}^2(\Omega)}^2 \\ &\quad = \int_\Omega |\phi_0^\epsilon - 1|^2 |(T + c_1)^{-1} w|^2 \mathrm{d}s\, \mathrm{d}u \\ &\quad \le \frac{\delta^2}{\|(T + c_1)^{-1}\|^2} \|(T + c_1)^{-1}\|^2 \int_{\mathbb{R}} |w|^2\, \mathrm{d}s \le \delta^2.\end{aligned}$$

The remaining term can be estimated as follows:

$$\begin{aligned}&\Big\|(T + c_1)^{-1} P(\phi) - (T + c_1)^{-1} w\Big\|_{\mathrm{L}^2(\Omega)} \\ &\quad = \left(\int_\Omega |(T + c_1)^{-1}(w - P(\phi))|^2\, \mathrm{d}s\, \mathrm{d}u\right)^{1/2} \\ &\quad \le \Big\|(T + c_1)^{-1}\Big\| \left(\int_{\mathbb{R}} |w - P(\phi)|^2\, \mathrm{d}s\right)^{1/2}.\end{aligned}$$

From the discussion in Section 3.3 and (9), for $0 < \epsilon < \epsilon_0$, we have

$$\begin{aligned}\left(\int_{\mathbb{R}} |w(s) - P(\phi)(s)|^2 \,\mathrm{d}s\right)^{1/2} &= \left(\int_{\mathbb{R}} \left|\int_0^1 \phi(\phi_0^\epsilon - 1)\,\mathrm{d}u\right|^2 \mathrm{d}s\right)^{1/2} \\ &\leq \int_0^1 \left(\int_{\mathbb{R}} |\phi|^2 |\phi_0^\epsilon - 1|^2 \,\mathrm{d}s\right)^{1/2} \mathrm{d}u \\ &\leq \delta / \|(T + c_1)^{-1}\|.\end{aligned}$$

Hence, given $\delta > 0$, there exists ϵ_0 such that, if $0 < \epsilon < \epsilon_0$,

$$\Big\| \Big[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\Big](\phi) \\ - \Big[\pi_0^{-1} \circ (T + c_1)^{-1} \circ \pi_0 \oplus 0_{E^\perp}\Big](\phi)\Big\|_{\mathrm{L}^2(\Omega)} < 2\delta\,,$$

and the proof is complete. □

4. Three-dimensional forms

As in the planar case, the first step will be to "straighten" the tubular region via a unitary transformation U_ϵ,

$$\begin{array}{rccc} \mathrm{U}_\epsilon : \mathrm{L}^2(\Omega_\epsilon) & \to & \mathrm{L}^2(\Omega, \epsilon^2\beta_\epsilon \mathrm{d}y\,\mathrm{d}s) \\ \psi & \mapsto & v = \psi \circ \mathcal{L}_\epsilon \end{array}. \tag{31}$$

This leads to the operator $\mathfrak{A}_\epsilon = \mathrm{U}_\epsilon(-\Delta_R^{\Omega_\epsilon})\mathrm{U}_\epsilon^{-1}$, which is associated with the quadratic form $\mathfrak{a}_\epsilon$ given by

$$\mathfrak{a}_\epsilon(v) := \mathrm{F}_\epsilon(v \circ \mathcal{L}_\epsilon^{-1}), \qquad \mathrm{dom}\,\mathfrak{a}_\epsilon = H^1(\Omega);$$

see (11) for the definition of F_ϵ. We write the gradient of v in the form $(\nabla_y v, v')$, being v' the derivative with respect to the third variable $s \in \mathbb{R}$.

Now we present the action of the quadratic form $\mathfrak{a}_\epsilon$. For each $v \in \mathrm{dom}\,\mathfrak{a}_\epsilon$, put $\psi = \mathrm{U}_\epsilon^{-1} v$, so that

$$\begin{aligned}\int_{\Omega_\epsilon} |\nabla\psi(x)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}}\int_S (|\nabla v(s,y) \nabla\mathcal{L}_\epsilon^{-1}(y,s)|^2)\epsilon^2\beta_\epsilon \,\mathrm{d}y\,\mathrm{d}s \\ &= \epsilon^2 \int_{\mathbb{R}}\int_S \left[\frac{1}{\beta_\epsilon}\Big|v' + (\nabla_y v \cdot Ry)\tau\Big|^2 + \frac{\beta_\epsilon}{\epsilon^2}|\nabla_y v|^2\right] \mathrm{d}y\,\mathrm{d}s\end{aligned}$$

wherein R is the clockwise rotation matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

By following [2], let $y = y(t)$ be a piecewise $\mathrm{C}^1[0,1]$ parameterization of the boundary ∂S of S, counterclockwise oriented; then, with $\dot{y} = \frac{dy}{dt}$, one has the following parametrization

$$\begin{array}{rcl} \sigma_\epsilon : [0,1] \times \mathbb{R} & \to & \mathbb{R}^3 \\ (t,s) & \mapsto & \mathcal{L}_\epsilon(y(t),s) \end{array} \tag{32}$$

of the surface $\partial\Omega_\epsilon$ in $\mathbb{R}^3$, and since

$$\frac{\partial \sigma_\epsilon}{\partial t} \times \frac{\partial \sigma_\epsilon}{\partial s} = \begin{vmatrix} T & N & B \\ 0 & \epsilon(\dot{y}\cdot e_1) & \epsilon(\dot{y}\cdot e_2) \\ \beta_\epsilon(y(t),s) & -\tau y_2(t) & \tau y_1(t) \end{vmatrix}$$

one has

$$\left\| \frac{\partial \sigma_\epsilon}{\partial t} \times \frac{\partial \sigma_\epsilon}{\partial s} \right\| = \epsilon\Big(\sqrt{\beta_\epsilon^2 + \epsilon^2 \tau^2 (\dot{y}\cdot y)^2} \Big) = \epsilon(\beta_\epsilon + \epsilon^2 r_\epsilon) \tag{33}$$

for which a Taylor expansion of the square root gives, for the function $r_\epsilon(y,s)$,

$$r_\epsilon \geq 0 \quad \text{and} \quad \left| r_\epsilon - \frac{\tau^2}{2}(\dot{y}\cdot y)^2 \right| \leq \mathrm{C}_1 \epsilon\,. \tag{34}$$

It follows that the boundary integral is given by

$$\begin{aligned} &\int_{\partial\Omega_\epsilon} \tilde{\gamma} |\mathrm{tr}_\epsilon(\psi)|^2 \,\mathrm{d}\sigma_\epsilon(x) \\ &\quad = \int_{\mathbb{R}} \int_0^1 \tilde{\gamma}(\mathcal{L}_\epsilon(y(t),s)) |\mathrm{tr}_\epsilon(v \circ \mathcal{L}_\epsilon^{-1})(\mathcal{L}_\epsilon(y(t),s))|^2 \left\| \frac{\partial \sigma_\epsilon}{\partial t} \times \frac{\partial \sigma_\epsilon}{\partial s} \right\| \mathrm{d}t\,\mathrm{d}s \\ &\quad = \int_{\mathbb{R}} \int_0^1 \gamma(y(t),s) |\mathrm{tr}(v)(y(t),s)|^2 \epsilon(\beta_\epsilon(y(t),s) + \epsilon^2 r_\epsilon(y(t),s))\,\mathrm{d}t\,\mathrm{d}s \\ &\quad = \epsilon \int_{\mathbb{R}} \left(\int_{\partial S} \gamma |\mathrm{tr}(v)|^2 (\beta_\epsilon + \epsilon^2 r_\epsilon) \mathrm{d}\sigma(y) \right) \mathrm{d}s. \end{aligned} \tag{35}$$

Thus, for all $v \in \operatorname{dom} \mathfrak{a}_\epsilon$,

$$\mathfrak{a}_\epsilon(v) = \epsilon^2 \int_{\mathbb{R}} \int_S \left[\frac{1}{\beta_\epsilon} \Big| v' + (\nabla_y v \cdot Ry)\tau \Big|^2 + \frac{\beta_\epsilon}{\epsilon^2} |\nabla_y v|^2 \right] \mathrm{d}y \, \mathrm{d}s$$
$$+ \int_{\mathbb{R}} \int_0^1 \gamma(y(t), s) |\mathrm{tr}(v)(y(t), s)|^2 \epsilon (\beta_\epsilon(y(t), s) + \epsilon^2 r_\epsilon(y(t), s)) \, \mathrm{d}t \, \mathrm{d}s.$$

Since $\gamma \in \mathrm{L}^\infty(\partial\Omega)$, as in (33) and (34) we get

$$\begin{aligned} &\Bigg| \int_{\partial\Omega_\epsilon} \tilde{\gamma} |\mathrm{tr}_\epsilon(\psi)|^2 \, \mathrm{d}\sigma_\epsilon(x) \\ &\quad - \epsilon \int_{\mathbb{R}} \left(\int_{\partial S} \gamma |\mathrm{tr}(v)|^2 \Big(\beta_\epsilon + \frac{\epsilon^2 \tau^2}{2} (\dot{y} \cdot y)^2 \Big) \, \mathrm{d}\sigma(y) \right) \mathrm{d}s \Bigg| \\ &\quad \le \mathrm{C}_1 \epsilon^4 \int_{\mathbb{R}} \int_{\partial S} |\gamma| |\mathrm{tr}(v)|^2 \, \mathrm{d}\sigma(y) \mathrm{d}s. \end{aligned} \tag{36}$$

By virtue of (36), we define the quadratic form $\tilde{\mathfrak{a}}_\epsilon : H^1(\Omega, \epsilon^2 \beta_\epsilon \, \mathrm{d}y \mathrm{d}s) \to \mathbb{R}$,

$$\tilde{\mathfrak{a}}_\epsilon(v) := \epsilon^2 \int_{\mathbb{R}} \int_S \left[\frac{1}{\beta_\epsilon} \Big| v' + (\nabla_y v \cdot Ry)\tau \Big|^2 + \frac{\beta_\epsilon}{\epsilon^2} |\nabla_y v|^2 \right] \mathrm{d}y \, \mathrm{d}s$$
$$+ \int_{\mathbb{R}} \left(\int_{\partial S} \epsilon \gamma |\mathrm{tr}(v)|^2 \Big(\beta_\epsilon + \frac{\epsilon^2 \tau^2}{2} (\dot{y} \cdot y)^2 \Big) \, \mathrm{d}\sigma(y) \right) \mathrm{d}s$$

and we have

$$|\mathrm{F}_\epsilon(v \circ \mathcal{L}_\epsilon^{-1}) - \tilde{\mathfrak{a}}_\epsilon(v)| \le \epsilon^4 \, \mathrm{C}_2 \|v\|_{1,2}^2 \, . \tag{37}$$

Next, we consider the unitary transformation

$$\mathrm{V}_\epsilon : \mathrm{L}^2(\Omega, \epsilon^2 \beta_\epsilon \, \mathrm{d}y \, \mathrm{d}s) \to \mathrm{L}^2(\Omega) : \{ v \mapsto (\sqrt{\epsilon^2 \beta_\epsilon}) v \} \tag{38}$$

and we investigate the asymptotic behavior of the family of quadratic forms $\{b_\epsilon : \operatorname{dom} b_\epsilon \to \mathbb{R}\}_{\epsilon > 0}$, in $\mathrm{L}^2(\Omega)$, given by

$$b_\epsilon(v) := \tilde{\mathfrak{a}}_\epsilon(\mathrm{V}_\epsilon^{-1} v), \qquad \operatorname{dom} b_\epsilon = \mathrm{V}_\epsilon(\operatorname{dom} \tilde{\mathfrak{a}}_\epsilon) = H^1(\Omega),$$

whose associated operator is $B_\epsilon = \mathrm{V}_\epsilon \tilde{\mathfrak{A}}_\epsilon \mathrm{V}_\epsilon^{-1}$, where $\tilde{\mathfrak{A}}_\epsilon$ is associated with $\tilde{\mathfrak{a}}_\epsilon$. A direct computation gives

$$\begin{aligned} b_\epsilon(v) = &\int_\Omega \frac{1}{\beta_\epsilon^2} \left| v' + \tau(\nabla_y v \cdot Ry) - \frac{v}{2\beta_\epsilon}\Big(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry)\Big)\right|^2 \mathrm{d}y\,\mathrm{d}s \\ &+ \frac{1}{\epsilon^2}\int_\Omega |\nabla_y v|^2 \,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_{\mathbb{R}} \left(\int_{\partial S} \gamma |\mathrm{tr}(v)|^2 \,\mathrm{d}\sigma(y)\right) \mathrm{d}s \\ &+ \int_\Omega |v|^2 \frac{k^2}{4\beta_\epsilon^2}\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{\epsilon^2}\int_\Omega \mathrm{Re}\left(\nabla_y v \cdot \frac{\bar{v}}{\beta_\epsilon}\nabla_y \beta_\epsilon\right) \mathrm{d}y\,\mathrm{d}s \\ &+ \epsilon \int_{\mathbb{R}} \frac{\tau^2}{2}\left(\int_{\partial S} \gamma \frac{|\mathrm{tr}(v)|^2}{\beta_\epsilon}(\dot{y}\cdot y)^2 \,\mathrm{d}\sigma(y)\right) \mathrm{d}s. \end{aligned}$$

Finally, we introduce the quadratic form $\widehat{b}_\epsilon$, which corresponds to a simpler version of b_ϵ, $\operatorname{dom} \widehat{b}_\epsilon = H^1(\Omega)$, with action

$$\begin{aligned} \widehat{b}_\epsilon(v) := &\int_\Omega \left| v' + \tau(\nabla_y v \cdot Ry) - \frac{v}{2\beta_\epsilon}\Big(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry)\Big)\right|^2 \mathrm{d}y\,\mathrm{d}s \\ &+ \frac{1}{\epsilon^2}\int_\Omega |\nabla_y v|^2 \,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_{\mathbb{R}} \left(\int_{\partial S} \gamma |\mathrm{tr}(v)|^2 \,\mathrm{d}\sigma(y)\right) \mathrm{d}s \\ &+ \int_\Omega |v|^2 \frac{k^2}{4}\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{\epsilon^2}\int_\Omega \mathrm{Re}\left(\nabla_y v \cdot \frac{\bar{v}}{\beta_\epsilon}\nabla_y \beta_\epsilon\right) \mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

4.1. Estimates for straight tubes

Analogously to Section 3.1, we get other positive constants, also denoted by c_1, c_2, related to the family of quadratic forms $\tilde{b}_\epsilon = \widehat{b}_\epsilon + c_1$, whose properties are listed in Lemma 4.10 and Proposition 4.11.

Lemma 4.10. *Under the regularity assumptions $\tau \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$ and $k, \alpha \in W^{2,\infty}(\mathbb{R}) \cap \mathrm{C}^2(\mathbb{R})$, there exist positive constants c_1, c_2 such that $\tilde{b}_\epsilon = \widehat{b}_\epsilon + c_1$ is a closed and lower bounded form by c_2 (i.e., $\tilde{b}_\epsilon > c_2$); moreover, $\tilde{b}_\epsilon(v) \geq (2\epsilon)^{-2}\|\nabla_y v\|_2^2$ for all $v \in \operatorname{dom} \tilde{b}_\epsilon$.*

Proof. We are going to show that there exist positive constants c_1, c_2, independent of ϵ, such that $b_\epsilon(\phi) + c_1\|\phi\|_2^2 \geq c_2\|\phi\|_2^2$, which implies that the operator $B_\epsilon + c_1$ is strictly positive. We will use that $1/2 < \beta_\epsilon < 3/2$ for ϵ small enough.

After some calculations, we get

$$\epsilon \int_{\mathbb{R}} \frac{\tau^2}{2} \left(\int_{\partial S} \frac{\gamma}{\beta_\epsilon} |\mathrm{tr}(v)|^2 (\dot{y} \cdot y)^2 \, \mathrm{d}\sigma(y) \right) \mathrm{d}s = I_1^\epsilon(v) + I_2^\epsilon(v)$$

where

$$\begin{aligned} I_1^\epsilon(v) &= \frac{\epsilon}{2} \int_\Omega \frac{\tau^2}{\beta_\epsilon} \Big(\nabla_y |v|^2 \cdot (\alpha y_2^2, \alpha y_1^2) \Big) \, \mathrm{d}y \, \mathrm{d}s \\ I_2^\epsilon(v) &= \frac{\epsilon^2}{2} \int_\Omega \frac{k \tau^2 \alpha}{\beta_\epsilon^2} |v|^2 \, \mathrm{d}y \, \mathrm{d}s. \end{aligned} \tag{39}$$

Now we estimate each of the terms in (39). We claim that there exists $\mathrm{C}_3 > 0$ such that

$$\begin{aligned} &\epsilon \int_{\mathbb{R}} \frac{\tau^2}{2} \left(\int_{\partial S} \frac{\gamma}{\beta_\epsilon} |\mathrm{tr}(v)|^2 (\dot{y} \cdot y)^2 \, \mathrm{d}\sigma(y) \right) \mathrm{d}s \\ &\qquad \geq -\mathrm{C}_3 \left[\int_\Omega \big(|v|^2 + |\nabla_y v|^2 \big) \, \mathrm{d}y \, \mathrm{d}s \right]. \end{aligned} \tag{40}$$

Indeed,

$$|I_1^\epsilon(v)| = \epsilon \mathrm{C}_1 \left[\int_\Omega \big(|v|^2 + |\nabla_y v|^2 \big) \, \mathrm{d}y \, \mathrm{d}s \right]$$

where $\mathrm{C}_1 = 1 + 2\|\tau^2\|_\infty \|\alpha\|_\infty$. On the other hand, for $0 < \epsilon < \epsilon_1 < \frac{1}{\sqrt{2\mathrm{C}_1}}$, we get

$$|I_1^\epsilon(v)| \leq \epsilon \, \mathrm{C}_1 \int_\Omega \left(|v|^2 + \frac{1}{2\epsilon^2} |\nabla_y v|^2 \right) \mathrm{d}y \, \mathrm{d}s. \tag{41}$$

For the other term we have

$$|I_2^\epsilon(v)| \leq 2\epsilon \mathrm{C}_2 \int_\Omega |v|^2 \, \mathrm{d}y \, \mathrm{d}s \tag{42}$$

with $\mathrm{C}_2 = 1 + \|\alpha\|_\infty \|k\|_\infty \|\tau^2\|_\infty$. Then we can take $\mathrm{C}_3 = \mathrm{C}_1 + 2\mathrm{C}_2$ so that (40) holds.

Use integration by parts to establish the equality

$$\begin{aligned} &-\frac{1}{\epsilon^2} \int_\Omega \mathrm{Re} \left(\nabla_y \bar{v} \cdot \frac{v}{\beta_\epsilon} \nabla_y \beta_\epsilon \right) \mathrm{d}y \, \mathrm{d}s \\ &\qquad = -\int_\Omega \frac{k^2}{2\beta_\epsilon^2} |v|^2 \, \mathrm{d}y \, \mathrm{d}s + \frac{1}{\epsilon} \int_{\partial\Omega} \frac{k}{2\beta_\epsilon} |\mathrm{tr}(v)|^2 \nu_1 \, \mathrm{d}\sigma. \end{aligned} \tag{43}$$

By the very definition of γ (see (2)) it is found that

$$\frac{1}{\epsilon}\int_{\partial\Omega}\gamma|\mathrm{tr}(v)|^2\,\mathrm{d}\sigma = \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S}\gamma^s_{\alpha_k}|\mathrm{tr}(v)|^2\,\mathrm{d}\sigma(y)\right)\mathrm{d}s - \frac{1}{\epsilon}\int_{\partial\Omega}\frac{k}{2}|\mathrm{tr}(v)|^2\nu_1\,\mathrm{d}\sigma, \tag{44}$$

where $\gamma^s_{\alpha_k}$ is given explicitly in (53). From (43)-(44), we have that

$$\int_{\partial\Omega}\frac{y_1k^2}{2\beta_\epsilon}|\mathrm{tr}(v)|^2\nu_1\,\mathrm{d}\sigma \geq 0,$$

then from (40) get the inequality

$$b_\epsilon(v) \geq \frac{1}{2\epsilon^2}\int_\Omega|\nabla_y v|^2\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S}\gamma^s_{\alpha_k}|\mathrm{tr}(v)|^2\,\mathrm{d}\sigma(y)\right)\mathrm{d}s - (\mathrm{C}_3 + 2\|k\|^2_\infty)\int_\Omega|v|^2\,\mathrm{d}y\,\mathrm{d}s + \left(\frac{1}{2\epsilon^2} - \mathrm{C}_3\right)\int_\Omega|\nabla_y v|^2\,\mathrm{d}y\,\mathrm{d}s. \tag{45}$$

A similar argument as in Lemma 3.4 produces the lower bound

$$\frac{1}{2\epsilon^2}\int_\Omega|\nabla_y v|^2\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S}\gamma^s_{\alpha_k}|\mathrm{tr}(v)|^2\,\mathrm{d}\sigma(y)\right)\mathrm{d}s \geq -4(\|\alpha_k\|^2_\infty + \|\alpha\|^2_\infty)\int_\Omega|v|^2\,\mathrm{d}y\,\mathrm{d}s.$$

Finally, it follows from this last inequality and (45) that

$$b_\epsilon(v) \geq -\mathrm{C}_4\int_\Omega|v|^2\,\mathrm{d}y\,\mathrm{d}s + \mathrm{C}_5\int_\Omega|\nabla_y v|^2\,\mathrm{d}y\,\mathrm{d}s$$

where $\mathrm{C}_4 = \mathrm{C}_3 + 4\|\alpha_k\|^2_\infty + 4\|\alpha\|^2_\infty + 2\|k\|^2_\infty$ and $\mathrm{C}_5 = \left(\frac{1}{2\epsilon^2} - \mathrm{C}_3\right)$. Now, by choosing $c_1 = 2c_2$, with $c_2 = \mathrm{C}_4$, one has

$$b_\epsilon(v) + c_1\|v\|^2_2 \geq c_2\|v\|^2_2, \quad \forall\ v \in \mathrm{dom}\, a_\epsilon;$$

since $\mathrm{C}_5 > 0$, for $0 < \epsilon < \epsilon_2 < 1/2\sqrt{\mathrm{C}_3}$, we have $\mathrm{C}_5 > (2\epsilon)^{-2}$. This means that for $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$,

$$b_\epsilon(v) + c_1\|v\|^2 \geq (2\epsilon)^{-2}\|\nabla_y v\|^2_2, \quad \forall\ v \in \mathrm{dom}\, a_\epsilon. \tag{46}$$

For the quadratic form $\widehat{b}_\epsilon$, we perform similar estimates. It is worth mentioning that we can choose the same constants c_1, c_2 as above when we deal with $\widehat{b}_\epsilon$. □

Proposition 4.11. *Let c_1, c_2 be the constants in Lemma 4.10. Denote by $B_\epsilon, \tilde{B}_\epsilon$ the operators associated with $b_\epsilon \geq -c_2$ and $\tilde{b}_\epsilon \geq c_2$, respectively. Then, for ϵ small enough, there exist $\Lambda, \tilde{\Lambda} > 0$ so that*

$$\left|(b_\epsilon + c_1)(v) - \tilde{b}_\epsilon(v)\right| \leq (\epsilon\Lambda)\, \tilde{b}_\epsilon(v), \qquad v \in \operatorname{dom} b_\epsilon,$$

$$\left\|(B_\epsilon + c_1)^{-1} - \tilde{B}_\epsilon^{-1}\right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \leq \epsilon\tilde{\Lambda}.$$

Proof. First, we have that $\|\beta_\epsilon^{-2} - 1\|_\infty \leq \tilde{\mathrm{C}}\epsilon$, with $\tilde{\mathrm{C}} > 0$ depending only on $\|k\|_\infty$. Letting

$$I_\epsilon(v) = \int_\Omega \left| v' + \tau(\nabla_y v \cdot Ry) - \frac{v}{2\beta_\epsilon}\Big(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry)\Big)\right|^2 \mathrm{d}y\, \mathrm{d}s \tag{47}$$

one has

$$\begin{aligned}\left|(b_\epsilon + c_1)(v) - \tilde{b}_\epsilon(v)\right| \leq (\tilde{\mathrm{C}}\epsilon)&\left[I_\epsilon(v) + \int_\Omega \frac{k^2}{4}|v^2|\, \mathrm{d}y\, \mathrm{d}s\right] \\ &+ \epsilon\left|\int_{\mathbb{R}} \frac{\tau^2}{2}\left(\int_{\partial S} \gamma \frac{|\mathrm{tr}(v)|^2}{\beta_\epsilon}(\dot{y}\cdot y)^2\, \mathrm{d}\sigma(y)\right) \mathrm{d}s\right|.\end{aligned}$$

By Lemma 4.10 (see also (41)-(42)), we can infer the estimates (for ϵ small enough)

$$\begin{aligned}&\left|\int_{\mathbb{R}} \frac{\tau^2}{2}\left(\int_{\partial S} \gamma \frac{|\mathrm{tr}(v)|^2}{\beta_\epsilon}(\dot{y}\cdot y)^2\, \mathrm{d}\sigma(y)\right) \mathrm{d}s\right| \\ &\qquad \leq \left[2\mathrm{C}_3 \int_\Omega |v|^2\, \mathrm{d}y\, \mathrm{d}s + \int_\Omega \frac{1}{2\epsilon^2}|\nabla_y v|^2\, \mathrm{d}y\, \mathrm{d}s\right]\end{aligned} \tag{48}$$

$$\begin{aligned}(2\mathrm{C}_3 - c_1)\int_\Omega |v|^2\, \mathrm{d}y\, \mathrm{d}s \leq \frac{1}{2\epsilon^2}&\int_\Omega |\nabla_y v|^2\, \mathrm{d}y\, \mathrm{d}s \\ &+ \epsilon^{-1}\int_{\mathbb{R}}\left(\int_{\partial S} \gamma|\mathrm{tr}(v)|^2\, \mathrm{d}\sigma(y)\right) \mathrm{d}s \\ &- \epsilon^{-2}\int_\Omega \mathrm{Re}\left(\nabla_y v \cdot \frac{\bar{v}}{\beta_\epsilon}\nabla_y \beta_\epsilon\right) \mathrm{d}y\, \mathrm{d}s.\end{aligned} \tag{49}$$

Therefore, for ϵ small enough, the inequalities (48)-(49) imply

$$\begin{aligned}\left|(b_\epsilon + c_1)(v) - \tilde{b}_\epsilon(v)\right| &\leq (\tilde{\mathrm{C}}\epsilon)\left[I_\epsilon(v) + \int_\Omega \frac{k^2}{4}|v^2|\,\mathrm{d}y\,\mathrm{d}s\right] \\ &\quad + \epsilon\left[\tilde{b}_\epsilon(v) - \left(I_\epsilon(v) + \int_\Omega \frac{k^2}{4}|v^2|\,\mathrm{d}y\,\mathrm{d}s\right)\right] \\ &\leq \epsilon(\tilde{\mathrm{C}}+1)\tilde{b}_\epsilon(v).\end{aligned}$$

By applying Theorem 1 in [7], the proof of the proposition follows. □

4.2. Effective potential and operators: square cross-section

We introduce an effective potential function V_{eff} with corresponding *effective* operator for our Robin tubes, in a similar way we have done for strips, with technical details left to Appendix B. For the restriction of the quadratic form $\tilde{b}_\epsilon$ to $\mathrm{d}_\epsilon = \{wu_0^\epsilon; w \in H^1(\mathbb{R})\} \sqsubseteq \mathrm{H}_\epsilon$, we have

$$\tilde{b}_\epsilon(wu_0^\epsilon) = \int_{\mathbb{R}} \left(|w'(s)|^2 + [V_{\text{eff}}^\epsilon(s) + c_1]|w(s)|^2\right)\mathrm{d}s \tag{50}$$

and the following uniform convergence holds

$$V_{\text{eff}}^\epsilon \to V_{\text{eff}} = -2\alpha^2 - \alpha k, \qquad \text{as } \epsilon \to 0.$$

By identifying d_ϵ with $H^1(\mathbb{R})$ via the unitary transformation π_ϵ (26), for simplicity we consider the restricted form $\tilde{q}_\epsilon = \tilde{b}_\epsilon|\mathrm{d}_\epsilon \geq c_2$ in $\mathrm{L}^2(\mathbb{R})$,

$$\tilde{q}_\epsilon(w) = \int_{\mathbb{R}} \left(|w'|^2 + |w|^2[V_{\text{eff}}^\epsilon + c_1]\right)\mathrm{d}s, \qquad \operatorname{dom}\xi_\epsilon = H^1(\mathbb{R}),$$

and let $T_{\tilde{q}_\epsilon}$ be the self-adjoint operator associated with $\tilde{q}_\epsilon$, that is,

$$T_{\tilde{q}_\epsilon}(w) = -w'' + [V_{\text{eff}}^\epsilon + c_1]w, \qquad \operatorname{dom} T_{\tilde{q}_\epsilon} = H^2(\mathbb{R}).$$

Since $V_{\text{eff}}^\epsilon \to V_{\text{eff}}$, it is natural to define the form $\tilde{q} : H^1(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$, $\tilde{q} \geq c_2$, by

$$\tilde{q}(w) = \int_{\mathbb{R}} \left(|w'|^2 + |w|^2[V_{\text{eff}} + c_1]\right)\mathrm{d}s$$

with associated self-adjoint *effective* operator $[T + c_1]$, where

$$T = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + V_{\text{eff}}, \qquad \operatorname{dom} T = H^2(\mathbb{R}). \tag{51}$$

We have the following auxiliary theorem for the process of reduction of dimension, whose proof is based on estimates similar to those presented in proof of Theorem 3.7. The family $\{\mathrm{H}_\epsilon\}_{\epsilon>0}$ of closed subspaces of $\mathrm{L}^2(S\times\mathbb{R})$ is introduced in Section 4.3.

Theorem 4.12. *Suppose $\tau\in W^{1,\infty}(\mathbb{R})\cap\mathrm{C}^1(\mathbb{R})$ and $k,\alpha\in W^{2,\infty}(\mathbb{R})\cap\mathrm{C}^2(\mathbb{R})$; then the following convergence holds true:*

$$\left\|T_{\tilde{q}_\epsilon}^{-1}-(T+c_1)^{-1}\right\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}))}\to 0,\quad \epsilon\to 0.$$

Corollary 4.13. *Consider the restriction $\tilde{q}_\epsilon=\tilde{b}_\epsilon|_{\mathrm{d}_\epsilon}$, $\mathrm{d}_\epsilon\sqsubseteq\mathrm{H}_\epsilon$, and the associated self-adjoint operator $\tilde{Q}_\epsilon\geq c_2$. Then, $\tilde{Q}_\epsilon=\pi_\epsilon^{-1}\circ(T_{\tilde{q}_\epsilon})\circ\pi_\epsilon$, and*

$$\left\|\left[\tilde{Q}_\epsilon^{-1}\oplus 0\right]-\left[\pi_\epsilon^{-1}\circ(T+c_1)^{-1}\circ\pi_\epsilon\oplus 0\right]\right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))}\to 0,\quad \epsilon\to 0.$$

Proof. It is entirely analogous to the proof of Corollary 3.8. □

Lemma 4.14. *Let $(T+c_1)^{-1}:\mathrm{L}^2(\mathbb{R})\to\mathrm{L}^2(\mathbb{R})$ be as in Theorem 4.12. Then,*

$$\Big\|\left[\pi_\epsilon^{-1}\circ(T+c_1)^{-1}\circ\pi_\epsilon\oplus 0\right] - \left[\pi_0^{-1}\circ(T+c_1)^{-1}\circ\pi_0\oplus 0_{E^\perp}\right]\Big\|_{\mathcal{B}(\mathrm{L}^2(\Omega))}\to 0,\quad \epsilon\to 0,$$

where 0 is the null operator on the subspace $\mathrm{H}_\epsilon^\perp$ and $0_{E^\perp}$ the null operator on $E^\perp$.

Proof. Analogous to the proof of Lemma 3.9. □

4.3. Robin Laplacian on square

In order to apply the technique for dimensional reduction of [12], we need a "good choice" of a family of closed subspaces $\{\mathrm{H}_\epsilon\}_{\epsilon>0}$ of $\mathcal{H}=\mathrm{L}^2(\Omega)$, $\Omega=S\times\mathbb{R}$ (straight tube). Denote by u_0^ϵ the normalized eigenfunction associated with the lowest eigenvalue $\lambda_{0,\epsilon}^S<0$ (see ahead) of the Robin Laplacian in such cross-section S (see (52)), we pick

$$\mathrm{H}_\epsilon=\{w(s)u_0^\epsilon(y,s);w\in\mathrm{L}^2(\mathbb{R})\}.$$

Since the boundary ∂S is piecewise $\mathrm{C}^1[0,1]$, in (35) we take the parametrization $y(t)=(t,0)\cup(1,t)\cup(1,t)\cup(0,t)$. We shall refer to problem (52) as the (cross-section) Robin problem with boundary parameter $\alpha_k = \alpha + \frac{k}{2}$,

$$\text{(52)} \qquad \begin{cases} -\Delta_y u = \lambda u, & \text{in } S \\ \dfrac{\partial u}{\partial \vec{\nu}} + (\epsilon \gamma^s_{\alpha_k}) u = 0, & \text{in } \partial S \end{cases}$$

where

$$\text{(53)} \qquad \gamma^s_{\alpha_k}(y_1, y_2) = \begin{cases} -\alpha_k(s), & (y_1, y_2) \in (0,1] \times \{0\}, \\ \alpha(s), & (y_1, y_2) \in \{1\} \times (0,1], \\ \alpha_k(s), & (y_1, y_2) \in [0,1) \times \{1\}, \\ -\alpha(s), & (y_1, y_2) \in \{0\} \times [0,1). \end{cases}$$

Note that we can recast the boundary condition in (52) as

$$\begin{cases} -\dfrac{\partial u}{\partial y_2}(y_1, 0) - \epsilon \alpha_k(s) u(y_1, 0) = 0 \\ \dfrac{\partial u}{\partial y_2}(y_1, 1) + \epsilon \alpha_k(s) u(y_1, 1) = 0 \end{cases} \qquad \begin{cases} -\dfrac{\partial u}{\partial y_1}(0, y_2) - \epsilon \alpha(s) u(0, y_2) = 0 \\ \dfrac{\partial u}{\partial y_1}(1, y_2) + \epsilon \alpha(s) u(1, y_2) = 0 \end{cases}$$

By the definition of $\gamma^s_{\alpha_k}$ in (53), we have

$$u_0^\epsilon(s, y) = \phi_0^\epsilon(s, y_1) \psi_0^\epsilon(s, y_2)$$

with

$$\phi_0^\epsilon(s, y_1) = c_\epsilon(s) e^{-\alpha_k(s) y_1 \epsilon} \quad \text{and} \quad \psi_0^\epsilon(s, y_2) = c_\epsilon(s) e^{-\alpha(s) y_2 \epsilon}$$

being the corresponding normalized eigenfunctions of $-\Delta^{I_1}_{\epsilon\alpha_k(s)}, -\Delta^{I_2}_{\epsilon\alpha(s)}$, where $I_i = I, i = 1, 2$ ($c_\epsilon(s)$ is a normalization parameter; see (24)). Since $S = I_1 \times I_2$, we have

$$\lambda^S_{0,\epsilon}(s) = \lambda^{I_1}_{0,\epsilon}(s) + \lambda^{I_2}_{0,\epsilon}(s) = -(\epsilon \alpha_k(s))^2 - (\epsilon \alpha(s))^2.$$

In Proposition 4.15 we give additional information about the eigenfunctions of the Robin Laplacian $-\Delta^S_R$ on the square cross-section; it was motivated by Proposition 1, page 264, in [20].

Define the Robin Laplacian $-\Delta^S_R$ as the unique self-adjoint operator in $\mathrm{L}^2(S, \mathrm{d}y)$ associated with the quadratic form

$$b(u) = \int_S |\nabla u|^2 \mathrm{d}y + \int_{\partial S} (\epsilon \gamma^s_{\alpha_k}) |u|^2 \, \mathrm{d}\sigma(y), \qquad \operatorname{dom} b = H^1(S).$$

Proposition 4.15. *Let $S = I \times I$ and $\gamma^s_{\alpha_k}$ as in* (53). *Then,*

$$D_R = \left\{ u; u \in \mathrm{C}^\infty(\overline{S}) \text{ with } \frac{\partial u}{\partial \nu}(y) + \epsilon\gamma^s_{\alpha_k}(y)u(y) = 0 \text{ in } \partial S \right\}$$

is a core of the operator $-\Delta^S_R$, and if $u \in D_R$ then

$$-\Delta^S_R u = -\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2}.$$

Proof. Initially, since $\mathrm{C}^\infty_0(S) \subset D_R$, then $D_R \sqsubseteq \mathrm{L}^2(S)$. Consider the symmetric operator $B = -\Delta$, $\operatorname{dom} B = D_R$; an integration by parts gives

$$\begin{aligned}(u, Bu) = \int_S \bar{u}(-\Delta u)\mathrm{d}y &= \int_S |\nabla u|^2 \mathrm{d}y - \int_{\partial S} \bar{u}\frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma(y) \\ &= \int_S |\nabla u|^2 \mathrm{d}y + \int_{\partial S} \epsilon\gamma^s_{\alpha_k}|u|^2\,\mathrm{d}\sigma(y).\end{aligned}$$

Since $\{\phi_n\}_{n=0}^\infty$ is an orthonormal basis of $\mathrm{L}^2(I)$ constituted of eigenfunctions of $-\Delta^I_{\epsilon\alpha_k(s)}$, see Section 3.2, then $\{\phi_n(y_1)\phi_m(y_2)\}_{m,n=0}^\infty$ is an orthonormal basis of $\mathrm{L}^2(S)$ formed by eigenfunctions of B. By Theorem 2.2.10 in [4], B is essentially self-adjoint and its closure $\bar{B}$ is its (unique) self-adjoint extension.

Now, consider the closed and lower bounded sesquilinear form

$$b(u, v) = \int_S \overline{\nabla u(y)}\nabla v(y)\mathrm{d}y + \int_{\partial S} (\epsilon\gamma^s_{\alpha_k})\overline{\mathrm{tr}(u)}\mathrm{tr}(v)\,\mathrm{d}\sigma(y), \quad u, v \in H^1(S).$$

By definition,

$$b(u, v) = (u, -\Delta^S_R v), \quad \forall\ u \in H^1(S), v \in \operatorname{dom}(-\Delta^S_R).$$

But, for each $v \in D_R \subset H^2(S) \subset \operatorname{dom} b$, we have

$$\begin{aligned}b(u, v) &= \int_S \overline{\nabla u}\nabla v\mathrm{d}y + \int_{\partial S} (\epsilon\gamma^s_{\alpha_k})\overline{\mathrm{tr}(u)}\mathrm{tr}(v)\,\mathrm{d}\sigma(y) \\ &= \int_S \bar{u}(-\Delta v)\mathrm{d}y + \int_{\partial S} \bar{u}(\nabla v \cdot \nu)\,\mathrm{d}\sigma(y) + \int_{\partial S} \bar{u}\epsilon\gamma^s_{\alpha_k} v\,\mathrm{d}\sigma(y) \\ &= \int_S \bar{u}(-\Delta v)\mathrm{d}y.\end{aligned}$$

Then it follows that $v \in \operatorname{dom}(-\Delta^S_R)$ and $-\Delta^S_R|_{\operatorname{dom} B} = -\Delta$, thus $\bar{B} \subset -\Delta^S_R$, and we obtain that $\bar{B} = -\Delta^S_R$. □

By Proposition 4.15, for each fixed s, $u_0^\epsilon(y,s)$ is an eigenfunction of our cross-section Robin Laplacian $-\Delta_R^S$, since $u_0^\epsilon(\cdot,s)\in D_R$. Recall that u_0^ϵ is associated with the first eigenvalue given by

$$\lambda_{0,\epsilon}^S(s)=\lambda_{0,\epsilon}^{I_1}(s)+\lambda_{0,\epsilon}^{I_2}(s).$$

Furthermore, the second eigenvalue λ_1^S is

$$\lambda_1^S(s)=\lambda_1^{I_1}+\lambda_{0,\epsilon}^{I_2}(s)=\pi^2-(\epsilon\alpha(s))^2.$$

5. Intermediate convergences

For each $\phi\in\operatorname{dom}\tilde{a}_\epsilon$, we can write $\phi=w(s)\phi_0^\epsilon+\phi_\perp(s,u)$, with $w\in H^1(\mathbb{R})$, $\phi_\perp\in H^1(\mathbb{R}\times I)\cap\mathcal{H}_\epsilon^\perp$. We may decompose the quadratic form $\tilde{a}_\epsilon$ as follows:

$$\tilde{a}_\epsilon(\phi)=\tilde{a}_\epsilon(w\phi_0^\epsilon)+\tilde{a}_\epsilon(\phi_\perp)+2\mathrm{Re}[\tilde{a}_\epsilon(w\phi_0^\epsilon,\phi_\perp)],\qquad \phi\in\operatorname{dom}\tilde{a}_\epsilon.$$

Suppose, for a moment, that the family $\{\tilde{a}_\epsilon\}_{\epsilon>0}$, satisfies the following estimates, where $\mathcal{M}$ is a positive constant,

$$\tilde{a}_\epsilon(\phi_\epsilon)\geq c_2\|\phi_\epsilon\|_2^2\,,\quad \forall\ \phi_\epsilon=w\phi_0^\epsilon\in d_\epsilon:=H^1(\mathbb{R}\times I)\cap\mathcal{H}_\epsilon; \tag{54}$$

$$\tilde{a}_\epsilon(\phi^\epsilon)\geq\frac{\pi^2}{\epsilon^2}\|\phi^\epsilon\|_2^2\,,\quad \forall\ \phi^\epsilon=\phi_\perp\in d^\epsilon:=H^1(\mathbb{R}\times I)\cap\mathcal{H}_\epsilon^\perp; \tag{55}$$

$$|\,\tilde{a}_\epsilon(\phi_\epsilon,\phi^\epsilon)|^2\leq(\mathcal{M}\epsilon^2)\tilde{a}_\epsilon(\phi_\epsilon)\tilde{a}_\epsilon(\phi^\epsilon),\qquad \phi=\phi_\epsilon+\phi^\epsilon\in\operatorname{dom}\tilde{a}_\epsilon. \tag{56}$$

Similarly, suppose for the family $\{\tilde{b}_\epsilon\}_{\epsilon>0}$, with $\psi_\epsilon=wu_0^\epsilon$ and $\psi^\epsilon=\psi_\perp$, that there exist other constants c_2 and $\mathcal{M}'$ for which

$$\tilde{b}_\epsilon(\psi_\epsilon)\geq c_2\|\psi_\epsilon\|_2^2\,,\quad \forall\ \psi_\epsilon\in\mathrm{d}_\epsilon:=H^1(S\times\mathbb{R})\cap\mathrm{H}_\epsilon; \tag{57}$$

$$\tilde{b}_\epsilon(\psi^\epsilon)\geq\frac{\pi^2}{\epsilon^2}\|\psi^\epsilon\|_2^2\,,\quad \forall\ \psi^\epsilon\in\mathrm{d}^\epsilon:=H^1(S\times\mathbb{R})\cap\mathrm{H}_\epsilon^\perp; \tag{58}$$

$$|\,\tilde{b}_\epsilon(\psi_\epsilon,\psi^\epsilon)|^2\leq(\mathcal{M}'\epsilon^2)\tilde{b}_\epsilon(\psi_\epsilon)\tilde{b}_\epsilon(\psi^\epsilon),\qquad \psi=\psi_\epsilon+\psi^\epsilon\in\operatorname{dom}\tilde{b}_\epsilon. \tag{59}$$

Then, by invoking Proposition 3.1 in [12], we have Theorems 5.16 and 5.17 below, whose details of the proofs are left to Appendix A.

Theorem 5.16. *There exists $\tilde{D} > 0$ such that, for ϵ small enough,*

$$\left\| \tilde{A}_\epsilon^{-1} - \left[Q_\epsilon^{-1} \oplus 0 \right] \right\|_{\mathcal{B}(\mathcal{H})} \le \tilde{D}\epsilon, \qquad \textit{with } \mathcal{H} = \mathrm{L}^2(\mathbb{R} \times I),$$

where 0 *is the null operator on the subspace* $\mathcal{H}_\epsilon^\perp$*,* $\tilde{A}_\epsilon$ *the operator associated with* $\tilde{a}_\epsilon$*, and* Q_ϵ *(see Corollary 3.8) the operator associated with* $q_\epsilon = \tilde{a}_\epsilon | d_\epsilon$*.*

Theorem 5.17. *There exists $\tilde{D} > 0$ such that, for ϵ small enough,*

$$\left\| \tilde{B}_\epsilon^{-1} - \left[\tilde{Q}_\epsilon^{-1} \oplus 0 \right] \right\|_{\mathcal{B}(\mathcal{H})} \le \tilde{D}\epsilon, \qquad \textit{with } \mathcal{H} = \mathrm{L}^2(S \times \mathbb{R}),$$

where 0 *is the null operator on the subspace* $\mathrm{H}_\epsilon^\perp$*,* $\tilde{B}_\epsilon$ *the operator associated with* $\tilde{b}_\epsilon$*, and* $\tilde{Q}_\epsilon$ *(see Corollary 4.13) the operator associated with* $\tilde{q}_\epsilon = \tilde{b}_\epsilon | \mathrm{d}_\epsilon$*.*

It is important to note that Theorem 5.16 (planar strip) allows us to derive a kind of norm convergence of resolvents, i.e., rigorously we can give an answer to the question of how is the approach to effective operators whose potential is expressed in terms of our Robin boundary conditions and geometrically induced terms from the original model. In this sense, we say that T_ϵ in $\mathrm{L}^2(\Omega)$ converges to T in $\mathrm{L}^2(\mathbb{R})$ in "norm resolvents sense," where $T = -\frac{\mathrm{d}^2}{\mathrm{d}s} + V^{\mathrm{eff}}$. For Theorem 5.17 we have a similar interpretation for our tubes.

6. Proofs of Theorems 2.1 and 2.2

We begin with some results that actually implement the dimensional reduction, the first one for strips and the second one for tubes.

Theorem 6.18. *Consider the self-adjoint operator* T_ϵ *in* $\mathrm{L}^2(\Omega)$ *(see Section 3) unitarily equivalent to the Robin Laplacian operator* $-\Delta_R^{\Omega_\epsilon}$ *in* $\mathrm{L}^2(\Omega_\epsilon)$*. If* T *denotes the self-adjoint operator in* $\mathrm{L}^2(\mathbb{R})$ *given by* (5) *or* (30)*, then*

$$\left\| (\mathrm{T}_\epsilon + c_1)^{-1} - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0 \right] \right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \longrightarrow 0, \quad \epsilon \to 0,$$

where 0 *is the null operator on the subspace* $\mathcal{H}_\epsilon^\perp$*.*

Proof. We are going to use the same symbol $\|\cdot\|$ to indicate all involved norms. By the triangle inequality, Proposition 3.5, Corollary 3.8 and Theorem 5.16, we get

$$\begin{aligned}&\left\|(\mathrm{T}_\epsilon + c_1)^{-1} - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\right]\right\| \\ &\quad\leq \left\|(\mathrm{T}_\epsilon + c_1)^{-1} - \tilde{A}_\epsilon^{-1}\right\| + \left\|\tilde{A}_\epsilon^{-1} - \left[Q_\epsilon^{-1} \oplus 0\right]\right\| \\ &\quad\quad + \left\|\left[Q_\epsilon^{-1} \oplus 0\right] - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\right]\right\|\end{aligned}$$

and since each term tends to zero as $\epsilon \to 0$, the result follows. □

Theorem 6.19. *Consider the self-adjoint operator B_ϵ in $\mathrm{L}^2(\Omega)$ (see Section 4) associated with $b_\epsilon \geq -c_2$. If T denotes the self-adjoint operator in $\mathrm{L}^2(\mathbb{R})$ given by* (3) *or* (51)*, then*

$$\left\|(B_\epsilon + c_1)^{-1} - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0^\perp\right]\right\|_{\mathcal{B}(\mathrm{L}^2(\Omega))} \longrightarrow 0, \quad \epsilon \to 0,$$

where $0^\perp$ is the null operator on the subspace $\mathrm{H}_\epsilon^\perp$.

Proof. Let $\tilde{B}_\epsilon, \tilde{Q}_\epsilon$ be the unique self-adjoint operators associated, respectively, with $\tilde{b}_\epsilon \geq c_2$, $\tilde{q}_\epsilon := \tilde{b}_\epsilon|_{d_\epsilon}$. By triangle inequality,

$$\begin{aligned}&\left\|(B_\epsilon + c_1)^{-1} - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\right]\right\| \\ &\quad\leq \left\|(B_\epsilon + c_1)^{-1} - \tilde{B}_\epsilon^{-1}\right\| + \left\|\tilde{B}_\epsilon^{-1} - \left[\tilde{Q}_\epsilon^{-1} \oplus 0\right]\right\| \\ &\quad\quad + \left\|\left[\tilde{Q}_\epsilon^{-1} \oplus 0\right] - \left[\pi_\epsilon^{-1} \circ (T + c_1)^{-1} \circ \pi_\epsilon \oplus 0\right]\right\|\end{aligned}$$

and an application of Proposition 4.11, Corollary 4.13 and Theorem 5.17 completes the proof. □

Note that Theorem 2.1 follows by combining Theorem 6.18 and Lemma 3.9, whereas Theorem 6.19 and Lemma 4.14 prove Theorem 2.2.

Appendix A. Technicalities

A.1. Proof of Theorem 5.16

By definition, inequality (54) holds. Relation (55) will be obtained by the minimax principle, since $\phi_0^\epsilon \perp \phi_\perp$ in $\mathrm{L}^2(I)$ for almost all $s \in \mathbb{R}$. By recalling the action of the form $\tilde{a}_\epsilon$ for each $\phi_\perp \in \mathcal{H}_\epsilon^\perp$, we have (by chosen c_1 as in Lemma 3.4)

$$\tilde{a}_\epsilon(\phi_\perp) \geq \frac{1}{\epsilon^2}\int_{\mathbb{R}}\left[\int_0^1 |\partial_u\phi_\perp|^2\,\mathrm{d}u + \epsilon\alpha_k(s)\Big(|\mathrm{tr}(\phi_\perp)(s,1)|^2 - |\mathrm{tr}(\phi_\perp)(s,0)|^2\Big)\right]\mathrm{d}s.$$

By the minimax principle (see Theorem 11.4.28 in [4]) and Theorem 3.6

$$\tilde{a}_\epsilon(\phi_\perp) \geq \frac{\lambda_1^I}{\epsilon^2}\int_\Omega |\phi_\perp|^2\,\mathrm{d}s\,\mathrm{d}u \tag{A.1}$$

here the quantity $\lambda_1^I = \pi^2$ (see Section 3.2) is the second eigenvalue of the Robin Laplacian operator $-\Delta^I_{\epsilon\alpha_k(s)}$ with boundary condition

$$-\psi'(0) - \epsilon\alpha_k(s)\psi(0) = 0 \quad \text{and} \quad \psi'(1) + \epsilon\alpha_k(s)\psi(1) = 0.$$

Nondiagonal part:

The goal now is to check (56) (nondiagonal part). Given $\phi \in H^1(\Omega)$, we can write $\phi = w\phi_0^\epsilon + \eta$, where $\eta = \phi_\perp$. Consider the family of eigenfunctions $\{\phi_0^\epsilon\}_{\epsilon>0}$; we denote by I_1^ϵ the sesquilinear form

$$I_\epsilon^1(\phi,\psi) = \int_\Omega \partial_s\bar{\phi}\partial_s\psi\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon^2}\int_\Omega \partial_u\bar{\phi}\partial_u\psi\,\mathrm{d}s\,\mathrm{d}u + \frac{1}{\epsilon}\int_{\partial\Omega}\alpha_k\mathrm{tr}(\bar{\phi})\mathrm{tr}(\psi)\nu_2\,\mathrm{d}\sigma$$

and since $\int_I(\partial_s\eta)\phi_0^\epsilon\,\mathrm{d}u = -\int_I \eta(\partial_s\phi_0^\epsilon)\,\mathrm{d}u,$ a.e. $s \in \mathbb{R}$, and $|\partial_s\phi_0^\epsilon| \leq \mathrm{C}|\phi_0^\epsilon|$ with $\mathrm{C} > 0$ independent of ϵ, we have

$$|I_\epsilon^1(w\phi_0^\epsilon,\eta)| \leq \int_\Omega |w'\phi_0^\epsilon\partial_s\eta + w\partial_s\phi_0^\epsilon\partial_s\eta|\,\mathrm{d}s\,\mathrm{d}u \leq \mathrm{C}(\|w\|_{1,2}\|\eta\|_2 + \epsilon\|w\|_2\|\partial_s\eta\|_2).$$

Thus, by Lemma 3.4 and (A.1), there exists $\mathrm{M} > 0$ (independent of ϵ) such that

$$|I_\epsilon^1(w\phi_0^\epsilon,\eta)| \leq (\epsilon\mathrm{M})(\tilde{a}_\epsilon)[\phi_\epsilon]^{1/2}(\tilde{a}_\epsilon)[\phi^\epsilon]^{1/2}.$$

Let

$$I_\epsilon^2(\phi,\psi) = \frac{1}{\epsilon}\int_\Omega \frac{k}{2\beta_\epsilon}[\bar{\phi}\partial_u\psi + \partial_u\bar{\phi}\psi]\,\mathrm{d}s\,\mathrm{d}u - \frac{1}{\epsilon}\int_{\partial\Omega}\frac{k}{2}\mathrm{tr}(\bar{\phi})\mathrm{tr}(\psi)\nu_2\,\mathrm{d}\sigma;$$

upon integration by parts we obtain

$$\int_{\partial\Omega} \frac{k}{2} \mathrm{tr}(\bar{\phi})\mathrm{tr}(\psi)\nu_2 \,\mathrm{d}\sigma = \int_{\Omega} \frac{k}{2}[\bar{\phi}\partial_u\psi + \partial_u\bar{\phi}\psi]\,\mathrm{d}s\,\mathrm{d}u$$

and so

$$I_\epsilon^2(w\phi_0^\epsilon, \eta) = \int_\Omega \frac{uk^2}{2\beta_\epsilon}[w\phi_0^\epsilon\partial_u\eta + \eta\partial_u(w\phi_0^\epsilon)]\,\mathrm{d}s\,\mathrm{d}u.$$

Since $k(s)$ and $\beta_\epsilon(s)$ are bounded functions, there exists $\mathrm{C} > 0$ such that for ϵ small enough (after combining with Lemma 3.4) we get

$$|I_\epsilon^2(w\phi_0^\epsilon, \eta)| \leq \mathrm{C}\int_\Omega |w\phi_0^\epsilon\partial_u\eta|\,\mathrm{d}s\,\mathrm{d}u \leq (\epsilon\mathrm{C})\,\tilde{a}_\epsilon[w\phi_0^\epsilon]^{1/2}\tilde{a}_\epsilon[\eta]^{1/2}.$$

We finally obtain

$$|\tilde{a}_\epsilon(w\phi_0^\epsilon, \eta)|^2 \leq (\mathcal{M}\epsilon^2)\tilde{a}_\epsilon[w\phi_0^\epsilon]\tilde{a}_\epsilon[\eta].$$

Thus, it is enough to invoke Proposition 3.1 in [12] to complete the proof of Theorem 5.16.

A.2. Proof of Theorem 5.17

First we check the conditions (57), (58) and (59); then we complete the proof of the theorem by applying Proposition 3.1 in [12].

• **Estimate for the diagonal part:** Let $\eta \in \mathrm{d}^\epsilon$; then there exists $\mu > 0$ so that, for ϵ small enough,

$$\tilde{b}_\epsilon(\eta) \geq \frac{\mu}{\epsilon^2}\|\eta\|_2^2.$$

Indeed, let λ_1^S be the second eigenvalue of the Robin Laplacian $-\Delta_R^S$ on S and pick $\eta \in H^1(\Omega) \cap \mathcal{H}_\epsilon^\perp$. By choosing $c_1 > 0$ as in Lemma 4.10, we get the inequality

$$\begin{aligned}\tilde{b}_\epsilon(\eta) &\geq \frac{1}{\epsilon^2}\int_\Omega |\nabla_y\eta|^2\,\mathrm{d}s\,\mathrm{d}y + \frac{1}{\epsilon}\int_{\partial\Omega}\gamma_{\alpha_k}^s|\eta|^2\,\mathrm{d}\sigma(y,s) + \|\alpha\|_\infty^2\int_\Omega |\eta|^2\,\mathrm{d}y\,\mathrm{d}s\\ &= \frac{1}{\epsilon^2}\int_{\mathbb{R}}\left[\int_S |\nabla_y\eta|^2\,\mathrm{d}y + \int_{\partial S}\epsilon\gamma_{\alpha_k}^s|\eta|^2\,\mathrm{d}\sigma(y)\right]\,\mathrm{d}s + \|\alpha\|_\infty^2\int_\Omega |\eta|^2\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

Since $\lambda_1^S = \lambda_1^I - \epsilon^2|\alpha(s)|^2$, by the minimax principle (see Theorem 11.4.28 in [4]) and Proposition 4.15 we have

$$\int_S |\nabla_y \eta|^2 \,\mathrm{d}y + \int_{\partial S} \epsilon \gamma_{\alpha_k}^s |\eta|^2 \,\mathrm{d}\sigma(y) \geq \left(\lambda_1^I - \epsilon^2 \|\alpha\|_\infty^2\right) \int_S |\eta|^2 \mathrm{d}y.$$

Thus,

$$\tilde{b}_\epsilon(\eta) \geq \frac{\lambda_1^I}{2\epsilon^2} \int_\Omega |\eta|^2 \,\mathrm{d}y \,\mathrm{d}s.$$

• **Claim 1:** For ϵ small enough the inequality $I_\epsilon(v) + c_2\|v\|_2^2 \geq \|w'\|_2^2$ is satisfied for each $v \in \mathrm{d}_\epsilon$. Thus, $\tilde{b}(wu_0^\epsilon) \geq \|w'\|^2_{\mathrm{L}^2(\mathbb{R})}$.

Proof. (Claim 1) Note first that, by the proof of Lemma 4.10,

$$\tilde{b}_\epsilon(v) + c_2\|v\|_2^2 \geq I_\epsilon(v),$$

see (47) for the definition of I_ϵ. Clearly, we have

$$\begin{aligned} I_\epsilon(v) \geq & \int_\Omega |v'|^2 \,\mathrm{d}s \,\mathrm{d}y \\ & + 2\mathrm{Re} \int_\Omega v' \left[\tau(\nabla_y \bar{v} \cdot Ry) - \bar{v}\frac{1}{2\beta_\epsilon}(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry))\right] \mathrm{d}y \,\mathrm{d}s. \end{aligned}$$

For $v = wu_0^\epsilon$, we find $\int_\Omega |v'|^2 \,\mathrm{d}s \,\mathrm{d}y \geq \int_\mathbb{R} |w'|^2 \,\mathrm{d}s$. We will estimate the second term above, which consists of two steps:

• **Step 1:** For $v = wu_0^\epsilon$ and ϵ small enough we have

$$\begin{aligned} \int_\Omega 2v'\tau(\nabla_y \bar{v} \cdot Ry) \,\mathrm{d}s \,\mathrm{d}y &= \int_\Omega |w|^2 |u_0^\epsilon|^2 \left[\tau((\epsilon\alpha_k, \epsilon\alpha) \cdot Ry)\right]' \,\mathrm{d}y \,\mathrm{d}s \\ &\geq -\frac{c_2}{2} \int_\mathbb{R} |w|^2 \,\mathrm{d}y \,\mathrm{d}s. \end{aligned} \tag{A.2}$$

• **Step 2:** For $v = wu_0^\epsilon$ and ϵ small enough,

$$-2\mathrm{Re} \int_\mathbb{R} \int_S v'\bar{v} \left[\frac{1}{2\beta_\epsilon}(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry))\right] \mathrm{d}y \,\mathrm{d}s \geq -\frac{c_2}{2} \int_\mathbb{R} |w|^2 \,\mathrm{d}y \,\mathrm{d}s. \tag{A.3}$$

Indeed, we have

$$
\begin{aligned}
-2\mathrm{Re}\int_\Omega v'\bar{v}\left[\frac{1}{2\beta_\epsilon}(\beta_\epsilon'+\tau(\nabla_y\beta_\epsilon\cdot Ry)\right]\mathrm{d}y\,\mathrm{d}s\\
=\int_\Omega |w|^2|u_0^\epsilon|^2\left[\frac{1}{2\beta_\epsilon}(\beta_\epsilon'+\tau(\nabla_y\beta_\epsilon\cdot Ry)\right]'\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

and since k,τ,k',τ',k'' are bounded we have that $\psi_\epsilon',\psi_\epsilon\to 0,\ \epsilon\to 0$, uniformly, where

$$
\psi_\epsilon=\frac{1}{2\beta_\epsilon}(\beta_\epsilon'+\tau(\nabla_y\beta_\epsilon\cdot Ry)). \tag{A.4}
$$

So, we have checked Step 2. Thus, for each $v\in \mathrm{d}_\epsilon$, we obtain by (A.2) and (A.3), that $I_\epsilon(v)+c_2\|v\|_2^2\geq \|w'\|_2^2$, since $\tilde{b}_\epsilon(v)-I_\epsilon(v)+c_2\|v\|_2^2\geq 0$ then $\tilde{b}_\epsilon(v)\geq \|w'\|_2^2$, for ϵ small enough. □

• **Estimate for the nondiagonal part:** We need to verify condition (59). Given $v\in\operatorname{dom}\tilde{b}_\epsilon$ we put $v=wu_0^\epsilon+\eta$ with $w\in H^1(\mathbb{R})$ and $\eta=\phi_\perp\in H^1(\Omega)\cap\mathcal{H}_\epsilon^\perp$.

We note that (58) follows by the definition of $\tilde{b}_\epsilon$. Claim 1 and (57) will be freely used. Consider the quadratic form $\tilde{b}_\epsilon(wu_0^\epsilon,\eta)$, and recall that $wu_0^\epsilon\perp\eta$, for all $w\in H^1(\mathbb{R})$; by an integration by parts we may write

$$
\tilde{b}_\epsilon(wu_0^\epsilon,\eta)=I_\epsilon(wu_0^\epsilon,\eta)+J_\epsilon(wu_0^\epsilon,\eta)
$$

with I_ϵ and J_ϵ given by

$$
\begin{aligned}
I_\epsilon(wu_0^\epsilon,\eta)&=\int_\Omega\left[(\bar{w}u_0^\epsilon)'+\tau\bar{w}(\nabla_y u_0^\epsilon\cdot Ry)+\bar{w}u_0^\epsilon\psi_\epsilon\right]\\
&\qquad\times\left[\eta'+\tau(\nabla_y\eta\cdot Ry)+\eta\psi_\epsilon\right]\mathrm{d}y\,\mathrm{d}s\\
J_\epsilon(wu_0^\epsilon,\eta)&=\frac{1}{\epsilon^2}\int_\Omega\bar{w}\nabla_y(u_0^\epsilon)\nabla_y\eta\,\mathrm{d}s\,\mathrm{d}y+\frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S}\gamma_{\alpha_k}^s\bar{w}u_0^\epsilon\eta\,\mathrm{d}\sigma(y)\right)\mathrm{d}s\\
&\quad+\int_\Omega\frac{uk^2}{2\beta_\epsilon}[wu_0^\epsilon\partial_{y_1}\eta+\eta\partial_{y_1}(wu_0^\epsilon)]\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

Now we estimate each one of the above terms.

• J_ϵ**-Estimate:** By the definition of γ and proceeding as in (A.1) we get

$$|J_\epsilon(wu_0^\epsilon,\eta)| \le (\epsilon\tilde{\mathrm{D}})\tilde{b}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{b}_\epsilon[\eta]^{1/2}$$

where $\tilde{\mathrm{D}} > 0$ is independent of $\epsilon > 0$.

• I_ϵ**-Estimate:** Since $\tau \in W^{1,\infty}(\mathbb{R}) \cap \mathrm{C}^1(\mathbb{R})$ and $k \in W^{2,\infty}(\mathbb{R}) \cap \mathrm{C}^2(\mathbb{R})$ then we have that $\psi_\epsilon \in \mathrm{C}^1(\Omega) \cap W^{1,\infty}(\Omega)$, with $\|\psi_\epsilon\|_{1,\infty} < \mathrm{C}$, where $\mathrm{C} > 0$ is independent of ϵ.

Since $wu_0^\epsilon \perp \eta$, we have

$$\int_S u_0^\epsilon \eta' \,\mathrm{d}y = -\int_S (u_0^\epsilon)'\eta \,\mathrm{d}y, \qquad \text{a.e.} \quad s \in \mathbb{R};$$

also note that $\nabla_y(u_0^\epsilon) = -\epsilon u_0^\epsilon(\alpha_k, \alpha)$, $|(u_0^\epsilon)''| \le C|u_0^\epsilon|$, $|(u_0^\epsilon)'| \le C|u_0^\epsilon|$. Since $(\bar{w}u_0^\epsilon)' = \bar{w}'u_0^\epsilon + \bar{w}(u_0^\epsilon)'$, and keeping in mind the above observations, we should estimate only three types of integrals in I_ϵ, namely:

• I_1**-Estimate:** Using integration by parts, we get

$$\int_\Omega \bar{w}(u_0^\epsilon)'\eta' \,\mathrm{d}y\,\mathrm{d}s = -\int_\Omega \bar{w}'u_0^\epsilon\eta \,\mathrm{d}y\,\mathrm{d}s - \int_\Omega \bar{w}(u_0^\epsilon)''\eta \,\mathrm{d}y\,\mathrm{d}s$$

since $\alpha, \alpha', \alpha''$ are uniformly bounded, we get

$$\left|\int_\Omega \bar{w}(u_0^\epsilon)'\eta' \,\mathrm{d}y\,\mathrm{d}s\right| \le (\epsilon D)\ \tilde{a}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{a}_\epsilon[\eta]^{1/2}$$

where $D > 0$ is independent of ϵ.

• I_2**-Estimate:** Upon integration by parts

$$\begin{aligned}\int_\Omega \bar{w}\tau(\nabla_y u_0^\epsilon \cdot Ry)\eta' \,\mathrm{d}y\,\mathrm{d}s = &-\int_\Omega \bar{w}'\tau(\nabla_y u_0^\epsilon \cdot Ry)\eta \,\mathrm{d}y\,\mathrm{d}s \\ &-\int_\Omega \bar{w}\tau'(\nabla_y u_0^\epsilon \cdot Ry)\eta \,\mathrm{d}y\,\mathrm{d}s \\ &-\int_\Omega \bar{w}\tau(\nabla_y u_0^\epsilon \cdot Ry)'\eta \,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

then

$$\left|\int_\Omega \bar{w}\tau(\nabla_y u_0^\epsilon \cdot Ry)\eta' \,\mathrm{d}y\,\mathrm{d}s\right| \le (D\epsilon)\tilde{a}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{a}_\epsilon[\eta]^{1/2}.$$

• I_3**-Estimate:** Upon integration by parts,

$$\int_\Omega \bar{w}u_0^\epsilon \psi_\epsilon \eta' \,\mathrm{d}y\,\mathrm{d}s = -\int_\Omega [\bar{w}' u_0^\epsilon \psi_\epsilon + \bar{w}(u_0^\epsilon)'\psi_\epsilon + \bar{w}u_0^\epsilon(\psi_\epsilon)']\eta \,\mathrm{d}y\,\mathrm{d}s$$

and so

$$\left|\int_\Omega \bar{w}u_0^\epsilon \psi_\epsilon \eta' \,\mathrm{d}y\,\mathrm{d}s\right| \le (D\epsilon)\tilde{a}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{a}_\epsilon[\eta]^{1/2}.$$

Thus, we may write

$$|I_\epsilon(wu_0^\epsilon, \eta)| \le (\tilde{D}\epsilon)\tilde{b}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{b}_\epsilon[\eta]^{1/2}$$

for ϵ small enough. Consequently,

$$|\tilde{b}_\epsilon(wu_0^\epsilon, \eta)|^2 \le (\mathcal{M}'\epsilon^2)\tilde{b}_\epsilon[wu_0^\epsilon]^{1/2}\tilde{b}_\epsilon[\eta]^{1/2}$$

where $\mathcal{M}' > 0$ is independent of ϵ (for ϵ small enough).

Appendix B. Effective three-dimensional potential

By considering each integral in $\tilde{b}_\epsilon$ we will be able to find out the (intermediate) effective potential $V_{\mathrm{eff}}^\epsilon$, which arises after evaluating $\tilde{b}_\epsilon(wu_0^\epsilon)$. Note that the integral over the region S will be regarded as function of the variable s. Our Robin boundary conditions (particularly the expression of the first cross-section eigenfunction) combined with the symmetry of the cross-section will result in the vanishing of all terms with torsion as $\epsilon \to 0$.

One has

$$\begin{aligned}\widehat{b}_\epsilon(v) = &\int_\Omega \left| v' + \tau(\nabla_y v \cdot Ry) - \frac{v}{2\beta_\epsilon}\Big(\beta_\epsilon' + \tau(\nabla_y \beta_\epsilon \cdot Ry)\Big)\right|^2 \mathrm{d}s\,\mathrm{d}y \\ &+ \frac{1}{\epsilon^2}\int_\Omega |\nabla_y v|^2 \,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S} \gamma_{\alpha_k}^s |v|^2 \,\mathrm{d}\sigma(y)\right)\mathrm{d}s \\ &+ \int_\Omega |v|^2 \frac{k^2}{4}\,\mathrm{d}y\,\mathrm{d}s + \frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar{v}\partial_{y_1} v)\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{\epsilon}\int_{\partial\Omega}\frac{k}{2}|\mathrm{tr}(v)|^2 \nu_1 \,\mathrm{d}\sigma\end{aligned}$$

and so

$$\int_\Omega |v|^2 \frac{k^2}{4}\,\mathrm{d}s\,\mathrm{d}y = \int_{\mathbb{R}} |w|^2 \frac{|k|^2}{4}\,\mathrm{d}s \tag{B.1}$$

$$\text{(B.2)}\qquad \begin{aligned} &\frac{1}{\epsilon^2}\int_\Omega |\nabla_y v|^2\,\mathrm{d}s\,\mathrm{d}y + \frac{1}{\epsilon}\int_{\mathbb{R}}\left(\int_{\partial S}\gamma^s_{\alpha_k}|v|^2\,\mathrm{d}\sigma(y)\right)\mathrm{d}s\\ &\quad = -\int_{\mathbb{R}}(\alpha_k^2+\alpha^2)|w|^2\,\mathrm{d}s \end{aligned}$$

$$\text{(B.3)}\qquad \begin{aligned} &\frac{1}{\epsilon}\int_\Omega \frac{k}{\beta_\epsilon}\mathrm{Re}(\bar v\partial_{y_1}v)\,\mathrm{d}y\,\mathrm{d}s - \frac{1}{\epsilon}\int_{\partial\Omega}\frac{k}{2}|\mathrm{tr}(v)|^2\nu_1\,\mathrm{d}\sigma\\ &\quad = \int_{\mathbb{R}} k\alpha_k|w|^2\left[1-\int_I \frac{|u_0^\epsilon|^2}{\beta_\epsilon}\,\mathrm{d}u\right]\mathrm{d}s \end{aligned}$$

Now we replace v by wu_0^ϵ in $I_\epsilon(v)$, see (47) for the definition of I_ϵ,

$$\begin{aligned} I_\epsilon(v) &= \int_\Omega |v'|^2\,\mathrm{d}s\,\mathrm{d}y + \int_\Omega \tau^2|(\nabla_y v\cdot Ry)|^2\,\mathrm{d}s\,\mathrm{d}y\\ &\quad + 2\mathrm{Re}\int_\Omega v'\tau(\nabla_y\bar v\cdot Ry)\,\mathrm{d}s\,\mathrm{d}y + \int_\Omega |v|^2|\psi_\epsilon|^2\,\mathrm{d}s\,\mathrm{d}y\\ &\quad - 2\mathrm{Re}\int_\Omega v'\bar v\big[\psi_\epsilon\big]\,\mathrm{d}s\,\mathrm{d}y - 2\mathrm{Re}\int_\Omega \tau(\nabla_y v\cdot Ry)\bar v\big[\psi_\epsilon\big]\,\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

we obtain that

$$\text{(B.4)}\qquad I_\epsilon(v) = \int_{\mathbb{R}}\left[|w'|^2 + |w|^2\left(\epsilon G_1(s) + \int_S\left[(u_0^\epsilon)'\right]^2\mathrm{d}y\right)\right]\mathrm{d}s$$

where $G_1(s)$ is bounded (in $\mathbb{R}$).

Note that the functions that multiply $|w|^2$ in (B.3)-(B.4) converge uniformly to zero as $\epsilon\to 0$. Therefore, the uniform convergence of the potential $V_{\mathrm{eff}}^\epsilon$ comes from the expressions in (B.1)-(B.2). To compute the value of the expressions $I_\epsilon(v)$ we have used integration by parts and that

$$2\mathrm{Re}\int_\Omega (w'\bar w)u_0^\epsilon(u_0^\epsilon)'\,\mathrm{d}s\,\mathrm{d}y = 0.$$

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