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Limiting case of an isoperimetric inequality with radial densities and applications

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We prove a sharp isoperimetric inequality with radial densities whose functional counterpart corresponds to a limiting case for the exponents of the Il'in (or Caffarelli-Kohn-Nirenberg) inequality in L^1. We show how the latter applies to obtain an optimal critical Sobolev weighted norm improvement to one of the L^1 weighted Hardy inequalities of [29]. Further applications include an L^p version with the best constant of the functional analogue of this isoperimetric inequality and also a weighted Pólya-Szegö inequality.

1. Introduction

1.1. Overview, notation and central results

The Gagliardo-Nirenberg inequality in its sharp form states that

$$(1.1) \quad \int_{\mathbb{R}^n} |\nabla f| \,\mathrm{d}x \geq n\omega_n^{1/n}\bigg(\int_{\mathbb{R}^n} |f|^{n/(n-1)} \,\mathrm{d}x\bigg)^{1-1/n} \qquad \forall\ f \in C_c^1(\mathbb{R}^n),$$

Here and throughout the whole paper, ω_n is the volume of a unit ball in $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, and $|\cdot|$ denotes both absolute value of numbers and length of vectors. Also, whenever $G \subseteq \mathbb{R}^n$ is open, then $C_c^1(G)$ stands for the set of continuously differentiable scalar functions with compact support in G and $C_c^1(G;\mathbb{R}^n)$ for all n-vector functions with $C_c^1(G)$ components. Inequality (1.1) was proved by Maz'ya in [23] and independently by Federer and Fleming in [14]. It was shown in both papers that it is equivalent to the classical isoperimetric inequality in $\mathbb{R}^n$. Nowadays this equivalence is formulated as follows (see [13, Section 5.6.2]): (1.1) is equivalent to

$$\mathcal{P}(E) \geq n\omega_n^{1/n}\mathcal{L}^n(E)^{1-1/n}, \tag{1.2}$$

valid for any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ with finite perimeter $\mathcal{P}(E)$; that is

$$\mathcal{P}(E) := \sup\left\{ \int_E \operatorname{div}\varphi \, \mathrm{d}x : \ \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \ |\varphi| \leq 1 \text{ in } \mathbb{R}^n \right\} < \infty,$$

Recall that equality holds in (1.2) if $E = B_R(x_0) := \{x \in \mathbb{R}^n : \ |x - x_0| < R\}$, any $R > 0$, any $x_0 \in \mathbb{R}^n$.

Maz'ya and Shaposhnikova in [26, Corollary 5.1]- (see also [25, Corollary in Section 4.8]), have obtained the best constant in the following scale invariant weighted generalization of (1.1), the weights being powers of the distance to the origin:

if $0 \leq a < n-1$ and $an/(n-1) \leq b \leq a+1$, then

$$\int_{\mathbb{R}^n} \frac{|\nabla f|}{|x|^a} \, \mathrm{d}x \geq \mathcal{C}_{n,a,b} \left(\int_{\mathbb{R}^n} \frac{|f|^{(n-b)/(n-1-a)}}{|x|^b} \, \mathrm{d}x \right)^{(n-1-a)/(n-b)} \quad \forall\, f \in C_c^1(\mathbb{R}^n), \tag{1.3}$$

where $\mathcal{C}_{n,a,b} := \big(n\omega_n(n-b)^{(n-1-a)/(1+a-b)}\big)^{(1+a-b)/(n-b)}$. The case $a = 0$ has been established earlier in [24]. Without the sharp constant, inequality (1.3) goes back to Il'in; see [22, Theorem 1.4, pg 367]. It is a subcase of the Caffarelli-Kohn-Nirenberg interpolation inequality; see [8] and also the exhaustive work of Rabier [32]. As with (1.1), the sharp estimate (1.3) has the isoperimetric counterpart

$$\mathcal{P}(E;|x|^{-a}) \geq \mathcal{C}_{n,a,b}\big(\mathcal{L}^n(E;|x|^{-b})\big)^{(n-1-a)/(n-b)}, \tag{1.4}$$

valid for any $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ satisfying $\mathcal{P}(E;|x|^{-a}) < \infty$. Here we have set (see [4])

$$\mathcal{P}(E;|x|^{-\alpha}) := \sup\left\{ \int_E \operatorname{div}\varphi \, \mathrm{d}x : \ \varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \ |\varphi(x)| \leq |x|^{-\alpha} \text{ in } \mathbb{R}^n \right\},$$

$$\mathcal{L}^n(E;|x|^{-b}) := \int_E |x|^{-b} \, \mathrm{d}x.$$

Taking $a = b = 0$ in (1.4) we recover (1.2). If $a \neq 0$ then

- if $b \neq a+1$, equality in (1.4) (resp. (1.3) formulated in $BV(\mathbb{R}^n)$) holds if $E = B_R(0)$ (resp. $f = \chi_{B_R(0)}$), that is to say the ball has to be centered at the origin. This is an effect of having weighted both sides by powers of the distance to the origin. The obvious changes go through if the distance is taken from a point $x_0 \in \mathbb{R}^n$.
- if $b = a+1$, the same as above is true for (1.4) but equality in (1.3) is additionally achieved for any nonnegative, radially decreasing function f.

Section 3 of this paper is devoted to an extension of this last case (see the first application in Section 1.2).

Several mathematicians have shown interest on different aspects of isoperimetric inequalities involving various kinds of weights; see for instance [5], [6], [7], [10], [11], [12], [15], [18], [20], [27] and [28]. The work [3] contains an extended list of the relevant references. Moreover, it gives new isoperimetric inequalities of the type (1.3) and (1.4) in the case $0 > a > b-1$ (see Theorem 1.1-(iii) and (iv) there) and applications to Caffarelli-Kohn-Nirenberg inequalities. Let us mention that the preceded work [12] falls within this particular range for the parameters ($n = 2$, $b = 0$ and $0 > a > -1$) and uses different methods. However, it seems that the above mentioned paper [26] by Maz'ya and Shaposhnikova, where (1.4) is established, was unnoticed in the corresponding to radial weights recent literature (see for instance the reference given in [3] for inequality (1.4), which is stated as Theorem 1.1-(ii) there).

The aim of this work is to investigate the end point case $a = n-1$ in (1.4) and (1.3) (this amounts to $l + N = 0$ in the notation of [3]). Note the parameter assumptions for (1.3) to be valid force b to equal n in this case, so the right hand side of (1.3) is infinite unless f is supported away from the origin. We provide sharp substitutes for (1.4) and (1.3) by logarithmically correcting the weights in both of their sides. More precisely, let

$$X(t) := (1 - \log t)^{-1}, \quad t \in (0,1],$$

and observe $\lim_{t\to 0^+} X(t) = 0$. Standard calculus shows that given $R > 0$, for any $\delta \in (0, R]$ one has

$$\int_{B_\delta(0)} |x|^{-n} X^{1+\theta}\Big(\frac{|x|}{R}\Big)\, \mathrm{d}x = n\omega_n \int_0^\delta r^{-1} X^{1+\theta}\Big(\frac{r}{R}\Big)\, \mathrm{d}r < \infty$$

if and only if $\quad \theta > 0.$

Keeping this in mind we read the basic result of the paper

Theorem A. *Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, containing the origin and set $R_\Omega := \sup_{x\in\Omega} |x|$. For all $\gamma \in (0, n-1]$ and any $f \in C_c^1(\Omega)$, it holds that*

(1.5)

$$\int_\Omega \frac{|\nabla f|}{|x|^{n-1}} X^\gamma\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x \geq \mathfrak{C}_{n,\gamma}\bigg(\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x\bigg)^{1-1/n},$$

with the best constant $\mathfrak{C}_{n,\gamma} := n\omega_n^{1/n}\big(\gamma/(n-1)\big)^{1-1/n}$.

Remark 1.1. Clearly, (1.5) fails for $\gamma = 0$. On the other hand, the restriction $\gamma \leq n-1$ is not essential. Note for instance that $0 \leq X(t) \leq 1$ for all $t \in [0,1]$, so the right hand side will further decrease upon increasing the exponent on X. Of course the constant ceases to be optimal. We don't search for the best constant in case $\gamma > n-1$ here. Our aim is to demonstrate that taking $\gamma > 0$ smaller and smaller, the weight $X^{1+\gamma n/(n-1)}(|x|/R_\Omega)$ relaxes less the singularity at 0 of the weight $|x|^{-n}$, but the estimate fails for $\gamma = 0$. A similar open ended condition for the parameter indicating the power on a logarithmic correction has appeared recently in the Leray-Trudinger estimate of [31]. See [1], [16], [19] and [30] for examples where the range of γ has to be in a closed interval for the purposes there.

Our isoperimetric inequality with radial densities will be a direct consequence of Theorem A: with

$$\mathcal{L}^n\big(E; |x|^{-n}X^{1+\gamma n/(n-1)}(|x|/R_\Omega)\big) := \int_E |x|^{-n}X^{1+\gamma n/(n-1)}(|x|/R_\Omega)\,\mathrm{d}x,$$

whenever $E \subseteq \Omega$ is $\mathcal{L}^n$-measurable, we have

Corollary A. *Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, containing the origin and set $R_\Omega := \sup_{x\in\Omega} |x|$. For all $\gamma \in (0, n-1]$ there holds*

$$\begin{aligned}(1.6)\qquad &\mathcal{P}\big(E; |x|^{1-n}X^\gamma(|x|/R_\Omega)\big)\\ &\quad\geq \mathfrak{C}_{n,\gamma}\Big(\mathcal{L}^n\big(E; |x|^{-n}X^{1+\gamma n/(n-1)}(|x|/R_\Omega)\big)\Big)^{1-1/n},\end{aligned}$$

for any $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ satisfying $\mathcal{P}(E; |x|^{1-n}X^\gamma(|x|/R_\Omega)) :=$

$$\sup\bigg\{\int_E \operatorname{div}\varphi\,\mathrm{d}x :\ \varphi \in C_c^1(\Omega;\mathbb{R}^n),\ |\varphi(x)| \leq |x|^{1-n}X^\gamma(|x|/R_\Omega) \text{ in } \Omega\bigg\} < \infty.$$

Moreover, equality holds if E is a ball centered at the origin.

1.2. Applications

The choice $b = a + 1$ in (1.3) leads to the following L^1 weighted Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|\nabla f|}{|x|^{s-1}}\, dx \geq (n-s) \int_{\mathbb{R}^n} \frac{|f|}{|x|^s}\, dx \quad \forall\, f \in C_c^1(\mathbb{R}^n), \tag{1.7}$$

whenever $s \in [1, n)$ (here we have switched from "b" to "s" to be consistent with the notation in [29]). Actually, it takes only an integration by parts to see that for any $s \in \mathbb{R} \setminus \{n\}$ there holds

$$\int_{\mathbb{R}^n} \frac{|\nabla f|}{|x|^{s-1}}\, dx \geq |n-s| \int_{\mathbb{R}^n} \frac{|f|}{|x|^s}\, dx, \tag{1.8}$$

valid for all $f \in C_c^1(\mathbb{R}^n)$ if $s < n$, or all $f \in C_c^1(\mathbb{R}^n \setminus \{0\})$ if $s > n$. Moreover, it is easy to check when $s < n$, that equality holds for any nonnegative, radially decreasing function. In contrast, if $s > n$ it is known (see [29, Section 2.1]) that the constant appearing in (1.8) is again optimal but the inequality itself can be improved. More precisely, let Ω be any domain in $\mathbb{R}^n$ containing the origin. For $s \geq n$ set

$$I[f] := \int_{\Omega} \frac{|\nabla f|}{|x|^{s-1}}\, dx - (s-n) \int_{\Omega} \frac{|f|}{|x|^s}\, dx, \quad f \in C_c^1(\Omega \setminus \{0\}). \tag{1.9}$$

We will establish the following improvement for (1.8).

Theorem B *Let Ω be a bounded domain in $\mathbb{R}^n$ containing the origin, and set $R_\Omega := \sup_{x\in\Omega} |x|$. Then for all $\gamma > 0$, $s \geq n$, and any $f \in C_c^1(\Omega \setminus \{0\})$, it holds that*

$$\begin{aligned} I[f] \geq{}& \frac{\gamma}{R_\Omega^{s-n}} \int_\Omega \frac{|f|}{|x|^n} X^{1+\gamma}\Big(\frac{|x|}{R_\Omega}\Big)\, dx \\ &+ \frac{\mathfrak{C}_{n,\gamma}}{R_\Omega^{s-n}} \left(\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big)\, dx \right)^{1-1/n}, \end{aligned} \tag{1.10}$$

where the second term on the right fails to appear when $\gamma = 0$.

Remark 1.2. For the first term on the right, we already know (see [29, Remark 2.6]) that it fails to appear when $\gamma = 0$. Hence the above inequality includes the optimal homogeneous weighted norm improvement and the optimal critical Sobolev weighted norm improvement at the same time.

Other applications include a p-version of Theorem A with $p \in (1, n)$ (see Corollary 4.1 and Theorem 4.3 for the best constant) and also a weighted Pólya-Szegö inequality (see Theorem 5.4).

Throughout the rest of this paper, Ω denotes a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, containing the origin and $R_\Omega := \sup_{x\in\Omega} |x|$. Furhermore, $\mathcal{L}^n$ stands for the Lebesgue measure in $\mathbb{R}^n$ and σ for the $n-1$-dimensional Hausdorff measure in $\mathbb{R}^n$. $B_r(x)$ is the open ball in $\mathbb{R}^n$ having radius $r > 0$ and centre at $x \in \mathbb{R}^n$; $\partial B_r(x)$ is its boundary. In particular $\mathbb{B}^n := B_1(0)$ and $\mathbb{S}^{n-1} := \partial B_1(0)$. Also, $\omega_n := \mathcal{L}^n\big(B_1(x)\big)$ and so $\sigma\big(\partial B_1(x)\big) = n\omega_n$.

2. Proof of Theorem A and Corollary A

The proof is based on the ideas in [9] and [21], as applied in [3] and [1]. We start noting that since $0 \in \Omega$, given $f \in C_c^1(\Omega)$ we have $f \in C_c^1(B_{R_\Omega}(0))$ and hence it suffices to establish (1.5) with $\Omega = B_{R_\Omega}(0)$. Moreover, being invariant under scaling, it is enough to establish it for $R_\Omega = 1$.

Consider the transformation $\overline{\mathbb{B}^n} \ni x \mapsto (t, \theta) \in [1, \infty] \times \mathbb{S}^{n-1}$ given by

$$t := |x|^{1-n}, \quad \theta := |x|^{-1}x.$$

Then $t_{x_i} = -(n-1)t^{n/(n-1)}\theta_i$, $\theta_{x_i} = t^{1/(n-1)}(e_i - \theta_i\theta)$, and writing $g(t, \theta)$ for $f(x)$ we have

$$\begin{aligned} f_{x_i} &= g_t t_{x_i} + \nabla_\theta g \cdot \theta_{x_i} \\ &= -(n-1)t^{n/(n-1)} g_t \theta_i + t^{1/(n-1)} g_{\theta_i}. \end{aligned}$$

Altogether,

$$|\nabla f| = (n-1)t^{n/(n-1)}\Big(g_t^2 + \big((n-1)t\big)^{-2}|\nabla_\theta g|^2\Big)^{1/2}.$$

Also, the absolute value of the determinant of the Jacobian matrix of this transformation is

$$J(t, \theta) = (n-1)^{-1} t^{-1-n/(n-1)},$$

therefore, with $\mathcal{A} := \big\{g \in C^1([1,\infty)\times\mathbb{S}^{n-1})\setminus\{0\} :\ g(1,\theta)=0\big\}$, we have

(2.1)

$$n\omega_n^{1/n} = \inf_{\substack{f\in C_c^1(\mathbb{B}^n)\\ f\not\equiv 0}} \frac{\int_{\mathbb{B}^n}|\nabla f|\ \mathrm{d}x}{\Big(\int_{\mathbb{B}^n}|f|^{n/(n-1)}\ \mathrm{d}x\Big)^{1-1/n}}$$

$$= (n-1)^{1-1/n}\inf_{g\in\mathcal{A}} \frac{\int_1^\infty\int_{\mathbb{S}^{n-1}} t^{-1}\Big(g_t^2+\big((n-1)t\big)^{-2}|\nabla_\theta g|^2\Big)^{1/2}\ \mathrm{d}\sigma(\theta)\mathrm{d}t}{\Big(\int_1^\infty\int_{\mathbb{S}^{n-1}} t^{-1-n/(n-1)}|g|^{n/(n-1)}\ \mathrm{d}\sigma(\theta)\mathrm{d}t\Big)^{1-1/n}}.$$

Next we define

$$\mathfrak{C} := \inf_{\substack{f\in C_c^1(\mathbb{B}^n)\\ f\not\equiv 0}} \frac{\int_{\mathbb{B}^n}|x|^{1-n}|\nabla f|X^\gamma(|x|)\ \mathrm{d}x}{\Big(\int_{\mathbb{B}^n}|x|^{-n}|f|^{n/(n-1)}X^{1+\gamma n/(n-1)}(|x|)\ \mathrm{d}x\Big)^{1-1/n}}.$$

Consider this time the transformation $\overline{\mathbb{B}^n}\ni x\mapsto(\tau,\theta)\in[1,\infty]\times\mathbb{S}^{n-1}$ given by

$$\tau := X^{-\gamma}(|x|) = (1-\log|x|)^\gamma, \quad \theta := |x|^{-1}x.$$

Then $\tau_{x_i} = -\gamma\tau^{1-1/\gamma}e^{\tau^{1/\gamma}-1}\theta_i$, $\theta_{x_i} = e^{\tau^{1/\gamma}-1}(e_i-\theta_i\theta)$, and writing $h(\tau,\theta)$ for $f(x)$ we get

$$\begin{aligned} f_{x_i} &= h_\tau\tau_{x_i} + \nabla_\theta h\cdot\theta_{x_i}\\ &= e^{\tau^{1/\gamma}-1}\big(-\gamma\tau^{1-1/\gamma}h_\tau\theta_i + h_{\theta_i}\big). \end{aligned}$$

These imply

$$|\nabla f| = e^{\tau^{1/\gamma}-1}\Big(\gamma^2\tau^{2(1-1/\gamma)}h_\tau^2 + |\nabla_\theta h|^2\Big)^{1/2}.$$

The absolute value of the determinant of the Jacobian matrix of the transformation is

$$J(\tau,\theta) = \frac{1}{\gamma}\tau^{1/\gamma-1}e^{-n(\tau^{1/\gamma}-1)},$$

and taking into account that the transformation says $|x| = e^{1-\tau^{1/\gamma}}$ and $X^\gamma(|x|) = \tau^{-1}$, an elementary computation implies

(2.2)
$$\mathfrak{C} = \gamma^{1-1/n}\inf_{h\in\mathcal{A}} \frac{\int_1^\infty\int_{\mathbb{S}^{n-1}} \tau^{-1}\Big(h_\tau^2+\big(\gamma\tau^{1-1/\gamma}\big)^{-2}|\nabla_\theta h|^2\Big)^{1/2}\ \mathrm{d}\sigma(\theta)\mathrm{d}\tau}{\Big(\int_1^\infty\int_{\mathbb{S}^{n-1}} \tau^{-1-n/(n-1)}|h|^{n/(n-1)}\ \mathrm{d}\sigma(\theta)\mathrm{d}\tau\Big)^{1-1/n}}.$$

To compare the two infima in (2.1) and (2.2) we observe that since $\gamma \in (0, n-1]$ and $\tau \geq 1$, we know

$$\gamma\tau^{1-1/\gamma} \leq (n-1)\tau.$$

Hence we may combine these equations to conclude

$$\mathfrak{C} \geq n\omega_n^{1/n}\Big(\frac{\gamma}{n-1}\Big)^{1-1/n}.$$

The rest is a routine procedure. Supposing that $R \in (0, R_\Omega)$ we choose $E = B_R(0)$ and $f = \chi_{B_R(0)}$, the characteristic function of $B_R(0)$. Translating for the moment $|\nabla f|\ \mathrm{d}x$ as the variation measure of f, it is known in this case that both

$$\int_\Omega \frac{|\nabla f|}{|x|^{n-1}} X^\gamma\Big(\frac{|x|}{R_\Omega}\Big)\ \mathrm{d}x \quad \text{and} \quad \mathcal{P}\big(E; |x|^{1-n}X^\gamma(|x|/R_\Omega)\big),$$

are equal to

$$\begin{aligned}\int_{\partial B_R(0)} |x|^{1-n}X^\gamma(|x|/R_\Omega)\ \mathrm{d}\sigma &= R^{1-n}X^\gamma(R/R_\Omega)\sigma\big(\partial B_R(0)\big)\\ &= n\omega_n X^\gamma(R/R_\Omega).\end{aligned}$$

On the other hand

$$\begin{aligned}\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big)\ \mathrm{d}x &= \int_{B_R(0)} \frac{1}{|x|^n} X^{1+\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big)\ \mathrm{d}x\\ &= n\omega_n \int_0^R r^{-1}X^{1+\gamma n/(n-1)}\Big(\frac{r}{R_\Omega}\Big)\ \mathrm{d}r.\end{aligned}$$

Noting that $\frac{\mathrm{d}}{\mathrm{d}r}\big[X^{\gamma n/(n-1)}(r/R_\Omega)\big] = \big(\gamma n/(n-1)\big)r^{-1}X^{1+\gamma n/(n-1)}(r/R_\Omega)$, we conclude

$$\begin{aligned}\bigg(\int_\Omega \frac{|f|^{n/(n-1)}}{|x|^n} X^{1+\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big)\mathrm{d}x\bigg)^{1-1/n} &= \Big(\omega_n\frac{n-1}{\gamma}\Big)^{1-1/n} X^\gamma(R/R_\Omega)\\ &= \frac{n\omega_n X^\gamma(R/R_\Omega)}{\mathfrak{C}_{n,\gamma}},\end{aligned}$$

as required. This together with a standard mollification of the $BV(\Omega)$ function $\chi_{B_R(0)}$ shows that the constant $\mathfrak{C}_{n,\gamma}$ in (1.5) is the best possible. At the same time we have shown that equality holds in (1.6) whenever E is a ball centered at the origin. $\square$

3. Proof of Theorem B

By pushing further an argument of [29] we deduce here the following improvement for (1.8)

Proposition 3.1. *For all $s \geq n$, $\gamma \geq 0$ and any $f \in C_c^1(\Omega \setminus \{0\})$, it holds that*

$$I[f] \geq \frac{\gamma}{R^{s-n}} \int_\Omega \frac{|f|}{|x|^n} X^{1+\gamma}\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x + \frac{1}{R^{s-n}} \int_\Omega \frac{|\nabla f|}{|x|^{n-1}} X^{\gamma}\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x.$$

Proof. Let $\varphi \in C^1(\Omega \setminus \{0\}; \mathbb{R}^n)$. Integrating by parts we easily get

$$\int_\Omega |\nabla f||\varphi| \, \mathrm{d}x \geq \int_\Omega |f| \mathrm{div}\varphi \, \mathrm{d}x, \tag{3.1}$$

for all $f \in C_c^1(\Omega \setminus \{0\})$. Choosing

$$\varphi(x) = -\Big[1 - \Big(\frac{|x|}{R_\Omega}\Big)^{s-n} X^{\gamma}\Big(\frac{|x|}{R_\Omega}\Big)\Big]|x|^{-s}x, \quad x \in \Omega \setminus \{0\},$$

there holds $|\varphi(x)| = \big(1 - (|x|/R_\Omega)^{s-n} X^{\gamma}(|x|/R_\Omega)\big)|x|^{1-s}$ for $x \in \Omega \setminus \{0\}$, hence

$$\int_\Omega |\nabla f||\varphi| \, \mathrm{d}x = \int_\Omega \frac{|\nabla f|}{|x|^{s-1}} \, \mathrm{d}x - \frac{1}{R^{s-n}} \int_\Omega \frac{|\nabla f|}{|x|^{n-1}} X^{\gamma}\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x. \tag{3.2}$$

On the other hand,

$$\mathrm{div}(v) = (s-n)|x|^{-s} + \gamma R_\Omega^{n-s}|x|^{-n} X^{1+\gamma}\big(|x|/R_\Omega\big), \qquad x \in \Omega \setminus \{0\},$$

thus

$$\int_\Omega |f| \, \mathrm{div}\, \varphi \, \mathrm{d}x = (s-n) \int_\Omega \frac{|f|}{|x|^s} \, \mathrm{d}x + \frac{\gamma}{R_\Omega^{s-n}} \int_\Omega \frac{|f|}{|x|^n} X^{1+\gamma}\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x. \tag{3.3}$$

The proposition now follows by inserting (3.2) and (3.3) in (3.1). □

Proof of Theorem B. The inequality of Theorem B readily follows from Theorem A and Proposition 3.1. In order to establish the optimality assertion, consider the function

$$f_\delta(x) := \chi_{B_\eta \setminus B_\delta}(x);\ x \in \mathbb{R}^n, \tag{3.4}$$

where, for any $r > 0$, by B_r we denote (until the end of this proof), the open ball of radius r centered at the origin. Here $0 < \delta < \eta < R_\Omega$ and η is fixed. It is easily seen that the distributional gradient of f_δ is given by

$$\nabla f_\delta = \vec{\nu}_{\partial B_\delta} \delta_{\partial B_\delta} - \vec{\nu}_{\partial B_\eta} \delta_{\partial B_\eta},$$

where, for any $r > 0$, $\vec{\nu}_{\partial B_r}$ stands for the outward pointing unit normal vector field along ∂B_r, and by $\delta_{\partial B_r}$ we denote the Dirac measure on ∂B_r. Also the variation measure of f_δ is

$$|\nabla f_\delta|\ \mathrm{d}x = \delta_{\partial B_\delta} + \delta_{\partial B_\eta}.$$

Using (3.4) we showed in [29, Remark 2.4 and Remark 2.6] that the constant $s - n$ in the inequality $I[f] \geq 0$ for all $f \in C_c^1(\Omega \setminus \{0\})$ (see (1.9) for the definition of $I[f]$) is optimal and that if $\gamma = 0$, the first term on the right of (1.10) fails to appear. In the same fashion we have

$$\begin{aligned}
&\frac{I[f_\delta]}{\Big(\int_\Omega |f_\delta|^{n/(n-1)} |x|^{-n} X(|x|/R_\Omega)\ \mathrm{d}x \Big)^{1-1/n}} \\
&\quad = \frac{\delta^{1-s}\sigma(\partial B_\delta) + \eta^{1-s}\sigma(\partial B_\eta) - (s-n)n\omega_n \int_\delta^\eta r^{n-1-s}\ \mathrm{d}r}{\Big(n\omega_n \int_\delta^\eta r^{-1} X(r/R_\Omega)\ \mathrm{d}r \Big)^{1-1/n}} \\
&\quad = (n\omega_n)^{1/n} \frac{2\eta^{n-s}}{\Big(\log\big(X(\eta/R_\Omega)\big) - \log\big(X(\delta/R_\Omega)\big) \Big)^{1-1/n}} \\
&\quad = o_\delta(1).
\end{aligned}$$

A standard mollification of the $BV(\Omega)$ function f_δ applies to see that the above computation holds in the limit. This shows that if $\gamma = 0$, the second term on the right of (1.10) fails to appear. □

4. Limiting case in the Caffarelli-Kohn-Nirenberg inequality

Performing a weighted variant of a classical argument of [14] (see also [13, Section 4.5.1, pg 140]) we obtain next a substitute for the end point case ($a = n - p$) of the parameters in the following p-version of (1.3):

if $1 \le p < n$, $0 \le a < n - p$ and $an/(n-p) \le b \le a + p$, then

$$\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|^a}\, \mathrm{d}x \ge \mathcal{C}_{n,p,a,b} \left(\int_{\mathbb{R}^n} \frac{|f|^{(n-b)p/(n-p-a)}}{|x|^b}\, \mathrm{d}x \right)^{(n-p-a)/(n-b)} \tag{4.1}$$
$$\forall\ f \in C^1_c(\mathbb{R}^n).$$

This was also established by Il'in in [22] and reproved later in [8] as a particular case of a multiplicative embedding inequality with weights; the Cafarelli-Kohn-Nirenberg inequality. The best constant in (4.1) when $p > 1$ goes back at least to [21, Lemma 3.3]. It can also be found in [26, Corollary 5.1] as a particular case of a more general inequality involving Lorenz spaces; see [26, Theorem 5.2].

Corollary 4.1. *If $1 \le p < n$, then for all $\alpha \in (1-p, n+1-2p]$ and any $f \in C^1_c(\Omega)$, it holds that*

$$\begin{aligned} &\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\alpha\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \\ &\qquad \ge C^p_{n,\alpha,p} \left(\int_\Omega \frac{|f|^{np/(n-p)}}{|x|^n} X^{1+(\alpha+p-1)n/(n-p)}\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \right)^{1-p/n}, \end{aligned} \tag{4.2}$$

where $C_{n,\alpha,p} = (n-p)n\omega_n^{1/n}\big((\alpha+p-1)/(n-p)\big)^{1-1/n}/\big(p(n-1)\big)$.

Proof. For $p = 1$ this is (1.5). Let $1 < p < n$. We replace $|f|$ by $|f|^\theta$ in (1.5), where $\theta > 1$ will be selected below. With $\beta := 1 + \gamma n/(n-1)$, we find

$$\begin{aligned} &\mathfrak{C}_{n,\gamma} \left(\int_\Omega \frac{|f|^{\theta n/(n-1)}}{|x|^n} X^\beta\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \right)^{1-1/n} \le \theta \int_\Omega \frac{|f|^{\theta-1}|\nabla f|}{|x|^{n-1}} X^\gamma\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \\ &= \theta \int_\Omega \Big\{ \frac{|f|^{\theta-1}}{|x|^{n(p-1)/p}} X^{\beta(p-1)/p}\Big(\frac{|x|}{R_\Omega}\Big) \Big\} \Big\{ \frac{|\nabla f|}{|x|^{(n-p)/p}} X^{\gamma-\beta(p-1)/p}\Big(\frac{|x|}{R_\Omega}\Big) \Big\}\, \mathrm{d}x \\ &\le \theta \left(\int_\Omega \frac{|f|^{(\theta-1)p/(p-1)}}{|x|^n} X^\beta\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \right)^{1-1/p} \left(\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^{\gamma p-\beta(p-1)}\Big(\frac{|x|}{R_\Omega}\Big)\, \mathrm{d}x \right)^{1/p}. \end{aligned}$$

Choose θ so that $\theta n/(n-1) = (\theta-1)p/(p-1)$. Then $\theta n/(n-1) = np/(n-p)$. Thus

$$\frac{\mathfrak{C}_{n,\gamma}}{\theta}\left(\int_\Omega \frac{|f|^{np/(n-p)}}{|x|^n} X^\beta\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x\right)^{1/p-1/n} \leq \left(\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\alpha\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x\right)^{1/p},$$

where we have set $\alpha := \gamma p - \beta(p-1)$. Since $\beta = 1 + \gamma n/(n-1)$, we have

$$\gamma = (\alpha + p - 1)(n-1)/(n-p).$$

From this, condition $0 < \gamma \leq n-1$ is translated to $1 - p < \alpha \leq n + 1 - 2p$ and also

$$\beta = 1 + \gamma n/(n-1) = 1 + (\alpha + p - 1)n/(n-p),$$

as required. □

Remark 4.2. The above estimate was known only for $\alpha = 0$ when $p = 2$ (see [16, Lemma 3.2]) and for $\alpha = 2 - p$ when $p \in (1,2)$ (see [19, Section 3]).

The best constant in (4.2) for $\alpha = 0$ when $p = 2$ was found in [1], while for $\alpha = 2 - p$ when $p \in (1,2)$ and $n \geq 3$ can be extracted from the computations performed in [19, Section 3]. We can obtain the best constant for the whole range of the parameters n, α, p introduced in the above corollary, by arguing as in the proof of Theorem A.

Theorem 4.3. *If $1 \leq p < n$, then for all $\alpha \in (1-p, n+1-2p]$, the best constant in* (4.2) *is given by*

$$\mathfrak{S}_{n,\alpha,p} = \Big(\frac{\alpha + p - 1}{n-p}\Big)^{1-1/n} S_{n,p},$$

where $S_{n,p}$ is the best constant in the Sobolev inequality (see [2] and [33]).

Proof. To avoid repetition with the proof of Theorem A, we present only the basic tasks towards the proof of (4.2) and leave their verification to the reader. As before, we may assume $f \in C_c^1(\mathbb{B}^n)$. Consider the transformation

$\overline{\mathbb{B}^n} \ni x \mapsto (t, \theta) \in [1, \infty] \times \mathbb{S}^{n-1}$ given by

$$t := |x|^{p-n}, \quad \theta := |x|^{-1}x.$$

Then, working as in the proof of Theorem A, we obtain

(4.3)

$$\begin{aligned} S^p_{n,p} &= \inf_{\substack{f\in C^1_c(\mathbb{B}^n)\\ f\not\equiv 0}} \frac{\int_{\mathbb{B}^n} |\nabla f|^p \, \mathrm{d}x}{\Big(\int_{\mathbb{B}^n} |f|^{np/(n-p)} \, \mathrm{d}x\Big)^{1-p/n}} \\ &= (n-p)^{p(n-1)/n} \inf_{g\in\mathcal{A}} \frac{\int_1^\infty \int_{\mathbb{S}^{n-1}} t^{p-2}\Big(g_t^2 + \big((n-p)t\big)^{-2}|\nabla_\theta g|^2\Big)^{p/2} \, \mathrm{d}\sigma(\theta)\mathrm{d}t}{\Big(\int_1^\infty \int_{\mathbb{S}^{n-1}} t^{-1-n/(n-p)}|g|^{np/(n-p)} \, \mathrm{d}\sigma(\theta)\mathrm{d}t\Big)^{1-p/n}}, \end{aligned}$$

where $\mathcal{A}$ is as in the proof of Theorem A.

Next we define

$$\mathfrak{S}^p_{n,\alpha,p} := \inf_{\substack{f\in C^1_c(\mathbb{B}^n)\\ f\not\equiv 0}} \frac{\int_{\mathbb{B}^n} |x|^{p-n}|\nabla f|^p X^\alpha(|x|) \, \mathrm{d}x}{\Big(\int_{\mathbb{B}^n} |x|^{-n}|f|^{np/(n-p)} X^{1+(\alpha+p-1)n/(n-p)}(|x|) \, \mathrm{d}x\Big)^{1-p/n}}.$$

Consider this time the transformation $\overline{\mathbb{B}^n} \ni x \mapsto (\tau, \theta) \in [1, \infty] \times \mathbb{S}^{n-1}$ given by

$$\tau := X^{-\alpha-p+1}(|x|) = (1 - \log|x|)^{\alpha+p-1}, \quad \theta := |x|^{-1}x.$$

It is not difficult to see that this gives

(4.4)

$$\begin{aligned} &\frac{\mathfrak{S}^p_{n,\alpha,p}}{(\alpha+p-1)^{p(1-1/n)}} \\ &= \inf_{h\in\mathcal{A}} \frac{\int_1^\infty \int_{\mathbb{S}^{n-1}} \tau^{p-2}\Big(h_\tau^2 + \big((\alpha+p-1)\tau^{1-1/(\alpha+p-1)}\big)^{-2}|\nabla_\theta h|^2\Big)^{p/2} \, \mathrm{d}\sigma(\theta)\mathrm{d}\tau}{\Big(\int_1^\infty \int_{\mathbb{S}^{n-1}} t^{-1-n/(n-p)}|h|^{np/(n-p)} \, \mathrm{d}\sigma(\theta)\mathrm{d}t\Big)^{1-p/n}}. \end{aligned}$$

To compare the two infima in (4.3) and (4.4) we observe that since $\alpha \in (1-p, n+1-2p]$ and $\tau \geq 1$, we know

$$(\alpha+p-1)\tau^{1-1/(\alpha+p-1)} \leq (n-p)\tau.$$

Hence we may combine these equations to conclude

$$\mathfrak{S}_{n,\alpha,p} \geq \Big(\frac{\alpha+p-1}{n-p}\Big)^{1-1/n} S_{n,p}.$$

It remains to show that the reverse inequality is also true. This is a consequence of the fact that the infimum in (4.3) is attained by radially symmetric functions (see [2] and [33]), which in turn means the value of the infimum in (4.3) is the same if we consider admissible functions depending on t only. With this in mind we read

$$\mathfrak{S}_{n,\alpha,p} \leq \mathfrak{S}^{(\mathrm{radial})}_{n,\alpha,p} = \Big(\frac{\alpha+p-1}{n-p}\Big)^{1-1/n} S^{(\mathrm{radial})}_{n,p} = \Big(\frac{\alpha+p-1}{n-p}\Big)^{1-1/n} S_{n,p},$$

and the proof is complete. □

Remark 4.4. The analogous remark of Remark 1.1 applies to the above theorem.

5. A Pólya-Szegö inequality

In this section, whenever $x \in \Omega \setminus \{0\}$ and $\gamma \in (0, n-1]$, we write for convenience

$$v(x) := |x|^{-n} X^{1+\gamma n/(n-1)}(|x|/R_\Omega),$$
$$w(x) := |x|^{1-n} X^{\gamma}(|x|/R_\Omega).$$

Definition 5.1. For any $\mathcal{L}^n$-measurable $E \subseteq \Omega$ we define successively:

(i) $E^\star \subset \mathbb{R}^n$ to be the ball centered at the origin and satisfying

$$\mathcal{L}^n(E; v) = \mathcal{L}^n(E^\star; v), \tag{5.1}$$

(ii) $\chi^\star_E : \mathbb{R}^n \mapsto \{0,1\}$ to be the characteristic function of $E^\star$; that is $\chi^\star_E := \chi_{E^\star}$.

(iii) the v-weighted rearrangement of a Borel measurable $f : \Omega \to \mathbb{R}$ given by

$$f^\star(x) := \int_0^\infty \chi^\star_{\{|f|>t\}}(x)\ \mathrm{d}t.$$

Remark 5.2. Since Ω contains the origin, given $E \subseteq \Omega$, we know $E^\star \subseteq B_{R_\Omega}(0)$. Hence the radius of $E^\star$ never exceeds R_Ω. As a consequence, the argument of X on the right hand side of (5.1) is well defined.

Remark 5.3. The function $f^\star$ is nonnegative, measurable, radial and radially non-increasing. Moreover, $\{|f|>t\}^\star=\{f^\star>t\}$ for all $t\geq 0$, which together with (5.1) implies the weighted equimeasurability formula

$$\mathcal{L}^n\big(\{|f|>t\};v\big)=\mathcal{L}^n\big(\{f^\star>t\};v\big)\quad \text{for all } t\geq 0. \tag{5.2}$$

The two statements of Corollary A together with (5.1) imply

$$\mathcal{P}\big(E;w\big)\geq \mathfrak{C}_{n,\gamma}\big(\mathcal{L}^n(E;v)\big)^{1-1/n}=\mathfrak{C}_{n,\gamma}\big(\mathcal{L}^n(E^\star;v)\big)^{1-1/n}=\mathcal{P}\big(E^\star;w\big).$$

In particular, if $E\subseteq\Omega$ is a sufficiently smooth domain this reads

$$\int_{\partial E} w\ \mathrm{d}\sigma\geq\int_{\partial E^\star} w\ \mathrm{d}\sigma. \tag{5.3}$$

With this at hand, performing a weighted variant of a standard argument (we follow the presentation of [17, Theorem 3.1] for this), we establish here the following Pólya-Szegö inequality with radial density.

Theorem 5.4. *Let $f\in C^1_c(\Omega)$, $p\geq 1$ and $\gamma\in(0,n-1]$. Then*

$$\int_\Omega\frac{|\nabla f|^p}{|x|^{n-p}}X^\vartheta\Big(\frac{|x|}{R_\Omega}\Big)\ \mathrm{d}x\geq\int_{\Omega^\star}\frac{|\nabla f^\star|^p}{|x|^{n-p}}X^\vartheta\Big(\frac{|x|}{R_\Omega}\Big)\ \mathrm{d}x, \tag{5.4}$$

where ϑ is given by $\vartheta:=\gamma p-\big(1+\gamma n/(n-1)\big)(p-1)$.

Proof. We first recall some facts from basic geometric measure theory. In particular we are going to use the coarea formula; [25, Theorem 1.2.4]. It asserts that

$$\int_\Omega|\nabla u|\Phi\ \mathrm{d}x=\int_0^\infty\int_{\{|u|=t\}}\Phi\ \mathrm{d}\sigma\mathrm{d}t, \tag{5.5}$$

whenever Φ is a Borel measurable nonnegative function in Ω and $u:\Omega\to\mathbb{R}$ is Lipschitz. Rademacher's theorem asserts u is differentiable $\mathcal{L}^n$-a.e. in Ω and setting $\mathcal{N}_u:=\{x\in\Omega:\ \nabla u(x)=0\}$, we may take $\Phi=\chi_{\mathcal{N}_u}$ in (5.5) to obtain

$$0=\int_{\mathcal{N}_u}|\nabla u|\ \mathrm{d}x=\int_0^\infty\sigma\big(\{|u|=t\}\cap\mathcal{N}_u\big)\ \mathrm{d}t.$$

Hence $\sigma\big(\{|u|=t\}\cap\mathcal{N}_u\big)=0$ for $\mathcal{L}^1$-a.e. $t\geq 0$. This implies (5.5) can take the form

$$\int_{\{|u|>s\}} g\ \mathrm{d}x=\int_s^\infty\int_{\{|u|=t\}}\frac{g}{|\nabla u|}\ \mathrm{d}\sigma\mathrm{d}t\quad \text{for } \mathcal{L}^1\text{-a.e. } s\geq 0, \tag{5.6}$$

whenever $g \in L^1(\Omega)$ is nonnegative (simply choose $\Phi = g\chi_{\{|u|>s\}}/(|\nabla u| + \varepsilon)$, $\varepsilon > 0$, in (5.5), and then take the limit as $\varepsilon \to 0$ using the monotone convergence theorem). We performed the above analysis in order to show that the assumption $\operatorname{essinf}|\nabla u| > 0$ for (5.6) to hold true (see [13, Proposition 3, Section 3.4.4]) is redundant. Observe finally that (5.6) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{L}^n\big(\{|u| > t\}; g\big)\Big] = -\int_{\{|u|=t\}} \frac{g}{|\nabla u|}\,\mathrm{d}\sigma \quad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0. \tag{5.7}$$

To prove the theorem we take $u = f$ and $\Phi = |\nabla f|^{p-1}|x|^{p-n}X^{\vartheta}(|x|/R_\Omega)$ in (5.5) to obtain

$$\begin{aligned}
\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}}X^{\vartheta}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x &= \int_0^\infty \int_{\{|f|=t\}} \frac{|\nabla f|^{p-1}}{|x|^{n-p}}X^{\vartheta}(|x|/R_\Omega)\,\mathrm{d}\sigma(x)\mathrm{d}t \\
&\geq \int_0^\infty \bigg(\int_{\{|f|=t\}} w\,\mathrm{d}\sigma\bigg)^p \bigg(\int_{\{|f|=t\}} \frac{1}{|\nabla f|}v\,\mathrm{d}\sigma\bigg)^{1-p} \mathrm{d}t \\
&\geq \int_0^\infty \bigg(\int_{\{f^\star=t\}} w\,\mathrm{d}\sigma\bigg)^p \bigg(-\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{L}^n\big(\{|f| > t\}; v\big)\Big]\bigg)^{1-p} \mathrm{d}t
\end{aligned}$$

where we have applied first Hölder's inequality to reach the middle line, and then (5.3) together with (5.7) for $g = v$ to reach the last line. Note that the level sets of f are smooth enough by virtue of Sard's lemma. Hence, using (5.2)

$$\begin{aligned}
&\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}}X^{\vartheta}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x \\
&\qquad \geq \int_0^\infty \bigg(\int_{\{f^\star=t\}} w\,\mathrm{d}\sigma\bigg)^p \bigg(-\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{L}^n\big(\{f^\star > t\}; v\big)\Big]\bigg)^{1-p} \mathrm{d}t \\
&\qquad = \int_0^\infty \bigg(\int_{\{f^\star=t\}} w\,\mathrm{d}\sigma\bigg)^p \bigg(\int_{\{f^\star=t\}} \frac{1}{|\nabla f^\star|}v\,\mathrm{d}\sigma\bigg)^{1-p} \mathrm{d}t,
\end{aligned}$$

because of (5.7) for $g = v$ but with $u = f^\star$ this time. On the other hand, taking $u = f^\star$ and $\Phi = |\nabla f^\star|^{p-1}|x|^{p-n}X^{\vartheta}(|x|/R_\Omega)$ in (5.5)

$$\begin{aligned}
\int_{\Omega^\star} \frac{|\nabla f^\star|^p}{|x|^{n-p}}X^{\vartheta}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x &= \int_0^\infty \int_{\{f^\star=t\}} \frac{|\nabla f^\star|^{p-1}}{|x|^{n-p}}X^{\vartheta}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}\sigma(x)\mathrm{d}t \\
&= \int_0^\infty \bigg(\int_{\{f^\star=t\}} w\,\mathrm{d}\sigma\bigg)^p \bigg(\int_{\{f^\star=t\}} \frac{1}{|\nabla f^\star|}v\,\mathrm{d}\sigma\bigg)^{1-p} \mathrm{d}t,
\end{aligned}$$

since $|\nabla f^\star|$ is constant on the level sets of $f^\star$. $\square$

As a final result we will apply Theorem 5.4 to produce an embedding inequality in weighted Lorentz spaces (see [25, Section 4.8]). This is an improvement of the embedding implied by Corollary 4.1 with $\alpha = \vartheta$. Recall that the weighted Lorentz space $L(P, Q; v)$, $P > 1, Q \geq 1$, is defined through the seminorm

$$[f]_{L(P,Q;v)} := \left(\int_0^{\mathcal{L}^n(\Omega;v)} \big(f^*(s)\big)^Q \, \mathrm{d}(s^{Q/P}) \right)^{1/Q},$$

where

$$f^*(s) := \int_0^\infty \chi_{\left[0, \mathcal{L}^n(\{|f|>t\};v)\right)}(s) \, \mathrm{d}t.$$

Corollary 5.5. *Under the assumptions of Theorem 5.4, there holds*

$$\int_\Omega \frac{|\nabla f|^p}{|x|^{n-p}} X^\vartheta\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x \geq \omega_n^{p/n} \Big(\frac{\gamma}{n-1}\Big)^{p(1-1/n)} \Big(\frac{n-p}{p}\Big)^{p-1} \frac{n}{p} [f]^p_{L(np/(n-p),p;v)}.$$

Proof. By the fact that

$$\frac{\mathrm{d}}{\mathrm{d}r}\big[X^{\gamma n/(n-1)}(r/R_\Omega)\big] = \big(\gamma n/(n-1)\big) r^{-1} X^{1+\gamma n/(n-1)}(r/R_\Omega),$$

we find first

$$\mathcal{L}^n\big(B_{|x|}(0); v\big) = \frac{n-1}{\gamma} \omega_n X^{\gamma n/(n-1)}\Big(\frac{|x|}{R_\Omega}\Big), \quad x \in \Omega.$$

Hence, noting $f^\star(x) = f^*\big(\mathcal{L}^n\big(B_{|x|}(0); v\big)\big)$, we compute

$$\begin{aligned}
&\int_{\Omega^\star} \frac{|\nabla f^\star|^p}{|x|^{n-p}} X^\vartheta\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x \\
&= \int_{\Omega^\star} \frac{\big|[f^*]'\big(\mathcal{L}^n\big(B_{|x|}(0); v\big)\big) \nabla_x \mathcal{L}^n\big(B_{|x|}(0); v\big)\big|^p}{|x|^{n-p}} X^\vartheta\Big(\frac{|x|}{R_\Omega}\Big) \, \mathrm{d}x \\
&= (n\omega_n)^{p+1} \int_0^R \Big|[f^*]'\Big(\frac{n-1}{\gamma}\omega_n X^{\gamma n/(n-1)}\Big(\frac{r}{R_\Omega}\Big)\Big)\Big|^p X^{\gamma p+1+\gamma n/(n-1)}\Big(\frac{r}{R_\Omega}\Big) \, \frac{\mathrm{d}r}{r} \\
&= (n\omega_n^{1/n})^p \Big(\frac{\gamma}{n-1}\Big)^{p(1-1/n)} \int_0^{\mathcal{L}^n(\Omega^\star;v)} \Big|\frac{\mathrm{d}}{\mathrm{d}s}[f^*(s)]\Big|^p s^{p(1-1/n)} \, \mathrm{d}s,
\end{aligned}$$

where R is the radius of the ball $\Omega^\star$. The one dimensional weighted Hardy inequality (see [25, Section 1.3.1]) applies to deduce

$$\int_{\Omega^\star} \frac{|\nabla f^\star|^p}{|x|^{n-p}} X^{\vartheta}\Big(\frac{|x|}{R_\Omega}\Big)\,\mathrm{d}x \\ \geq \omega_n^{p/n}\Big(\frac{\gamma}{n-1}\Big)^{p(1-1/n)}\Big(\frac{n-p}{p}\Big)^p \int_0^{\mathcal{L}^n(\Omega^\star;v)} \big(f^*(s)\big)^p s^{-p/n}\,\mathrm{d}s,$$

which is the desired estimate. □

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