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Positive mass theorem for initial data sets with corners along a hypersurface

Aghil Alaee and Shing-Tung Yau

We prove positive mass theorem with angular momentum and charges for axially symmetric, simply connected, maximal, complete initial data sets with two ends, one designated asymptotically flat and the other either (Kaluza-Klein) asymptotically flat or asymptotically cylindrical, for 4-dimensional Einstein-Maxwell theory and 5-dimensional minimal supergravity theory which metrics fail to be C^1 and second fundamental forms and electromagnetic fields fail to be C^0 across an axially symmetric hypersurface Σ. Furthermore, we remove the completeness and simple connectivity assumptions in this result and prove it for manifold with boundary such that the mean curvature of the boundary is non-positive.

1. Introduction

In 2007, Dain proved a positive mass theorem with angular momentum $\mathcal{J}$ [17],

$$m \geq \sqrt{|\mathcal{J}|}, \tag{1.1}$$

for 3-dimensional, smooth, axially symmetric, simply connected, complete, maximal initial data sets for vacuum Einstein equations with two ends, one

designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. Moreover, he proved that the rigidity of inequality (1.1) holds for the canonical slice of the extreme Kerr spacetime. The main physical motivation of this inequality is by the standard picture of gravitational collapse through the final state conjecture and the weak cosmic censorship conjecture [16]. From then, the main objective of research in this direction was to remove the unnecessary assumptions in the Dain inequality and extend it to different physical theories and higher dimensions.

The conditions of axial symmetry, vacuum, and two ends are essential to have a non-zero conserved angular momentum. However, Chruściel, Li, and Weinstein [14] extend the proof to initial data sets for multiple black holes with non-negative energy density condition. Moreover, Dain, Khuri, Weinstein, and Yamada [18] replaced the vacuum energy flux condition with vanishing energy flux in the direction of axial symmetry. On the other hand, inequality (1.1) has also been generalized to the Einstein-Maxwell theory by Costa [15], that is

$$m^2 \geq \frac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2}, \tag{1.2}$$

where $Q = \sqrt{Q_e^2 + Q_b^2}$ such that Q_e and Q_b are the electric and magnetic charges, respectively. Moreover, the equality holds for canonical slice of the extreme Kerr-Newman spacetime.

In [36], Schoen and Zhou developed an alternative proof of (charged-) Dain inequality using convexity of reduced harmonic energy. Moreover, Zhou [40] treated the near maximal initial data sets and Cha and Khuri [12] extended the proof of inequality to non-maximal initial data sets assuming a system of equations admits a solution. Then Khuri and Wienstein [28] proved these inequalities with optimal asymptotic decay conditions for multiple charge black hole initial data sets.

In general, the topological censorship theorem says that the domain of outer communication of asymptotically flat black holes must be simply connected [20, 21]. But there is no topological restriction in the inside region of a horizon. This shows that the simple connectivity and two ends assumptions in above inequalities are only technical restriction and not physical features of black holes. Therefore, recently, Khuri, Bryden, and Sokolowsky [11] showed that the strict inequality of (1.2) is true for initial data sets with minimal axially symmetric boundary ∂M^3 without simple connectivity assumption.

In higher dimensions, a generalization of Hawking topology theorem, by Galloway and Schoen [22], states that the cross sections of the event horizon of black holes must be positive Yamabe type. In particular, in five dimensions the only admissible horizon topologies are 3-sphere (or its quotient), $S^1 \times S^2$, or connected sum of these cases. For higher dimensional black holes similar to 3-dimensional black holes, we need to impose additional axial symmetry to have well-defined conserved total angular momenta. In order to accommodate the desired amount of axial symmetry as well as asymptotic flatness, the dimension of spacetime must be restricted to five. Therefore, the first author, Khuri, and Kunduri [3] proved a positive mass theorem with angular momentum and charges for black holes with 3-sphere horizon topology in the minimal supergravity theory,

$$m \geq \frac{27\pi}{8} \frac{(\mathcal{J}_1 + \mathcal{J}_2)^2}{\left(2m + \sqrt{3}|Q|\right)^2} + \sqrt{3}|Q|, \tag{1.3}$$

where $\mathcal{J}_i$ are angular momenta corresponding to $U(1)^2$-symmetry and Q is an electric charge. Moreover, equality holds for canonical slice of the extreme charged Myers-Perry spacetime. In particular, if $Q = 0$, we derive the inequality in the vacuum [1], that is

$$m^3 \geq \frac{27\pi}{8} \left(|\mathcal{J}_1| + |\mathcal{J}_2|\right)^2, \tag{1.4}$$

where the rigidity is canonical slice of the extreme Myers-Perry spacetime. The inequality for black ring with horizon topology $S^1 \times S^2$ is only proved in the Einstein theory because of lack of an explicit extreme black hole solution with two nonzero angular momenta in the minimal supergravity. In particular, if the topology of initial data set is $M^4 = S^2 \times B^2 \# \mathbb{R}^4$ and $\mathcal{J}_1 \geq \mathcal{J}_2$, where $\mathcal{J}_1$ is angular momentum in direction of S^1 and $\mathcal{J}_2$ is angular momentum in the direction of S^2, then the first author, Khuri, and Kunduri [2] also proved the following inequality

$$m^3 \geq \frac{27\pi}{4} |\mathcal{J}_2||\mathcal{J}_1 - \mathcal{J}_2|.$$

Moreover, if $\mathcal{J}_1 > \mathcal{J}_2$, then equality holds if and only if the initial data set arise from the canonical slice of the extreme Pomeransky-Seńkov black ring spacetime. A remarkable feature of this result is that the manifold is not

doubling of a Cauchy surface in the domain of outer communication. Moreover, the second end has asymptotic topology $S^1 \times S^2$ with different decay conditions on the metric components which separate the Kaluza-Klein-asymptotically flat from asymptotically cylindrical.

The purpose of this paper is to prove all of the above inequalities without the assumption of smoothness. Similar to above inequalities, the proof of positive mass theorem by Schoen and the second author was for smooth initial data sets [34, 35]. However, Bray showed that the Riemannian positive mass theorem holds for smooth metric which are Lipschitz on minimal hypersurfaces [10]. Moreover, Miao [32] extend the Riemannian positive mass theorem for metrics with corners across a hypersurface Σ. On the other hand, in the proof of positivity of quasi-local masses by Shi and Tam [37] and Liu and the second author [31], an analogous result holds for spin Riemannian manifold. This result has been generalized for a class of Lipschitz metrics where the complement of some singular set S of metrics has Minkowski dimension less than $n/2$ by Lee [29]. More recently, Lee and LeFloch [30] proved a Riemannian positive mass theorem for spin manifolds which metrics have the Sobolev regularity $C^0 \cap W^{1,n}$. For initial data sets with dominant energy condition, Shibuya [38] showed that the positive mass theorem still holds if it has a causal corner, i.e., dominant energy condition does not violate along corner. Moreover, the authors and Khuri [5] extended the Miao smoothing to a gluing of the Jang deformation of initial data sets with dominant energy condition and proved geometric inequalities for spacetime quasi-local masses. A similar smooth gluing of the Jang deformation developed in the proof of a localized spacetime Penrose inequality by the authors and Lesourd [9].

The present article is concerned with axially symmetric initial data sets with corners across an axially symmetric hypersurface Σ. We show that an initial data set can be deform to initial data set with C^2 metric, C^1 second fundamental form, and C^1 electromagnetic field such that it is also a solution of Maxwell equations, has vanishing energy flux in the direction of symmetries, and has non-negative energy density. Moreover, the angular momentum and charges are conserved along this deformation and its mass converges to the mass of the initial data set with corners. This is enough to implement the current results and achieve above inequalities for non-smooth initial data sets. Finally, we prove the rigidity cases for all of these inequalities.

2. Statement of main results

In order to state the main result we first discuss the appropriate setting and note that we only consider three and four dimensional initial data sets, that is $n = 3, 4$. An initial data set (M^n, g, k, E, B) of 4-dimensional Einstein-Maxwell theory and 5-dimensional minimal supergravity theory are consists of a n-dimensional Riemannian manifold M^n, with metric g, second fundamental form k, electric field E, and magnetic $(n-2)$-form B which is a solution of the following constraint equations.

$$16\pi\mu = R(g) - |k|_g^2 + (\mathrm{tr}_g k)^2 - \frac{2}{(n-2)^2}|E|_g^2 - \frac{2}{(n-2)^3}|B|_g^2, \tag{2.1}$$

$$8\pi J = \mathrm{div}_g\left(k - (\mathrm{tr}_g k)g\right) + \frac{2}{(n-2)^2} \star (B \wedge E), \tag{2.2}$$

$$\mathrm{div}_g E = \frac{n-3}{\sqrt{3}} \star (B \wedge B), \qquad \mathrm{div}_g B = 0, \tag{2.3}$$

where $\star$ is the Hodge star operator with respect to metric g, μ is energy density, and J is energy flux of non-electromagnetic fields. To have a conserved angular momentum, we need axially symmetric initial data set. In particular, we have the following definition.

Definition 2.1. *An initial data set (M^n, g, k, E, B) is axially symmetric if there exists a $U(1)^{n-2}$ subgroup within the group of isometries of the Riemannian manifold (M^n, g) so that the Lie derivative of initial data set vanishes along axial symmetry, that is*

$$\mathcal{L}_{\eta_{(l)}} g = \mathcal{L}_{\eta_{(l)}} k = \mathcal{L}_{\eta_{(l)}} E = \mathcal{L}_{\eta_{(l)}} B = 0, \tag{2.4}$$

where $\eta_{(l)}$, for $l = 1, n-2$, are generators of $U(1)^{n-2}$ group.

The initial data set for an isolated system consist of a Riemannian manifold with asymptotically flat end which means there exists a sub-manifold $M_{\mathrm{end}} \subset M^n$ diffeomorphic to $R^n \setminus B_r(0)$ such that in local coordinate on M_{end} obtained from $R^n \setminus B_r(0)$, we have the following fall-off conditions.

$$g_{ij} = \delta_{ij} + o_s(r^{-\frac{n-2}{2}}), \qquad \partial g_{ij} \in L^2(M_{\mathrm{end}}), \qquad k_{ij} = o_{s-1}(r^{-\frac{n}{2}}), \tag{2.5}$$

$$B^i, E^i = o_{s-1}(r^{-\frac{n}{2}}), \quad B_{ij} = o_{s-1}(r^{-\frac{n}{2}}), \quad \mu, J, J(\eta_{(l)}) \in L^1(M_{\mathrm{end}}), \tag{2.6}$$

for some $s \geq 5$. The assumption $s \geq 5$ is needed for existence of the Brill coordinate system [13] in three dimensions. These fall-off conditions ensure

that the ADM energy, angular momentum, and total electric and magnetic charges are well-defined. The ADM energy of asymptotically flat initial data sets is defined by

$$m = \frac{1}{16\pi} \int_{S^{n-1}_\infty} (g_{ij,i} - g_{ii,j})\, \nu^j \tag{2.7}$$

where S^n_∞ is a coordinate sphere at infinity with unit outer normal ν. The total angular momentum of corresponding rotational symmetry $\eta_{(l)}$ is defined by

$$\mathcal{J}_l = \frac{1}{8\pi} \int_{S^{n-1}_\infty} (k_{ij} - (\mathrm{tr}_g k) g_{ij})\, \nu^j \eta^i_{(l)}. \tag{2.8}$$

Moreover, the total electric and magnetic charges of initial data sets are defined by the following flux integrals at infinity

$$Q_e = \frac{1}{4(n-2)^2\pi} \int_{S^{n-1}_\infty} E_i \nu^i, \qquad Q_b = \frac{1}{4\pi} \int_{S^2_\infty} B_i \nu^i. \tag{2.9}$$

Note that Q_b is only defined for $n = 3$. For $n = 4$, there is no total magnetic charge because B is a 2-form and we cannot integrate it over S^3_∞. However, for black ring initial data set we have $H_2(M^4) \neq 0$ and there exist a local dipole charge, see [4] for definitions of total electric charge and local dipole charge in the minimal supergravity. For non-smooth initial data set we consider the following class with corners along an axially symmetric hypersurface Σ which is a generalization of analogous definition for Riemannian manifolds with corners by Miao [32, Definition 1].

Definition 2.2. *An axially symmetric initial data set $(\mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ admitting corners across an axially symmetric hypersurface Σ is defined to be $\mathcal{G} = (g_-, g_+)$, $\mathcal{K} = (k_-, k_+)$, $\mathcal{E} = (E_-, E_+)$, and $\mathcal{B} = (B_-, B_+)$ where (g_-, k_-, E_-, B_-) and (g_+, k_+, E_+, B_+) are initial data sets on Ω and $M \backslash \bar{\Omega}$ respectively such that the metrics are C^2 up to the boundary and second fundamental form, electric field, and magnetic $(n-2)$-form are C^1 up to the boundary. Moreover, they reduced the same metric, charge potentials, and twist potentials on Σ.*

Note that the second fundamental form, electric field, and magnetic $(n-2)$-form can be discontinuous on Σ. The continuity of potentials are related to conservation of angular momentum and charges for homologous surfaces to S^{n-1}_∞.

Theorem 2.3. *Let $(M^n, \mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ be a n-dimensional axially symmetric, simply connected, maximal initial data set satisfying constraint equations* (2.1), (2.2), *and* (2.3) *and admitting corners across an axially symmetric hypersurface Σ with two ends, one designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. Suppose that $\mu \geq 0$ and $J(\eta_{(l)}) = 0$, for $l = 1, n-2$, in Ω and $M^n \setminus \bar{\Omega}$, and*

$$H_-(\Sigma, g_-) \geq H_+(\Sigma, g_+) \tag{2.10}$$

where $H_-(\Sigma, g_-)$ and $H_-(\Sigma, g_-)$ represent the mean curvature of Σ in $(\bar{\Omega}, g_-)$ and $(M^n \setminus \Omega, g_-)$ with respect to unit normal vectors pointing to the designated asymptotically flat region.

(a) If $n = 3$, then

$$m^2 \geq \frac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2}.$$

Moreover, equality holds if and only if $(\mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ is isometric to the canonical slice of an extreme Kerr-Newman spacetime.

(b) If $n = 4$ and $H_2(M^4) = 0$, then

$$m \geq \frac{27\pi}{8} \frac{(\mathcal{J}_1 + \mathcal{J}_2)^2}{\left(2m + \sqrt{3}|Q|\right)^2} + \sqrt{3}|Q|$$

Moreover, equality holds if and only if $(\mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ is isometric to the canonical slice of an extreme charge-Myers-Perry spacetime.

(c) If $n = 4$, $\mathcal{E} = \mathcal{B} = 0$, $M^4 \cong S^2 \times B^2 \# \mathbb{R}^4$, $\mathcal{J}_1 \geq \mathcal{J}_2$, and the second end is either Kaluza-Klein-asymptotically flat or asymptotically cylindrical, then

$$m^3 \geq \frac{27\pi}{4} |\mathcal{J}_2| |\mathcal{J}_1 - \mathcal{J}_2|$$

Moreover, if $\mathcal{J}_1 > \mathcal{J}_2$, then equality holds if and only if $(\mathcal{G}, \mathcal{K})$ arise from the canonical slice of an extreme Pomeransky-Seńkov black ring spacetime.

For black ring initial data set, the manifold is $M^4 \cong S^2 \times B^2 \# \mathbb{R}^4$ [8, 27]. In this manifold axially symmetric hypersurfaces Σ are diffeomorphic to 3-sphere and $S^1 \times S^2$. Moreover, the second end has topology $S^1 \times S^2$ which can geometrically divide to Kaluza-Klein-asymptotically flat or asymptotically cylindrical.

It should be pointed out that the hypotheses used in Theorem 2.3 is strong, but it is necessary to address the rigidity cases. In general, we remove simple connectivity and completeness for non-smooth initial data sets in three and four dimensions and prove strict inequalities which are generalization of [11].

Corollary 2.4. *Consider the initial data set $(M^n, \mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ in Theorem 2.3 that have a boundary ∂M^n with non-positive mean curvature, with respect to unit normal vectors pointing to the designated asymptotically flat region, instead of a second end and without simple connectivity assumption. Assume the outermost minimal surface Σ_{min} in M^n has one component and enclosed by corner Σ.*

(a) If $n = 3$, then the strict inequality in Theorem 2.3-(a) holds.

(b) If M^4 is spin, $\pi_1(\Sigma_{min}) = 0$, and $H_2(M^4 \setminus W) = 0$, where $\partial W = \partial M^n \cup \Sigma_{min}$, then the strict inequality in Theorem 2.3-(b) holds..

(c) If M^4 is spin, $\mathcal{E} = \mathcal{B} = 0$, $\pi_1(\Sigma_{min}) = \mathbb{Z}$, $H_2(M^4 \setminus W) = \mathbb{Z}$ and $\mathcal{J}_1 \geq \mathcal{J}_2$, then the strict inequality in Theorem 2.3-(c) holds.

The organization of this paper is as follows. In Section 3, first we construct potentials for our initial data sets. Then we show any axially symmetric maximal initial data sets have related $(t - \phi^i)$ symmetric initial data sets with same mass, angular momentum, and charges. Furthermore, we deform initial data set to construct a C^2 metric and C^1 second fundamental, electric field and magnetic $(n-2)$-form such that it is a solution of Maxwell equations and has vanishing energy flux in the direction of axial symmetry. In section 4, we construct a conformal transformation of initial data set such that the conformal data has non-negative energy density as well as to be a solution of Maxwell equations and has vanishing energy flux in the direction of axial symmetry. Finally, we prove the main results.

3. Smooth deformations of initial data sets across Σ

In this section, we assume M^n is a simply connected, asymptotically flat Riemannian manifold with two ends. Before deforming the initial data smoothly we need to construct potentials that characterize angular momentums and charges of initial data sets and we have two remarks about our setting.

Remark 3.1. *The Hodge star operator $\star$ is an isomorphism from p-form on (M, g) to $(p-1)$-form and defined by $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \star 1 = \langle \alpha, \beta \rangle \, dV_g$ for*

p-forms α and β. In particular, $\star^2\alpha = (-1)^{p(n-p)}\alpha$, $\iota_X \star \alpha = \star(\alpha \wedge X)$, and $div_g X = (-1)^{n+1} \star d \star X$, where X is a vector field and for dual 1-form we use same notation.

Remark 3.2. *In order to analyze three and four dimensional initial data sets together, we define vectors with components 1 and $n-2$ for 1-forms, vector fields, and functions. In particular, we write the vector $\eta = (\eta_{(1)}, \eta_{(n-2)})^T$ for generators of $U(1)^{n-2}$ symmetry of n-dimensional initial data sets. Then if $n = 3$, $\eta = \eta_{(1)}$ and if $n = 4$ we have $\eta = (\eta_{(1)}, \eta_{(2)})^T$.*

3.1. Potentials

Consider the initial data set $(M^n, \mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ on region Ω and $M^n \setminus \bar{\Omega}$. Define the following 1-form $\Upsilon_\pm = (\Upsilon^1_\pm, \Upsilon^{n-2}_\pm)^T$ from magnetic $(n-2)$-form $B_\pm$ as following

$$\Upsilon_\pm = \iota_\eta \star_\pm B_\pm = \left(\iota_{\eta_{(1)}} \star_\pm B_\pm, \iota_{\eta_{(n-2)}} \star_\pm B_\pm\right)^T, \tag{3.1}$$

where ι is the interior product and $\star_\pm$ is the Hodge star operation with respect to metrics $g_\pm$. Then the divergence free property of the magnetic $n-2$-form equation (2.3) and Cartan's magic formula shows that the 1-form Υ is closed and by the Poincaré lemma, it is exact. Thus there exist potentials

$$d\psi_\pm = \left(d\psi^1_\pm, d\psi^{n-2}_\pm\right)^T = \Upsilon_\pm. \tag{3.2}$$

Since $B_\pm$ is invariant under the $U(1)^{n-2}$ symmetry, these potentials are invariant under $U(1)^{n-2}$ symmetry. We define another 1-form $\mathcal{Q}_\pm = \iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} \star_\pm E_\pm - \frac{n-3}{\sqrt{3}}\psi^T_\pm \mathfrak{J} d\psi_\pm$ where

$$\mathfrak{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

Observe that vector $\psi^T\mathfrak{J}$ is orthogonal to vector ψ. Then with Cartan's magic formula, Remark 3.1, and Maxwell equations (2.3), we have

$$
\begin{aligned}
d\mathcal{Q}_{\pm} &= d\iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} \star_{\pm} E_{\pm} - \frac{n-3}{\sqrt{3}} d\left(\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) \\
&= \iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} d \star_{\pm} E_{\pm} - \frac{n-3}{\sqrt{3}} d\left(\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) \\
&= \iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} \left(-\frac{n-3}{\sqrt{3}} B_{\pm}\wedge B_{\pm}\right) - \frac{n-3}{\sqrt{3}} d\left(\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) \\
&= -\frac{n-3}{\sqrt{3}}\left(\iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}}\left(B_{\pm}\wedge B_{\pm}\right) + d\left(\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right)\right),
\end{aligned}
\tag{3.4}
$$

where we take $\star_{\pm}$ of Maxwell equation (2.3) to get third line. For $n = 3$, this expression vanishes. Assuming $n = 4$, then using $\star_{\pm}B_{\pm}\wedge\star_{\pm}B_{\pm} = B_{\pm}\wedge B_{\pm}$ for a 2-form on 4-manifold and definition of $\psi_{\pm}$ in (3.2), the expression also vanishes. Hence $\mathcal{Q}_{\pm}$ is closed and there exist a global electric potential $\chi_{\pm}$ so that

$$
d\chi_{\pm} = \iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} \star_{\pm} E_{\pm} - \frac{n-3}{\sqrt{3}}\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}. \tag{3.5}
$$

Moreover, it immediately follows that this potential is invariant under the $U(1)^{n-2}$ symmetry. Next we define another 1-form

$$
\begin{aligned}
\Xi_{\pm} = {} & (n-2)\star_{\pm}\left(k_{\pm}(\eta)\wedge\eta_{(1)}\wedge\eta_{(n-2)}\right) \\
& - \psi_{\pm}\left(d\chi_{\pm} + \frac{n-3}{3\sqrt{3}}\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) - (n-4)\,\chi_{\pm}d\psi_{\pm}
\end{aligned}
\tag{3.6}
$$

Then using again Cartan's magic formula, Remark 3.1, $J(\eta_{(l)}) = 0$, the momentum constraint equation (2.2), and (3.5), we have

$$
\begin{aligned}
d\Xi_{\pm} &= (n-2)d\star_{\pm}\left(k_{\pm}(\eta)\wedge\eta_{(1)}\wedge\eta_{(n-2)}\right) \\
&\quad - d\psi_{\pm}\wedge\left(d\chi_{\pm} + \frac{n-3}{\sqrt{3}}\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) \\
&\quad - (n-4)\,d\chi_{\pm}\wedge d\psi_{\pm} \\
&= (n-2)\iota_{\eta_{(n-2)}}\iota_{\eta_{(1)}} \star_{\pm}\star_{\pm}d\star_{\pm} k_{\pm}(\eta) \\
&\quad - d\psi_{\pm}\wedge\left(d\chi_{\pm} + \frac{n-3}{\sqrt{3}}\psi_{\pm}^{T}\mathfrak{J}d\psi_{\pm}\right) \\
&\quad - (n-4)\,d\chi_{\pm}\wedge d\psi_{\pm}
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&= \frac{-2}{(n-2)} \iota_{\eta_{(n-2)}} \iota_{\eta_{(1)}} \left(\star_{\pm} E_{\pm} \wedge d\psi_{\pm} \right) \\
&\quad - d\psi_{\pm} \wedge \left(d\chi_{\pm} + \frac{n-3}{\sqrt{3}} \psi_{\pm}^{T} \mathfrak{J} d\psi_{\pm} \right) \\
&\quad - (n-4)\, d\chi_{\pm} \wedge d\psi_{\pm} \\
&= \frac{-2}{(n-2)} \left(d\chi_{\pm} + \frac{n-3}{\sqrt{3}} \psi_{\pm}^{T} \mathfrak{J} d\psi_{\pm} \right) \wedge d\psi_{\pm} \\
&\quad - d\psi_{\pm} \wedge \left(d\chi_{\pm} + \frac{n-3}{\sqrt{3}} \psi_{\pm}^{T} \mathfrak{J} d\psi_{\pm} \right) \\
&\quad - (n-4)\, d\chi_{\pm} \wedge d\psi_{\pm} = 0
\end{aligned}
$$

Hence, there exist twist potentials

$$
d\zeta_{\pm} = \Xi_{\pm} \tag{3.8}
$$

where $\zeta_{\pm} = (\zeta_{\pm}^{1}, \zeta_{\pm}^{n-2})^{T}$. We assume the potentials are continuous on Σ and define the following functions

$$
\zeta := \begin{cases} \zeta_{-} & \text{on} \quad \Omega \\ \zeta_{-} = \zeta_{+} & \text{on} \quad \Sigma \\ \zeta_{+} & \text{on} \quad M \backslash \bar{\Omega} \end{cases}, \quad \psi := \begin{cases} \psi_{-} & \text{on} \quad \Omega \\ \psi_{-} = \psi_{+} & \text{on} \quad \Sigma \\ \psi_{+} & \text{on} \quad M \backslash \bar{\Omega} \end{cases}, \tag{3.9}
$$

and

$$
\chi := \begin{cases} \chi_{-} & \text{on} \quad \Omega \\ \chi_{-} = \chi_{+} & \text{on} \quad \Sigma \\ \chi_{+} & \text{on} \quad M \backslash \bar{\Omega} \end{cases}. \tag{3.10}
$$

In particular, a computation using definition of angular momentum (2.8) and charges (2.9) shows that, see [3, 18] for details,

$$
\begin{aligned}
\mathcal{J}_{l} &= \frac{\pi^{n-3}}{4} \left(\zeta_{l}|_{\Gamma_{-}} - \zeta_{l}|_{\Gamma_{+}} \right), \\
Q_{e} &= \frac{\pi^{n-3}}{2^{n-2}} \left(\chi|_{\Gamma_{-}} - \chi|_{\Gamma_{+}} \right), \\
Q_{b} &= \frac{1}{2} \left(\psi|_{\Gamma_{-}} - \psi|_{\Gamma_{+}} \right),
\end{aligned} \tag{3.11}
$$

where Γ_{+} and Γ_{-} are two asymptotic parts of axis of rotation Γ. Since the value of potentials at axis is the same in $\bar{\Omega}$ and $M^{n} \backslash \Omega$, charges and angular

momentum are conserve quantities for any surface homologous to coordinate sphere S^{n-1} at infinity of $M^n \setminus \Omega$.

3.2. $(t - \phi^i)$-symmetric data

In this section, we show that every axially symmetric maximal data set has related $(t-\phi^i)$-symmetric data with same mass, angular momentum, and charges. Consider axially symmetric, C^2 initial data set (M^n, g, k, E, B) with two ends. Then there exist a global Brill coordinate (ρ, z, ϕ^l) such that $\eta_{(l)} = \frac{\partial}{\partial \phi^l}$. This has been proved in three dimensions [13] and it is conjectured to be true in four dimensions [1]. In particular, the metric of axially symmetric initial data sets takes the form

$$g = e^{-2U+2\alpha}\left(d\rho^2 + dz^2\right) + e^{-2U}\lambda_{ij}\left(d\phi^i + A^i_a dy^a\right)\left(d\phi^j + A^j_a dy^a\right), \tag{3.12}$$

for some functions U, α, A^i_a, and a symmetric positive definite matrix $\lambda = [\lambda_{ij}]$ with $\det\lambda = \rho^2$, all independent of (ϕ^1, ϕ^{n-2}) with asymptotic fall-off in [1, 28]. Note that α in equation (3.12) is equal to $\alpha - \log(2\sqrt{\rho^2+z^2})$ in [1]. Moreover, the coordinates should take values in the following ranges $\rho \in [0, \infty)$, $z \in \mathbb{R}$, and $\phi^i \in [0, 2\pi]$, for $i = 1, n-2$. The transformation to spherical coordinate in three dimensions is $\rho = r\sin\theta$ and $z = r\cos\theta$ and in four dimensions is $\rho = \frac{r^2}{2}\sin 2\theta$ and $z = \frac{r^2}{2}\cos 2\theta$. Consider the global frame

$$e_1 = e^{U-\alpha}\left(\partial_\rho - A^i_\rho \partial_{\phi^i}\right), \quad e_2 = e^{U-\alpha}\left(\partial_z - A^i_z\partial_{\phi^i}\right), \quad e_{i+2} = e^U \partial_{\phi^i}, \tag{3.13}$$

for $i = 1, n-2$, with dual co-frame

$$\theta^1 = e^{-U+\alpha}d\rho, \quad \theta^2 = e^{-U+\alpha}dz, \quad \theta^{i+2} = e^{-U}\left(d\phi^i + A^i_a dy^a\right), \tag{3.14}$$

for $i = 1, n-2$. For arbitrary axially symmetric function $\omega = (\omega^1, \omega^{n-2})^T$, we define a 1-form

$$\mathcal{P}^\omega = \frac{1}{\det\Lambda} \star \left(d\omega \wedge \eta_{(1)} \wedge \eta_{(n-2)}\right) \tag{3.15}$$

where $\mathcal{P}^\omega = \left(\mathcal{P}^\omega_1, \mathcal{P}^\omega_{n-2}\right)^T$, $\Lambda = [\Lambda_{ij}] = e^{-2U}[\lambda_{ij}]$ and $\star$ is Hodge star operation with respect to g. Then we have the following decomposition for electric 1-form and second fundamental form

$$E = \mathcal{P}^\chi + \frac{n-3}{\sqrt{3}}\psi^T \mathfrak{J}\mathcal{P}^\psi + \hat{E}, \tag{3.16}$$

$$k(e_i,e_j)=\frac{1}{(n-2)}\left(\eta^T(e_i)\Lambda^{-1}\mathcal{P}(e_j)+\eta^T(e_j)\Lambda^{-1}\mathcal{P}(e_i)\right)+\pi(e_i,e_{j+2}), \tag{3.17}$$

where $i,j=1,\cdots,4$, $\pi(e_k,e_{l+2})=0$, $\hat{E}(e_k)=0$ for $k,l=1,2$ and

$$\mathcal{P}=\mathcal{P}^{\zeta}+\psi\left(\mathcal{P}^{\chi}+\frac{n-3}{3\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^{\psi}\right)+(n-4)\,\chi\mathcal{P}^{\psi} \tag{3.18}$$

For magnetic $(n-2)$-form we have different decompositions related to the dimension. If $n=3$, then we have

$$B=\mathcal{P}^{\psi}+\hat{B}, \tag{3.19}$$

where $\hat{B}(e_k)=0$ for $k=1,2$. If $n=4$, the Killing symmetry implies $\star B(\eta_{(i)},\eta_{(j)})=0$ which leads to $B(e_1,e_2)=0$, and

$$B=B(e_1,e_{k+2})\theta^1\wedge\theta^{k+2}+B(e_2,e_{k+2})\theta^2\wedge\theta^{k+2}+\hat{B}(e_3,e_4)\theta^3\wedge\theta^4, \tag{3.20}$$

such that

$$\begin{aligned}B(e_1,e_{i+2})(t)&=-e^{2U-\alpha}\epsilon^j{}_i\partial_z\psi^j(t),\\ B(e_2,e_{i+2})(t)&=e^{2U-\alpha}\epsilon^j{}_i\partial_\rho\psi^j(t),\end{aligned} \tag{3.21}$$

where $\epsilon_{\delta ij}=\sqrt{\det\lambda}\varepsilon_{ij}$ is volume form associated to λ and $\varepsilon_{ij}=0,\pm1$.

In general, the 2-dimensional distribution $\mathcal{D}^2$ orthogonal to $\eta_{(l)}$ is not integrable for axially symmetric initial data sets. However, the Killing vector fields $\eta_{(l)}$ have 0-dimensional and 1-dimensional fix points and assuming $Ric_g(\eta,\partial_{y^a})=0$, for $a=1,2$ and $(y^1,y^2)=(\rho,z)$, the result of Wald [39, Theorem 7.1.1] implies $\mathcal{D}^2$ is integrable. In particular, if $\mathcal{D}^2$ is integrable, we have the following divergence-free one-form, see [6, Section 2.2] for 4-dimensional case and [39, Appendix C] for 3-dimensional case.

Lemma 3.3. *Let $(M,\bar{g})$ be a n-dimensional Riemannian manifold with $U(1)^{n-2}$ isometry subgroup for $n=3,4$. If 2-dimensional distribution $\mathcal{D}^2$ orthogonal to $\eta_{(l)}$ is integrable, then the following 1-form is divergence free*

$$\mathcal{P}^{\omega}=\frac{1}{\det\Lambda}\star_{\bar{g}}\left(d\omega\wedge\eta_{(1)}\wedge\eta_{(n-2)}\right), \tag{3.22}$$

where $\mathcal{P}^{\omega}=(\mathcal{P}^{\omega_1},\mathcal{P}^{\omega_{n-2}})^T$, $\Lambda=[\Lambda_{ij}]=[\bar{g}(\eta_{(i)},\eta_{(j)})]$ is Gram matrix of the Killing fields, and $\omega=(\omega_1,\omega_{n-2})^T$ is axially symmetric function.

Proof. Since 2-dimensional distribution orthogonal to $\eta_{(l)}$ is integrable, by Frobenius theorem we have $\nabla^i\eta^j = \frac{1}{2}\left(\nabla^i\Lambda\Lambda^{-1}\eta^j - \nabla^j\Lambda\Lambda^{-1}\eta^i\right)$, where ∇ is covariant derivative with respect to $\bar{g}$ and $\eta = (\eta_1, \eta_{n-2})^T$. A straightforward computation then shows that

$$\epsilon_{ijkl}\nabla^i\eta^j_{(1)}\eta^k_{(n-2)} + \epsilon_{ijkl}\eta^j_{(1)}\nabla^i\eta^k_{(n-2)} = \nabla^i \log\det\Lambda\epsilon_{ijkl}\eta^j_{(1)}\eta^k_{(n-2)}, \tag{3.23}$$

where ϵ_{ijkl} is volume form with respect to $\bar{g}$. It follows that

$$\begin{aligned}\operatorname{div}_{\bar{g}}\mathcal{P}^\omega &= -\frac{1}{\det\Lambda}\nabla^i \log\det\Lambda\epsilon_{ijkl}\eta^j_{(1)}\eta^k_{(n-2)}\nabla^l\omega \\ &\quad + \frac{1}{\det\Lambda}\epsilon_{ijkl}\nabla^i\eta^j_{(1)}\eta^k_{(n-2)}\nabla^l\omega \\ &\quad + \frac{1}{\det\Lambda}\epsilon_{ijkl}\eta^j_{(1)}\nabla^i\eta^k_{(n-2)}\nabla^l\omega \\ &\quad + \frac{1}{\det\Lambda}\epsilon_{ijkl}\eta^j_{(1)}\eta^k_{(n-2)}\nabla^i\nabla^l\omega \\ &= 0.\end{aligned} \tag{3.24}$$

The last term vanishes because $\nabla^i\nabla^l\omega = \nabla^l\nabla^i\omega$ and ϵ_{ijkl} is antisymmetric in i and l. □

The class of initial data which has the above condition is called $(t-\phi^i)$-symmetric. This class was defined in three dimensions by Gibbons [23] and was generalized to higher dimensions in [19]. We extend the definition to initial data sets with electromagnetic field.

Definition 3.4. *An axially symmetric initial data set $(M^n, \bar{g}, \bar{k}, \bar{E}, \bar{B})$ is $(t-\phi^i)$-symmetric if $\phi^i \to -\phi^i$, we have $\bar{g} \to \bar{g}$, $\bar{k} \to -\bar{k}$, $\bar{E} \to \bar{E}$, and $\bar{B} \to (-1)^{n-1}\bar{B}$.*

Then we have the following proposition.

Proposition 3.5. *Suppose (M^n, g, k, E, B) is an axially symmetric, maximal initial data set with $\mu \geq 0$ and $J(\eta_{(l)}) = 0$. Then there exists a $(t-\phi^i)$-symmetric initial data set $(M^n, \bar{g}, \bar{k}, \bar{E}, \bar{B})$ with same mass, angular momentum, and charges. Moreover, $\bar{\mu} \geq \mu$ and $\bar{J}(\eta_{(l)}) = 0$.*

Proof. Consider the axially symmetric, maximal initial data set (M^n, g, k, E, B). Then related $(t-\phi^i)$ initial data set $(M^n, \bar{g}, \bar{k}, \bar{E}, \bar{B})$ is obtained as follows. The metric $\bar{g}$ obtained from g by setting $A^i_a = 0$ in equation (3.12). Then we have new global frame $\{e_i\}$ equal to the frame (3.13) without A^i_a terms. Moreover, setting $\pi = \hat{E} = \hat{B} = 0$ in equations (3.17), (3.16),

(3.19), and (3.20), we get $\bar{k}$, $\bar{E}$ and $\bar{B}$, respectively. We define the energy density and energy flux as following

$$16\pi\bar{\mu} = 16\pi\mu + |\pi|_g^2 + |\hat{E}|_g + |\hat{B}|_g + \frac{1}{4}e^{-2\alpha}\lambda_{ij}\left(\partial_\rho A_z^i - \partial_z A_\rho^i\right)\left(\partial_\rho A_z^j - \partial_z A_\rho^j\right), \tag{3.25}$$

$$8\pi\bar{J} = \mathrm{div}_{\bar{g}}\bar{k} + \frac{2}{(n-2)^2}\star_{\bar{g}}\left(\bar{B}\wedge\bar{E}\right). \tag{3.26}$$

It follows that $(M^n, \bar{g}, \bar{k}, \bar{E}, \bar{B})$ is a solution of Hamiltonian constraint equation with $\bar{\mu} \geq \mu$. Next we need to show that $\bar{J}(\eta_{(l)}) = 0$. Since 2-dimensional distribution orthogonal to $\eta_{(l)}$ is integrable, the 1-forms $\mathcal{P}^\chi$, $\mathcal{P}^\psi$, and $\mathcal{P}^\zeta$ defined in Lemma 3.3 with respect to metric $\bar{g}$ are divergence-free. Using axially symmetry condition, the divergence of $\mathcal{P}$ is

$$\begin{aligned}
\mathrm{div}_{\bar{g}}\mathcal{P} &= \sum_{i=1}^{2} e_i(\psi)\left(\mathcal{P}^\chi(e_i) + \frac{n-3}{3\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^\psi(e_i)\right) \\
&\quad + \frac{n-3}{3\sqrt{3}}\psi\sum_{i=1}^{2} e_i\left(\psi^T\right)\mathfrak{J}\mathcal{P}^\psi(e_i) \\
&\quad + (n-4)\sum_{i=1}^{2} e_i(\chi)\,\mathcal{P}^\psi(e_i) \\
&= \sum_{i=1}^{2} e_i(\psi)\left(\mathcal{P}^\chi(e_i) + \frac{n-3}{\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^\psi(e_i)\right) \\
&\quad + (n-4)\sum_{i=1}^{2} e_i(\chi)\,\mathcal{P}^\psi(e_i) \\
&= \sum_{i=1}^{2} e_i(\psi)\left((5-n)\mathcal{P}^\chi(e_i) + \frac{n-3}{\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^\psi(e_i)\right) \\
&= \frac{2}{(n-2)}\sum_{i=1}^{2} e_i(\psi)\left(\mathcal{P}^\chi(e_i) + \frac{n-3}{\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^\psi(e_i)\right),
\end{aligned} \tag{3.27}$$

where we used $e_i(\chi)\,\mathcal{P}^\psi(e_i) = -e_i(\psi)\,\mathcal{P}^\chi(e_i)$. The last equality is true if $n = 3, 4$. On the the hand, we have

$$\begin{aligned}
i_{\eta_{(l)}}\star_{\bar{g}}\left(\bar{B}\wedge\bar{E}\right) &= -\left\langle d\psi^l, \bar{E}\right\rangle_{\bar{g}} \\
&= -\sum_{i=1}^{2} e_i(\psi^l)\left(\mathcal{P}^\chi(e_i) + \frac{n-3}{\sqrt{3}}\psi^T\mathfrak{J}\mathcal{P}^\psi(e_i)\right).
\end{aligned} \tag{3.28}$$

Combining (3.27) and (3.28) shows that

$$\mathrm{div}_{\bar{g}}\mathcal{P} = -\frac{2}{(n-2)} i_{\eta_l} \star_{\bar{g}} (B \wedge E)\,. \tag{3.29}$$

Observe that $\bar{k}$ has the following decomposition

$$\bar{k}(e_i, e_j) = \frac{1}{(n-2)} \left(\eta^T(e_i)\Lambda^{-1}\mathcal{P}(e_j) + \eta^T(e_j)\Lambda^{-1}\mathcal{P}(e_i)\right), \tag{3.30}$$

where $i, j = 1, \cdots, 4$ and clearly $\mathrm{tr}_{\bar{g}}\bar{k} = 0$. Then

$$\mathrm{div}_{\bar{g}}\left(\bar{k}(\eta_{(l)})\right) = \left(\mathrm{div}_{\bar{g}}\bar{k}\right)(\eta_{(l)}) + \frac{1}{2}\left\langle k, \mathcal{L}_{\eta_{(l)}}\bar{g}\right\rangle = \left(\mathrm{div}_{\bar{g}}\bar{k}\right)(\eta_{(l)}). \tag{3.31}$$

Since $\bar{g}\left(\mathcal{P}, \eta_{(l)}\right) = 0$, we have

$$\bar{k}(\eta_{(l)}, \cdot) = \frac{1}{(n-2)}\mathcal{P}. \tag{3.32}$$

Now from equations (3.29), (3.31), and (3.32), we have

$$\bar{J}(\eta_{(l)}) = \left(\mathrm{div}_{\bar{g}}\bar{k}\right)(\eta_{(l)}) + \frac{2}{(n-2)^2} i_{\eta_{(l)}} \star_{\bar{g}} \left(\bar{B} \wedge \bar{E}\right) = 0, \tag{3.33}$$

for $l = 1, n-2$. Next we need to show Maxwell equations for initial data set $(M^n, \bar{g}, \bar{k}, \bar{E}, \bar{B})$. Let $\mathcal{P}^\chi$ be the 1-form given in Lemma 3.3 with respect to metric $\bar{g}$. Then using $\hat{E} = 0$ and (3.16), the divergence of $\bar{E}$ is

$$\mathrm{div}_{\bar{g}}\bar{E} = \mathrm{div}_{\bar{g}}\mathcal{P}^\chi + \frac{n-3}{\sqrt{3}}\mathrm{div}_{\bar{g}}\left(\psi^T\mathfrak{J}\mathcal{P}^\psi\right). \tag{3.34}$$

Using definition of $\bar{B}$, equation (3.21), and Lemma 3.3, we have

$$\begin{aligned}\mathrm{div}_{\bar{g}}(\psi^T\mathfrak{J}\mathcal{P}^\psi) &= \sum_{i=1}^{2} e_i\left(\psi^T\right)\mathfrak{J}\mathcal{P}^\psi(e_i) \\ &= \sum_{i=1}^{2}\left(e_i\left(\psi^1\right)\mathcal{P}^{\psi^2}(e_i) - e_i\left(\psi^2\right)\mathcal{P}^{\psi^1}(e_i)\right) \\ &= \star_{\bar{g}}\left(\bar{B} \wedge \bar{B}\right)\end{aligned} \tag{3.35}$$

In light of this, and the fact that the first term of (3.34) vanishes by Lemma (3.3), it follows that

$$\mathrm{div}_{\bar{g}}\bar{E} = \frac{n-3}{\sqrt{3}} \star_{\bar{g}} \left(\bar{B} \wedge \bar{B}\right). \tag{3.36}$$

Note that when $n = 3$, similarly we can show $\mathrm{div}_{\bar{g}}\bar{B} = 0$, but for $n = 4$, we have magnetic 2-form in global frame $\{e_i\}$. Then, we have

$$\begin{aligned}
\star_{\bar{g}}\bar{B} &= \epsilon_{\bar{g}2(j+2)}{}^{1(i+2)}\bar{B}(e_1, e_{i+2})\theta^2 \wedge \theta^{j+2} \\
&\quad + \epsilon_{\bar{g}1(j+2)}{}^{2(i+2)}\bar{B}(e_2, e_{i+2})\theta^1 \wedge \theta^{j+2} \\
&= \epsilon_{\bar{g}_j}{}^i\bar{B}(e_1, e_{i+2})\theta^2 \wedge \theta^{j+2} - \epsilon_{\bar{g}_j}{}^i\bar{B}(e_2, e_{i+2})\theta^1 \wedge \theta^{j+2} \\
&= -\epsilon_{\delta_j}{}^i\epsilon_\delta{}^j{}_i e_2(\psi^j)\theta^2 \wedge \theta^{j+2} - \epsilon_{\delta_j}{}^i\epsilon_\delta{}^j{}_i e_1(\psi^j)\theta^1 \wedge \theta^{j+2} \\
&= -e_2(\psi^j)\theta^2 \wedge \theta^{j+2} - e_1(\psi^j)\theta^1 \wedge \theta^{j+2}.
\end{aligned} \tag{3.37}$$

Since $e_1(e_2(\psi)) = e_2(e_1(\psi))$, we have $d \star_{\bar{g}} \bar{B} = 0$ which is equivalent to $\mathrm{div}_{\bar{g}}\bar{B} = 0$.

Since charge and twist potentials of axially symmetric and related $(t - \phi^i)$ symmetric data sets are the same, in light of equation (3.11), angular momentum and charges do not change. Moreover, a computation in [7, 13] shows that the ADM mass is only depends on U and α. Since both initial data sets have same U and α, they have the same mass. This complete the proof. □

Remark 3.6. *Assume $(M^n, \mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$ is axially symmetric maximal initial data with corners along an axially symmetric hypersurface Σ as in Theorem 2.3. In each region $(\bar{\Omega}, g_-)$ and $(M^n \setminus \Omega, g_+)$ with related Brill coordinate (ρ, z, ϕ^i), if the axially symmetric surface is constant $r = r_0$ surface Σ, then the mean curvatures are*

$$H_\pm(\Sigma, g_\pm) = \left.\frac{2(n-2)/r + \partial_r\left(\alpha_\pm - 2(n-2)U_\pm\right)}{\sqrt{e^{-2U_\pm + 2\alpha_\pm} + e^{-2U_\pm}\lambda_{ij}A^i_\pm A^j_\pm}}\right|_{r=r_0}, \tag{3.38}$$

where $A^i_\pm = r^{n-3}\left(A^i_{\pm\rho}\sin(n-2)\theta + A^i_{\pm z}\cos(n-2)\theta\right)$.

Remark 3.7. *The $(t - \phi^i)$ symmetric initial data set $(M^n, \bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ with corners across an axially symmetric hypersurface Σ involves functions $\Psi = (U, \lambda_{ij}, \zeta^1, \zeta^{n-2}, \chi, \psi^1, \psi^{n-2})$ and α. Moreover, since it encodes all geometrical and physical properties of related axially symmetric maximal initial*

data set $(M^n, \mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$*, in Section 3.3, we study smooth deformations of this data set.*

3.3. Smooth deformation

Consider the $(t-\phi^i)$ symmetric initial data set $(M^n, \bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ with corners along an axially symmetric hypersurface Σ. For this initial data set, the 2-dimensional distribution $\mathcal{D}^2$ orthogonal to $\eta_{(l)}$ is integrable globally. Then we consider the orbit space $O = M^n/U(1)^{n-2}$ and we modify the differential structure on O such that the metric on neighborhood of $\Sigma/U(1)^{n-2}$ has Gaussian normal form. The orbit space has two region $O_- = \bar{\Omega}/U(1)^{n-2}$ and $O_+ = (M \backslash \Omega)/U(1)^{n-2}$. The metric of orbit space is

$$\pi^* (q_\pm) = \bar{g}_\pm - \Lambda^{kl}_\pm \eta_{(k)} \eta_{(l)}, \tag{3.39}$$

where $\pi : M \to O_\pm$ is projection onto orbit space. The orbit space is a 2-dimensional smooth manifold with boundary and corners [25, Proposition 1] which is diffeomorphic to upper-half plane by the Riemann mapping theorem and $\Sigma/U(1)^{n-2}$ is a semi-circle in this upper-half plane. Given $\epsilon > 0$, let $V_-^{2\epsilon}$ and $V_+^{2\epsilon}$ be neighborhood of $\Sigma/U(1)^{n-2}$ in (O_-, q_-) and (O_+, q_+), respectively. Moreover, Let

$$\begin{aligned} &\Phi_- : (-2\epsilon, 0] \times \Sigma/U(1)^{n-2} \to V_-^{2\epsilon}, \\ &\Phi_+ : [0, 2\epsilon) \times \Sigma/U(1)^{n-2} \to V_+^{2\epsilon}, \end{aligned} \tag{3.40}$$

be diffeomorphisms such that the pullback metrics are

$$\Phi_-^*(q_-) = dt^2 + q_{-\theta}(t,\theta) d\theta^2, \qquad \Phi_+^*(q_+) = dt^2 + q_{+\theta}(t,\theta) d\theta^2. \tag{3.41}$$

Note that level sets of the distance function $d_\pm : O_\pm \to \mathbb{R}$ to Σ provide such diffeomorphisms. Identifying $V = V_-^{2\epsilon} \cup V_+^{2\epsilon}$ with $(-2\epsilon, 2\epsilon) \times \Sigma/U(1)^{n-2}$, we define a differential structure with open covering $\{O_-, O_+, V\}$ on O and denote the new smooth manifold $\tilde{O}$. In particular, metric on $V = (-2\epsilon, 2\epsilon) \times \Sigma/U(1)^{n-2}$ takes the form

$$q = dt^2 + q_\theta(t,\theta) d\theta^2, \tag{3.42}$$

where $q = \Phi_-^*(q_-)$ for $t \leq 0$ and $q = \Phi_+^*(q_+)$ for $t \geq 0$. Moreover, the metric $\mathcal{G}$ on related region $(-2\epsilon, 2\epsilon) \times \Sigma$ on $\tilde{M}$ is a continuous metric $\bar{g}$ as following

$$\bar{g} = dt^2 + \gamma(t,\theta) = dt^2 + q_\theta(t,\theta) d\theta^2 + \Lambda_{ij}(t,\theta) d\phi^i d\phi^j, \tag{3.43}$$

where $\eta_{(l)}=\frac{\partial}{\partial\phi^l}$. Then $\gamma(t)=g_{ij}(t,\theta)dx^idx^j$, for $(x^1,x^2,x^{n-1})=(\theta,\phi^1,\phi^{n-2})$, is a path of metrics on Σ. Suppose $\mathcal{S}^i(\Sigma)$ is Banach space of C^i symmetric $(0,2)$ tensors on Σ and $\mathcal{M}^i(\Sigma)$ is open and convex subset of $\mathcal{S}^i(\Sigma)$ consisting of C^i metrics. Then we have the following path of metrics

$$\gamma:(-2\epsilon,2\epsilon)\to\mathcal{M}^2(\Sigma)\hookrightarrow\mathcal{M}^1(\Sigma)\hookrightarrow\mathcal{M}^0(\Sigma). \tag{3.44}$$

By assumption, γ is a continuous path in $\mathcal{M}^2(\Sigma)$ and piecewise C^1 path in $\mathcal{M}^1(\Sigma)$. Given $0<\delta\ll\epsilon$, we define the deforming path for $s\in(-\epsilon,\epsilon)$

(3.45)
$$\gamma_\delta(s)=\int_{\mathbb{R}}\gamma(s-\sigma_\delta(s)t)\phi(t)dt=\begin{cases}\int_{\mathbb{R}}\gamma(t)\left(\frac{1}{\sigma_\delta(s)}\phi(\frac{s-t}{\sigma_\delta(s)})\right)dt & \sigma_\delta(s)>0\\ \gamma(s) & \sigma_\delta(s)=0\end{cases},$$

where $\phi(s)\in C_c^\infty([-1,1])$ is standard mollifier on $\mathbb{R}$ with $0\le\phi(s)\le1$ and $\int_{-1}^1\phi ds=1$. Moreover, $\sigma_\delta(t)=\delta^2\sigma(t/\delta)$ for the cut-off function $\sigma(t)\in C_c^\infty([-\frac{1}{2},\frac{1}{2}])$ with definition

$$\sigma(t)=\begin{cases}\sigma(t)=\frac{1}{100} & |t|\le\frac{1}{4}\\ 0<\sigma(t)\le\frac{1}{100} & \frac{1}{4}<|t|<\frac{1}{2}\end{cases}. \tag{3.46}$$

Then γ_δ has the following properties [32].

Lemma 3.8. *The deform metric γ_δ has the following properties*

(a) γ_δ is C^2 path in $\mathcal{M}^0(\Sigma)$ and a C^1 path in $\mathcal{M}^1(\Sigma)$.

(b) γ_δ is C^0 path in $\mathcal{M}^2(\Sigma)$ which is uniformly close to γ and agrees with γ outside $(-\frac{\delta}{2},\frac{\delta}{2})$.

(c) $\|\gamma_\delta(s)-\gamma(s)\|_{\mathcal{M}^0(\Sigma)}\le L\delta^2$ for $s\in(-\epsilon,\epsilon)$.

Now we define the deformation metric

$$\bar{g}_\delta=\begin{cases}dt^2+\gamma_\delta(t) & (t,x)\in(-\epsilon,\epsilon)\times\Sigma\\ \bar{g} & (t,x)\notin(-\epsilon,\epsilon)\times\Sigma\end{cases}. \tag{3.47}$$

Before deforming $(\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$ smoothly on $(-\epsilon,\epsilon)\times\Sigma$, we define the following frame for metric $\bar{g}_\delta$ on $(-2\epsilon,2\epsilon)\times\Sigma$

$$e_1=\partial_t,\quad e_2=\frac{\partial_\theta}{\sqrt{\bar{g}_\delta(\partial_\theta,\partial_\theta)}}\quad e_{i+2}=\partial_{\phi^i},\quad \text{for } i=1,n-2, \tag{3.48}$$

with dual frame $\{\theta^i\}$ such that

$$\bar{g}_\delta = (\theta^1)^2 + (\theta^2)^2 + \Lambda_{\delta ij}\theta^{i+2}\theta^{j+2}. \tag{3.49}$$

Suppose ω represents the potentials in the initial data set, that is $\omega = \zeta, \chi$, and ψ, then we define

$$\omega_\delta(t,x) = \int_{\mathbb{R}} \omega(t - \sigma_\delta(t)s)\phi(s)ds. \tag{3.50}$$

This implies $\omega_\delta = \omega$ for $(t,x) \notin (-\epsilon,\epsilon)\times\Sigma$. Moreover, since ω approaches constant on axis Γ and $\int_{\mathbb{R}}\phi(t)dt = 1$, we have $\omega_\delta(t)|_\Gamma = \omega(t)|_\Gamma$. We define deformations for each components of extrinsic curvature, electric and magnetic $(n-2)$-form in frame $\{e_i\}$. Then using equations (3.16), (3.17), (3.19), (3.20), (3.49), and (3.50) we obtain the following expressions

$$\begin{aligned}(n-2)\bar{k}_\delta(e_1,e_{i+2})(t) &= -\frac{1}{\det\Lambda_\delta\sqrt{\bar{g}_\delta(\partial_\theta,\partial_\theta)}}\\ &\times\left(\partial_\theta\zeta^i_\delta + \psi^i_\delta\left(\partial_\theta\chi_\delta + \frac{n-3}{3\sqrt{3}}\psi^T_\delta\mathfrak{J}d\psi_\delta\right) + (n-4)\,\chi_\delta\partial_\theta\psi^i_\delta\right)(t),\end{aligned} \tag{3.51}$$

$$\begin{aligned}(n-2)\bar{k}_\delta(e_2,e_{i+2})(t) &= \frac{1}{\det\Lambda_\delta}\\ &\times\left(\partial_t\zeta^i_\delta + \psi^i_\delta\left(\partial_t\chi_\delta + \frac{n-3}{3\sqrt{3}}\psi^T_\delta\mathfrak{J}\partial_t\psi_\delta\right) + (n-4)\,\chi_\delta\partial_t\psi^i_\delta\right)(t),\end{aligned} \tag{3.52}$$

for $i = 1, 2$ and

$$\bar{E}_\delta(e_1)(t) = \frac{1}{\sqrt{\bar{g}_\delta(\partial_\theta,\partial_\theta)\det\Lambda_\delta}}\left(\partial_\theta\chi_\delta + \frac{n-3}{\sqrt{3}}\psi^T_\delta\mathfrak{J}\partial_\theta\psi_\delta\right)(t), \tag{3.53}$$

$$\bar{E}_\delta(e_2)(t) = -\frac{1}{\sqrt{\det\Lambda_\delta}}\left(\partial_t\chi_\delta + \frac{n-3}{\sqrt{3}}\psi^T_\delta\mathfrak{J}\partial_t\psi_\delta\right)(t). \tag{3.54}$$

For magnetic $(n-2)$-form we have different decomposition in each dimensions. If $n = 3$, then we have

$$\begin{aligned}\bar{B}_\delta(e_1)(t) &= \frac{1}{\sqrt{\bar{g}_\delta(\partial_\theta,\partial_\theta)\det\Lambda_\delta}}\partial_\theta\psi_\delta(t),\\ \bar{B}_\delta(e_2)(t) &= -\frac{1}{\sqrt{\det\Lambda_\delta}}\partial_t\psi_\delta(s)(t)\end{aligned} \tag{3.55}$$

For $n = 4$, we have

$$\begin{aligned} \bar{B}_\delta(e_1, e_{i+2})(t) &= -\frac{1}{\sqrt{\bar{g}_\delta(\partial_\theta, \partial_\theta)}} \epsilon_\delta{}^j{}_i \partial_\theta \psi_\delta^j(t), \\ \bar{B}_\delta(e_2, e_{i+2})(t) &= \epsilon_\delta{}^j{}_i \partial_t \psi_\delta^j(t) \end{aligned} \tag{3.56}$$

where $\epsilon_{\delta ij} = \sqrt{\det \Lambda_\delta}\varepsilon_{ij}$ is volume form associated to Λ_δ and $\varepsilon_{ij} = 0, \pm 1$.

Lemma 3.9. *The deform initial data set $(\tilde{M}^n, \bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ has the following properties.*

(a) Each components of $\bar{k}_\delta$, $\bar{E}_\delta$, and $\bar{B}_\delta$ are C^1 path in $C^0(\Sigma)$ and C^0 path in $C^1(\Sigma)$. Moreover, $(\bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ agrees with $(\bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ outside $(-\frac{\delta}{2}, \frac{\delta}{2})$.

(b) $\bar{E}_\delta$, $\bar{B}_\delta$ and $\bar{k}_\delta$ are bounded by constants depends on $(\bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$, and γ and not δ.

Proof. (a) Using definition of initial data set follows that the components $\bar{k}_{ij}, \bar{E}_i, \bar{B}_{ij} : (-\epsilon, \epsilon) \to C^{1-l}(\Sigma)$ are C^l away from Σ for $l = 0, 1$. Then, the components of deform initial data sets $\bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta : (-\epsilon, \epsilon) \to C^{1-l}(\Sigma)$ are C^l away from $[-\frac{\delta^2}{100}, \frac{\delta^2}{100}]$. For $t \in (-\frac{\delta}{4}, \frac{\delta}{4})$, we have $\sigma_\delta(t) = \frac{\delta^2}{100}$ and we get standard mollification of potentials and metric which are smooth. Moreover, for $|t| > \frac{\delta}{2}$, we have $\sigma_\delta(t) = 0$, $\gamma_\delta = \gamma$, $\psi_\delta = \psi$, $\chi_\delta = \chi$, and $\zeta_\delta = \zeta$. Therefore, $(\bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ agrees with $(\bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$.
(b) By Lemma 3.8 and definition of $\bar{E}_\delta$, $\bar{B}_\delta$ and $\bar{k}_\delta$, we only need to show that derivative of deform potentials are bounded by constants depend on $(\bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ and γ. Observe that

$$\partial_t \omega_\delta(t, \theta) = \int_{\mathbb{R}} \omega'\left(t - \sigma_\delta(t)s\right)\left(1 - s\delta\sigma'(\frac{t}{\delta})\right)\phi(s)ds \tag{3.57}$$

for $\omega_\delta = \zeta_\delta, \chi_\delta$, and ψ_δ. Clearly this is bounded by constants depend on $(\bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$. □

Next, we show behavior of energy density, energy flux, and Maxwell equations of deform initial data set and we prove the following generalization of [32, Proposition 3.1] to axially symmetric initial data set.

Proposition 3.10. *Let $(\tilde{M}^n, \bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ be a n-dimensional $(t - \phi^i)$-symmetric, simply connected initial data set admitting corners across axially symmetric hypersurface Σ. Then there exists a family of 4-tuple $(\bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$, where $\bar{g}_\delta$ is C^2 and $\bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta$ are C^1, for $0 < \delta \leq \delta_0$ on*

$\tilde{M}^n$ so that $\bar{g}$ is uniformly close to $\bar{g}$ on $\tilde{M}^n$. Moreover, $(\bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta) = (\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ outside $\mathcal{O}_\delta = (-\frac{\delta}{2}, \frac{\delta}{2}) \times \Sigma$ and the energy density satisfies

$$\bar{\mu}_\delta(t,x) = O(1), \quad \text{for} \quad (t,x) \in \left\{ -\frac{\delta^2}{100} < |t| \leq \frac{\delta}{2} \right\} \times \Sigma \tag{3.58}$$

and for $(t,x) \in [-\frac{\delta^2}{100}, \frac{\delta^2}{100}] \times \Sigma$, we have

$$\bar{\mu}_\delta(t,x) = O(1) + \{H(\Sigma, \bar{g}_-)(x) - H(\Sigma, \bar{g}_+)(x)\} \left\{ \frac{100}{\delta^2} \phi(\frac{100t}{\delta^2}) \right\}, \tag{3.59}$$

where $O(1)$ represents quantities that are bounded by constants depending only on $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$, but not on δ. Moreover, for all $t \in \mathbb{R}$, we have $tr_{\bar{g}_\delta} \bar{k}_\delta(t)$, $\bar{J}_\delta(\eta_{(l)})(t) = 0$, and

$$div_{\bar{g}_\delta} \bar{B}_\delta(t) = 0, \qquad div_{\bar{g}_\delta} \bar{E}_\delta(t) = \frac{n-3}{\sqrt{3}} \star_{\bar{g}_\delta} \left(\bar{B}_\delta \wedge \bar{B}_\delta \right)(t), \tag{3.60}$$

Furthermore, the angular momentum and charges of $(\tilde{M}^n, \bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ are conserved and equal to angular momentum and charges of $(\tilde{M}^n, \bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$.

Proof of Proposition 3.10. For $t \notin (-\frac{\delta}{2}, \frac{\delta}{2})$ or outside $\mathcal{O}_\delta$, the initial data $(\bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ is same as $(\tilde{M}^n, \bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$, thus the result hold. Let $t \in (-\epsilon, \epsilon)$. First, we show energy density $\bar{\mu}_\delta$ satisfies in equations (3.58) and (3.59). The Gauss equation and evolution of mean curvature of Σ leads to

$$R_{\bar{g}_\delta} = \text{scal}_\delta - \left(H_\delta^2 + |A_\delta|^2 \right) - 2 D_\nu H_\delta \tag{3.61}$$

where ν is unit normal vector field on Σ, A_δ and H_δ are the second fundamental form and mean curvature on Σ, and scal_δ is scalar curvature of hypersurface Σ. Then, substitution equation (3.61) in the Hamiltonian constraint equation (2.1), we obtain

$$\begin{aligned} 16\pi \bar{\mu}_\delta = \text{scal}_\delta - \left(H_\delta^2 + |A_\delta|^2 \right) - 2 D_\nu H_\delta - |\bar{k}_\delta|^2_{\bar{g}_\delta} \\ - \frac{2}{(n-2)^2} |\bar{E}_\delta|^2_{\bar{g}_\delta} - \frac{2}{(n-2)^3} |\bar{B}_\delta|^2_{\bar{g}_\delta}. \end{aligned} \tag{3.62}$$

The Lemma 3.9 implies $\bar{k}_\delta$, $\bar{B}_\delta$, and $\bar{E}_\delta$ are bounded by constants depends on $\bar{\mathcal{K}}$, $\bar{\mathcal{E}}$, $\bar{\mathcal{B}}$, and γ. Then, in light of [32, Proposition 3.1], we have (3.58)

and (3.59). Based on structure of $(t-\phi^i)$ symmetric data and Proposition 3.5, the deform initial data set is maximal, $\bar{J}_\delta(\eta_{(l)})=0$, and is a solution of field equations in (3.60). Finally, conservation of angular momentum and charges for any homologous surface to S^{n-1}_∞ follows from field equations (3.60) and [18, Lemma 2.1] for 3-dimensional case and [3, Section 4] for 4-dimensional case. □

4. Proof of main results

In order to prove the main theorem, we need to show that under appropriate conformal transformation the energy density of the deform initial data set is non-negative and conservation of angular momentum and charges hold. Let C, C_1, C_2, and C_3 be constants depend on initial data set $(\bar{\mathcal{G}},\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$. Consider the following jump on the mean curvatures of an axially symmetric hypersurface Σ within $(t-\phi^i)$-symmetric data set

$$H_-(\Sigma,\bar{g}_-)\geq H_+(\Sigma,\bar{g}_+). \tag{4.1}$$

By Remark 3.6 and definition of $(t-\phi^i)$-symmetric data set, this is equivalent to $H_-(\Sigma,g_-)\geq H_+(\Sigma,g_+)$ for related axially symmetric data set. Assume the initial data set has one designated asymptotically flat ∞ and the other either asymptotically cylindrical ∞_1 or boundary $\partial\tilde{M}^n$. Then we have the following proposition.

Proposition 4.1. *Let $\bar{g}_\delta$ be a C^2 $(t-\phi^i)$-symmetric asymptotically flat metric with one designated asymptotically flat ∞ and the other either asymptotically cylindrical ∞_1 or boundary $\partial\tilde{M}^n$, and $c_n=\frac{n-2}{4(n-1)}$, then*

$$\begin{cases}\Delta_{\bar{g}_\delta}u_\delta+c_n\bar{\mu}_{\delta-}u_\delta=0 & on\quad \tilde{M}^n\\ u_\delta=1 & on\quad \infty\\ u_\delta=1 & on\quad \infty_1\end{cases} \tag{4.2}$$

$$and\quad \begin{cases}\Delta_{\bar{g}_\delta}u_\delta+c_n\bar{\mu}_{\delta-}u_\delta=0 & on\quad \tilde{M}^n\\ u_\delta=1 & on\quad \infty\\ \frac{\partial u_\delta}{\partial n}=0 & on\quad \partial\tilde{M}^n\end{cases}$$

have a unique axially symmetric C^2 solution $u_\delta\geq 1$ on $\tilde{M}^n$ so that, $\lim_{\delta\to 0}||u_\delta-1||_{L^\infty(\tilde{M}^n)}=0$, $u_\delta=1+\frac{A_\delta}{|x|^{n-2}}+O_1(|x|^{1-n})$ for some constant A_δ as $|x|\to\infty$, and $\frac{\partial u_\delta}{\partial n}=0$ at ∞_1.

Proof. By equations (3.58) and (3.59) for $\bar{\mu}_{\delta-}$ and jump in the mean curvature in (4.1), we have

$$\begin{cases} \bar{\mu}_{\delta-} = 0 & \text{outside } \mathcal{O}_\delta \\ |\bar{\mu}_{\delta-}| \leq C_1 & \text{inside } \mathcal{O}_\delta. \end{cases} \tag{4.3}$$

Since outside a compact set $\bar{\mu}_{\delta-} = 0$ and the first eigenvalue of Laplacian is zero on asymptotically cylindrical end ∞_1, the solution of (4.2) can have asymptotically constant behavior at asymptotically cylindrical end ∞_1. Moreover, asymptotically cylindrical end yields to exponential decay for solutions and we obtain $\frac{\partial u_\delta}{\partial n} = 0$ at ∞_1. Then using [34, Lemma 3.3], we get existence of a unique C^2 positive solution with asymptotic $u_\delta = 1 + \frac{A_\delta}{|x|^{n-2}} + O(|x|^{1-n})$ at ∞ such that

$$A_\delta = \frac{1}{\omega_n} \lim_{\delta \to 0} \int_{\tilde{M}} \left(-|\nabla_{\bar{g}_\delta} u_\delta|^2 + c_n \bar{\mu}_{\delta-} u_\delta^2 \right) dx_{\bar{g}_\delta}, \tag{4.4}$$

where $dx_{\bar{g}_\delta}$ is volume form with respect to $\bar{g}_\delta$ and ω_n is volume of unit $(n-1)$-dimensional unit sphere. Since u_δ is superharmonic, by strong maximum principle the minimum is at ends or boundary. For Dirichlet problem in (4.2), it is clear that $u_\delta \geq 1$. For other problem, if the minimum is at asymptotically flat end ∞, then $u_\delta \geq 1$. But if the minimum is at $\partial \tilde{M}^n$ and less than one, then using Hopf maximum principle we should have $\nu(u_\delta) < 0$, which is a contradiction. Therefore, $u_\delta \geq 1$. Finally, using [32, Proposition 4.1], we have $\lim_{\delta \to 0} ||u_\delta - 1||_{L^\infty(\tilde{M}^n)} = 0$. □

Next we define the following conformal transformation

$$\tilde{g}_\delta = u_\delta^{\frac{4}{n-2}} \bar{g}_\delta, \qquad \tilde{k}_\delta = u_\delta^{-2} \bar{k}_\delta, \qquad \tilde{E}_\delta = u_\delta^{-2} \bar{E}_\delta \qquad \tilde{B}_\delta = u_\delta^{-2} \bar{B}_\delta \tag{4.5}$$

where $\bar{g}_\delta$ and $\bar{k}_\delta$ are $(0,2)$-tensor, $\bar{E}_\delta$ is a 1-form, and $\bar{B}_\delta$ is a $(n-2)$-form. Then we have

Proposition 4.2. *Consider the conformal initial data set $(\tilde{g}_\delta, \tilde{k}_\delta, \tilde{E}_\delta, \tilde{B}_\delta)$ in equation (4.5). Then the metric is C^2, $\tilde{k}_\delta, \tilde{E}_\delta$, and $\tilde{B}_\delta$ are C^1, and it has non-negative energy density and vanishing energy flux in the direction of symmetries. Furthermore, its mass converges to the mass of $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ and angular momentum and charges are the same as angular momentum and charges of $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$.*

Proof. It follows from Proposition 3.10 and Proposition 4.1 that the metric $\tilde{g}_\delta$ is C^2 and $(\tilde{k}_\delta, \tilde{E}_\delta, \tilde{B}_\delta)$ are C^1. Assuming $n = 3, 4$, we have

$$\tilde{\mu}_\delta = R(\tilde{g}_\delta) - |\tilde{k}_\delta|^2_{\tilde{g}_\delta} - \frac{2}{(n-2)^2}|\tilde{E}_\delta|^2_{\tilde{g}_\delta} - \frac{2}{(n-2)^3}|\tilde{B}_\delta|^2_{\tilde{g}_\delta} \tag{4.6}$$

$$= u_\delta^{\frac{4}{2-n}} \left(\bar{\mu}_{\delta+} + \left(1 - u_\delta^{4\left(\frac{n-1}{2-n}\right)}\right) |\bar{k}_\delta|^2_{\bar{g}_\delta} + \frac{2}{(n-2)^2}\left(1 - u_\delta^{-4}\right)|\bar{E}_\delta|^2_{\bar{g}_\delta} \right. \tag{4.7}$$

$$\left. + \frac{2}{(n-2)^3}\left(1 - u_\delta^{2(n-6)}\right)|\bar{B}_\delta|^2_{\bar{g}_\delta} \right). \tag{4.8}$$

Since $u_\delta \geq 1$, we have $\tilde{\mu}_\delta \geq 0$. By asymptotic of u_δ we have

$$\begin{aligned} m_{ADM}(\tilde{g}_\delta) &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S^2_r} \left((\tilde{g}_\delta)_{ij,i} - (\tilde{g}_\delta)_{ii,j}\right) \tilde{\nu}^j d\tilde{S}_\delta \\ &= \frac{1}{16\pi} \lim_{r\to\infty} \int_{S^2_r} \left((\bar{g}_\delta)_{ij,i} - (\bar{g}_\delta)_{ii,j}\right) \nu^j dS_\delta \\ &\quad - \frac{1}{2\pi(n-2)} \lim_{r\to\infty} \int_{S^2_r} (u_\delta)_{,j} \nu^j dS_\delta, \\ &= m_{ADM}(\bar{g}_\delta) + \frac{2}{n-2} A_\delta \end{aligned} \tag{4.9}$$

where $\tilde{\nu} = u_\delta^{-\frac{2}{n-2}}\nu$ and $d\tilde{S}_\delta = u_\delta^{\frac{2(n-1)}{n-2}} dS_\delta$. Then using equation (4.2) and Hölder inequality we have

$$\begin{aligned} \lim_{\delta\to 0} A_\delta &\leq C \lim_{\delta\to 0} \int_{\tilde{M}} |\nabla_{\bar{g}_\delta} u_\delta|^2 dx_{\bar{g}_\delta} \\ &\quad + C \lim_{\delta\to 0} \left(\int_{\mathcal{O}_\delta} |\bar{\mu}_{\delta-}|^{n/2} dx_{\bar{g}_\delta} \right)^{2/n} \left(\int_{\mathcal{O}_\delta} u_\delta^{\frac{2n}{n-2}} dx_{\bar{g}_\delta} \right)^{\frac{n-2}{2n}} \\ &\leq C \lim_{\delta\to 0} \left(\int_{\tilde{M}} |\bar{\mu}_{\delta-}|^{n/2} dx_{\bar{g}_\delta} \right)^{2/n} \left(\int_{\tilde{M}} |u_\delta - 1|^{\frac{2n}{n-2}} dx_{\bar{g}_\delta} \right)^{\frac{n-2}{2n}} \\ &\quad + C \lim_{\delta\to 0} \left(\int_{\tilde{M}} |\bar{\mu}_{\delta-}|^{\frac{2n}{n+2}} dx_{\bar{g}_\delta} \right)^{\frac{n+2}{2n}} \left(\int_{\tilde{M}} |u_\delta - 1|^{\frac{2n}{n-2}} dx_{\bar{g}_\delta} \right)^{\frac{n-2}{n}} \\ &\quad + C \lim_{\delta\to 0} \left(\int_{\mathcal{O}_\delta} |\bar{\mu}_{\delta-}|^{n/2} dx_{\bar{g}_\delta} \right)^{2/n} \left(\int_{\mathcal{O}_\delta} u_\delta^{\frac{2n}{n-2}} dx_{\bar{g}_\delta} \right)^{\frac{n-2}{2n}}. \end{aligned} \tag{4.10}$$

Combining this with equation (4.3) and Proposition 4.1, we get $\lim_{\delta\to 0} A_\delta = 0$. Therefore,

$$\lim_{\delta\to 0} m_{ADM}(\tilde{g}_\delta) = m_{ADM}(\bar{\mathcal{G}}) \tag{4.11}$$

Next, observe that

$$\begin{aligned} \mathrm{div}_{\tilde{g}_\delta}\tilde{k}_\delta &= u^{-\frac{2(n-4)}{n-2}}\mathrm{div}_{\bar{g}_\delta}\bar{k}_\delta,\\ \mathrm{div}_{\tilde{g}_\delta}\tilde{B}_\delta &= u^{-\frac{2(n-4)}{n-2}}\mathrm{div}_{\bar{g}_\delta}\bar{B}_\delta,\\ \mathrm{div}_{\tilde{g}_\delta}\tilde{E}_\delta &= u^{-\frac{2n}{n-2}}\mathrm{div}_{\bar{g}_\delta}\bar{E}_\delta, \end{aligned} \tag{4.12}$$

and $\star_{\tilde{g}_\delta}\alpha = u^{\frac{2n-4p}{n-2}}\star_{\bar{g}_\delta}\alpha$ for a p-form α. Therefore, we have

$$\tilde{J}_\delta = u^{-\frac{2(n-4)}{n-2}}\bar{J}_\delta, \quad \mathrm{div}_{\tilde{g}_\delta}\tilde{B}_\delta = 0, \quad \mathrm{div}_{\tilde{g}_\delta}\tilde{E}_\delta = \frac{n-3}{\sqrt{3}}\star_{\tilde{g}_\delta}\left(\tilde{B}_\delta\wedge\tilde{B}_\delta\right). \tag{4.13}$$

Hence $\tilde{J}_\delta(\eta_{(l)}) = 0$ and it shows that angular momentum and charges are conserved as in Proposition 3.10. □

Now we have all tools to prove Theorem 2.3.

Proof of Theorem 2.3. First we prove the strict inequality and then the rigidity cases. From Proposition 3.5, the axially symmetric initial data set $(\mathcal{G},\mathcal{K},\mathcal{E},\mathcal{B})$ has same mass, angular momentum and charges as related $(t-\phi^i)$-symmetric data set $(\bar{\mathcal{G}},\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$. Assume $(\bar{\mathcal{G}},\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$ has an outermost minimal surface $\Sigma_{\min}$ that enclosed by Σ. Then by [11, Proposition 3.1.], $\Sigma_{\min}$ is axially symmetric. We cut the initial data from $\Sigma_{\min}$ and consider $(\bar{\mathcal{G}},\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$ with outermost minimal surface boundary $\Sigma_{\min}$. By proposition 4.1, the initial data $(\tilde{g}_\delta,\tilde{k}_\delta,\tilde{E}_\delta,\tilde{B}_\delta)$ has same angular momentum and charges as $(\bar{\mathcal{G}},\bar{\mathcal{K}},\bar{\mathcal{E}},\bar{\mathcal{B}})$ with axially symmetric minimal surface boundary $\Sigma_{\min}$.

For parts (a) and (b), since $\Sigma_{\min}$ is minimal surface boundary, we double the initial data set. Then the mean curvature of two sides coincide and the metric is Lipschitz across $\Sigma_{\min}$. Moreover, mass, angular momentum, and charges are conserved through this doubling. Then we deform again the initial data set and obtain a complete initial data set with two asymptotically flat ends and denote it by $(\tilde{g}_\delta,\tilde{k}_\delta,\tilde{E}_\delta,\tilde{B}_\delta)$. Therefore, by [1–3, 15], since $\tilde{g}_\delta$ is C^2, $\tilde{k}_\delta,\tilde{E}_\delta$, and $\tilde{B}_\delta$ are C^1, and it has non-negative energy density and vanishing energy flux in the direction of symmetries, the mass, angular momentum, and charge inequality hold for initial data set $(\tilde{g}_\delta,\tilde{k}_\delta,\tilde{E}_\delta,\tilde{B}_\delta)$, that is

$$m_{ADM}(\tilde{g}_\delta)^2 > \frac{Q^2+\sqrt{Q^4+4\mathcal{J}^2}}{2}, \qquad \text{for } n=3, \tag{4.14}$$

if $M^4 = \mathbb{R}^3 \times S^3$, we have

$$m_{ADM}(\tilde{g}_\delta) > \frac{27\pi}{8} \frac{(\mathcal{J}_1 + \mathcal{J}_2)^2}{\left(2m_{ADM}(\tilde{g}_\delta) + \sqrt{3}|Q|\right)^2} + \sqrt{3}|Q|, \quad \text{for } n = 4, \tag{4.15}$$

Note that the inequality is not sharp because the initial data set has two asymptotically flat ends. For part (c), we do not need to double the initial data set and we can apply [2] with topology $M^4 = S^2 \times B^2 \# \mathbb{R}^4$ and we obtain

$$m_{ADM}(\tilde{g}_\delta)^3 > \frac{27\pi}{4} |\mathcal{J}_2||\mathcal{J}_1 - \mathcal{J}_2|, \qquad \text{for } n = 4. \tag{4.16}$$

It follows from Proposition 4.2 that

$$m_{ADM}(\mathcal{G}) = m_{ADM}(\bar{\mathcal{G}}) = \lim_{\delta \to 0} m_{ADM}(g_\delta) = \lim_{\delta \to 0} m_{ADM}(\tilde{g}_\delta). \tag{4.17}$$

Therefore, the strict inequalities hold for initial data set $(\mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B})$.

For the rigidity case, the initial data set has asymptotically cylindrical end ∞_1. We establish the rigidity for 3-dimensional initial data sets and 4-dimensional cases are similar. Assume rigidity in (1.2), then

$$\begin{aligned} \tfrac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2} &= m_{ADM}(\mathcal{G})^2 = m_{ADM}(\bar{\mathcal{G}})^2 \\ &= \lim_{\delta \to 0} m_{ADM}(g_\delta)^2 = \lim_{\delta \to 0} m_{ADM}(\tilde{g}_\delta)^2. \end{aligned} \tag{4.18}$$

The deform data $(\bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ agrees with $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ outside small neighborhood $\mathcal{O}_\delta$ of Σ according to Proposition 3.10. Since the metric $\bar{g}_\delta$ is C^2, there exist a global coordinate system (ρ, z, ϕ) [13] such that it depends on δ and the conformal metric $\tilde{g}_\delta = u_\delta^4 \bar{g}_\delta$ takes the following form

$$\tilde{g}_\delta = u_\delta^4 e^{-2U_\delta + 2\alpha_\delta} \left(d\rho^2 + dz^2\right) + \rho^2 u_\delta^4 e^{-2\bar{U}_\delta} d\phi^2, \tag{4.19}$$

where $\rho \in [0, \infty)$, $z \in \mathbb{R}$, and $\phi \in [0, 2\pi)$. Moreover, it has the following fall-off at infinity

$$U_\delta = O(r^{-1/2-\kappa}), \qquad \alpha_\delta = O(r^{-1/2-\kappa}), \qquad \text{for } \kappa > 0. \tag{4.20}$$

Furthermore, $\alpha_\delta = 0$ on the axis $\Gamma = \{\rho = 0\}$. Set $V_\delta = U_\delta - 2 \log u_\delta$. The scalar curvature of the metric has the following simple expression

$$2e^{-2V_\delta + 2\alpha_\delta} R(\tilde{g}_\delta) = 8\Delta V_\delta - 4\Delta_{\rho,z}\alpha_\delta - 4|\nabla V_\delta|^2, \tag{4.21}$$

where Δ is the Laplacian with respect to flat metric $\delta_3 = d\rho^2 + dz^2 + \rho^2 d\phi^2$ on $\mathbb{R}^3$ and $\Delta_{\rho,z}$ is Laplacian with respect to flat metric $\delta_2 = d\rho^2 + dz^2$ on $\mathbb{R}^2$. Therefore, using the Hamiltonian constraint equation (2.1) and maximality condition $\mathrm{tr}_{\tilde{g}_\delta}\tilde{k}_\delta = 0$, the ADM mass has the following simple expression [28]

$$m_{ADM}(\tilde{g}_\delta) = I(\Psi_\delta) + \int_{\mathbb{R}^3} e^{-2V_\delta + 2\alpha_\delta}\tilde{\mu}_\delta dx, \tag{4.22}$$

where $\Psi_\delta = (V_\delta, \zeta_\delta, \chi_\delta, \psi_\delta)$, dx is the volume form with respect to δ_3, and $I(\Psi_\delta)$ is the reduced harmonic energy from $\mathbb{R}^3\backslash\Gamma \to \mathbb{H}^2_{\mathbb{C}}$, that is

$$\begin{aligned} I(\Psi_\delta) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\Bigg(&|\nabla V_\delta|^2 + \frac{e^{4V_\delta}}{\rho^4}|\nabla\zeta_\delta + \chi_\delta\nabla\psi_\delta - \psi_\delta\nabla\chi_\delta|^2 \\ &+ \frac{e^{2V_\delta}}{\rho^4}\left(|\nabla\chi_\delta|^2 + |\nabla\psi_\delta|^2\right)\Bigg)dx. \end{aligned} \tag{4.23}$$

Assume $\Psi_0 = (U_0, \zeta_0, \chi_0, \psi_0)$ is the harmonic map of the canonical slice of extreme Kerr-Newman black hole (g_0, k_0, E_0, B_0) such that $I(\Psi_0) = \frac{Q^2+\sqrt{Q^4+4\mathcal{J}^2}}{2}$. Then, applying the Schoen and Zhou gap inequality [36], we obtain

$$I(\Psi_\delta) - I(\Psi_0) \geq C\left(\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi_0)\, dx\right)^{1/3}, \tag{4.24}$$

Combing this with equations (4.18) and (4.22) shows that

$$\begin{aligned} 0 &= \lim_{\delta\to 0}\left(m_{ADM}(\tilde{g}_\delta)^2 - \tfrac{Q^2+\sqrt{Q^4+4\mathcal{J}^2}}{2}\right) \\ &\geq \lim_{\delta\to 0}\left(I(\Psi_\delta)^2 - \tfrac{Q^2+\sqrt{Q^4+4\mathcal{J}^2}}{2}\right) + \lim_{\delta\to 0}\left(\int_{\mathbb{R}^3} e^{-2V_\delta+2\alpha_\delta}\tilde{\mu}_\delta dx\right)^2 \\ &\geq C\lim_{\delta\to 0}\left(\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi_0)\, dx\right)^{2/3} \\ &\quad + \lim_{\delta\to 0}\left(\int_{\mathbb{R}^3} e^{-2V_\delta+2\alpha_\delta}\tilde{\mu}_\delta dx\right)^2. \end{aligned} \tag{4.25}$$

It follows that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi_0)\, dx = 0, \qquad \lim_{\delta\to 0}\int_{\mathbb{R}^3} e^{-2V_\delta+2\alpha_\delta}\tilde{\mu}_\delta dx = 0. \tag{4.26}$$

By Remark 3.7 and Proposition 3.5, $\Psi = (U, \zeta, \chi, \psi)$ and α characterize $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$. Next we show that $\Psi = \Psi_0$. Observe that

$$\begin{aligned}(4.27)\qquad &\Big(\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi) - \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)\Big)^6 - \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)\\ &\quad\le \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi_0) - \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)\\ &\quad\le \Big(\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi) + \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)\Big)^6 - \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0).\end{aligned}$$

If we show that

$$(4.28)\qquad \lim_{\delta\to 0}\int_{\mathbb{R}^3} \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi)\, dx = 0,$$

then in light of equation (4.27), we can pass the limit through integral

$$(4.29)\qquad 0 = \lim_{\delta\to 0}\int_{\mathbb{R}^3} \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi_0)\, dx = \int_{\mathbb{R}^3} \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)\, dx = 0.$$

Using the triangle inequality and $u_\delta \ge 1$, a direct computation produces

(4.30)

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi) &\le \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((V_\delta, \zeta_\delta, \chi_\delta, \psi_\delta), (U, \zeta_\delta, \chi_\delta, \psi_\delta))\\ &\quad + \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U, \zeta_\delta, \chi_\delta, \psi_\delta), (U, \zeta, \chi_\delta, \psi_\delta))\\ &\quad + \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U, \zeta, \chi_\delta, \psi_\delta), (U, \zeta, \chi, \psi_\delta))\\ &\quad + \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U, \zeta, \chi, \psi_\delta), (U, \zeta, \chi, \psi))\\ &\le C\bigg(|U - V_\delta| + \frac{e^{2U}}{\rho^2}\left(|\zeta - \zeta_\delta| + |\psi_\delta||\chi - \chi_\delta| + |\chi||\chi - \chi_\delta|\right)\\ &\quad + \frac{e^U}{\rho}\left(|\chi - \chi_\delta| + |\psi - \psi_\delta|\right)\bigg).\end{aligned}$$

Since $(U_\delta, \zeta_\delta, \chi_\delta, \psi_\delta) = (U, \zeta, \chi, \psi)$ outside $\mathcal{O}_\delta$ by Proposition 3.10, we have

$$\begin{aligned}(4.31)\qquad \int_{\mathbb{R}^3} \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_\delta, \Psi)\, dx &\le C\int_{\mathbb{R}^3} |\log u_\delta|^6 dx + C\int_{\mathcal{O}_\delta} |U - U_\delta|^6 dx\\ &\quad + C\int_{\mathcal{O}_\delta} \frac{e^{12U}}{\rho^{12}}(|\zeta - \zeta_\delta|^6 + |\psi_\delta|^6|\chi - \chi_\delta|^6 + |\chi|^6|\chi - \chi_\delta|^6) dx\\ &\quad + C\int_{\mathcal{O}_\delta} \frac{e^{6U}}{\rho^6}(|\chi - \chi_\delta|^6 + |\psi - \psi_\delta|^6) dx,\end{aligned}$$

By Lemma 3.9 and Lemma 3.8, as $\delta \to 0$, all terms converge to 0 except the first term. But, since $u_\delta \ge 1$ is a solution of (4.2) and using the Sobolev

inequality and Hölder inequality similar to [32, Proposition 4.1], we have

$$
\begin{aligned}
C_2 \int_{\mathbb{R}^3} |\log u_\delta|^6 dx &\leq \int_{\mathbb{R}^3} e^{-3U_\delta+\alpha_\delta} |\log u_\delta|^6 dx = \int_{\tilde{M}^3} |\log u_\delta|^6 dx_{\bar{g}_\delta} \\
&\leq \int_{\tilde{M}^3} |u_\delta - 1|^6 dx_{\bar{g}_\delta} \\
&\leq C \left(\int_{\tilde{M}^3} |\bar{\mu}_{\delta-}|^{6/5} dx_{\bar{g}_\delta} \right)^5 \to 0,
\end{aligned} \tag{4.32}
$$

as $\delta \to 0$. Combining this with (4.31) and (4.28), it follows that $\Psi = \Psi_0$. Next, using equation (4.26) and $V_\delta = U_\delta - 2\log u_\delta$, we have

$$
\begin{aligned}
0 &= \lim_{\delta\to 0} \int_{\mathbb{R}^3} e^{-2V_\delta+2\alpha_\delta} \tilde{\mu}_\delta dx \\
&\geq \lim_{\delta\to 0} \int_{\mathbb{R}^3} e^{-2U_\delta+2\alpha_\delta} \bar{\mu}_{\delta+} dx = \lim_{\delta\to 0} \int_{\tilde{M}^3} e^{U_\delta} \bar{\mu}_{\delta+} dx_{\bar{g}_\delta} \\
&= \lim_{\delta\to 0} \int_{\mathcal{O}_\delta} e^{U_\delta} \bar{\mu}_{\delta+} dx_{\bar{g}_\delta} + \lim_{\delta\to 0} \int_{\tilde{M}^3\setminus\mathcal{O}_\delta} e^{U} \bar{\mu} dx_{\bar{g}} \geq 0,
\end{aligned} \tag{4.33}
$$

where the last equality follows from $\bar{g}_\delta = \bar{g}$ and $\bar{\mu}_{\delta+} = \bar{\mu}$ outside $\mathcal{O}_\delta$. This shows that $\bar{\mu} = 0$ away from Σ.

We claim that if there exist a strict jump of mean curvature across Σ, that is $H_+(\Sigma, \bar{g}_+)(x) > H_-(\Sigma, \bar{g}_-)(x)$ for some $x \in \Sigma$, we get a contradiction to (4.33). Assume there is a strict jump of mean curvature across Σ. Since mean curvatures are continuous functions on Σ, there exist a compact set $\mathcal{C} \subset \Sigma$ such that

$$
H_+(\Sigma, \bar{g}_+)(x) - H_-(\Sigma, \bar{g}_-)(x) \geq \beta, \qquad \forall x \in \mathcal{C}, \tag{4.34}
$$

for some constant $\beta > 0$. Then by Proposition 3.10 and equation (4.3), we have

$$
\bar{\mu}_{\delta+}(t,x) \geq \beta \left\{ \tfrac{100}{\delta^2} \phi(\tfrac{100t}{\delta^2}) \right\} - C_1, \qquad \forall (t,x) \in [-\tfrac{\delta^2}{100}, \tfrac{\delta^2}{100}] \times \mathcal{C} \subset \mathcal{O}_\delta. \tag{4.35}
$$

which suggests that the energy density of $(\bar{g}_\delta, \bar{k}_\delta, \bar{E}_\delta, \bar{B}_\delta)$ and $(\tilde{g}_\delta, \tilde{k}_\delta, \tilde{E}_\delta, \tilde{B}_\delta)$ has a fixed amount of concentration on $\mathcal{C}$. Therefore, we get the following estimate which is a contradiction to equation (4.33)

$$
\lim_{\delta\to 0} \int_{\mathcal{O}_\delta} e^{U_\delta} \bar{\mu}_{\delta+} dx_{\bar{g}_\delta} \geq \beta C_3 |\mathcal{C}| > 0, \tag{4.36}
$$

where $|\mathcal{C}|$ is the measure of compact set $\mathcal{C}$. Hence, $H_+(\Sigma, \bar{g}_+)(x) = H_-(\Sigma, \bar{g}_-)(x)$ for all $x \in \Sigma$. Next, we show that $\alpha = \alpha_0$. By constraint equation for the initial data $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ on $\tilde{M}^3 \backslash \bar{\Omega}$ and $\bar{\mu} = 0$ away from Σ, we have

$$
\begin{aligned}
R(\bar{\mathcal{G}}) &= |\bar{\mathcal{K}}|^2_{\bar{\mathcal{G}}} + 2|\bar{\mathcal{E}}|^2_{\bar{\mathcal{G}}} + 2|\bar{\mathcal{B}}|^2_{\bar{\mathcal{G}}} \\
&= 2\frac{e^{6U_+ - 2\alpha_+}}{\rho^4} |\nabla\zeta_+ + \chi_+ \nabla\psi_+ - \psi_+ \nabla\chi_+|^2 \\
&\quad + 2\frac{e^{4U_+ - 2\alpha_+}}{\rho^4} \left(|\nabla\chi_+|^2 + |\nabla\psi_+|^2\right) \\
&= 2\frac{e^{6U_0 - 2\alpha_+}}{\rho^4} |\nabla\zeta_0 + \chi_0 \nabla\psi_0 - \psi_0 \nabla\chi_0|^2 \\
&\quad + 2\frac{e^{4U_0 - 2\alpha_+}}{\rho^4} \left(|\nabla\chi_0|^2 + |\nabla\psi_0|^2\right) \\
&= e^{2\alpha_0 - 2\alpha_+} R(g_0)
\end{aligned} \tag{4.37}
$$

Then scalar curvature equation (4.21) becomes

$$
\begin{aligned}
2e^{-2U_0 + 2\alpha_+} R(\bar{\mathcal{G}}) &= 8\Delta U_+ - 4\Delta_{\rho,z}\alpha_+ - 4|\nabla U_+|^2 \\
&= 8\Delta U_0 - 4\Delta_{\rho,z}\alpha_+ - 4|\nabla U_0|^2 \\
&= 2e^{-2U_0 + 2\alpha_0} R(g_0) + 4\Delta_{\rho,z}\left(\alpha_0 - \alpha_+\right)
\end{aligned} \tag{4.38}
$$

Combining this with (4.37) yields $\Delta_{\rho,z}\left(\alpha_0 - \alpha_+\right) = 0$ on $\tilde{M}^3 \backslash \bar{\Omega}$. Multiplying it to $(\alpha_0 - \alpha_+)$ and integrating over O_+ and applying integration by parts we have

$$
\int_{O_+} |\nabla\left(\alpha_0 - \alpha_+\right)|^2 d\rho dz = -\int_{\Sigma/U(1)} \left(\alpha_0 - \alpha_+\right) \partial_r \left(\alpha_0 - \alpha_+\right). \tag{4.39}
$$

Similarly, we have $\Delta_{\rho,z}\left(\alpha_0 - \alpha_+\right) = 0$ on Ω which yields to

$$
\int_{O_-} |\nabla\left(\alpha_0 - \alpha_-\right)|^2 d\rho dz = \int_{\Sigma/U(1)} \left(\alpha_0 - \alpha_-\right) \partial_r \left(\alpha_0 - \alpha_-\right). \tag{4.40}
$$

Putting this together with equation (3.38) yield to the following expression

$$
\begin{aligned}
&\int_{O_+} |\nabla\left(\alpha_0 - \alpha_+\right)|^2 d\rho dz + \int_{O_-} |\nabla\left(\alpha_0 - \alpha_-\right)|^2 d\rho dz \\
&\qquad = \int_{\Sigma/U(1)} \left(\alpha_0 - \alpha\right) \left(H_+(\Sigma, \bar{g}_+) - H_-(\Sigma, \bar{g}_-)\right) e^{-U_0 + \alpha},
\end{aligned} \tag{4.41}
$$

where we used $\alpha = \alpha_- = \alpha_+$ on Σ. Since the mean curvature is the same on Σ, we have $\nabla(\alpha_0 - \alpha_+) = 0$ on O_+ and $|\nabla(\alpha_0 - \alpha_-)| = 0$ on O_-. Thus $\alpha_0 - \alpha$ is a constant away from Σ. In light of vanishing boundary condition at axis to avoid conical singularity, we have $\alpha = \alpha_0$ away from Σ. Moreover, since α is continuous across Σ, $\alpha = \alpha_0$ on $\tilde{M}^3$. Then $(\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ is isometric to (g_0, k_0, E_0, B_0) and $\bar{\mu} = 0$ on whole $\tilde{M}^3$. Therefore, by equation (3.25), we have

$$\hat{B} = \hat{E} = \pi = \mu = 0, \qquad \partial_\rho A_z = \partial_z A_\rho \tag{4.42}$$

It follows that 1-form $A_\rho d\rho + A_z dz$ is closed on $\mathbb{R}^3$, so there exist a potential f such that $\partial_\rho f = A_\rho$ and $\partial_z f = A_z$. Thus under change of coordinates $\tilde{\phi} = \phi^i + f(\rho, z)$, the metric takes the form

$$g = e^{-2U_0 + 2\alpha_0}\left(d\rho^2 + dz^2\right) + \rho e^{-2U_0} d\tilde{\phi}^2, \tag{4.43}$$

which implies $(\mathcal{G}, \mathcal{K}, \mathcal{E}, \mathcal{B}) = (\bar{\mathcal{G}}, \bar{\mathcal{K}}, \bar{\mathcal{E}}, \bar{\mathcal{B}})$ is isometric to canonical slice of extreme Kerr-Newman spacetime. □

Next we use Theorem 2.3 and prove the Corollary 2.4.

Proof of Corollary 2.4. Since ∂M^n has non-negative mean curvature and it is boundary of an asymptotically flat manifold, by barrier argument there exist an outermost minimal surface $\Sigma_{\min}$ in M^n and by [11, Proposition 3.1.], $\Sigma_{\min}$ is axially symmetric. Assume $W \subset M^n$ such that $\partial W = \partial M^n \cup \Sigma_{\min}$. In three dimensions using [26, Lemma 4.1], we have $M^3 \setminus W$ is diffeomorphic to the complement of a finite number of open 3-balls in $\mathbb{R}^3$ and it is simply connected.

In four dimensions, first we show $M^4 \setminus W$ is simply connected. Assume $\pi_1(M^4 \setminus W) \neq 0$, then by residual finiteness, it admits a finite nontrivial universal cover. Thus by asymptotically flatness it must have finite number of ends which by barrier argument for the mean curvature it includes a minimal surface. Clearly this minimal surface is not contain entirely in the boundary and so its projection violate outermost minimal surface condition for $\Sigma_{\min}$ and we get a contradiction. Therefore, $M^4 \setminus W$ is simply connected. Moreover, since it has isometry subgroup $U(1)^2$ and spin by Orlik and Raymond [33] and Hollands and Ishibashi [24], it should be diffeomorphic to $\mathbb{R}^4 \# l(S^2 \times S^2)$, for $l \in \mathbb{N}$, minus 4-manifold with boundaries $S^1 \times S^2$ or S^3 (its quotients).

(a) Since $\Sigma_{\min}$ has only one component, $M^3 \setminus W$ is diffeomorphic to $\mathbb{R}^3$ minus a 3-ball. Then we deform it by Proposition 3.10 and Proposition 4.1

and we get a C^2 metric with C^1 second fundamental form and electromagnetic fields. Following proof of Theorem 2.3-(a) we get the strict inequality.

(b) If M^4 is spin, $\pi_1(\Sigma_{\min}) = 0$, and $H_2(M^4 \backslash W) = 0$, we get $M^4 \backslash W$ is diffeomorphic to R^4 minus a 4-ball. Following proof of Theorem 2.3-(b) we get the strict inequality.

(c) If M^4 is spin, $\mathcal{E} = \mathcal{B} = 0$, $\pi_1(\Sigma_{\min}) = \mathbb{Z}$, and $H_2(M^4 \backslash W) = \mathbb{Z}$, we get $M^4 \backslash W$ is diffeomorphic to $\mathbb{R}^4 \# (S^2 \times B^2)$. Then we deform it by Proposition 3.10 and Proposition 4.1 and we get a C^2 metric with C^1 second fundamental form. This is similar to initial data of non-extreme black ring and we can apply [2] and get the result. □

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Center of Mathematical Sciences and Applications
Harvard University, Cambridge, MA 02138, USA
E-mail address: `aghil.alaee@cmsa.fas.harvard.edu`

Department of Mathematics, Harvard University
Cambridge, MA 02138, USA
Current address:
Yau Mathematical Sciences Center, Tsinghua University
Haidian District, Beijing 100084, China
E-mail address: `yau@math.harvard.edu`

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