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Extremally Ricci pinched G_2-structures on Lie groups

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Only two examples of extremally Ricci pinched G_2-structures can be found in the literature and they are both homogeneous. We study in this paper the existence and structure of such very special closed G_2-structures on Lie groups. Strong structural conditions on the Lie algebra are proved to hold. As an application, we obtain three new examples of extremally Ricci pinched G_2-structures and that they are all necessarily steady Laplacian solitons. The deformation and rigidity of such structures are also studied.

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1. Introduction

A G_2-*structure* on a 7-dimensional differentiable manifold M is a positive (or definite) differential 3-form on M. Each G_2-structure φ defines a Riemannian metric g on M together with an orientation and (M,φ) is called *homogeneous* if its automorphism group $\mathrm{Aut}(M,\varphi) := \{f \in \mathrm{Diff}(M) : f^*\varphi = \varphi\}$ acts transitively on M.

As is well known, torsion-free (or parallel) G_2-structures (i.e. $d\varphi = 0$ and $d * \varphi = 0$) produce Ricci flat Riemannian metrics with holonomy contained in G_2. Homogeneous torsion-free G_2-structures are therefore necessarily flat by [AK]. In the case that φ is closed, the only torsion that survives is a 2-form τ and one has that,

$$d\varphi = 0, \qquad \tau = - * d * \varphi, \qquad d * \varphi = \tau \wedge \varphi, \qquad d\tau = \Delta\varphi.$$

Closed G_2-structures play an important role as natural candidates to deform toward a torsion-free one via the *Laplacian flow* $\frac{\partial}{\partial t}\varphi(t) = \Delta\varphi(t)$, introduced back in 1992 by R. Bryant in [B] (see [Lo] for an account of recent advances). In the homogeneous case, closed G_2-structures are only allowed on non-compact manifolds (see [PR]) and examples on non-solvable Lie groups were given in [FR3].

A closed G_2-structure is said to be *extremally Ricci-pinched* (ERP for short) when

$$d\tau = \tfrac{1}{6}|\tau|^2\varphi + \tfrac{1}{6} * (\tau \wedge \tau),$$

one of the ways in which $d\tau$ can quadratically depend on τ. It is proved in [B, (4.66)] that this is the only way in the compact case. In the homogeneous case, the only other possibility for a quadratic dependence is to have $d\tau = \frac{1}{7}|\tau|^2\varphi$ (i.e. φ an eigenform of Δ), though the existence of such structures is still an open problem (see [L3, Lemma 3.4] and [L4]). ERP G_2-structures were introduced by R. Bryant in [B, Remark 13] and owe their name to the fact that they are precisely the structures at which equality holds in the following estimate for closed G_2-structures on a compact manifold M obtained in [B, Corollary 3]:

$$\int_M \mathrm{scal}^2 *1 \leq 3 \int_M |\mathrm{Ric}|^2 * 1.$$

This estimate does not hold in general in the homogeneous case, examples of closed G_2-structures on solvable Lie groups such that $\mathrm{scal}^2 > 3|\mathrm{Ric}|^2$ were found in [L3]. In [FR2], it is proved that the Laplacian flow solution starting at an ERP G_2-structure φ is simply given by

$$\varphi(t) = \varphi + c(t)d\tau, \qquad c(t) = \tfrac{6}{|\tau|^2}\left(e^{\frac{|\tau|^2}{6}t} - 1\right),$$

from which follows that the set of ERP G_2-structures is invariant under the Laplacian flow and the solutions are always eternal.

Until now, only two examples of ERP G_2-structures were known and they are both (locally) homogeneous: one on the homogeneous space $\mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2/\mathrm{SU}(2)$ (see [B, Example 1]), or alternatively, on the solvable Lie group given in [CI, Section 6.3] (see also [L3, Examples 4.13, 4.10]), and a second one on a unimodular solvable Lie group given in [L3, Example 4.7]. It is worth highlighting that both examples are also steady Laplacian solitons, that is, they evolve under the Laplacian flow in the following silly way: there is a one-parameter family $f(t) \in \mathrm{Diff}(M)$ such that the Laplacian flow solution starting at φ is given by $\varphi(t) = f(t)^*\varphi$.

Motivated by this major lack of examples, we study in this paper left-invariant ERP G_2-structures on Lie groups, in which the G_2-structure can be identified with a positive 3-form on the Lie algebra. Our aim is to show that the condition produces quite strong structure constraints on the Lie algebra (see Section 4).

We first introduce some notation. Given a real vector space $\mathfrak{g}$ with basis $\{e_1, \dots, e_7\}$, we consider the positive 3-form

$$\begin{aligned} \varphi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \\ &= \omega_3 \wedge e^3 + \omega_4 \wedge e^4 + \omega_7 \wedge e^7 + e^{347}, \end{aligned} \tag{1}$$

where $\omega_7 := e^{12} + e^{56}$, $\omega_3 := e^{26} - e^{15}$ and $\omega_4 := e^{16} + e^{25}$, and let θ denote the usual representation of $\mathfrak{gl}_4(\mathbb{R})$ on $\Lambda^2\mathbb{R}^4$. Two Lie groups endowed with G_2-structures (G, φ) and (G', φ') are called *equivariantly equivalent* if there is a Lie group isomorphism $f : G \to G'$ such that $\varphi = f^*\varphi'$.

We are now ready to state our main result (see Theorem 4.7 and Proposition 4.9).

Theorem 1.1. *Every Lie group endowed with a left-invariant ERP G_2-structure is equivariantly equivalent, up to scaling, to a (G, φ) with torsion $\tau = e^{12} - e^{56}$, where φ is as in* (1)*, and the following conditions hold for the Lie algebra $\mathfrak{g}$ of G:*

(i) $\mathfrak{h} := \mathrm{sp}\{e_1, \dots, e_6\}$ *is a unimodular ideal.*

(ii) $\mathfrak{g}_0 := \mathrm{sp}\{e_7, e_3, e_4\}$ *is a Lie subalgebra and* $\mathfrak{g}_1 := \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ *is an abelian ideal of* $\mathfrak{g}$. *In particular,* $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ *and* $\mathfrak{g}$ *is solvable.*

(iii) $\mathfrak{h}_1 := \mathrm{sp}\{e_3, e_4\}$ *is an abelian subalgebra; in particular* $\mathfrak{h} = \mathfrak{h}_1 \ltimes \mathfrak{g}_1$.

(iv) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_7$, $\theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_3$ *and* $\theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_4$.

(v) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\omega_7 + \theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\omega_3 + \theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\omega_4 = \tau + (\mathrm{tr}\,\mathrm{ad}\, e_7|_{\mathfrak{g}_0})\omega_7$.

Conversely, if $\mathfrak{g}$ *satisfies (i)-(v), then* (G, φ) *is an ERP* G_2*-structure with torsion* $\tau = e^{12} - e^{56}$.

As a first application, we obtain the following geometric consequence.

Corollary 1.2. *Any left-invariant ERP* G_2*-structure on a Lie group is a steady Laplacian soliton and its underlying metric is an expanding Ricci soliton.*

It is worth pointing out that the converse of the above corollary does not hold. Indeed, an example of a simply connected solvable Lie group endowed with a steady Laplacian soliton that is not an ERP G_2-structure is exhibited in [FR3].

Structurally, it follows from Theorem 1.1 that the Lie algebra $\mathfrak{g}$ of any ERP (G, φ) is determined by the 2×2 matrix $A_1 := \mathrm{ad}\, e_7|_{\mathfrak{h}_1}$ and the three 4×4 matrices $A := \mathrm{ad}\, e_7|_{\mathfrak{g}_1}$, $B = \mathrm{ad}\, e_3|_{\mathfrak{g}_1}$, $C := \mathrm{ad}\, e_4|_{\mathfrak{g}_1}$. The Jacobi condition is equivalent to

$$[A, B] = aB + cC, \quad [A, C] = bB + dC, \quad [B, C] = 0, \qquad A_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].$$

It must be stressed that conditions (iv) and (v) are really demanding on these matrices.

In Section 5, we exhibit three new examples of ERP G_2-structures on Lie groups and obtain further refinements for the algebraic structure of $\mathfrak{g}$ by using the structural theorem on solvsolitons [L1, Theorem 4.8]. We prove that there are only three possibilities for the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ and that the following conditions must hold in each case:

- $\mathfrak{n} = \mathfrak{g}_1$: this is equivalent to $\mathfrak{g}$ unimodular and one has that $A_1 = 0$, the matrices A, B, C are all symmetric, they pairwise commute and $\left\{\sqrt{3}A, \sqrt{3}B, \sqrt{3}C\right\}$ is orthonormal. In particular, $\mathfrak{g}$ is isomorphic to the Lie algebra of [L3, Example 4.7], a result previously obtained in [FR2].

- $\mathfrak{n} = \mathbb{R}e_4 \oplus \mathfrak{g}_1$: A, B are symmetric, $[A, B] = 0$, C is nilpotent and $a = b = c = 0$. We found two new examples in this case, with $\mathfrak{n}$ 2-step and 3-step nilpotent, respectively.
- $\mathfrak{n} = \mathfrak{h}$: A_1 and A are normal and B, C nilpotent. A new example is given with $\mathfrak{n}$ 4-step nilpotent.

Lastly, we study in Section 6 deformations and rigidity of ERP G_2-structures on Lie groups by using the moving-bracket approach. We have obtained that the five known examples are all rigid.

We believe that the present paper paves the way toward achieving a complete classification of ERP G_2-structures on Lie groups, which will be the object of further research.

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2. Preliminaries

2.1. Linear algebra

Given a real vector space $\mathfrak{g}$ with basis $\{e_1, \dots, e_7\}$, we consider the positive 3-form

$$\varphi = \omega \wedge e^7 + \rho^+ = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{2}$$

where

$$\omega := e^{12} + e^{34} + e^{56}, \qquad \rho^+ := e^{135} - e^{146} - e^{236} - e^{245}.$$

The usual notation $e^{ij\cdots}$ to indicate $e^i \wedge e^j \wedge \cdots$ will be freely used throughout the paper. Note that $\omega \wedge \rho^+ = 0$. We have that $\{e_1, \dots, e_7\}$ is an oriented orthonormal basis with respect to the inner product $\langle\cdot,\cdot\rangle$ and orientation vol determined by φ, i.e.

$$\langle X, Y\rangle \operatorname{vol} = \frac{1}{6} i_X(\varphi) \wedge i_Y(\varphi) \wedge \varphi, \qquad \forall X, Y \in \mathfrak{g}. \tag{3}$$

The almost-complex structure J defined on the subspace $\mathfrak{h} := \operatorname{sp}\{e_1, \dots, e_6\}$ by $\omega = \langle J\cdot,\cdot\rangle$ is given by $Je_i = e_{i+1}$, $i = 1, 3, 5$, and we set

$$\rho^- := *_{\mathfrak{h}}\rho^+ = e^{145} + e^{136} + e^{235} - e^{246}.$$

Let $\theta : \mathfrak{gl}(\mathfrak{h}) \longrightarrow \operatorname{End}(\Lambda^k\mathfrak{h}^*)$ denote the representation obtained as the derivative of the natural left $\operatorname{GL}(\mathfrak{h})$-action on each $\Lambda^k\mathfrak{h}^*$ (i.e. $h\cdot\alpha = \alpha(h^{-1}\cdot,\dots,h^{-1}\cdot)$), which is given for each $B\in\mathfrak{gl}(\mathfrak{h})$ by,

$$\theta(B)\gamma = \frac{d}{dt}\Big|_0 e^{tB}\cdot\gamma = -\left(\gamma(B\cdot,\dots,\cdot)+\cdots+\gamma(\cdot,\dots,B\cdot)\right), \quad \forall\gamma\in\Lambda^k\mathfrak{h}^*.$$

The following technical lemma contains some useful information on the linear algebra involved in subsequent computations.

Lemma 2.1. *Let $* : \Lambda^k\mathfrak{g}^* \longrightarrow \Lambda^{7-k}\mathfrak{g}^*$ and $*_{\mathfrak{h}} : \Lambda^k\mathfrak{h}^* \longrightarrow \Lambda^{6-k}\mathfrak{h}^*$ be the Hodge star operators determined by the ordered bases $\{e_1,\dots,e_7\}$ and $\{e_1,\dots,e_6\}$, respectively.*

(i) $*\gamma = *_{\mathfrak{h}}\gamma\wedge e^7$, *for any* $\gamma\in\Lambda^k\mathfrak{h}^*$.

(ii) $*(\gamma\wedge e^7) = (-1)^k *_{\mathfrak{h}}\gamma$, *for any* $\gamma\in\Lambda^k\mathfrak{h}^*$.

(iii) $*_{\mathfrak{h}}\omega = \frac{1}{2}\omega\wedge\omega$ *and* $*_{\mathfrak{h}}(\omega\wedge\omega) = 2\omega$.

(iv) $*^2 = id$ *and* $*_{\mathfrak{h}}^2 = (-1)^k id$ *on* $\Lambda^k\mathfrak{h}^*$.

(v) $*\varphi = \frac{1}{2}\omega\wedge\omega + \rho^-\wedge e^7 = e^{3456}+e^{1256}+e^{1234}-e^{2467}+e^{2357}+e^{1457}+e^{1367}$.

(vi) $\theta(A)*_{\mathfrak{h}} + *_{\mathfrak{h}}\theta(A^t) = -(\operatorname{tr}A)*_{\mathfrak{h}}$ *on* $\Lambda\mathfrak{h}^*$, *for any* $A\in\mathfrak{gl}(\mathfrak{h})$.

Proof. Parts (i)-(v) follow easily (see e.g. [L2, Lemmas 5.11, 5.12]) and to prove part (vi), we first recall that

$$\alpha\wedge *_{\mathfrak{h}}\beta = \langle\alpha,\beta\rangle *_{\mathfrak{h}} 1, \qquad *_{\mathfrak{h}}1 = e^1\wedge\cdots\wedge e^6, \qquad \forall\alpha,\beta\in\Lambda^k\mathfrak{h}^*.$$

Thus, for any $\alpha\in\Lambda^p\mathfrak{h}^*$ and $\beta\in\Lambda^{6-p}\mathfrak{h}^*$, one has

$$\begin{aligned}\langle\alpha,\theta(A)*_{\mathfrak{h}}\beta\rangle *_{\mathfrak{h}} 1 &= \langle\theta(A^t)\alpha, *_{\mathfrak{h}}\beta\rangle *_{\mathfrak{h}} 1 = \theta(A^t)\alpha\wedge *_{\mathfrak{h}}^2\beta = (-1)^p\theta(A^t)\alpha\wedge\beta\\ &= (-1)^{p+1}(\operatorname{tr}A)\alpha\wedge\beta + (-1)^{p+1}\alpha\wedge\theta(A^t)\beta\\ &= -(\operatorname{tr}A)\alpha\wedge *_{\mathfrak{h}} *_{\mathfrak{h}}\beta - \alpha\wedge *_{\mathfrak{h}} *_{\mathfrak{h}}\theta(A^t)\beta\\ &= \langle\alpha, -(\operatorname{tr}A)*_{\mathfrak{h}}\beta - *_{\mathfrak{h}}\theta(A^t)\beta\rangle *_{\mathfrak{h}} 1,\end{aligned}$$

concluding the proof of the lemma. □

Recall that $\theta(B)$ is a derivation of the algebra $\Lambda\mathfrak{h}^*$ and that $\theta(B)e^{1\cdots 6} = -(\operatorname{tr}B)e^{1\cdots 6}$. We consider the 14-dimensional simple Lie group

$$G_2 := \{h\in\operatorname{GL}_7(\mathbb{R}) : h\cdot\varphi = \varphi\}\subset\operatorname{SO}(7),$$

where φ is as in (2). Let $\mathfrak{g}_2$ denote the Lie algebra of G_2. The spaces of 2-forms and 3-forms on $\mathfrak{g}$ respectively decompose into irreducible G_2-representations as follows,

$$\Lambda^2\mathfrak{g}^* = \Lambda^2_7\mathfrak{g}^* \oplus \Lambda^2_{14}\mathfrak{g}^*, \qquad \Lambda^3\mathfrak{g}^* = \Lambda^3_1\mathfrak{g}^* \oplus \Lambda^3_7\mathfrak{g}^* \oplus \Lambda^3_{27}\mathfrak{g}^*,$$

where subscript numbers are the dimensions. A description of each of these irreducible components (see e.g. [B, (2.14)]) can be obtained by considering different suitable G_2-equivariant linear maps. For example, the kernel of the map $\Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^6\mathfrak{g}^*$, $\alpha \mapsto \alpha \wedge *\varphi$ must be $\Lambda^2_{14}\mathfrak{g}^*$. On the other hand, the map $\Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^*$, $\alpha \mapsto *(\alpha\wedge\varphi)$ is necessarily a multiple of the identity and one obtains that such a multiple is -1 by evaluating at $e^{12} - e^{34}$. This implies that

$$\Lambda^2_{14}\mathfrak{g}^* = \{\alpha \in \Lambda^2\mathfrak{g}^* : \alpha\wedge *\varphi = 0\} = \{\alpha \in \Lambda^2\mathfrak{g}^* : \alpha\wedge\varphi = - * \alpha\}. \tag{4}$$

Since $\Lambda^2_{14}\mathfrak{g}^*$ is, as a G_2-representation, equivalent to the adjoint representation $\mathfrak{g}_2$, any nonzero $\tau \in \Lambda^2_{14}\mathfrak{g}^*$ can be diagonalized, in the sense that there exists an oriented orthonormal basis $\{e_1, \dots, e_7\}$ of $\mathfrak{g}$ such that φ is as in (2) and

$$\tau = a\, e^{12} + b\, e^{34} + c\, e^{56}, \qquad a+b+c=0, \qquad a \geq b \geq 0 > c. \tag{5}$$

In particular,

$$\begin{aligned} \tau\wedge\tau &= 2ab\, e^{1234} + 2ac\, e^{1256} + 2bc\, e^{3456}, \\ \tau\wedge\tau\wedge\tau &= 6abc\, e^{123456}, \\ |\tau\wedge\tau| &= |\tau|^2 = a^2+b^2+c^2. \end{aligned} \tag{6}$$

2.2. The Lie group G_μ

Let $\mathfrak{g}$ be a Lie algebra of dimension 7 and assume that $\mathfrak{g}$ has a 6-dimensional ideal $\mathfrak{h}$. Consider a basis $\{e_1, \dots, e_7\}$ of $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{sp}\{e_1, \dots, e_6\}$, so $\mathfrak{g} = \mathfrak{h} \rtimes \mathbb{R}e_7$. The Lie bracket μ of $\mathfrak{g}$ is therefore given by

$$\mu = \lambda + \mu_A, \tag{7}$$

where λ is the Lie bracket of $\mathfrak{h}$ (extended to $\mathfrak{g}$ by $\lambda(\mathfrak{g}, e_7) = 0$) and μ_A is the Lie bracket defined for some $A \in \mathrm{Der}(\mathfrak{h})$ by

$$\mu_A(e_7, v) = Av, \qquad \mu_A(v,w) = 0, \qquad \forall v,w \in \mathfrak{h}.$$

Let G_μ denote the simply connected Lie group with Lie algebra $(\mathfrak{g}, \mu)$. Note that G_μ is solvable if and only if the Lie algebra $(\mathfrak{h}, \lambda)$ is solvable, and it is nilpotent if and only if $(\mathfrak{h}, \lambda)$ is nilpotent and A is a nilpotent linear map. Denote by H_λ the simply connected Lie group with Lie algebra $(\mathfrak{h}, \lambda)$ and by G_A the simply connected Lie group with Lie algebra $(\mathfrak{g}, \mu_A)$.

Some properties of the differentials of forms on these Lie groups are given in the following lemma.

Lemma 2.2. *Let d_μ, d_λ, d_A denote the differentials of left-invariant k-forms on the Lie groups G_μ, H_λ and G_A, respectively.*

(i) $d_\mu = d_\lambda + d_A$, *for any* $\gamma = \alpha + \beta \wedge e^7 \in \Lambda^k \mathfrak{g}^*$, $\alpha \in \Lambda^k \mathfrak{h}^*$, $\beta \in \Lambda^{k-1}\mathfrak{h}^*$,

$$d_\mu \gamma = d_\lambda \alpha + d_\lambda \beta \wedge e^7 + d_A \alpha,$$

and $d_A \alpha = (-1)^k \theta(A)\alpha \wedge e^7$.

(ii) $d_\mu e^7 = 0$, $d_\lambda e^7 = 0$, $d_A e^7 = 0$ *and* $d_A(\alpha \wedge e^7) = 0$, *for all* $\alpha \in \Lambda^k \mathfrak{h}^*$.

(iii) $d_\lambda \circ \theta(D) = \theta(D) \circ d_\lambda$ *for any* $D \in \mathrm{Der}(\mathfrak{h})$.

(iv) $d_\mu * e^i = (-1)^i \operatorname{tr}(\operatorname{ad}_\mu e_i) e^{1\dots7}$, *for any* $i = 1, \dots, 7$.

(v) $d_\mu^*|_{\Lambda^k \mathfrak{g}^*} = (-1)^{k+1} * d_\mu *$ *and* $d_\lambda^*|_{\Lambda^k \mathfrak{h}^*} = * d_\lambda *$ *for any* k.

Proof. Parts (i) and (ii) clearly hold (see e.g. [L2, Lemma 5.12]), parts (iv) and (v) are straightforward computations and part (iii) follows from the fact that $\theta(D)$ is precisely minus the Lie derivative $\mathcal{L}_{X_D}$, where X_D is the vector field on H_λ attached to D. □

2.3. Subgroups of G_2

In our study of ERP G_2-structures in Section 4, we need to compute the stabilizer of the 2-form $\tau := e^{12} - e^{56}$ in G_2, as well as inside the subgroup $U_\mathfrak{h} \subset G_2$ leaving $\mathfrak{h}$ invariant. It is well known (see e.g. [VM, Lemma 2.2.2]) that

$$U_\mathfrak{h} := \{h \in G_2 : h(\mathfrak{h}) \subset \mathfrak{h}\} = \left[\begin{array}{c|c} 1 & \\ \hline & \mathrm{SU}(3) \end{array}\right] \cup \left[\begin{array}{c|c} 1 & \\ \hline & \mathrm{SU}(3) \end{array}\right] \tilde{g}, \tag{8}$$

where $\tilde{g} = \mathrm{Dg}(-1, 1, -1, 1, -1, 1, -1)$ and $\mathrm{SU}(3)$ is defined by using J. Any matrix in this section will be written in terms of the basis $\{e_7, e_3, e_4, e_1, e_2, e_5, e_6\}$.

Lemma 2.3. *The subgroup of* G_2

$$U_{\mathfrak{h},\tau} := \{h \in G_2 : h(\mathfrak{h}) \subset \mathfrak{h},\ h \cdot \tau = \tau\},$$

is given by $U_{\mathfrak{h},\tau} = U_0 \cup U_0 g$, *where*

$$U_0 := \left\{ \begin{bmatrix} 1 &&& \\ & h_1 && \\ && h_2 & \\ &&& h_3 \end{bmatrix} : h_i \in \mathrm{SO}(2),\ h_1 h_2 h_3 = I \right\}, \quad g := \begin{bmatrix} -1 &&&&&& \\ & 1 &&&&& \\ && -1 &&&& \\ &&&&& 1 & 0 \\ &&&&& 0 & -1 \\ &&& -1 & 0 && \\ &&& 0 & 1 && \end{bmatrix}.$$

Proof. Since for any $h \in \mathrm{O}(7)$, $h \cdot \tau = \tau$ if and only if

$$hJ_1h^{-1} = J_1, \qquad J_1 := \begin{bmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 &&&& \\ &&& 0 & -1 && \\ &&& 1 & 0 && \\ &&&&& 0 & 1 \\ &&&&& -1 & 0 \end{bmatrix},$$

it is not hard to see that

$$U_0 = U_{\mathfrak{h},\tau} \cap \left[\begin{array}{c|c} 1 & \\ \hline & \mathrm{SU}(3) \end{array}\right].$$

On the other hand, by using that $\tilde{g} \in G_2$ and $\tilde{g} \cdot \tau = -\tau$ (see (8)), we obtain that

$$g = \begin{bmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 1 &&&& \\ &&&&& 1 & 0 \\ &&&&& 0 & 1 \\ &&& -1 & 0 && \\ &&& 0 & -1 && \end{bmatrix} \tilde{g} \in U_{\mathfrak{h},\tau} \cap \left[\begin{array}{c|c} 1 & \\ \hline & \mathrm{SU}(3) \end{array}\right] \tilde{g}.$$

Now if $h = f\tilde{g} \in U_{\mathfrak{h},\tau}$, where $fe_7 = e_7$ and $f_0 := f|_{\mathfrak{h}} \in \mathrm{SU}(3)$, then

$$f_0 = \begin{bmatrix} f_1 && \\ & 0 & f_2 \\ & f_3 & 0 \end{bmatrix}, \qquad f_1 f_2 f_3 = -1,$$

as f_0 must commute with $J|_{\mathfrak{h}}$ and $J_1|_{\mathfrak{h}}$, and therefore,

$$h = \begin{bmatrix} 1 &&& \\ & f_1 && \\ && f_2 & \\ &&& -f_3 \end{bmatrix} g \in U_0 g, \quad \text{that is,} \quad U_{\mathfrak{h},\tau} \cap \left[\begin{array}{c|c} 1 & \\ \hline & \mathrm{SU}(3) \end{array}\right] \tilde{g} = U_0 g,$$

concluding the proof. □

Other subgroups of G_2 we will need to consider are

$$U_{\mathfrak{g}_1} := \{h \in G_2 : h(\mathfrak{g}_1) \subset \mathfrak{g}_1\}, \tag{9}$$

where $\mathfrak{g}_1 := \mathrm{sp}\{e_1, e_2, e_5, e_6\}$, whose Lie algebra is well-known (see e.g. [VM]) to be given by

$$\mathfrak{u}_{\mathfrak{g}_1} = \left\{ \begin{bmatrix} 0 & c & -b & & & & \\ -c & 0 & a & & & & \\ b & -a & 0 & & & & \\ & & & 0 & -d & b-e & -f \\ & & & d & 0 & -c+f & -e \\ & & & -b+e & c-f & 0 & -a+d \\ & & & f & e & a-d & 0 \end{bmatrix} : a,b,c,d,e,f \in \mathbb{R} \right\},$$

and the corresponding subgroup stabilizing τ,

$$U_{\mathfrak{g}_1,\tau} := \{h \in G_2 : h(\mathfrak{g}_1) \subset \mathfrak{g}_1,\ h \cdot \tau = \tau\}, \tag{10}$$

with Lie algebra,

$$\mathfrak{u}_{\mathfrak{g}_1,\tau} = \left\{ \begin{bmatrix} 0 & c & -b & & & & \\ -c & 0 & a & & & & \\ b & -a & 0 & & & & \\ & & & 0 & -d & \frac{1}{2}b & -\frac{1}{2}c \\ & & & d & 0 & -\frac{1}{2}c & -\frac{1}{2}b \\ & & & -\frac{1}{2}b & \frac{1}{2}c & 0 & -a+d \\ & & & \frac{1}{2}c & \frac{1}{2}b & a-d & 0 \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}.$$

3. Closed G_2-structures

A G_2-*structure* on a 7-dimensional differentiable manifold M is a differential 3-form $\varphi \in \Omega^3 M$ such that φ_p is *positive* on T_pM for any $p \in M$, that is, φ_p can be written as in (2) with respect to some basis $\{e_1, \dots, e_7\}$ of T_pM. Each G_2-structure φ defines a Riemannian metric g on M and an orientation $\mathrm{vol} \in \Omega^7 M$ (unique up to scaling) as in (3). Thus φ also determines a Hodge star operator $* : \Omega M \longrightarrow \Omega M$ and the *Hodge Laplacian operator* $\Delta : \Omega^k M \longrightarrow \Omega^k M$, $\Delta := d^*d + dd^*$, where $d^* : \Omega^{k+1} M \longrightarrow \Omega^k M$, $d^* = (-1)^{k+1} * d *$, is the adjoint of d. The *torsion forms* of a G_2-structure φ on M are the components of the *intrinsic torsion* $\nabla\varphi$, where ∇ is the Levi-Civita connection of the metric g. They can be defined as the unique differential forms $\tau_i \in \Omega^i M$, $i = 0, 1, 2, 3$, such that

$$d\varphi = \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \qquad d * \varphi = 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi. \tag{11}$$

Two manifolds endowed with G_2-structures (M, φ) and (M', φ') are called *equivalent* if there exists a diffeomorphism $f : M \longrightarrow M'$ such that $\varphi = f^*\varphi'$.

In the case of a closed G_2-structure φ on a 7-manifold M, the only torsion that survives is a the 2-form $\tau := \tau_2$ and one therefore has that,

$$(12) \quad d\varphi = 0, \qquad \tau = d^*\varphi = - * d * \varphi, \qquad d * \varphi = \tau \wedge \varphi, \qquad d\tau = \Delta\varphi.$$

In particular, φ is torsion-free (or parallel) if and only if $\tau = 0$. Since $\tau \in \Omega^2_{14}M$ (see e.g. [B, Proposition 1]), all the useful conditions (4)-(6) hold for τ at each $p \in M$.

In this paper, we study left-invariant G_2-structures on Lie groups, which allows us to work at the Lie algebra level as in Section 2. The G_2-structure is determined by a positive 3-form on the Lie algebra $\mathfrak{g}$, which will be most of the times the one given in (2). Two Lie groups endowed with left-invariant G_2-structures (G, φ) and (G', φ') are called *equivariantly equivalent* if there exists a Lie group isomorphism $F : G \longrightarrow G'$ such that $\varphi = f^*\varphi'$, where $f := dF|_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is the corresponding Lie algebra isomorphism.

Definition 3.1. (G_μ, φ) is the Lie group G_μ defined in Section 2.2 endowed with the left-invariant G_2-structure determined by the positive 3-form φ on $\mathfrak{g}$ given in (2).

Recall from (7) that G_μ depends only on the Lie bracket λ on $\mathfrak{h} = \mathrm{sp}\{e_1, \dots, e_6\}$ and the map $A \in \mathrm{Der}(\mathfrak{h}, \lambda)$.

Proposition 3.2. *Any Lie group endowed with a left-invariant closed G_2-structure is equivariantly equivalent to (G_μ, φ) for some $\mu = \lambda + \mu_A$ such that the ideal $(\mathfrak{h}, \lambda)$ is unimodular.*

Remark 3.3. If the Lie group is not unimodular, then $\mathfrak{h} = \{X \in \mathfrak{g} : \operatorname{tr} \operatorname{ad}_\mu X = 0\}$. On the other hand, the pair (ω, ρ^+) defines an SU(3)-structure on the Lie algebra $\mathfrak{h}$.

Proof. Let (G, ψ) be a Lie group G endowed with a closed G_2-structure ψ. If $\mathfrak{g}$ is not unimodular, then we can take the codimension-one ideal $\mathfrak{h}$ of $\mathfrak{g}$ as above and in the case when $\mathfrak{g}$ is unimodular, it follows from the classification obtained in [FR3, Main Theorem] that there exists a codimension-one ideal $\mathfrak{h}$. Thus there exists an orthonormal basis $\{e_1, \dots, e_7\}$ of $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{sp}\{e_1, \dots, e_6\}$ and φ can be written as in (2). It therefore follows that if $\lambda := \mu|_{\mathfrak{h}\times\mathfrak{h}}$ and $A := \operatorname{ad} e_7|_{\mathfrak{h}}$, then the Lie bracket μ of $\mathfrak{g}$ is given by $\mu = \lambda + \mu_A$, concluding the proof. □

We therefore focus from now on on G_2-structures of the form (G_μ, φ). According to Lemma 2.2, (i), for (G_μ, φ) one has that,

$$d_\mu\varphi = d_\lambda\rho^+ + d_\lambda\omega \wedge e^7 - \theta(A)\rho^+ \wedge e^7, \tag{13}$$

$$d_\mu * \varphi = d_\lambda\omega \wedge \omega + d_\lambda\rho^- \wedge e^7 + \theta(A)\omega \wedge \omega \wedge e^7. \tag{14}$$

Thus (G_μ, φ) is closed if and only if the following two conditions hold:

$$d_\lambda\omega = \theta(A)\rho^+, \qquad d_\lambda\rho^+ = 0. \tag{15}$$

We now compute the torsion in terms of λ and A, which is the only data varying.

Proposition 3.4. *The torsion* 2*-form* τ_μ *of a closed* G_2*-structure* (G_μ, φ) *is given by* $\tau_\mu = \tau_\lambda + \tau_A$*, where*

$$\tau_\lambda := - *_{\mathfrak{h}} (d_\lambda\omega \wedge \omega) \wedge e^7 - *_{\mathfrak{h}} d_\lambda\rho^-, \qquad \tau_A := (\operatorname{tr} A)\omega + \theta(A^t)\omega.$$

Furthermore, $d_\lambda\omega \wedge \omega = -\theta(A)\omega \wedge \rho^+$*.*

Proof. It follows from (2) that

$$\begin{aligned} *d_A * \varphi &= * \left(\frac{1}{2}\theta(A)(\omega \wedge \omega) \wedge e^7\right) = * \left(\theta(A) *_{\mathfrak{h}} \omega \wedge e^7\right) \\ &= * \left(-(\operatorname{tr} A) *_{\mathfrak{h}} \omega \wedge e^7 - *_{\mathfrak{h}}\theta(A^t)\omega \wedge e^7\right) = -(\operatorname{tr} A)\omega - \theta(A^t)\omega, \end{aligned}$$

and

$$d_\lambda\omega \wedge \omega = \theta(A)\rho^+ \wedge \omega = \theta(A)\omega \wedge \rho^+.$$

Using Lemma 2.1, (v) we now compute,

$$\begin{aligned} d_\mu * \varphi &= d_\lambda\omega \wedge \omega + d_\lambda\rho^- \wedge e^7 + d_A * \varphi, \\ *d_\mu * \varphi &= * (d_\lambda\omega \wedge \omega) + *_{\mathfrak{h}} d_\lambda\rho^- + *d_A * \varphi \\ &= *_{\mathfrak{h}} (d_\lambda\omega \wedge \omega) \wedge e^7 + *_{\mathfrak{h}} d_\lambda\rho^- - (\operatorname{tr} A)\omega - \theta(A^t)\omega, \end{aligned}$$

from which the desired formula follows. □

Straightforwardly, one obtains that the above proposition and Lemma 2.2 give that for any closed G_2-structure (G_μ, φ),

$$\begin{aligned} d_\mu \tau_\mu = & -d_\lambda *_\mathfrak{h} (d_\lambda \omega \wedge \omega) \wedge e^7 - d_\lambda *_\mathfrak{h} d_\lambda *_\mathfrak{h} \rho^+ - \theta(A) *_\mathfrak{h} d_\lambda \rho^- \wedge e^7 \\ & + (\operatorname{tr} A) d_\lambda \omega + (\operatorname{tr} A)\theta(A)\omega \wedge e^7 + \theta(A)\theta(A^t)\omega \wedge e^7 \\ & + d_\lambda \theta(A^t)\omega. \end{aligned} \tag{16}$$

The following result shows that two left-invariant G_2-structures on non-isomorphic Lie groups can indeed be equivalent, in spite of they are not equivariantly equivalent. This generalizes [L2, Proposition 5.6] beyond the almost-abelian case and the proof also follows the lines of [H, Proposition 2.5].

Proposition 3.5. *Let (G_μ, φ) be a G_2-structure as above with $\mu = \lambda + \mu_A$. If $D \in \mathfrak{su}(3) \cap \operatorname{Der}(\mathfrak{h}, \lambda)$, $[D, A] = 0$ and we set $\mu_1 := \lambda + \mu_{A+D}$, then the G_2-structures (G_μ, φ) and (G_{μ_1}, φ) are equivalent.*

Remark 3.6. The hypothesis on the matrix D means precisely that

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}_2 \cap \operatorname{Der}(\mathfrak{g}, \mu), \ \mathfrak{g}_2 \cap \operatorname{Der}(\mathfrak{g}, \mu_1).$$

Note that the Lie groups G_μ, G_{μ_1} are in general not isomorphic. For instance, if μ is not unimodular, then the spectra of D and A must coincide up to scaling in order to μ and μ_1 be isomorphic.

Proof. Let $\mathfrak{g}_\mu$ denote the Lie algebra $(\mathfrak{g}, \mu)$ of G_μ. We consider the Lie group

$$F := \operatorname{Aut}(G_{\mu_1}) \cap \operatorname{Aut}(G_{\mu_1}, \varphi) \simeq \operatorname{Aut}(\mathfrak{g}_{\mu_1}) \cap G_2,$$

with Lie algebra $\mathfrak{f} := \operatorname{Der}(\mathfrak{g}_{\mu_1}) \cap \mathfrak{g}_2$, the homomorphism $\alpha : \mathfrak{g}_{\mu_1} \longrightarrow \mathfrak{f}$ defined by

$$\alpha(e_7) = \begin{bmatrix} -D & 0 \\ 0 & 0 \end{bmatrix}, \qquad \alpha|_\mathfrak{h} \equiv 0,$$

and denote also by α the corresponding Lie group homomorphism $G_{\mu_1} \longrightarrow F$. If $L : G_{\mu_1} \longrightarrow \operatorname{Aut}(G_{\mu_1}, \varphi)$ is the left-multiplication morphism, then

$$G_1 := \{L_s \circ \alpha(s) : s \in G_{\mu_1}\} \subset \operatorname{Aut}(G_{\mu_1}, \varphi),$$

is a subgroup. Indeed, using that $G_{\mu_1} = \exp \mathbb{R}e_7 \ltimes \exp \mathfrak{h}$, one has for $s = ah$, $t = bg$ that,

$$\begin{aligned} L_s \circ \alpha(s) \circ L_t \circ \alpha(t) &= L_s \circ L_{\alpha(s)(t)}\alpha(s)\alpha(b) = L_{s\alpha(s)(t)}\alpha(s)\alpha(b\alpha(a)(g)) \\ &= L_{s\alpha(s)(t)}\alpha(s)\alpha(\alpha(a)(b)\alpha(a)(g)) = L_{s\alpha(s)(t)}\alpha(s\alpha(s)(t)). \end{aligned}$$

Thus G_1 is a connected and closed Lie subgroup of $\mathrm{Aut}(G_{\mu_1}, \varphi)$ since $s \mapsto L_s \circ \alpha(s)$ is continuous and proper. Furthermore, G_1 acts simply and transitively on G_{μ_1} by automorphisms of φ, so the diffeomorphism $f : G_1 \longrightarrow G_{\mu_1}$, $f(L_s \circ \alpha(s)) := (L_s \circ \alpha(s))(e) = s$ defines an equivalence between the left-invariant G_2-structures $(G_1, f^*\varphi)$ and (G_{μ_1}, φ). On the other hand, the Lie algebra of G_1 is given by

$$\mathfrak{g}_0 := \{dL|_e X + \alpha(X) : X \in \mathfrak{g}\} \subset \mathrm{Lie}\left(\mathrm{Aut}(G_{\mu_1}, \varphi)\right),$$

and if $X = X_\mathfrak{h} + ae_7$, $Y = Y_\mathfrak{h} + be_7$ belong to $\mathfrak{g}$, then

$$\begin{aligned} &[dL|_e X + \alpha(X), dL|_e Y + \alpha(Y)] \\ &\quad = dL|_e \mu_1(X,Y) + dL|_e \alpha(X) Y - dL|_e \alpha(Y) X + \alpha([X,Y]) \\ &\quad = dL|_e \left(a(A+D)Y_\mathfrak{h} - b(A+D)X_\mathfrak{h} + \lambda(X_\mathfrak{h}, Y_\mathfrak{h}) - aDY_\mathfrak{h} + bDX_\mathfrak{h} + 0\right) \\ &\quad = dL|_e \left(aDY_\mathfrak{h} - bDX_\mathfrak{h} + \lambda(X_\mathfrak{h}, Y_\mathfrak{h})\right) \\ &\quad = dL|_e \mu(X,Y) = (dL|_e + \alpha)\mu(X,Y). \end{aligned}$$

This shows that $df|_e^{-1} = dL|_e + \alpha : \mathfrak{g}_\mu \longrightarrow \mathfrak{g}_0$ is a Lie algebra isomorphism and so (G_μ, φ) is equivalent to $(G_1, f^*\varphi)$, concluding the proof. □

Remark 3.7. By replacing φ by an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, $\mathrm{Aut}(G_{\mu_1}, \varphi)$ by $\mathrm{Iso}(G_{\mu_1}, \langle\cdot,\cdot\rangle)$ and G_2 by $\mathrm{O}(\mathfrak{g}, \langle\cdot,\cdot\rangle)$, the following Riemannian version can be proved in exactly the same way as above for any dimension: $(G_\mu, \langle\cdot,\cdot\rangle)$ is isometric to $(G_{\mu_1}, \langle\cdot,\cdot\rangle)$ for any $\mu = \lambda + \mu_A$, $\mu_1 = \lambda + \mu_{A+D}$ such that $D \in \mathfrak{so}(\mathfrak{h}, \langle\cdot,\cdot\rangle) \cap \mathrm{Der}(\mathfrak{h}, \lambda)$ and $[D, A] = 0$.

As an application of Proposition 3.5, one obtains that the one-parameter family of extremally Ricci pinched G_2-structures given in [FR2, Example 6.4] is pairwise equivalent.

3.1. ERP G_2-structures

The following nice estimate for a closed G_2-structure φ on a compact manifold M was proved by R. Bryant (see [B, Corollary 3]):

$$\int_M \mathrm{scal}^2 *1 \leq 3 \int_M |\operatorname{Ric}|^2 * 1,$$

and equality holds if and only if

$$d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau). \tag{17}$$

The factor of 3 on the right hand side, being much smaller than 7, shows that the metric is always far from being Einstein.

Definition 3.8. The distinguished closed G_2-structures for which condition (17) holds and $\tau \neq 0$ were called *extremally Ricci-pinched* (ERP for short) in [B, Remark 13].

We begin with some general results on such structures.

Proposition 3.9. *[B] Let (M, φ) be a manifold endowed with an ERP G_2-structure and assume that it is locally homogeneous. Then,*

(i) $\tau \wedge \tau \wedge \tau = 0$.

(ii) $d(\tau \wedge \tau) = 0$.

(iii) $d * (\tau \wedge \tau) = 0$.

(iv) $\operatorname{Ric}|_P = -\frac{1}{6}|\tau|^2 I$, $\operatorname{Ric}|_Q = 0$ *and* $\langle \operatorname{Ric} P, Q\rangle = 0$, *where*

$$P := \{X \in TM : \iota_X(\tau \wedge \tau) = 0\}, \qquad Q := \{X \in TM : \iota_X * (\tau \wedge \tau) = 0\},$$

and $\dim P = 3$, $\dim Q = 4$.

Proof. Parts (i), (ii) and (iii) follow from [B, (4.53)], [B, (4.55)] and [B, (4.51)], respectively, and the fact that $d|\tau|^2 = 0$ since M is locally homogeneous. If we write τ as in (5) at each $p \in M$, then $b = 0$ and $c = -a$ must hold, since $\tau \wedge \tau \wedge \tau = 0$ by (i), and so $\tau = a(e^{12} - e^{56})$. To prove part (iv), we consider the formula given in [L3, (16)] for $q = \frac{1}{6}$, so in terms of the ordered basis $\{e_7, e_3, e_4, e_1, e_2, e_5, e_6\}$,

$$\begin{aligned}\mathrm{Ric} &= -\frac{1}{6}|\tau|^2 I - \frac{1}{3}\tau^2 = -\frac{1}{3}a^2 I - \frac{1}{3}\,\mathrm{Dg}(-a^2, -a^2, 0, 0, -a^2, -a^2, 0)\\ &= -\frac{a^2}{3}\,\mathrm{Dg}(0,0,1,1,0,0,1) = -\frac{1}{6}|\tau|^2\,\mathrm{Dg}(0,0,1,1,0,0,1).\end{aligned}$$

As $\tau \wedge \tau = -2ae^{1256}$ and $*(\tau \wedge \tau) = -2ae^{347}$, it follows that $P = \mathrm{sp}\{e_7, e_3, e_4\}$, $Q = \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ and therefore (iv) holds, concluding the proof. □

4. Structure

Our aim in this section is to discover and prove structural results for extremally Ricci-pinched G_2-structures on Lie groups.

Recall from Section 3 that the Lie groups endowed with a G_2-structure of the form (G_μ, φ) (see Definition 3.1) cover the whole closed case up to equivariant equivalence (see Proposition 3.2). The Lie algebra of G_μ decomposes as $\mathfrak{g} = \mathbb{R}e_7 \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathrm{sp}\{e_1, \ldots, e_6\}$ is a unimodular ideal, and φ is always given as in (2).

The following proposition shows that under the ERP condition, the torsion 2-form can be diagonalized in a very convenient way relative to the Lie algebra structure, which is certainly the starting point toward the structure results we will obtain in this section.

Proposition 4.1. *Any Lie group endowed with a left-invariant ERP G_2-structure is equivariantly equivalent to (G_μ, φ), up to scaling, for some $\mu = \lambda + \mu_A$ with $(\mathfrak{h}, \lambda)$ unimodular and $\tau_\mu = e^{12} - e^{56}$.*

Remark 4.2. The SU(3)-structure (ω, ρ^+) on the Lie algebra $\mathfrak{h}$ is therefore *half-flat*, i.e. $d_\lambda\omega \wedge \omega = 0$ and $d_\lambda\rho^+ = 0$ (see Proposition 3.4 and (15)).

Proof. Let (G, φ) be a Lie group endowed with a left-invariant ERP G_2-structure. Consider the basis $\{e_1, \ldots, e_7\}$ of the Lie algebra $\mathfrak{g}$ of G such that φ has the form (2). Since the torsion form of (G, φ) τ belongs to $\Lambda^2_{14}\mathfrak{g}^*$, it follows from (5) and Proposition 3.9, (i) that it can be assumed to be given by $\tau = e^{12} - e^{56}$ (up to scaling). As a first consequence, $de^{347} = 0$ by

Proposition 3.9, (iii) and so if $c_{ijk} := \langle [e_i, e_j], e_k\rangle$, then

$$\begin{aligned} 0 = de^{347} &= de^3 \wedge e^{47} - e^3 \wedge de^4 \wedge e^7 + e^{34} \wedge de^7 \\ &= -\sum_{i=1,2,5,6} c_{i33} e^{i347} - \sum_{i=1,2,5,6} c_{i44} e^{i347} - \sum_{i=1,2,5,6} c_{i77} e^{i347} \\ &= -\sum_{i=1,2,5,6} \operatorname{tr} \operatorname{ad} e_i|_{\mathfrak{g}_0} e^{i347}, \end{aligned}$$

which implies that

$$\operatorname{tr} \operatorname{ad} e_i|_{\mathfrak{g}_0} = 0, \qquad \forall i = 1, 2, 5, 6, \tag{18}$$

where $\mathfrak{g}_0 := \operatorname{sp}\{e_3, e_4, e_7\}$ is a Lie subalgebra by Proposition 3.9, (ii). On the other hand, $\mathfrak{g}_1 := \operatorname{sp}\{e_1, e_2, e_5, e_6\}$ is also a subalgebra (see Proposition 3.9, (iii)) and hence by using Lemma 2.2, (i), we obtain that

$$d\tau = (\theta(\operatorname{ad} e_3|_{\mathfrak{g}_1})\tau) \wedge e^3 + (\theta(\operatorname{ad} e_4|_{\mathfrak{g}_1})\tau) \wedge e^4 + (\theta(\operatorname{ad} e_7|_{\mathfrak{g}_1})\tau) \wedge e^7 + d_{\mathfrak{g}_1}\tau,$$

where $d_{\mathfrak{g}_1} : \Lambda^k \mathfrak{g}_1^* \to \Lambda^{k+1}\mathfrak{g}_1^*$ denotes the exterior derivative of $\mathfrak{g}_1$. But the ERP condition on (G, φ) reads $d\tau = \frac{1}{3}\varphi - \frac{1}{3}e^{347}$, so it follows from (1) that $d_{\mathfrak{g}_1}\tau = 0$ and

$$\theta(\operatorname{ad} e_3|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_3, \qquad \theta(\operatorname{ad} e_4|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_4, \qquad \theta(\operatorname{ad} e_7|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_7.$$

This implies that the 2-forms $\tau, \omega_3, \omega_4, \omega_7$ are all closed on the 4-dimensional Lie algebra $\mathfrak{g}_1$ by using that the maps $\operatorname{ad} e_i|_{\mathfrak{g}_1}$ are derivations of $\mathfrak{g}_1$ (see Lemma 2.2, (iii)), from which it is easy to see with the help of a computer that $\mathfrak{g}_1$ is abelian. From this and (18), we obtain that $\mathfrak{g}_1$ is contained in the ideal $\mathfrak{u}$ of $\mathfrak{g}$ given by $\mathfrak{u} := \{X \in \mathfrak{g} : \operatorname{tr} \operatorname{ad} X = 0\}$. If G is not unimodular, then $\mathfrak{g} = \mathbb{R}X_0 \oplus \mathfrak{u}$ is an orthogonal decomposition for some $X_0 \in \langle e_3, e_4, e_7\rangle$, $|X_0| = 1$. It follows that there exists h in the group $U_{\mathfrak{g}_1,\tau}$ given in (10) such that $h(X_0) = e_7$. The map h therefore defines an equivariant equivalence between (G, φ) and (G_μ, φ), where $\mu := h \cdot [\cdot, \cdot]$, and we have that $h(\mathfrak{u}) = \mathfrak{h}$ (since h is orthogonal) and $\tau_\mu = h \cdot \tau = \tau$.

In the case when G is unimodular, it is proved in [FR2, Theorem 6.7] that $\mathfrak{g}$ must be isomorphic to certain solvable Lie algebra. In the present proof, we only use that $\mathfrak{g}$ is solvable and we argue as in the beginning of the proof of [FR2, Theorem 6.7]. Recall from Proposition 3.9, (iv) that $\operatorname{Ric} \leq 0$ and the kernel of Ric is $\mathfrak{g}_1$. The nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is therefore contained in $\mathfrak{g}_1$ by [D, Lemma 1] and since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. Hence $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, concluding the proof. $\square$

The following example shows that the above proposition is not valid in general for closed G_2-structures.

Example 4.3. Consider (G_μ, φ) with $\lambda(e_1, e_2) = e_3$, $\lambda(e_2, e_3) = 4e_5$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 \end{bmatrix},$$

with respect to the basis $\{e_1, \dots, e_6\}$. It is straightforward to check that $d_\mu\varphi = 0$ and $\tau_\mu = -2e^{12} - e^{16} - 4e^{34} - e^{37} + 6e^{56}$. Since $\mathfrak{h}$ is the nilradical of μ, the torsion 2-form τ_{μ_1} of any (G_{μ_1}, φ) equivariantly equivalent to (G_μ, φ) will satisfy that $\tau_{\mu_1}(e_7, \cdot)$ is not identically zero. Indeed, any orthogonal isomorphism between G_μ and G_{μ_1} must stabilize both $\mathfrak{h}$ and $\mathbb{R}e_7$.

The diagonalization of τ obtained in Proposition 4.1 makes the equivalence problem much simpler to tackle. Recall the subgroups $U_{\mathfrak{h},\tau}$ and $U_{\mathfrak{g}_1,\tau}$ of G_2 described in Section 2.3.

Proposition 4.4. *Assume that (G_{μ_1}, φ) and (G_{μ_2}, φ) have $\tau_{\mu_1} = \tau_{\mu_2} = e^{12} - e^{56}$. Then they are equivariantly equivalent if and only if $\mu_2 = h \cdot \mu_1$ for some $h \in U \subset G_2$, where*

(i) *$U = U_{\mathfrak{h},\tau}$ if they are not unimodular; and*

(ii) *$U = U_{\mathfrak{g}_1,\tau}$ if they are unimodular and $\mathfrak{g}_1 = \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ is their nilradical.*

Proof. In the non-unimodular case (i), $\mathfrak{h}$ is a characteristic ideal of both Lie algebras by Proposition 3.2 and so any equivariant equivalence h between them must leave $\mathfrak{h}$ invariant and stabilize τ, that is, $h \in U_{\mathfrak{h},\tau}$. On the other hand, part (ii) follows from the fact that h must leave $\mathfrak{g}_1$ invariant (i.e. $h \in U_{\mathfrak{g}_1}$) being $\mathfrak{g}_1$ the nilradical of both Lie algebras, and so $h \in U_{\mathfrak{g}_1,\tau}$ since $h \cdot \tau = \tau$ (see Section 2.3).

The converse easily follows from the fact that $U \subset G_2$. □

In the light of Proposition 4.1, we consider from now on a closed G_2-structure (G_μ, φ) such that

$$\tau := \tau_\mu = e^{12} - e^{56}. \tag{19}$$

In that case,

$$d_\lambda\omega \wedge \omega = 0, \tag{20}$$

by Proposition 3.4, $\tau \wedge \tau = -2e^{1256}$ and $*(\tau \wedge \tau) = -2e^{347}$. This implies that (G_μ, φ) is ERP if and only if

$$d_\mu \tau = \frac{1}{3}\varphi - \frac{1}{3}e^{347}, \tag{21}$$

which is equivalent by Lemma 2.2, (i) to

$$d_\lambda \tau = \frac{1}{3}\rho^+, \qquad \theta(A)\tau = \frac{1}{3}(e^{12} + e^{56}). \tag{22}$$

It follows from (15) and Lemma 2.2, (iii) that

$$\begin{aligned} d_\lambda \omega &= d_\lambda(e^{12} + e^{56}) + d_\lambda e^{34} = 3 d_\lambda \theta(A)\tau + d_\lambda e^{34} \\ &= 3\theta(A) d_\lambda \tau + d_\lambda e^{34} = \theta(A)\rho^+ + d_\lambda e^{34} = d_\lambda \omega + d_\lambda e^{34}, \end{aligned}$$

and consequently,

$$d_\lambda e^{34} = 0. \tag{23}$$

Some algebraic and geometric consequences of Proposition 3.9 follow.

Proposition 4.5. *If (G_μ, φ) is ERP with $\tau = e^{12} - e^{56}$, then,*

(i) $\mathfrak{g}_0 := \mathrm{sp}\{e_7, e_3, e_4\}$, $\mathfrak{g}_1 := \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ *and* $\mathfrak{h}_1 := \mathrm{sp}\{e_3, e_4\}$ *are Lie subalgebras of* $\mathfrak{g}$.

(ii) *The Ricci operator* Ric_μ *of* $(G_\mu, \langle\cdot,\cdot\rangle)$ *is diagonal with respect to* $\{e_i\}$ *and* $\mathrm{Ric}_\mu|_{\mathfrak{g}_0} = -\frac{1}{3}I$, $\mathrm{Ric}_\mu|_{\mathfrak{g}_1} = 0$.

(iii) *If Q_μ is the unique symmetric operator of $\mathfrak{g}$ such that $\theta(Q_\mu)\varphi = d_\mu \tau$, then*

$$\mathrm{Ric}_\mu = -\frac{1}{3}I - 2Q_\mu; \qquad \textit{in particular,} \quad Q_\mu|_{\mathfrak{g}_0} = 0, \quad Q_\mu|_{\mathfrak{g}_1} = -\frac{1}{6}I.$$

Remark 4.6. It follows from part (iii) that $Q_\mu \in \mathrm{Der}(\mathfrak{g})$, and in particular (G_μ, φ) is a steady Laplacian soliton (see [L2, Theorem 3.8]) and $(G_\mu, \langle\cdot,\cdot\rangle)$ is an expanding Ricci soliton (see [L1, (5)]), if and only if $\mathfrak{g}_1$ is an abelian ideal of $\mathfrak{g}$.

Proof. It is well known that the kernel of any closed k-form on a Lie algebra is a Lie subalgebra. Since $\tau \wedge \tau = -2e^{1256}$ and $*(\tau \wedge \tau) = -2e^{347}$, it follows from Proposition 3.9 (ii), (iii) that $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are Lie subalgebras of $\mathfrak{g}$. In particular, $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{g}_0$ is also a subalgebra. Parts (ii) and (iii) are direct consequences of [L3, (15)] (for $q = \frac{1}{6}$) and [L3, (12)]. □

We will now show that the ERP condition actually imposes much stronger constraints on the structure of the Lie algebra. Let us first introduce some notation. Consider

$$\mathfrak{sp}(\mathfrak{g}_1,\tau) := \{E \in \mathfrak{gl}(\mathfrak{g}_1) : -\theta(E)\tau = \tau(E\cdot,\cdot) + \tau(\cdot,E\cdot) = 0\}, \quad \tau = e^{12} - e^{56},$$

and note that $E \in \mathfrak{sp}(\mathfrak{g}_1,\tau)$ if and only if written in terms of the basis $\{e_1, e_2, e_5, e_6\}$,

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{15} & E_{16} \\ E_{21} & -E_{11} & E_{25} & E_{26} \\ E_{26} & -E_{16} & E_{55} & E_{56} \\ -E_{25} & E_{15} & E_{65} & -E_{55} \end{bmatrix}. \tag{24}$$

We also consider the following three matrices,

$$T_7 := \begin{bmatrix} -\frac{1}{3} & & & \\ & 0 & & \\ & & \frac{1}{3} & \\ & & & 0 \end{bmatrix}, \quad T_3 := \begin{bmatrix} & & & \frac{1}{3} \\ & & 0 & \\ & 0 & & \\ \frac{1}{3} & & & \end{bmatrix}, \quad T_4 := \begin{bmatrix} & & 0 & \\ & & & -\frac{1}{3} \\ 0 & & & \\ & -\frac{1}{3} & & \end{bmatrix}. \tag{25}$$

for which it is easy to check that

$$\theta(T_7)\tau = \frac{1}{3}\omega_7, \qquad \theta(T_3)\tau = \frac{1}{3}\omega_3, \qquad \theta(T_4)\tau = \frac{1}{3}\omega_4. \tag{26}$$

The following is our main structural result. Recall from Proposition 4.1 that any left-invariant ERP G_2-structure on a Lie group is equivariantly equivalent to some (G_μ,φ) with $\tau = e^{12} - e^{56}$ and $\mathfrak{h} = \mathrm{sp}\{e_1,\dots,e_6\}$ unimodular ($\mathfrak{g} = \mathbb{R}e_7 \oplus \mathfrak{h}$).

Theorem 4.7. *Let (G_μ,φ) be an ERP G_2-structure with $\tau = e^{12} - e^{56}$ and $\mathfrak{h}$ unimodular. Then, the following conditions hold:*

(i) $\mathfrak{g}_0 = \mathrm{sp}\{e_7, e_3, e_4\}$ *is a Lie subalgebra and* $\mathfrak{g}_1 = \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ *is an abelian ideal of* $\mathfrak{g}$. *In particular,* $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ *and* $\mathfrak{g}$ *is solvable.*

(ii) $\mathfrak{h}_1 = \mathrm{sp}\{e_3, e_4\}$ *is an abelian subalgebra (so* $\mathfrak{h} = \mathfrak{h}_1 \ltimes \mathfrak{g}_1$*).*

(iii) *There exist* $E, F, G \in \mathfrak{sp}(\mathfrak{g}_1,\tau)$ *such that*

$$A_2 = E + T_7, \quad B_2 = F + T_3, \quad C_2 = G + T_4,$$

where $A_1 := A|_{\mathfrak{h}_1}$, $A_2 := A|_{\mathfrak{g}_1}$, $B_2 := \mathrm{ad}\, e_3|_{\mathfrak{g}_1}$ *and* $C_2 := \mathrm{ad}\, e_4|_{\mathfrak{g}_1}$. *In particular,* $\mathrm{tr}\, A_2 = \mathrm{tr}\, B_2 = \mathrm{tr}\, C_2 = 0$ *and* $[B_2, C_2] = 0$.

Proof. We first prove part (iii). Recall that $\mathfrak{g}_0$, $\mathfrak{g}_1$ and $\mathfrak{h}_1$ are all Lie subalgebras of $\mathfrak{g}$. It was shown in the proof of Proposition 4.1 that,

$$\theta(A_2)\tau = \frac{1}{3}\omega_7, \qquad \theta(B_2)\tau = \frac{1}{3}\omega_3, \qquad \theta(C_2)\tau = \frac{1}{3}\omega_4,$$

and thus $A_2 - T_7$, $B_2 - T_3$ and $C_2 - T_4$ all belong to $\mathfrak{sp}(\mathfrak{g}_1, \tau)$ and the first assertion in part (iii) follows. Note that $\operatorname{tr} A_2 = \operatorname{tr} B_2 = \operatorname{tr} C_2 = 0$ and so $\lambda(e_3, e_4) = 0$ (i.e. $\mathfrak{h}_1$ abelian) follows from the fact that $\mathfrak{h}$ is unimodular, completing the proof of parts (ii) and (iii).

It was also obtained in the proof of Proposition 4.1 that $\mathfrak{g}_1$ is abelian. We now prove that $\mathfrak{g}_1$ is an ideal, which will conclude the proof of the theorem. If we set $B := \operatorname{ad} e_3|_{\mathfrak{h}}$ and $C := \operatorname{ad} e_4|_{\mathfrak{h}}$, then from (15),

$$\begin{aligned}
0 =& d_\lambda \rho^+ = d_\lambda \omega_3 \wedge e^3 + \omega_3 \wedge d_\lambda e^3 + d_\lambda \omega_4 \wedge e^4 + \omega_4 \wedge d_\lambda e^4 \\
=& d_{\mathfrak{g}_1}\omega_3 \wedge e^3 - \theta(C_2)\omega_3 \wedge e^{34} - \omega_3 \wedge \theta(B)e^3 \wedge e^3 - \omega_3 \wedge \theta(C)e^3 \wedge e^4 \\
& + d_{\mathfrak{g}_1}\omega_4 \wedge e^4 + \theta(B_2)\omega_4 \wedge e^{34} - \omega_4 \wedge \theta(B)e^4 \wedge e^3 - \omega_4 \wedge \theta(C)e^4 \wedge e^4 \\
=& \left(-\theta(C_2)\omega_3 + \theta(B_2)\omega_4\right) \wedge e^{34} - \big(\omega_3 \wedge \theta(B)e^3 + \omega_4 \wedge \theta(B)e^4\big) \wedge e^3 \\
& - \big(\omega_3 \wedge \theta(C)e^3 + \omega_4 \wedge \theta(C)e^4\big) \wedge e^4.
\end{aligned}$$

Since $\theta(B)e^3, \theta(B)e^4, \theta(C)e^3, \theta(C)e^4 \in \Lambda^1\mathfrak{g}_1^*$, it follows that

$$\begin{aligned}
0 &= \omega_3 \wedge \theta(B)e^3 + \omega_4 \wedge \theta(B)e^4 = \sum_{1,2,5,6} \big(\omega_3 \wedge c_{i33}e^i + \omega_4 \wedge c_{i34}e^i\big) \\
&= (c_{133} - c_{234})e^{126} + (c_{233} + c_{134})e^{125} + (c_{353} - c_{364})e^{256} \\
&\quad + (c_{363} + c_{354})e^{156},
\end{aligned}$$

and

$$\begin{aligned}
0 &= \omega_3 \wedge \theta(C)e^3 + \omega_4 \wedge \theta(C)e^4 = \sum \big(\omega_3 \wedge c_{i43}e^i + \omega_4 \wedge c_{i44}e^i\big) \\
&= (c_{143} - c_{244}e)e^{126} + (c_{243} + c_{144})e^{125} + (c_{453} - c_{464})e^{256} \\
&\quad + (c_{463} + c_{454})e^{156}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
0 &= d_\lambda e^{34} = d_\lambda e^3 \wedge e^4 - e^3 \wedge d_\lambda e^4 \\
&= -\big(\theta(B)e^3 + \theta(C)e^4\big) \wedge e^{34} = \sum_{1,2,5,6} (c_{3i3} + c_{4i4})e^{i34}.
\end{aligned}$$

Summarizing, we have obtained that

$$
\begin{aligned}
&c_{133} = -c_{144} = c_{234} = c_{243}, \qquad && c_{353} = c_{364} = -c_{454} = c_{463}, \\
&c_{134} = c_{143} = -c_{233} = c_{244}, \qquad && c_{354} = -c_{363} = c_{453} = c_{464}.
\end{aligned} \tag{27}
$$

As before,

$$
\begin{aligned}
0 &= \langle [\operatorname{ad} e_1, \operatorname{ad} e_2] e_4, e_3\rangle = \langle \operatorname{ad} e_1 \operatorname{ad} e_2 e_4 - \operatorname{ad} e_2 \operatorname{ad} e_1 e_4, e_3\rangle \\
&= \sum_{i=1}^{7} \langle c_{24i} \operatorname{ad} e_1(e_i) - c_{14i} \operatorname{ad} e_2(e_i), e_3\rangle = \sum_{i,j=1}^{7} \langle c_{24i} c_{1ij} e_j - c_{14i} c_{2ij} e_j, e_3\rangle \\
&= \sum_{i=1}^{7} (c_{24i} c_{1i3} - c_{14i} c_{2i3}) = c_{243} c_{133} + c_{244} c_{143} - c_{143} c_{233} - c_{144} c_{243} \\
&= 2(c_{133}^2 + c_{134}^2).
\end{aligned}
$$

In much the same way, one obtains that $0 = \langle [\operatorname{ad} e_5, \operatorname{ad} e_6](e_4), e_3\rangle = 2(c_{353}^2 + c_{354}^2)$. Thus, $c_{133} = c_{134} = c_{353} = c_{354} = 0$ and so it follows from (27) that $[\mathfrak{g}_1, \mathfrak{h}] \subset \mathfrak{g}_1$.

Therefore, it only remains to show that $[\mathfrak{g}_1, e_7] \subset \mathfrak{g}_1$. It follows from $\tau = e^{12} - e^{56}$, $\mathfrak{h}$ unimodular and $\mathfrak{g}_1$ abelian that

$$
\begin{aligned}
0 &= \langle e^{13}, \tau\rangle \operatorname{vol} = e^{13} \wedge *\tau = -e^{13} \wedge d * \varphi \\
&= -d(e^{13} \wedge *\varphi) + de^{13} \wedge *\varphi = -d(e^{123467}) + \langle de^{13}, \varphi\rangle \operatorname{vol} \\
&= \operatorname{tr}(\operatorname{ad} e_5) \operatorname{vol} + \langle de^1 \wedge e^3, \varphi\rangle \operatorname{vol} + \langle e^1 \wedge de^3, \varphi\rangle \operatorname{vol} = -c_{273}.
\end{aligned}
$$

In the same manner, we can see that $0 = c_{i7j}$ for each $i \in \{1, 2, 5, 6\}$ and $j \in \{3, 4, 7\}$. This implies that $\langle [\mathfrak{g}_1, e_7], \mathfrak{g}_0\rangle$ vanishes and so $\mathfrak{g}_1$ is an ideal, as desired. □

The following geometric consequence of Theorem 4.7 follows from Remark 4.6.

Corollary 4.8. *Any left-invariant ERP G_2-structure on a Lie group is both a steady Laplacian soliton and an expanding Ricci soliton.*

Recall that all the examples of Laplacian solitons found in [L2, N] are expanding.

We now give the converse of Theorem 4.7, which paves the way to the search for examples and eventually, to a full classification. In addition to (1), we denote by

$$
\overline{\omega_3} := e^{26} + e^{15}, \qquad \overline{\omega_4} := e^{16} - e^{25}.
$$

	τ	ω_7	ω_3	ω_4	$\overline{\omega_3}$	$\overline{\omega_4}$
T_7	$\frac{1}{3}\omega_7$	$\frac{1}{3}\tau$	0	$\frac{1}{3}\bar{\omega}_4$	0	$-\frac{1}{3}\omega_4$
T_3	$\frac{1}{3}\omega_3$	$\frac{1}{3}\overline{\omega_3}$	$\frac{1}{3}\tau$	0	$-\frac{1}{3}\omega_7$	0
T_4	$\frac{1}{3}\omega_4$	$\frac{1}{3}\overline{\omega_4}$	0	$\frac{1}{3}\tau$	0	$-\frac{1}{3}\omega_7$

Table 1: T_i-actions on 2-forms

Proposition 4.9. *Let μ denote a Lie bracket on $\mathfrak{g}$ whose only nonzero structural constants are given by A_1, A_2, B_2 and C_2 as in Theorem 4.7. Then (G_μ, φ) is ERP with $\tau_\mu = e^{12} - e^{56}$ if and only if there exist $E, F, G \in \mathfrak{sp}(\mathfrak{g}_1, e^{12} - e^{56})$ (see (24)) such that the following conditions hold:*

(i) *$A_2 = E + T_7$, $B_2 = F + T_3$ and $C_2 = G + T_4$, where the T_i's are defined as in (25).*

(ii) *$\theta(E^t)\omega_7 + \theta(F^t)\omega_3 + \theta(G^t)\omega_4 = -(\operatorname{tr} A_1)\omega_7$.*

Remark 4.10. The Jacobi condition for such a μ is equivalent to

(28) $$[A_2, B_2] = aB_2 + cC_2, \quad [A_2, C_2] = bB_2 + dC_2, \quad [B_2, C_2] = 0,$$
where $A_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.

Proof. We first suppose that (G_μ, φ) is ERP with $\tau_\mu = e^{12} - e^{56}$. Part (i) follows from Theorem 4.7. In order to prove (ii), we now proceed to compute τ_μ by using the formula given in Proposition 3.4 and Table 1 (recall from (20) that $d_\lambda\omega \wedge \omega = 0$):

$$\begin{aligned}
-*_{\mathfrak{h}} d_\lambda \rho^- &= -*_{\mathfrak{h}} (e^3 \wedge d_\lambda\omega_4 - e^4 \wedge d_\lambda\omega_3) \\
&= -*_{\mathfrak{h}} (e^{34} \wedge (\theta(C_2)\omega_4 + \theta(B_2)\omega_3)) \\
&= -*_{\mathfrak{g}_1} \theta(C_2)\omega_4 - *_{\mathfrak{g}_1}\theta(B_2)\omega_3 \\
&= \theta(C_2^t) *_{\mathfrak{g}_1} \omega_4 + \theta(B_2^t) *_{\mathfrak{g}_1} \omega_3 \\
&= \theta(C_2^t)\omega_4 + \theta(B_2^t)\omega_3 \\
&= \theta(G^t)\omega_4 + \theta(T_4)\omega_4 + \theta(F^t)\omega_3 + \theta(T_3)\omega_3 \\
&= \theta(G^t)\omega_4 + \theta(F^t)\omega_3 + \frac{2}{3}\left(e^{12} - e^{56}\right),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}(\operatorname{tr} A)\omega + \theta(A^t)\omega &= (\operatorname{tr} A_1)e^{34} + (\operatorname{tr} A_1)\omega_7 + \theta(A_2^t)\omega_7 + \theta(A_1^t)e^{34}\\ &= (\operatorname{tr} A_1)\omega_7 + \theta(E^t)\omega_7 + \frac{1}{3}(e^{12} - e^{56}).\end{aligned}$$

Thus part (ii) follows from the fact that $\tau_\mu = e^{12} - e^{56}$.

Conversely, assume that parts (i) and (ii) hold. Using part (i), (15) and Table 1 , it is easy to see that $d_\mu\varphi = 0$ if and only if

$$\begin{aligned}\theta(F)\omega_7 + a\omega_3 + c\omega_4 &= \theta(E)\omega_3 - \frac{1}{3}\overline{\omega_3},\\ \theta(G)\omega_7 + b\omega_3 + d\omega_4 &= \theta(E)\omega_4,\\ \theta(F)\omega_4 &= \theta(G)\omega_3.\end{aligned} \tag{29}$$

But straightforwardly, one obtains that these equalities respectively follow by just evaluating $\theta([A_2, B_2])$, $\theta([A_2, C_2])$ and $\theta([B_2, C_2])$ at τ and using the Jacobi condition (28). On the other hand, since

$$\begin{aligned}d_\lambda\omega \wedge \omega &= \frac{1}{2}d_\lambda(\omega \wedge \omega) = d_\lambda(e^{1234} + e^{3456} + e^{1256}) = d_\lambda(e^{1256})\\ &= \theta(B_2)e^{1256} \wedge e^3 + \theta(C_2)e^{1256} \wedge e^4\\ &= -\operatorname{tr} B_2 e^{12356} - \operatorname{tr} C_2 e^{12456} = 0,\end{aligned}$$

we obtain from part (ii) that $\tau_\mu = e^{12} - e^{56}$. It now follows from (22) and part (i) that (G_μ, φ) is ERP, which concludes the proof of the proposition. □

The strong conditions on the Ricci curvature imposed by ERP (see Proposition 4.5, (iii)) produce very useful constraints on the matrices involved.

Proposition 4.11. *If (G_μ, φ) is ERP with $\tau_\mu = e^{12} - e^{56}$, say $\mu = (A_1, A_2, B_2, C_2)$, then the following conditions hold:*

(i) $\operatorname{tr} S(A_1)^2 + \operatorname{tr} S(A_2)^2 = \frac{1}{3}$.

(ii) $\frac{1}{2}[A_2, A_2^t] + \frac{1}{2}[B_2, B_2^t] + \frac{1}{2}[C_2, C_2^t] = (\operatorname{tr} A_1)S(A_2)$.

(iii) $\operatorname{tr} S(A_2)S(B_2) = \operatorname{tr} S(A_2)S(C_2) = 0$.

(iv) $\begin{bmatrix} \operatorname{tr} S(B_2)^2 & \operatorname{tr} S(B_2)S(C_2) \\ \operatorname{tr} S(B_2)S(C_2) & \operatorname{tr} S(C_2)^2 \end{bmatrix} - \frac{1}{2}[A_1, A_1^t] + (\operatorname{tr} A_1)S(A_1) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

Proof. All the items follow from Proposition 4.5, (ii) by just applying the formula for the Ricci operator of a solvmanifold given in [L1, (25)]. □

We also note that if (G_μ, φ) is ERP with $\tau_\mu = e^{12} - e^{56}$, then $(G_\mu, \langle\cdot,\cdot\rangle)$ is a solvsoliton; indeed, in terms of the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,

$$\mathrm{Ric}_\mu = -\frac{1}{3}I + \begin{bmatrix} 0 & \\ & \frac{1}{3}I \end{bmatrix} \in \mathbb{R}I + \mathrm{Der}(\mu).$$

This allows us to use, in addition to Proposition 4.11, the structure theorem for solvsolitons [L1, Theorem 4.8].

5. Examples and structure refinements

Acording to Theorem 4.7, for any ERP (G_μ, φ) with $\tau = e^{12} - e^{56}$, $\mathfrak{g}_1 = \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ is an abelian ideal of the Lie algebra $(\mathfrak{g}, \mu)$. Thus the nilradical $\mathfrak{n}$ of $(\mathfrak{g}, \mu)$ contains $\mathfrak{g}_1$ and so $\dim \mathfrak{n} \geq 4$. Recall from Proposition 4.9 that the Lie bracket has always the form $\mu = (A_1, A, B, C)$ for certain matrices $A_1 \in \mathfrak{gl}_2(\mathbb{R})$ and $A, B, C \in \mathfrak{gl}_4(\mathbb{R})$ such that $[B, C] = 0$.

We can use Proposition 4.4 to consider the equivalence problem. The action of the group $U_{\mathfrak{h},\tau}$ on $\mu = (A_1, A, B, C)$ can be described as follows (see Section 2.3). If $h \in U_0$, say with $h_1 = \left[\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right]$, $x^2 + y^2 = 1$ and $h_2 := \begin{bmatrix} h_3 & 0 \\ 0 & h_4 \end{bmatrix}$, $h_3, h_4 \in \mathrm{SO}(2)$, then

$$h \cdot \mu = \left(h_1 A_1 h_1^{-1}, h_2 A h_2^{-1}, h_2(xB - yC)h_2^{-1}, h_2(yB + xC)h_2^{-1}\right), \tag{30}$$

and if $g_1 := \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ and $g_2 := \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right]$, then

$$g \cdot \mu = \left(-g_1 A_1 g_1^{-1}, -g_2 A g_2^{-1}, g_2 B g_2^{-1}, -g_2 C g_2^{-1}\right). \tag{31}$$

Let (G_μ, φ) be an ERP G_2-structure with $\tau = e^{12} - e^{56}$ and nilradical $\mathfrak{n}$, say $\mu = (A_1, A, B, C)$. If μ is unimodular, then $\mathfrak{n} = \mathfrak{g}_1$ (see Proposition 5.1 below) and in the non-unimodular case, $\mathfrak{g}_1 \subset \mathfrak{n} \subset \mathfrak{h}$. In any case, A_1 and A are necessarily normal matrices by [L1, Theorem 4.8].

In what follows, we separately study each of the cases $\dim \mathfrak{n} = 4, 5, 6$; note that μ can not be nilpotent since $\mathrm{Ric} \leq 0$ (see [W, M]).

5.1. Case $\dim \mathfrak{n} = 4$

In the unimodular case, some necessary algebraic conditions proved by I. Dotti [D] for $\operatorname{Ric} \leq 0$ give rise to the following characterization.

Proposition 5.1. *If (G_μ, φ) is ERP with $\tau = e^{12} - e^{56}$, say $\mu = (A_1, A, B, C)$, then the following conditions are equivalent:*

(i) *μ is unimodular (i.e. $\operatorname{tr} A_1 = 0$).*

(ii) *$A_1 = 0$ (in particular, A, B, C pairwise commute).*

(iii) *$\mathfrak{g}_1$ is the nilradical of μ (in particular, $\{A, B, C\}$ is linearly independent).*

Proof. Recall from Proposition 3.9, (iv) that $\operatorname{Ric} \leq 0$ and the kernel of Ric is $\mathfrak{g}_1$. If μ is unimodular, then the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is contained in $\mathfrak{g}_1$ by [D, Lemma 1], but $\mathfrak{g}_1 \subset \mathfrak{n}$ as $\mathfrak{g}_1$ is an abelian ideal of $\mathfrak{g}$, so $\mathfrak{n} = \mathfrak{g}_1$. Since the image of any derivation of a solvable Lie algebra is contained in the nilradical, we obtain that $A_1 = 0$. The remaining implications trivially hold. □

Proposition 5.2. *If (G_μ, φ) is ERP with $\tau = e^{12} - e^{56}$ and μ is unimodular, say $\mu = (0, A, B, C)$, then the 4×4 matrices A, B, C are all symmetric, they pairwise commute and the set $\left\{\sqrt{3}A, \sqrt{3}B, \sqrt{3}C\right\}$ is orthonormal.*

Remark 5.3. In particular, G_μ is isomorphic to the Lie group given in [L3, Example 4.7] and Example 5.4 below. This has been proved in [FR2, Theorem 6.7]. We note however that there could be other non-equivalent ERP G_2-structures on G_μ.

Proof. From equation (ii) in Proposition 4.11 (recall that $A_1 = 0$), we obtain that the matrices A, B, C are all normal, by just multiplying with each of the three terms (alternatively, one can just apply [L1, Theorem 4.8]). Thus $A, B, C, S(A), S(B), S(C)$ is a commuting family of normal 4×4 matrices, which are all non-zero by Proposition 4.11, (i) and (iv). The only possibility for this to happen is that they are all symmetric, and so the set $\left\{\sqrt{3}A, \sqrt{3}B, \sqrt{3}C\right\}$ is orthonormal by Proposition 4.11, (i), (iii) and (iv), as desired. □

Example 5.4. Consider $\mu_J := (0, A, B, C)$, where

$$A = \begin{bmatrix} -\frac{1}{6} & & & \\ & -\frac{1}{6} & & \\ & & \frac{1}{2} & \\ & & & -\frac{1}{6} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{6} & 0 & \frac{1}{3} \\ -\frac{\sqrt{2}}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{\sqrt{2}}{6} & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \end{bmatrix}.$$

It is straightforward to check that all the conditions in Proposition 4.9 hold for these matrices, thus (G_{μ_J}, φ) is an ERP G_2-structure with $\tau = e^{12} - e^{56}$, and also that the map

$$h := \frac{1}{6} \begin{bmatrix} 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & -\sqrt{6} & -2\sqrt{6} & 0 & 0 \\ -3\sqrt{2} & -3\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{6} & \sqrt{6} & 0 & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & -2\sqrt{3} & 2\sqrt{3} & -2\sqrt{3} & 0 & 0 \\ -2\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 & 0 & -2\sqrt{3} \end{bmatrix} \in G_2$$

defines an equivariant equivalence between (G_{μ_J}, φ) and [L3, Example 4.7].

The difficulty in finding new examples in this case relies on the complicated structure of the 4-dimensional group $U_{\mathfrak{g}_1,\tau}$ (see (10)) providing the equivariant equivalence.

5.2. Case $\dim \mathfrak{n} = 5$

By acting with $U_{\mathfrak{h},\tau}$ if necessary (see (30)), we can assume in this case that up to equivariant equivalence, $\mathfrak{n} = \mathbb{R}e_4 \oplus \mathfrak{g}_1$. Let (G_μ, φ) be an ERP G_2-structure with $\tau = e^{12} - e^{56}$ and $\mathfrak{n}$ as above, say $\mu = (A_1, A, B, C)$. It follows from [L1, Theorem 4.8] that A_1, A, B are normal and $[A, B] = 0$, and since $[e_7, \mathfrak{n}] \subset \mathfrak{n}$, one further obtains that

$$A_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right], \quad d \neq 0, \qquad [A, C] = dC, \qquad [B, C] = 0.$$

By acting with g if necessary as in (31), one can assume up to equivariant equivalence that $d > 0$.

The following two Lie brackets provide new examples of ERP G_2-structures (G_μ, φ) with $\tau = e^{12} - e^{56}$ and $\mathfrak{n} = \mathbb{R}e_4 \oplus \mathfrak{g}_1$ by Proposition 4.9.

Example 5.5. Consider $\mu_{M2} := (A_1, A, B, C)$, where

$$A_1 = \begin{bmatrix} 0 & \\ & \frac{1}{3} \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{1}{3} & & & \\ & 0 & & \\ & & 0 & \\ & & & \frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & -\frac{1}{6} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & & & \\ -\frac{1}{3} & 0 & & \\ \frac{1}{3} & 0 & 0 & \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Note that the nilradical $\mathfrak{n}$ is 3-step nilpotent.

Example 5.6. Consider $\mu_{M3} := (A_1, A, B, C)$, where

$$A_1 = \frac{1}{6} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{6} \end{bmatrix}, \quad A = \frac{1}{12} \begin{bmatrix} -2 & 0 & -\sqrt{2} & 0 \\ 0 & -2 & 0 & -\sqrt{2} \\ -\sqrt{2} & 0 & 2 & 0 \\ 0 & -\sqrt{2} & 0 & 2 \end{bmatrix},$$

$$B = \frac{1}{6} \begin{bmatrix} 0 & \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & -\sqrt{2} \\ 1 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad C = \frac{1}{12} \begin{bmatrix} -\sqrt{2} & 0 & 2-\sqrt{6} & 0 \\ 0 & \sqrt{2} & 0 & -2+\sqrt{6} \\ 2+\sqrt{6} & 0 & \sqrt{2} & 0 \\ 0 & -2-\sqrt{6} & 0 & -\sqrt{2} \end{bmatrix}.$$

The nilradical $\mathfrak{n}$ is 2-step nilpotent in this case.

By considering the possible forms for the normal matrices A and B under the condition given in Proposition 4.9, (i), it can be shown with some computer assistance that $[A, B] = 0$ never holds unless A and B are both symmetric.

5.3. Case $\dim \mathfrak{n} = 6$

We have that $\mathfrak{n} = \mathfrak{h}$ in this case, so B and C are nilpotent. Let (G_μ, φ) be an ERP G_2-structure with $\tau = e^{12} - e^{56}$ and nilradical $\mathfrak{n} = \mathfrak{h}$, say $\mu = (A_1, A, B, C)$. By using (30), we can assume that up to equivariant equivalence,

(i) either $A_1 = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, with $a \leq d$, $a + d > 0$ (in particular, $[A, B] = aB$, $[A, C] = dC$),

(ii) or $A_1 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, with $a > 0$, $b \neq 0$ (in particular, $[A, B] = aB - bC$, $[A, C] = bB + aC$).

Example 5.7. We now present in the format $\mu_B := (A_1, A, B, C)$ the example given by R. Bryant in [B, Example 1], as well as in [CI, Section 6.3] and [L3, Examples 4.13, 4.10]. Consider,

$$A_1 = \begin{bmatrix} \frac{1}{3} & \\ & \frac{1}{3} \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{1}{6} & & & \\ & -\frac{1}{6} & & \\ & & \frac{1}{6} & \\ & & & \frac{1}{6} \end{bmatrix}, \quad B = \begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & \frac{1}{3} & & \\ \frac{1}{3} & 0 & & \end{bmatrix}, \quad C = \begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ \frac{1}{3} & 0 & & \\ 0 & -\frac{1}{3} & & \end{bmatrix}.$$

Note that $\mathfrak{n}$ is 2-step nilpotent.

The following is a new example with a 4-step nilpotent nilradical of dimension 6.

Example 5.8. Consider $\mu_{M1} := (A_1, A, B, C)$, where

$$A_1 := \frac{1}{30}\begin{bmatrix} \sqrt{30} & 0 \\ 0 & 2\sqrt{30} \end{bmatrix}, \qquad A := \frac{1}{60}\begin{bmatrix} -10-\sqrt{30} & 0 & -2\sqrt{5} & 0 \\ 0 & -10+\sqrt{30} & 0 & -2\sqrt{5} \\ -2\sqrt{5} & 0 & 10-\sqrt{30} & 0 \\ 0 & -2\sqrt{5} & 0 & 10+\sqrt{30} \end{bmatrix},$$

$$B := \frac{1}{30}\begin{bmatrix} 0 & -\sqrt{5} & 0 & 5-\sqrt{30} \\ 5\sqrt{5} & 0 & 5 & 0 \\ 0 & 5+\sqrt{30} & 0 & \sqrt{5} \\ 5 & 0 & -5\sqrt{5} & 0 \end{bmatrix}, \quad C := \frac{1}{30}\begin{bmatrix} -\sqrt{5} & 0 & 5-\sqrt{30} & 0 \\ 0 & \sqrt{5} & 0 & -5+\sqrt{30} \\ 5+\sqrt{30} & 0 & \sqrt{5} & 0 \\ 0 & -5-\sqrt{30} & 0 & -\sqrt{5} \end{bmatrix}.$$

Remark 5.9. It is worth pointing out that the five examples we have given in this section (i.e. Examples 5.4, 5.5, 5.6, 5.7, 5.8) are pairwise non-equivalent (even up to scaling). Indeed, the underlying solvable Lie groups are pairwise non-isomorphic, and since they are all completely solvable, the corresponding left-invariant metrics can never be isometric up to scaling (see [A]).

6. Deformations and rigidity

We study in this section deformations and two notions of rigidity for ERP G_2-structures on Lie groups.

As in Section 2, we fix a 7-dimensional real vector space $\mathfrak{g}$ endowed with a basis $\{e_1, \dots, e_7\}$ and the positive 3-form defined in (2), whose associated inner product $\langle\cdot,\cdot\rangle$ is the one making the basis $\{e_i\}$ oriented and orthonormal.

Let $\mathcal{L} \subset \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ denote the algebraic subset of all Lie brackets on $\mathfrak{g}$ and for every $\mu \in \mathcal{L}$, denote by G_μ the simply connected Lie group with

Lie algebra $(\mathfrak{g}, \mu)$. Each $\mu \in \mathcal{L}$ will be identified with the left-invariant G_2-structure determined by φ on G_μ:

$$\mu \longleftrightarrow (G_\mu, \varphi).$$

The isomorphism class of μ, $\mathrm{GL}_7(\mathbb{R}) \cdot \mu$, therefore stands for the set of all left-invariant G_2-structures on G_μ, due to the equivariant equivalence,

$$(G_{h\cdot\mu}, \varphi) \simeq (G_\mu, \varphi(h\cdot, h\cdot, h\cdot)), \qquad \forall h \in \mathrm{GL}_7(\mathbb{R}).$$

Thus one has in $\mathcal{L}$, all together, all the Lie groups endowed with left-invariant G_2-structures. Note that two elements in $\mathcal{L}$ are equivariantly equivalent as G_2-structures if and only if they belong to the same G_2-orbit, and that they are in the same $\mathrm{O}(7)$-orbit if and only if they are equivariantly isometric as Riemannian metrics. Both assertions hold without the word 'equivariantly' for completely real solvable Lie brackets.

In this light, the following G_2-invariant algebraic subsets,

$$\begin{aligned} \mathcal{L}_c &:= \left\{\mu \in \mathcal{L} : d_\mu \varphi = 0\right\}, \\ \mathcal{L}_{erp} &:= \left\{\mu \in \mathcal{L}_c : d_\mu \tau_\mu = \tfrac{1}{6}|\tau_\mu|^2\varphi + \tfrac{1}{6} * (\tau_\mu \wedge \tau_\mu)\right\}, \end{aligned} \tag{32}$$

parametrize the spaces of all closed (or calibrated) and all ERP G_2-structures on Lie groups, respectively. Thus the quotient

$$\mathcal{L}_{erp}/G_2$$

parametrizes the set of all ERP G_2-structures on Lie groups, up to equivariant equivalence. Note that a given Lie group G_μ admits a closed (resp. ERP) G_2-structure if and only if the orbit $\mathrm{GL}_7(\mathbb{R}) \cdot \mu$ meets $\mathcal{L}_c$ (resp. $\mathcal{L}_{erp}$).

A C^1 curve $\mu : (-\epsilon, \epsilon) \longrightarrow \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ is said to be a *deformation* (of ERP G_2-structures) if $\mu(t) \in \mathcal{L}_{erp}$ for all t. Examples of deformations are given by $\mu(t) = h(t) \cdot \mu$, where $\mu \in \mathcal{L}_{erp}$ and $h(t) \in G_2$, which are trivial in the sense that the family $\{\mu(t)\}$ is in such case pairwise equivariantly equivalent. Given $\mu \in \mathcal{L}_{erp}$, let $\mathcal{T}_\mu\mathcal{L}_{erp}$ denote the set of all velocities $\mu'(0)$ such that $\mu(t)$ is a deformation with $\mu(0) = \mu$ (notice that $\mathcal{T}_\mu\mathcal{L}_{erp}$ is not necessarily a vector space). It follows that,

$$\mathfrak{g}_2 \cdot \mu \subset \mathcal{T}_\mu\mathcal{L}_{erp} \subset \overline{T}_\mu\mathcal{L}_{erp},$$

where $\mathfrak{g}_2 \cdot \mu$ coincides with the tangent space $T_\mu(G_2 \cdot \mu)$ and $\overline{T}_\mu\mathcal{L}_{erp}$ is the vector space determined by the linearization of both the Jacobi condition and the remaining equations defining $\mathcal{L}_{erp}$ given in (32).

It is therefore natural to call a $\mu \in \mathcal{L}_{erp}$ *equivariantly rigid* when

$$\mathfrak{g}_2 \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp}.$$

However, it is worth pointing out that according to Proposition 3.5, there might exist linear deformations of the form $\mu(t) = \mu + t\mu_D$, where D is a suitable derivation of μ. Such deformations are also trivial as $\mu(t)$ is equivalent to μ for all t, though in general they are not equivariantly equivalent. This shows that weaker notions of rigidity should also come into play.

In the case when $\mathfrak{h} = \mathrm{sp}\{e_1, \dots, e_6\}$ is an ideal of $\mu \in \mathcal{L}$, one has that $\mu = \lambda + \mu_A$ as in Section 2.2 and $\mu \leftrightarrow (G_\mu, \varphi)$ is indeed the structure we have studied in Sections 3 and 4. It follows from Proposition 3.2 that the G_2-orbit of any $\mu \in \mathcal{L}_c$ meets the algebraic subset

$$\mathcal{L}_{c,\mathfrak{h}} := \{\mu \in \mathcal{L}_c : \mu(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h},\ \operatorname{tr} \operatorname{ad}_\mu e_i = 0,\ i = 1, \dots, 6\},$$

and that the equivariant equivalence between non-unimodular elements in $\mathcal{L}_{c,\mathfrak{h}}$ is determined by the group $U_\mathfrak{h}$ given in (8). In the same vein, Proposition 4.1 asserts that any ERP G_2-structure $\mu \in \mathcal{L}_{erp}$ is equivariantly equivalent to an element in

$$\mathcal{L}_{erp,\mathfrak{h},\tau} := \{\mu \in \mathcal{L}_{c,\mathfrak{h}} : \tau_\mu = \tau\},$$

where $\tau := e^{12} - e^{56}$. In this case, the subgroups $U_{\mathfrak{h},\tau}, U_{\mathfrak{g}_1,\tau} \subset G_2$ computed in Section 2.3 are the ones providing equivariant equivalence among $\mathcal{L}_{erp,\mathfrak{h},\tau}$ in the non-unimodular and unimodular cases, respectively (see Proposition 4.4).

This motivates the study of deformations within $\mathcal{L}_{erp,\mathfrak{h},\tau}$. Analogously, for each $\mu \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ one has that,

$$\mathfrak{u} \cdot \mu = T_\mu(U \cdot \mu) \subset \mathcal{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau} \subset \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau},$$

where $\mathfrak{u}$, U are either $\mathfrak{u}_{\mathfrak{h},\tau}$, $U_{\mathfrak{h},\tau}$ or $\mathfrak{u}_{\mathfrak{g}_1,\tau}$, $U_{\mathfrak{g}_1,\tau}$, depending on whether μ is non-unimodular or unimodular. Here $\overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ is the linearization of the conditions defining $\mathcal{L}_{erp,\mathfrak{h},\tau}$ given in Proposition 4.9. Thus $\mu \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ is equivariantly rigid if and only if $\mathfrak{u} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$. Note that $\dim \mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu \leq 2$ and $\dim \mathfrak{u}_{\mathfrak{g}_1,\tau} \cdot \mu \leq 4$ for any μ.

According to the structural results proved in Section 4 (see Theorem 4.7 and Proposition 4.9), each $\mu \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ only depends on one 2×2 matrix A_1

and three 4×4 matrices A, B and C; in this way,

$$\mu = \lambda_{B,C} + \mu_{\left[\begin{smallmatrix} A_1 & 0 \\ 0 & A \end{smallmatrix}\right]}.$$

Thus any deformation $\mu(t) \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ such that $\mu(0) = \mu$ and $\mu'(0) = \overline{\mu}$ has the following form:

$$\mu = (A_1, A, B, C), \quad \mu(t) = (A_1(t), A(t), B(t), C(t)), \quad \overline{\mu} = (\overline{A}_1, \overline{A}, \overline{B}, \overline{C}),$$

$$A_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], \quad A_1(t) = \left[\begin{smallmatrix} a(t) & b(t) \\ c(t) & d(t) \end{smallmatrix}\right], \quad \overline{A}_1 = \left[\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{smallmatrix}\right].$$

It follows from Proposition 4.9 that a vector $\overline{\mu} = (\overline{A}_1, \overline{A}, \overline{B}, \overline{C})$ belongs to $\overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ if and only if the following conditions hold:

$$\overline{A}, \overline{B}, \overline{C} \in \mathfrak{sp}(\mathfrak{g}_1, \tau), \tag{33}$$

$$[\overline{A}, B] + [A, \overline{B}] = \overline{a}B + a\overline{B} + \overline{c}C + c\overline{C}, \tag{34}$$

$$[\overline{A}, C] + [A, \overline{C}] = \overline{b}B + b\overline{B} + \overline{d}C + d\overline{C}, \tag{35}$$

$$[\overline{B}, C] + [B, \overline{C}] = 0, \tag{36}$$

$$\theta(\overline{A}^t)\omega_7 + \theta(\overline{B}^t)\omega_3 + \theta(\overline{C}^t)\omega_4 = -(\overline{a} + \overline{d})\omega_7. \tag{37}$$

We now describe the linear deformations mentioned above. Given $\mu = (A_1, A, B, C) \in \mathcal{L}_{erp,\mathfrak{h},\tau}$, consider the vector space $\mathfrak{D}_\mu$ of all pairs $(D_1, D_2) \in \mathfrak{gl}_2(\mathbb{R}) \times \mathfrak{gl}_4(\mathbb{R})$ such that

$$[D_1, A_1] = 0, \quad [D_2, A] = 0, \quad [D_2, B] = rB + tC,$$
$$[D_2, C] = sB + uC, \quad D_1 = \left[\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right],$$

that is,

$$D := \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_2 \end{smallmatrix}\right]$$

defines a derivation of μ vanishing at e_7 (see Remark 3.6). It follows from Proposition 3.5 that each of these (D_1, D_2) satisfying that $D \in \mathfrak{su}(3)$ determines a linear deformation of μ given by $\mu(t) := \mu + t\mu_D$, or equivalently,

$$A_1(t) = A_1 + tD_1, \qquad A(t) = A + tD_2, \qquad B(t) \equiv B, \qquad C(t) \equiv C,$$

forming the vector space

$$\mathfrak{d}_\mu := \{\overline{\mu} = (D_1, D_2, 0, 0) : (D_1, D_2) \in \mathfrak{D}_\mu,\ D \in \mathfrak{su}(3)\} \subset \mathcal{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}.$$

This suggests the following weaker version of rigidity: $\mu \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ is said to be *rigid* if

$$\mathfrak{d}_\mu + \mathfrak{u} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}.$$

In the unimodular case, one always has that $\mathfrak{d}_\mu = 0$ and $\mathfrak{u}_{\mathfrak{g}_1,\tau} \cdot \mu$ is 4-dimensional. Indeed, any skew-symmetric derivation D of $\mu = (0, A, B, C)$ must stabilize the nilradical $\mathfrak{g}_1$ (see Proposition 5.1) and commute with the maximal abelian subalgebra $\mathrm{sp}\{A, B, C\} \subset \mathrm{sym}(4)$ (see Corollary 5.2), so $D = 0$. This implies that a unimodular $\mu \in \mathcal{L}_{erp,\mathfrak{h},\tau}$ is equivariantly rigid, if and only if it is rigid, if and only if $\dim \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau} = 4$.

In the non-unimodular case, $\mathfrak{D}_\mu = \mathrm{Der}(\mu) \cap \mathfrak{g}_2$ and it is easy to see that $\mathfrak{d}_\mu \perp \mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu$. Moreover, since $\mathrm{Der}(\mu) \cap \mathfrak{u}_{\mathfrak{h},\tau} \subset \mathfrak{d}_\mu$, one always has that $\dim(\mathfrak{d}_\mu + \mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu) \geq 2$.

By solving the linear system (33)-(37) and computing the derivations belonging to $\mathfrak{g}_2$ for all the examples given in Section 5, we obtain the following information:

- μ_J (Example 5.4): $\mathfrak{u}_{\mathfrak{g}_1,\tau} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ (4-dimensional), $\mathfrak{d}_\mu = 0$.
- μ_{M2} (Example 5.5): $\mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ (2-dimensional), $\mathfrak{d}_\mu = 0$.
- μ_{M3} (Example 5.6): $\mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ (2-dimensional), $\mathfrak{d}_\mu = 0$.
- μ_B (Example 5.7): $\mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu = 0$, $\mathfrak{d}_\mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ (2-dimensional).
- μ_{M1} (Example 5.8): $\mathfrak{u}_{\mathfrak{h},\tau} \cdot \mu = \overline{T}_\mu \mathcal{L}_{erp,\mathfrak{h},\tau}$ (2-dimensional), $\mathfrak{d}_\mu = 0$.

It follows that they are all equivariantly rigid, except for Example 5.7, which is only rigid.

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