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Longtime existence of Kähler-Ricci flow and holomorphic sectional curvature

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In this work, we obtain some sufficient conditions for the longtime existence of the Kähler-Ricci flow solution. Using the existence results, we generalize a result by Wu-Yau on the existence of Kähler-Einstein metric on noncompact complex manifolds.

1. Introduction

In this work we will discuss the existence of Kähler-Einstein metric on a complete noncompact Kähler manifold in terms of upper bound of holomorphic sectional curvature. In [22], Wu and Yau proved that if a compact complex manifold supports a Kähler metric with negative holomorphic sectional curvature, then it also supports a Kähler-Einstein metric with negative scalar curvature, under an additional assumption that the manifold is

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projective. Later, Tosatti and Yang [20] were able to remove the assumption of projectivity. Using Kähler-Ricci flow, Normura [10] recovered the result by proving that under the assumption that the holomorphic sectional curvature is bounded above by a negative constant, the metric can be deformed under the normalized Kähler-Ricci flow to a Kähler-Einstein metric with negative scalar curvature. In case that the holomorphic sectional curvature is quasi-negative, namely it is nonpositive and is negative somewhere, Diverio-Trapani [4] and Wu-Yau [23] proved that the canonical bundle is ample and hence the existence of the Kähler-Einstein metric follows by a well-known theorem of Yau [26].

In the noncompact case, it was proved by Wu and Yau [21] that if a noncompact complex manifold supports a complete Kähler metric with holomorphic sectional curvature bounded between two negative constants, then it also supports a complete Kähler-Einstein metric with negative scalar curvature. It is well-known that if the holomorphic sectional curvature is bounded then the curvature is bounded. In [18], the fourth author used Shi's Kähler-Ricci flow [15] for complete noncompact Kähler manifolds with bounded curvature to show that the Kähler metric mentioned above can also be deformed under the normalized Kähler-Ricci flow to a Kähler-Einstein metric with negative scalar curvature. In this work, we further generalize the results in [18].

First, we will give a rather general condition for a normalized Kähler-Ricci flow to converge to a Kähler-Einstein metric. We prove the following:

Theorem 1.1. *Suppose there is a complete noncompact Hermitian metric h on a complex manifold M^n compatible with the complex structure J such that the torsion $\widehat{T}$ and the holomorphic sectional curvature H_h satisfy*

$$H_h + \frac{n}{n+1}|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h \le -k \tag{1.1}$$

for some $k > 0$. Then any long-time complete solution to the normalized Kähler-Ricci flow $g(t)$ will converge in C^∞_{loc} to the unique Kähler-Einstein metric $g_\infty = -\mathrm{Ric}(g_\infty)$. In particular, there is no complete Ricci flat Kähler metric on M compatible with the same complex structure J.

Here $\widehat{\nabla}$ is the derivative with respect to the Chern connection of h. See [19] for more details on the Chern connection, its torsion and curvature. See also [25] for a related assumption on the Hermitian metric. For the definition of $|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h$, see (3.4).

By the theorem, to obtain a Kähler-Einstein metric, in some cases it is sufficient to obtain a longtime solution to the Kähler-Ricci flow. In this respect, we will prove the following:

Theorem 1.2. *Let (M^n, g_0) be a complete noncompact Kähler manifold and h be a fixed complete Hermitian metric on M such that the following hold.*

(i) *There exists a smooth exhaustion function $\rho \geq 1$ such that*

$$\limsup_{\rho\to\infty}\left[\frac{|\partial\rho|_h}{\rho}(1+|\widehat{\nabla} g_0|_h)+\frac{|\sqrt{-1}\partial\bar{\partial}\rho|_h}{\rho}\right]=0;$$

(ii) *the holomorphic sectional curvature of h and torsion $\widehat{T}$ of h satisfy*

$$H_h+\frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h\leq -k$$

for some constant $k \geq 0$; and

(iii) *there exists $\alpha > 1$ such that on M, $\alpha^{-1} g_0 \leq h \leq \alpha g_0$, $|\widehat{T}|_h \leq \alpha$.*

Then there is $\beta(n,\alpha) > 0$ such that the Kähler-Ricci flow has a complete solution $g(t)$ on $M \times [0, +\infty)$ with $g(0) = g_0$ and satisfies

$$\beta h \leq g(t)$$

on $M \times [0, +\infty)$.

It is known that if M has bounded curvature, then it will support an exhaustion function ρ with bounded gradient and Hessian [14, 17]. Hence if h is uniformly equivalent to a complete Hermitian metric with bounded Riemannian curvature and bounded torsion, then condition (i) in the theorem will be satisfied. See also a recent result in [6]. Therefore condition (i) is more general than the condition that the curvature is bounded for Kähler metrics.

Combining Theorems 1.1 and 1.2, we conclude that if (M^n, g_0) is a complete Kähler manifold, then there is a long-time solution of the normalized Kähler-Ricci flow which will converge to the Kähler-Einstein metric with negative scalar curvature in the following cases:

(a) The holomorphic sectional curvature is bounded above by $-k$ for some $k > 0$ and g_0 supports an exhaustion function with bounded gradient and bounded complex Hessian.

(b) There exists a complete Hermitian metric h so that g_0, h satisfy the conditions in Theorem 1.2 with $k > 0$.

(c) g_0 satisfies a Sobolev inequality, the curvature is bounded in some L^p sense and the holomorphic sectional curvature is bounded above by $-k$ for some $k > 0$. (See more precise statement in Corollary 5.2.)

In case g_0 has bounded curvature so that the holomorphic sectional curvature is bounded above by $-k < 0$ for some constant k, then the conditions in (c) will also be satisfied. Hence (a)–(c) are some generalizations to Wu-Yau's result [21].

The paper is organized as follows: In section 2, we will recall a short time existence result of the Chern-Ricci flow. In section 3, we will derive some a-priori estimates for the Chern-Ricci flow and apply them in section 4 to construct short time solution to the Chern-Ricci flow with estimate on existence time. In particular, we will use this to prove Theorem 1.2. In section 5, we will prove Theorem 1.1.

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2. A short time existence lemma

Let (M^n, g_0) be a complete noncompact Hermitian manifold with complex dimension n. In the following, connection and curvature will be referred to the Chern connection and curvature with respect to the Chern connection. When the torsion vanishes, the Chern connection coincides with the Levi-Civita connection. For basic facts on the Chern connection and curvature of Hermitian manifolds, we refer readers to [19] for example. In this section, we want to discuss the existence of the Chern-Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t} g_{i\bar{j}} = & -R_{i\bar{j}}; \\ g(0) = & g_0. \end{cases} \tag{2.1}$$

Here $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\det(g(t))$ is the Chern-Ricci curvature of $g(t)$. This equation is equivalent to the following parabolic complex Monge-Ampère equation:

$$
(2.2) \qquad \begin{cases} \frac{\partial}{\partial t}\psi = & \log \dfrac{(\omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar\partial\psi)^n}{\omega_0^n}; \\ \psi(0) = & 0. \end{cases}
$$

More precisely, if $g(t)$ is a solution to (2.1), let

$$
(2.3) \qquad \psi(x,t) = \int_0^t \log\left(\frac{\omega^n(x,s)}{\omega_0^n(x)}\right) ds,
$$

where $\omega(t)$ and ω_0 are the associated (1,1) forms of $g(t)$, g_0 respectively. Then ψ satisfies (2.2). One can see that $\omega(t) = \omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar\partial\psi$. Conversely, if ψ is a smooth solution to (2.2) so that $\omega_0 - t\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar\partial\psi > 0$, then $\omega(t)$ defined by the above relation satisfies (2.1). We will say that ψ is the solution of (2.2) corresponding to the solution $g(t)$ of (2.1).

Let us recall the following definition of bounded geometry:

Definition 2.1. *Let (M^n, g) be a complete Hermitian manifold. Let $k \geq 1$ be an integer and $0 < \alpha < 1$. g is said to have bounded geometry of order $k+\alpha$ if there are positive numbers r, κ_1, κ_2 such that at every $p \in M$ there is a neighbourhood U_p of p, and local biholomorphism ξ_p from $D(r)$, which is the Euclidean ball of radius r with center at the origin in $\mathbb{C}^n$, onto U_p with $\xi_p(0) = p$ satisfying the following properties:*

(i) *the pull back metric $\xi_p^*(g)$ satisfies:*

$$\kappa_1 g_e \leq \xi_p^*(g) \leq \kappa_2 g_e,$$

where g_e is the standard metric on $\mathbb{C}^n$; and

(ii) *the components $g_{i\bar j}$ of $\xi_p^*(g)$ in the natural coordinate of $D(r) \subset \mathbb{C}^n$ are uniformly bounded in the standard $C^{k+\alpha}$ norm in $D(r)$ independent of p.*

(M, g) is said to have bounded geometry of infinity order if instead of (ii) we have for any k, the k-th derivatives of $g_{i\bar j}$ in $D(r)$ are bounded by a constant independent of p. g is said to have bounded geometry of infinite order on a compact set Ω if (i) and (ii) are true for all k for all $p \in \Omega$.

In [9], it has been shown that when (M, g_0) has bounded geometry of infinite order, the Monge-Ampère equation (2.2) and hence the Chern-Ricci flow equation (2.1) has a short time solution on M.

Lemma 2.1 (see [1, 9]). *Let (M^n, g_0) be a complete noncompact Hermitian metric. Suppose g_0 has bounded geometry of infinite order, then* (2.1) *has a solution $g(t)$ on $M \times [0, S]$ for some $S > 0$ and there is a constant $C > 0$ such that $C^{-1} g_0 \le g(t) \le C g_0$.*

3. A-priori estimate for the Chern-Ricci flow

Let (M^n, g) be a Hermitian manifold. Under a local holomorphic coordinate system $(z_1, ..., z_n)$, the torsion tensor of g is defined by

$$T_{ij\bar{l}} = \partial_i g_{j\bar{l}} - \partial_j g_{i\bar{l}}.$$

Let $T_{ij}^k = g^{k\bar{l}} T_{ij\bar{l}}$, then $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ where Γ_{ij}^k is the Chern connection. T_{ij}^k is usually called the torsion. Here we use $T_{ij\bar{k}}$ to denote the torsion. The advantage is that it is invariant under the Chern-Ricci flow. If the torsion tensor $T = 0$, then g is Kähler. The curvature tensor of the Chern connection has components

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + g^{q\bar{p}} \frac{\partial g_{k\bar{p}}}{\partial z_i} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j}.$$

It can be checked easily that for $X, Y \in T^{1,0}M$, $R(X, \bar{X}, Y, \bar{Y})$ is real-valued. We introduce the following curvature condition.

Definition 3.1. *We say that (M, g) has holomorphic sectional curvature bounded above by κ if for any $p \in M$, $X \in T_p^{1,0}M$,*

$$R(X, \bar{X}, X, \bar{X}) \le \kappa |X|^4.$$

For notational convenience, we denote this condition by $H_g \le \kappa$.

Let $g(t)$ be a solution of the Chern-Ricci flow with initial metric $g(0) = g_0$ and h be another Hermitian metric on M. Now we wish to obtain some a-priori estimates for $g(t)$. First we list some evolution equations which are related to the Chern-Ricci flow. In this work, the Laplacian for a Hermitian metric g is defined by

$$\Delta u := g^{i\bar{j}} u_{i\bar{j}}. \tag{3.1}$$

This is the usual Laplacian for Kähler metric.

The following lemma concerns the evolution equation for the lower bound on evolving metric g with respect to fixed metric h while in [19, Proposition 3.1], Tosatti-Weinkove considered the upper bound of g, that is $\operatorname{tr}_h g$.

Lemma 3.1. *Let $\Lambda = \operatorname{tr}_g h = g^{i\bar{j}} h_{i\bar{j}}$. Then the evolution equation of Λ is given by*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda = (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}), \tag{3.2}$$

where

$$\begin{aligned}
(\mathbf{I}) &= -h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi^k_{pi} \overline{\Psi^l_{qj}} + 2\mathbf{Re}\left[g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \Psi^{\bar{s}}_{\bar{l}\bar{q}} (T_0)_{pi\bar{s}}\right];\\
(\mathbf{II}) &= g^{l\bar{k}} g^{j\bar{i}} g^{q\bar{p}} h_{j\bar{k}} (T_0)_{\bar{p}\bar{i}r}\left[\hat{T}_{ql\bar{s}} h^{r\bar{s}} - (T_0)_{ql\bar{s}} g^{r\bar{s}}\right]\\
&\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}}\left[\hat{\nabla}_p (T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}} (T_0)_{pi\bar{q}}\right];\\
(\mathbf{III}) &= g^{i\bar{j}} g^{p\bar{q}} \hat{R}_{p\bar{q}i\bar{j}}.
\end{aligned}$$

Here T_0 and $\hat{T}$ are the torsion of metric g_0 and h respectively and

$$\Psi^k_{ij} := \hat{\Gamma}^k_{ij} - \Gamma^k_{ij},$$

where $\hat{\Gamma}, \Gamma$ are the Chern connections of h and g respectively.

In particular,

$$(\mathbf{I}) \le h_{p\bar{r}} h_{c\bar{q}} h^{k\bar{a}} g^{s\bar{r}} g^{c\bar{d}} g^{i\bar{j}} g^{p\bar{q}} (T_0)_{si\bar{a}} (T_0)_{\bar{d}\bar{j}k}.$$

Moreover, the evolution equation of $\log\Lambda$ is given by:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\Lambda = (\mathbf{IV}) + \Lambda^{-1}\Big[(\mathbf{II}) + (\mathbf{III})\Big] \tag{3.3}$$

with

$$\begin{aligned}
(\mathbf{IV}) \le\ & \Lambda^{-1} h_{p\bar{r}} h_{c\bar{q}} h^{k\bar{a}} g^{s\bar{r}} g^{c\bar{d}} g^{i\bar{j}} g^{p\bar{q}} (T_0)_{si\bar{a}} (T_0)_{\bar{d}\bar{j}k}\\
&+ 2\Lambda^{-2}\mathbf{Re}\Big[h_{p\bar{r}} g^{a\bar{r}} g^{i\bar{l}} g^{p\bar{q}} (T_0)_{ia\bar{l}} \partial_{\bar{q}}\Lambda\Big].
\end{aligned}$$

Proof.

$$\partial_t \operatorname{tr}_g h = g^{i\bar{q}} g^{p\bar{j}} h_{i\bar{j}} R_{p\bar{q}}.$$

$$\begin{aligned}\Delta \operatorname{tr}_g h &= g^{i\bar{j}} g^{p\bar{q}} \nabla_{\bar{q}} \nabla_p h_{i\bar{j}} \\ &= g^{i\bar{j}} g^{p\bar{q}} \nabla_{\bar{q}} \left(\Psi_{pi}^k h_{k\bar{j}} \right) \\ &= g^{i\bar{j}} g^{p\bar{q}} \left[\left(R_{p\bar{q}i}^k - \hat{R}_{p\bar{q}i}^k \right) h_{k\bar{j}} + \Psi_{pi}^k \Psi_{\bar{q}\bar{j}}^{\bar{l}} h_{k\bar{l}} \right].\end{aligned}$$

Using the fact that the torsion T of $g(t)$ satisfies $T_{ij\bar{k}} = (T_0)_{ij\bar{k}}$, we have

$$\begin{aligned}R_{p\bar{q}i\bar{l}} &= R_{i\bar{l}p\bar{q}} - \nabla_p T_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} T_{pi\bar{q}} \\ &= R_{i\bar{l}p\bar{q}} - \nabla_p (T_0)_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} (T_0)_{pi\bar{q}}.\end{aligned}$$

Hence,

$$\begin{aligned}g^{i\bar{j}} g^{p\bar{q}} h_{k\bar{j}} R_{p\bar{q}i}^k &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} R_{p\bar{q}i\bar{l}} \\ &= g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[R_{i\bar{l}p\bar{q}} - \nabla_p (T_0)_{\bar{q}\bar{l}i} - \nabla_{\bar{l}} (T_0)_{pi\bar{q}} \right] \\ &= g^{i\bar{j}} g^{k\bar{l}} h_{k\bar{j}} R_{i\bar{l}} - g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\nabla_p (T_0)_{\bar{q}\bar{l}i} + \nabla_{\bar{l}} (T_0)_{pi\bar{q}} \right].\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta \right) \Lambda &= -h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi_{pi}^k \overline{\Psi_{qj}^l} + g^{i\bar{j}} g^{p\bar{q}} \hat{R}_{p\bar{q}i\bar{j}} \\ &\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\nabla_p (T_0)_{\bar{q}\bar{l}i} + \nabla_{\bar{l}} (T_0)_{pi\bar{q}} \right] \\ &= -h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi_{pi}^k \overline{\Psi_{qj}^l} + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\Psi_{pi}^r (T_0)_{\bar{q}\bar{l}r} + \Psi_{\bar{l}\bar{q}}^{\bar{s}} (T_0)_{pi\bar{s}} \right] \\ &\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\hat{\nabla}_p (T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}} (T_0)_{pi\bar{q}} \right] + g^{i\bar{j}} g^{p\bar{q}} \hat{R}_{p\bar{q}i\bar{j}} \\ &= -h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi_{pi}^k \overline{\Psi_{qj}^l} + 2\mathbf{Re} \left[g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \Psi_{\bar{l}\bar{q}}^{\bar{s}} (T_0)_{pi\bar{s}} \right] \\ &\quad + g^{l\bar{k}} g^{j\bar{i}} g^{q\bar{p}} h_{j\bar{k}} (T_0)_{\bar{p}\bar{i}r} \left[\hat{T}_{ql\bar{s}} h^{r\bar{s}} - (T_0)_{ql\bar{s}} g^{r\bar{s}} \right] \\ &\quad + g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \left[\hat{\nabla}_p (T_0)_{\bar{q}\bar{l}i} + \hat{\nabla}_{\bar{l}} (T_0)_{pi\bar{q}} \right] + g^{i\bar{j}} g^{p\bar{q}} \hat{R}_{p\bar{q}i\bar{j}}.\end{aligned}$$

From this, the first part of the lemma follows. Thus,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta \right) \log \Lambda &= \Lambda^{-1} \left(\frac{\partial}{\partial t} - \Delta \right) \Lambda + \Lambda^{-2} g^{i\bar{j}} \partial_i \Lambda \, \partial_{\bar{j}} \Lambda \\ &= \frac{1}{\Lambda} \left[(\mathbf{I}) + \frac{1}{\operatorname{tr}_g h} |\partial \Lambda|^2 \right] + \frac{1}{\Lambda} \Big[(\mathbf{II}) + (\mathbf{III}) \Big] \\ &= (\mathbf{IV}) + \frac{1}{\Lambda} \Big[(\mathbf{II}) + (\mathbf{III}) \Big].\end{aligned}$$

In case that g_0 is Kähler, it was shown by Yau [26] that the first bracket term is nonpositive. In the Hermitian case, we estimate the first bracket following the idea [19]. For any tensor C, we consider the following nonnegative quantity:

$$\begin{aligned}K &:= h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \left(\Psi^k_{pi} - \frac{\delta^k_i}{\operatorname{tr}_g h}\partial_p\Lambda + C^k_{pi}\right)\left(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{\delta^{\bar{l}}_{\bar{j}}}{\operatorname{tr}_g h}\partial_{\bar{q}}\Lambda + C^{\bar{l}}_{\bar{q}\bar{j}}\right)\\ &= h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} \Psi^k_{pi}\Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{1}{\operatorname{tr}_g h}|\partial \operatorname{tr}_g h|^2 + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi}\left(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - \frac{\delta^{\bar{l}}_{\bar{j}}}{\operatorname{tr}_g h}\partial_{\bar{q}}\Lambda\right)\\ &\quad + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^{\bar{l}}_{\bar{q}\bar{j}}\left(\Psi^k_{pi} - \frac{\delta^k_i}{tr_g h}\partial_p\Lambda\right) + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}}.\end{aligned}$$

Hence,

$$\begin{aligned}(\mathbf{I}) + \frac{1}{\Lambda}|\partial\Lambda|^2 &= -K + 2\mathbf{Re}\left[g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} h_{k\bar{j}} \Psi^{\bar{s}}_{\bar{l}\bar{q}} (T_0)_{pi\bar{s}}\right]\\ &\quad + 2\mathbf{Re}\left[h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi}\Psi^{\bar{l}}_{\bar{q}\bar{j}}\right]\\ &\quad + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}} - 2\Lambda^{-1}\mathbf{Re}\left[h_{k\bar{l}} g^{i\bar{l}} g^{p\bar{q}} C^k_{pi}\partial_{\bar{q}}\Lambda\right]\\ &= -K + h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} C^k_{pi} C^{\bar{l}}_{\bar{q}\bar{j}} - 2\Lambda^{-1}\mathbf{Re}\left[h_{k\bar{l}} g^{i\bar{l}} g^{p\bar{q}} C^k_{pi}\partial_{\bar{q}}\Lambda\right]\\ &\quad + 2\mathbf{Re}\left[\Psi^{\bar{l}}_{\bar{q}\bar{j}} g^{p\bar{q}} g^{i\bar{j}}\left(C^k_{pi} h_{k\bar{l}} + g^{k\bar{r}} h_{p\bar{r}} (T_0)_{ik\bar{l}}\right)\right].\end{aligned}$$

Therefore, if we choose the tensor C to be

$$C^q_{pi} = -g^{k\bar{r}} h^{q\bar{l}} h_{p\bar{r}} (T_0)_{ik\bar{l}},$$

then the last term vanished. Hence,

$$\begin{aligned}(\mathbf{IV}) \le\ &\Lambda^{-1} h_{p\bar{r}} h_{c\bar{q}} h^{k\bar{a}} g^{s\bar{r}} g^{c\bar{d}} g^{i\bar{j}} g^{p\bar{q}} (T_0)_{si\bar{a}} (T_0)_{\bar{d}\bar{j}k}\\ &+ 2\Lambda^{-2}\mathbf{Re}\Big[h_{p\bar{r}} g^{a\bar{r}} g^{i\bar{l}} g^{p\bar{q}} (T_0)_{ia\bar{l}}\partial_{\bar{q}}\Lambda\Big].\end{aligned}$$

The estimate ($\mathbf{I}$) follows the same line by considering a simpler quantity

$$K = h_{k\bar{l}} g^{i\bar{j}} g^{p\bar{q}} (\Psi^k_{pi} - C^k_{pi})(\Psi^{\bar{l}}_{\bar{q}\bar{j}} - C^{\bar{l}}_{\bar{q}\bar{j}}).$$

□

In [22], Wu-Yau made use of the Royden's Lemma [12] which relates the holomorphic sectional curvature with a bisectional curvature quantity. We follow their idea and generalize the Royden's Lemma for Hermitian metrics. In the following, $|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h(x)$ at a point x is defined as:

$$|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h = \max |\widehat{\nabla}_{\bar i}\widehat{T}_{jl\bar k}|, \tag{3.4}$$

where the maximum is taken over all unitary frames e_i of h at x. Define $|\widehat{\nabla}_{\bar\partial}T_0|_h$ similarly.

Lemma 3.2. *Let (M, h) be a Hermitian manifold and g be another Hermitian metric on M. Suppose that the holomorphic sectional curvature of h at x is bounded above by $\kappa(x)$. Suppose $\kappa(x) \le \kappa_0$. Then we have*

$$\begin{aligned} g^{i\bar j}g^{k\bar l}\hat R_{i\bar jk\bar l} \le{}& \left(\frac{n+1}{2n}\kappa + \frac12|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h\right)(\mathrm{tr}_g\, h)^2 \\ &+ \frac12\kappa_0\left[-\frac1n(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right]. \end{aligned}$$

Proof. Following the proof in [12] without appealing the symmetry of $\hat R$, we can deduce that at x,

$$g^{i\bar j}g^{k\bar l}\hat R_{i\bar jk\bar l} + g^{i\bar j}g^{k\bar l}\hat R_{i\bar lk\bar j} \le \kappa(\mathrm{tr}_g\, h)^2 + \kappa g^{i\bar j}g^{k\bar l}h_{k\bar j}h_{i\bar l}.$$

By the "Kähler" identity for the Chern curvature, e.g. see [19], we have

$$\begin{aligned} g^{i\bar j}g^{k\bar l}\hat R_{i\bar jk\bar l} &= g^{i\bar j}g^{k\bar l}(\hat R_{i\bar lk\bar j} - \hat\nabla_i\hat T_{\bar j\bar lk}) \\ &= \frac12 g^{i\bar j}g^{k\bar l}(\hat R_{i\bar lk\bar j} + \hat R_{i\bar jk\bar l} - \hat\nabla_i\hat T_{\bar j\bar lk}) \\ &\le \frac{\kappa}{2}(tr_g h)^2 + g^{i\bar j}g^{k\bar l}\left(\frac{\kappa}{2}h_{k\bar j}h_{i\bar l} - \frac12\hat\nabla_i\hat T_{\bar j\bar lk}\right) \\ &\le \frac12(\kappa(x)-\kappa_0)\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right] + \frac12(\mathrm{tr}_g\, h)^2|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h \\ &\quad + \frac12\kappa_0\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right] \\ &\le \frac12(\kappa(x)-\kappa_0)(1+\frac1n)(\mathrm{tr}_g\, h)^2 + \frac12(\mathrm{tr}_g\, h)^2|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h \\ &\quad + \frac12\kappa_0\left[(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right] \\ &= \left(\frac{n+1}{2n}\kappa + \frac12|\widehat{\nabla}_{\bar\partial}\widehat{T}|_h\right)(\mathrm{tr}_g\, h)^2 \\ &\quad + \frac12\kappa_0\left[-\frac1n(\mathrm{tr}_g\, h)^2 + g^{i\bar j}g^{k\bar l}h_{kj}h_{i\bar l}\right]. \end{aligned}$$

□

Combining this with Lemma 3.1, we have:

Corollary 3.1. *With the same assumptions and notation as in Lemma 3.1, suppose the holomorphic sectional curvature of h is bounded above by $\kappa(x)$ at x and suppose $\frac{n+1}{2n}\kappa(x) + \frac{1}{2}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|(x) \le \kappa_0$ for some $\kappa_0 \ge 0$ for all x. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda \le c(n)\left(\Lambda^4|T_0|_h^2 + \Lambda^3(|T_0|_h|\widehat{T}|_h + |\widehat{\nabla}_{\bar{\partial}}T_0|_h) + \Lambda^2\kappa_0\right)$$

for some constant $c(n) > 0$ depending only on n.

To get a C^0 estimate, it is useful to consider the Chern scalar curvature of $g(t)$ which gives us information on the derivatives of the volume form.

Lemma 3.3. *Under Chern-Ricci flow*

$$\frac{\partial}{\partial t}g = -\mathrm{Ric},$$

the Chern scalar curvature $R = g^{i\bar{j}}R_{i\bar{j}}$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)R = |\mathrm{Ric}|^2 \ge \frac{1}{n}R^2.$$

Proof.

$$\begin{aligned}\partial_t R &= \partial_t(g^{i\bar{j}}R_{i\bar{j}}) \\ &= R^{i\bar{j}}R_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}(\partial_t \log\det g) \\ &= |\mathrm{Ric}|^2 + \Delta R.\end{aligned}$$

The inequality can be observed by taking a coordinate chart at p such that $g_{i\bar{j}} = \delta_{ij}$ and $R_{i\bar{j}} = \lambda_i\delta_{ij}$. Then it follows immediately by Cauchy inequality. □

For later applications, we need the following maximum principle.

Lemma 3.4. *Let (M^n, h) be a complete noncompact Hermitian manifold satisfying condition: There exists a smooth positive real exhaustion function ρ such that $|\partial\rho|_h^2 + |\sqrt{-1}\partial\bar{\partial}\rho|_h \le C_1$. Suppose g_0 is another Hermitian metric uniformly equivalent to h and $g(t)$ is a solution to the Chern-Ricci flow* *with initial metric $g(0) = g_0$ on $M \times [0, S)$. Assume for any $0 < S_1 < S$, there is $C_2 > 0$ such that*

$$C_2^{-1} h \le g(t)$$

for $0 \le t \le S_1$. Let f be a smooth function on $M \times [0, S)$ which is bounded from above such that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \le 0$$

on $\{f > 0\}$. Suppose $f \le 0$ at $t = 0$, then $f \le 0$ on $M \times [0, S)$.

Proof. For any $\epsilon > 0$, if $\sup_{M\times[0,T]}(f - \epsilon\rho - 2\epsilon C_1 C_2 t) > 0$, then there is (x_0, t_0) with $t_0 > 0$ such that $f - \epsilon\rho - 2\epsilon C_1 C_2 t \le 0$ on $M \times [0, t_0]$ and $f - \epsilon\rho - 2\epsilon C_1 C_2 t = 0$ at (x_0, t_0). In particular, $f(x_0, t_0) > 0$. Hence at (x_0, t_0), we have

$$0 \le \left(\frac{\partial}{\partial t} - \Delta\right) (f - \epsilon\rho - 2\epsilon C_1 C_2 t) < 0,$$

which is impossible. Since ϵ is arbitrary, this completes the proof. □

Next we give a local estimate on the lower bound of the Chern scalar curvature. Note that here we do not need global bounds on the Hessian and gradient of the exhaustion function ρ. The estimate only depends on those bounds on a compact set.

Lemma 3.5. *Suppose h is a fixed Hermitian metric with a smooth positive real exhaustion function ρ and $g(t)$ is a solution to the Chern-Ricci flow on $M \times [0, S]$ with $g(t) \ge \alpha^{-1} h$ for some $\alpha > 1$. Then for any $0 < r_1 < r_2$, there exists $C > 0$ depending only on n, α and $\sup_{U_{r_2}}(|\partial\rho| + |\sqrt{-1}\partial\bar\partial\rho|)$ such that for any $x \in U_{r_1}$ and $t \in [0, S]$, we have*

$$R(x, t) \ge -\max\left\{C[(r_2 - r_1)^{-2} + 1], \sup_{\rho(y)<r_2} R_-(y, 0)\right\}.$$

Here R_- is the negative part of R and $U_r = \{x \in M : \rho(x) < r\}$.

Proof. Let ϕ be a cutoff function on $\mathbb{R}$ such that $\phi \equiv 1$ on $(-\infty, 1]$, vanishes outside $(-\infty, 2]$ and satisfies $\phi^{-1}|\phi'|^2 \le 100$ and $\phi'' \ge -100\phi$. Define

$$\Phi(x) = \phi\left(\frac{\rho(x) + r_2 - 2r_1}{r_2 - r_1}\right).$$

When the function ΦR achieves its local minimum at (x_0, t_0) in which we may assume $R(x_0, t_0) < 0$ and $t_0 > 0$, it satisfies the following.

$$\begin{aligned}
0 &\geq \left(\frac{\partial}{\partial t} - \Delta\right)(\Phi R) \\
&= \Phi \left(\frac{\partial}{\partial t} - \Delta\right) R - R\Delta\Phi - 2Re\left(g^{i\bar{j}}\partial_i \Phi \, \partial_{\bar{j}} R\right) \\
&\geq \frac{1}{n}\Phi R^2 - R\left[\frac{\phi''}{(r_2 - r_1)^2}|\partial\rho|^2 + \frac{\phi'}{r_2 - r_1}\Delta\rho - 2\frac{(\phi')^2}{(r_2 - r_1)^2\phi}|\partial\rho|^2\right] \\
&\geq \frac{1}{n}\Phi R^2 + CR[(r_2 - r_1)^{-2} + 1].
\end{aligned}$$

Hence, at its minimum point (x_0, t_0),

$$\Phi R \geq -C[(r_2 - r_1)^{-2} + 1].$$

The conclusion follows by the minimum principle. □

4. Existence of the Chern-Ricci flow

In this section, we will discuss the existence of the Chern-Ricci flow starting from a Hermitian metric with holomorphic sectional curvature bounded from above. We will give an estimate on the existence time. More generally, we will consider initial metric which is uniformly equivalent to a Hermitian metric with holomorphic sectional curvature bounded from above.

Lemma 4.1. *Let (M^n, g_0) be a Hermitian metric with bounded geometry of infinite order. Suppose g_0 is uniformly equivalent to a Hermitian metric h with holomorphic sectional curvature H_h and torsion $\widehat{T}$ satisfying: $H_h(x)$ bounded above by $\kappa(x)$ and $\frac{n+1}{n}\kappa(x) + |\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h(x) \leq \kappa_0$ for some $\kappa_0 \geq 0$ for all x, where $\widehat{\nabla}$ is the derivative of h with respect to the Chern connection. Assume*

$$\alpha^{-1}h \leq g_0 \leq \alpha h,$$

for some $\alpha > 1$. Then the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S]$ with the following properties:

(i) *There is a constant $c = c(n) > 0$ so that*

$$S \geq \frac{1}{3c(n\alpha + 1)^3\mathfrak{s}} =: S_1,$$

where

$$\mathfrak{s} = \sup_M \left(|T_0|_h^2 + |T_0|_h |\widehat{T}|_h + |\widehat{\nabla}_{\bar\partial} T_0|_h + \kappa_0 \right)$$

and T_0 *is the torsion of* g_0*; and*

(ii) $g(t)$ *is uniformly equivalent to* h *with*

$$\operatorname{tr}_g h \le \left(\frac{1}{(n\alpha+1)^{-3} - 3c\mathfrak{s}t} \right)^{\frac{1}{3}}$$

on $M \times [0, S_1]$.

Proof. (i) If $\mathfrak{s} = \infty$, then there is nothing to be proved. Suppose $\mathfrak{s} < \infty$, then by Lemma 2.1, there is a maximal $S > 0$ such that the Chern-Ricci flow has a solution $g(t)$ with $g(0) = g_0$ on $M \times [0, S)$ so that $g(t)$ is uniformly equivalent to g_0 on $[0, S']$ for all $S' < S$. Let $\Lambda = \operatorname{tr}_{g(t)} h$. By Corollary 3.1,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \Lambda \le & c_1 \left(\Lambda^4 |T_0|_h^2 + \Lambda^3 (|T_0|_h |\widehat{T}|_h + |\widehat{\nabla}_{\bar\partial} T_0|_h) + \Lambda^2 \kappa_0 \right) \\ \le & c_1 (\Lambda + 1)^4 \mathfrak{s} \end{aligned}$$

on $M \times [0, S]$. Here and below c_i will denote positive constants depending only on n. Let

$$v(t) = \left(\frac{1}{(n\alpha+1)^{-3} - 3c_2\mathfrak{s}t} \right)^{\frac{1}{3}}.$$

Then $v(t)$ is defined on $[0, S_1)$ with $S_1 = 1/\left[3c_2(n\alpha+1)^3\mathfrak{s}\right]$, with

$$\frac{dv}{dt} = c_2 \mathfrak{s} v^4$$

and $v(0) \ge (\Lambda + 1)|_{t=0}$. Suppose $S < S_1$. Since Λ and v are bounded on $[0, S']$ for all $0 < S' < S$, by Lemma 3.4 as in the proof of [9, Theorem 4.2], one can conclude that

$$\Lambda \le v(t) - 1$$

on $M \times [0, S)$. In particular,

$$h \le c_3 (v(t) - 1) g(t). \tag{4.1}$$

If $S < S_1$, then $v(t) \le C_1 < \infty$ on $[0, S]$ for some C_1. Hence $\Lambda \le C_1$ on $M \times [0, S)$.

On the other hand, since g_0 has bounded geometry of infinite order, by Lemma 3.5, we conclude that $R(x,t) \geq -C_2$ on $M \times [0,S)$ for some C_2. Since

$$\frac{\partial}{\partial t}\left(\log \frac{\det(g(t))}{\det(h)}\right) = -R \leq C_2,$$

we conclude that $\det(g(t)) \leq C_3 \det(h)$. Together with (4.1), we conclude that

$$C_3^{-1} g_0 \leq g(t) \leq C_3 g_0$$

on $M \times [0,S)$ for some $C_3 > 0$. Here we have used the fact that g_0 is uniformly equivalent to h. Using the fact that g_0 has bounded geometry of infinite order and by the local estimates of [13], $g(t)$ can be extended to be a solution of the Chern-Ricci flow which is uniformly equivalent to g_0 beyond S. Hence we have $S \geq S_1$. This proves (i).

(ii) Follows from (4.1). □

Let (M^n, h) be a complete noncompact Hermitian manifold satisfying the following:

(**a**) There exists smooth exhaustion $\rho \geq 1$, and constant $\beta > 0$ such that

$$|\partial\rho|_h + |\sqrt{-1}\partial\bar{\partial}\rho|_h \leq \beta\rho$$

if ρ is large enough.

(**b**) The holomorphic sectional curvature at x is bounded from above by $\kappa(x)$, and the torsion $\widehat{T}$ of h is such that

$$\frac{n+1}{n}\kappa + |\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \leq \kappa_0$$

for some $\kappa_0 \geq 0$.

Theorem 4.1. *Let (M^n, h) be a complete Hermitian metric as above. Let g_0 be another Hermitian metric with torsion T_0. Suppose the following are true:*

(i) $\alpha^{-1} g_0 \leq h \leq \alpha g_0$ *and* $|\widehat{T}|_h \leq \alpha$ *for some* $\alpha > 1$;

(ii) $|T_0|_h^2 + |\widehat{T}|_h |T_0|_h + |\widehat{\nabla}_{\bar{\partial}}(T_0)|_h \leq \beta$; *and*

(iii) $|\partial\rho|_h |\widehat{\nabla} g_0|_h \leq \beta\rho$ *for* ρ *large enough.*

Then there exist constants $c_1(n)$ depending only on n and $c_2(n,\alpha)$ depending only on n, α such that there is a solution $g(t)$ for the Chern-Ricci flow on $M \times [0,S)$ with $g(0) = g_0$, where

$$S = \frac{1}{3c_1(n\alpha+1)^3\mathfrak{s}},$$

and $\mathfrak{s} = \kappa_0 + c_2\beta(1+\beta)$. Moreover,

$$\operatorname{tr}_g h \le v(t) - 1$$

on $M \times [0,S)$, where

$$v(t) = \left(\frac{1}{(n\alpha+1)^{-3} - 3c_1\mathfrak{s}t}\right)^{\frac{1}{3}}.$$

We want to apply Lemma 4.1 to prove the theorem. However, in general it is not true that g_0 has bounded geometry of all order, we cannot apply Lemma 4.1 directly to obtain a solution of the Chern-Ricci flow. We now proceed as in [8, 9] to construct a Hermitian approximation.

Let $\tau \in (0, \frac{1}{8})$, $f : [0,1) \to [0,\infty)$ be the function:

$$f(s) = \begin{cases} 0, & s \in [0, 1-\tau]; \\ -\log\left[1 - \left(\dfrac{s-1+\tau}{\tau}\right)^2\right], & s \in (1-\tau, 1). \end{cases} \tag{4.2}$$

Let $\varphi \ge 0$ be a smooth function defined on $[0,1)$ such that

$$\varphi(s) = \begin{cases} 0, & s \in [0, 1-\tau+\tau^2]; \\ 1, & s \in (1-\tau+2\tau^2, 1). \end{cases} \tag{4.3}$$

and $\dfrac{2}{\tau^2} \ge \varphi' \ge 0$. Define

$$\mathfrak{F}(s) := \int_0^s \varphi(\tau) f'(\tau) d\tau.$$

From [8], we have:

Lemma 4.2. *Suppose $0 < \tau < \frac{1}{8}$. Then the function $\mathfrak{F} \ge 0$ defined above is smooth and satisfies the following:*

(i) *$\mathfrak{F}(s) = 0$ for $0 \le s \le 1 - \tau + \tau^2$.*

(ii) $\mathfrak{F}' \geq 0$ *and for any* $k \geq 1$, $\exp(-k\mathfrak{F})\mathfrak{F}^{(k)}$ *is uniformly bounded.*

(iii) *For any* $1 - 2\tau < s < 1$, *there is* $\widetilde{\tau} > 0$ *with* $0 < s - \widetilde{\tau} < s + \widetilde{\tau} < 1$ *such that*

$$1 \leq \exp(\mathfrak{F}(s + \widetilde{\tau}) - \mathfrak{F}(s - \widetilde{\tau})) \leq (1 + c_2\tau); \quad \widetilde{\tau} \exp(\mathfrak{F}(s_0 - \widetilde{\tau})) \geq c_3\tau^2$$

for some absolute constants $c_2 > 0, c_3 > 0$.

Fix $0 < \tau < \frac{1}{8}$. For any $\rho_0 > 0$, let U_{ρ_0} be the component of

$$\{x|\ \rho(x) < \rho_0\}$$

which contains a fixed point and ρ is the positive exhaustion function mentioned above. Hence U_{ρ_0} will exhaust M as $\rho_0 \to \infty$.

Let $\rho_i > 1$ be a sequence increasing to $+\infty$, let $F^{(i)}(x) = \mathfrak{F}(\rho(x)/\rho_i)$. Let $g_{0,i} = e^{2F^{(i)}} g_0$. In the following, $F^{(i)}$ will be denoted simply by F if there is no confusion.

Then $(U_{\rho_i}, g_{0,i})$ is a complete Hermitian metric, (e.g. see [5]) and $g_{i,0} = g_0$ on $\{\rho(x) < (1 - \tau + \tau^2)\rho_0\}$. Moreover, the new manifold has a very nice structure.

Lemma 4.3 ([9]). *For each* $\rho_i > 1$ *sufficiently large,* $(U_{\rho_i}, g_{0,i})$ *has bounded geometry of infinite order.*

In the following, we will estimate the torsion and the holomorphic sectional curvature after performing conformal change.

Lemma 4.4. *Let* g_0 *and* h *be as in Theorem 4.1. For* $i \to \infty$, *let* $g_{0,i}$ *be as in Lemma 4.3 and* $h_i = e^{2F}h$ *for the corresponding* $F = F^{(i)}$. *Let* $T_{0,i}$ *be the torsion of* $g_{0,i}$. *Then there is a constant* $c(n, \alpha)$ *depending only on* n *and* α *so that as* $i \to \infty$, *there*

(i) $|T_{0,i}|^2_{h_i} \leq c\beta(1 + \beta)$;

(ii) $|T_{0,i}|_{h_i} |\hat{T}^{(i)}|_{h_i} \leq c\beta(1 + \beta)$;

(iii) $|\widehat{\nabla}^{(i)}_{\bar{\partial}} T_{0,i}|_{h_i} \leq c\beta(1 + \beta)$, *where* $\widehat{\nabla}^{(i)}$ *is derivative with respect to the Chern connection of* h_i;

(iv)

$$\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar\partial} T_i|_{h_i}(x) \le \kappa_0 + c\beta(1+\beta),$$

where $\kappa_i(x)$ is the upper bound of holomorphic sectional curvature of h_i at x and T_i is the torsion of h_i.

Proof. In the following, c_i will denote a positive constant depending only on n, α.

(i)

$$\begin{aligned}(T_{0,i})_{pk\bar q} &= \partial_p(e^{2F}(g_0)_{k\bar q}) - \partial_k(e^{2F}(g_0)_{p\bar q})\\ &= 2e^{2F}(F_p(g_0)_{k\bar q} - F_k(g_0)_{p\bar q}) + e^{2F}(T_0)_{pk\bar q}\\ &= 2e^{2F}\rho_0^{-1}\mathfrak{F}'\left(\rho_p(g_0)_{k\bar q} - \rho_k(g_0)_{p\bar q}\right) + e^{2F}(T_0)_{pk\bar q}.\end{aligned} \tag{4.4}$$

Hence

$$|T_{0,i}|^2_{h_i} \le c_1\beta(1+\beta).$$

This proves (i). The proof of (ii) is similar using the assumption $|\widehat{T}|_h \le \alpha$.

(iii)

$$\begin{aligned}\widehat{\nabla}^{(i)}_{\bar l}(T_{0,i})_{pk\bar q} &= \widehat{\nabla}^{(i)}_{\bar l}[2e^{2F}\rho_0^{-1}\mathfrak{F}'\left(\rho_p(g_0)_{k\bar q} - \rho_k(g_0)_{p\bar q}\right) + e^{2F}(T_0)_{pk\bar q}]\\ &= 2e^{2F}\rho_0^{-1}\mathfrak{F}'(\rho_{p\bar l}(g_0)_{k\bar q} - \rho_{k\bar l}(g_0)_{p\bar q})\\ &\quad + (2e^{2F}\rho_0^{-2}\mathfrak{F}'' + 4e^{2F}\rho_0^{-2}(\mathfrak{F}')^2)(\rho_p\rho_{\bar l}g_{k\bar q} - \rho_k\rho_{\bar l}g_{p\bar q})\\ &\quad + 2e^{2F}\mathfrak{F}'\rho_0^{-1}\left(\rho_p\widehat{\nabla}^{(i)}_{\bar l}(g_0)_{k\bar q} - \rho_k\widehat{\nabla}^{(i)}_{\bar l}(g_0)_{p\bar q}\right)\\ &\quad + 2e^{2F}\rho_0^{-1}\mathfrak{F}'\rho_{\bar l}(T_0)_{pk\bar q} + e^{2F}\widehat{\nabla}^{(i)}_{\bar l}(T_0)_{pk\bar q}.\end{aligned} \tag{4.5}$$

Using the fact that

$$(\widehat{\Gamma}^{(i)} - \widehat{\Gamma})^l_{pq} = 2F_p\delta^l_q = 2\rho_0^{-1}\mathfrak{F}'\rho_p\delta^l_q$$

and hence

$$\begin{aligned}\widehat{\nabla}^{(i)}_{\bar l}(g_0)_{k\bar q} &= (\widehat{\nabla}^{(i)}_{\bar l} - \widehat{\nabla}_{\bar l})(g_0)_{k\bar q} + \widehat{\nabla}_{\bar l}(g_0)_{k\bar q}\\ &= -2\rho_0^{-1}\mathfrak{F}'\rho_{\bar l}(g_0)_{k\bar q} + \widehat{\nabla}_{\bar l}(g_0)_{k\bar q}.\end{aligned} \tag{4.6}$$

We may further infer that (iii) is true using the assumption $|\partial\rho|_h|\widehat{\nabla} g_0|_h \le \beta\rho$ and equivalence of g_0 and h.

Now we examine the holomorphic sectional curvature after conformal change. Let $e_1 \in T^{1,0}U_R$ be such that $|e_1|_{h_i} = 1$, $|e_1|_h = e^{-F}$. Let $\kappa(x)$ be the upper bound of the holomorphic sectional curvature of h at x,

$$
\begin{aligned}
\hat{R}_{1\bar{1}1\bar{1}} &= -\partial_1\partial_{\bar{1}}(e^{2F}h_{1\bar{1}}) + e^{-2F}h^{p\bar{l}}\partial_1(e^{2F}h_{1\bar{l}})\cdot\partial_{\bar{1}}(e^{2F}h_{p\bar{1}}) \\
&= -\partial_1(e^{2F}\partial_{\bar{1}}h_{1\bar{1}} + 2e^{2F}h_{1\bar{1}}F_{\bar{1}}) \\
&\quad + e^{-2F}h^{p\bar{l}}\left(e^{2F}\partial_1 h_{1\bar{l}} + 2\hat{h}_{1\bar{l}}F_1\right)\left(e^{2F}\partial_{\bar{1}}h_{p\bar{1}} + 2\hat{h}_{p\bar{1}}F_{\bar{1}}\right) \\
&= e^{2F}\tilde{R}_{1\bar{1}1\bar{1}} - 2\hat{h}_{1\bar{1}}F_{1\bar{1}} \\
&\le e^{-2F}\kappa - 2F_{1\bar{1}} \\
&\le e^{-2F}\kappa + c_2(\beta+\beta^2).
\end{aligned}
\tag{4.7}
$$

Estimate $|\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i}$ in a similar way as above, we may conclude that

$$
\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i} \le e^{-2F}\kappa_0 + c_3\beta(1+\beta).
$$

From this (iv) is true. □

Now we are able to construct a solution of the Chern-Ricci flow on M.

Proof of Theorem 4.1. For each sufficiently large ρ_i, $(U_{\rho_i}, g_{0,i})$ has bounded geometry by Lemma 4.3. By Lemma 4.4, using the notation in the lemma, we have:

$$
\frac{n+1}{n}\kappa_i(x) + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_i|_{h_i} \le \kappa_0 + c(\beta+\beta^2) =: \kappa_{0,i}.
$$

Let

$$
\mathfrak{s}_i := \sup_M \left(|T_{0,i}|^2_{h_i} + |T_{0,i}|_{h_i}|\widehat{T}_i|_{h_i} + |\widehat{\nabla}^{(i)}_{\bar{\partial}}T_{0,i}|_{h_i} + \kappa_{0,i}\right).
$$

Then by Lemma 4.4,

$$
\mathfrak{s}_i \le \kappa_0 + c\beta(1+\beta) =: \mathfrak{s}.
$$

By Lemma 4.1, there is a solution $g_i(t)$ on $U_{\rho_i}\times[0,S)$ with initial metric $g_{0,i}$ where

$$
S = \frac{1}{3c_1(n\alpha+1)^3\mathfrak{s}}
$$

for some constant $c_1 = c_1(n)$. Moreover, g_i is uniformly equivalent to $g_{0,i}$ and

$$
\operatorname{tr}_{g_i} h_i \le v(t) - 1 \tag{4.8}
$$

on $U_{\rho_i} \times [0, S)$ where

$$v(t) = \left(\frac{1}{(n\alpha+1)^{-3} - 3c_1\mathfrak{s}t}\right)^{\frac{1}{3}}.$$

Fix any compact subset $K \subset M$ and any $S' \in (0, S)$. Then for sufficiently large i, $g_i(t)$ is a solution of the Chern-Ricci flow defined on $U_{\rho_i} \supset U_{2r} \supset U_r \supset K$ for some large $r > 0$. By Lemma 3.5, for any $(x, t) \in K \times [0, S']$,

$$R_{g_i(t)} \geq -\max\left\{C(n, \alpha, \beta, S', r), \sup_{\rho(y)<2r} R_-(y, 0)\right\},$$

where we have used the fact that $h_i = h$ on U_{2r} for sufficiently large ρ_i. In particular, it is bounded from below uniformly. Since

$$\frac{\partial}{\partial t}\left(\log \frac{\det g_i(t)}{\det h}\right) = -R_{g_i(t)} \leq C(n, K, \alpha, \beta, g_0, S', h),$$

so on $K \times [0, S']$,

$$C(n, \alpha, \beta, S')h \leq g(t) \leq C(n, K, \alpha, \beta, g_0, S', h)h.$$

By the local estimate of the Chern-Ricci flow [13], for any $k \in \mathbb{N}$, there is $C(n, k, g_0, h, \beta, \alpha, K, S')$ such that for any $(x, t) \in K \times [0, S']$,

$$|\hat{\nabla}^k g_i(t)|_h \leq C(n, k, g_0, h, \beta, \alpha, K, S').$$

By taking diagonal subsequence and using Arzelà-Ascoli theorem, we may obtain a limiting solution of $g(t)$ defined on $M \times [0, S)$. The conclusion on $\operatorname{tr}_g h$ follows from (4.8). This completes the proof of the theorem. □

Next we apply Theorem 4.1 to prove Theorem 1.2. Let us restate the theorem:

Theorem 4.2. *Let (M^n, g_0) be a complete noncompact Kähler manifold and h be a fixed complete Hermitian metric on M such that the following hold.*

(i) *There exists a smooth exhaustion function $\rho \geq 1$ such that*

$$\limsup_{\rho\to\infty}\left[\frac{|\partial\rho|_h}{\rho}(1 + |\widehat{\nabla} g_0|_h) + \frac{|\sqrt{-1}\partial\bar{\partial}\rho|_h}{\rho}\right] = 0;$$

(ii) *the holomorphic sectional curvature of* h *and torsion* $\widehat{T}$ *of* h *satisfy*

$$H_h + \frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \le -k$$

for some constant $k \ge 0$; *and*

(iii) *there exists* $\alpha > 1$ *such that on* M, $\alpha^{-1}g_0 \le h \le \alpha g_0$, $|\widehat{T}|_h \le \alpha$.

Then there is $\tilde{\beta}(n,\alpha) > 0$ *such that the Kähler-Ricci flow has a complete solution* $g(t)$ *on* $M \times [0,+\infty)$ *with* $g(0) = g_0$ *and satisfies*

$$\tilde{\beta}h \le g(t)$$

on $M \times [0,+\infty)$.

Proof. By Theorem 4.1 and the assumptions, one can apply this theorem to g_0 with β arbitrarily small because the torsion T_0 of g_0 vanishes. Hence one can find solution $g_i(t)$ to the Chern-Ricci flow with $g_i(0) = g_0$ on $M \times [0, T_i]$ with $T_i \to \infty$. Moreover, $\operatorname{tr}_g h \le c(n,\alpha)$. Using the local estimate of scalar curvature in Lemma 3.5 as in the proof of Theorem 4.1, the results follow. □

5. Existence of the Kähler-Einstein metric

In this section, we discuss the existence of the Kähler-Einstein metric on M via the Kähler-Ricci flow. Let us recall Theorem 1.1:

Theorem 5.1. *Suppose there is a complete noncompact Hermitian metric* h *on a complex manifold* M^n *compatible with the complex structure* J *such that the torsion* $\widehat{T}$ *and the holomorphic sectional curvature* H_h *satisfy*

$$H_h + \frac{n}{n+1}|\widehat{\nabla}_{\bar{\partial}}\widehat{T}|_h \le -k \tag{5.1}$$

for some $k > 0$. *Then any long-time complete solution of the normalized Kähler-Ricci flow* $g(t)$ *will converge in* C^∞_{loc} *to the unique Kähler-Einstein metric* $g_\infty = -\mathrm{Ric}(g_\infty)$. *In particular, there is no complete Ricci flat Kähler metric on* M *compatible with the same complex structure* J.

Combining this theorem with Theorem 4.2, we have the following corollaries which generalize the result by Wu-Yau [21]:

Corollary 5.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold with holomorphic sectional curvature bounded above by a negative constant. Suppose M supports an exhaustion function with uniformly bounded gradient and uniformly bounded complex Hessian, which is the case if M has bounded curvature. Then M^n supports a unique Kähler Einstein metric with negative scalar curvature.*

Corollary 5.2. *Let (M^n, g_0) be a complete noncompact Kähler manifold with holomorphic sectional curvature bounded from above by a negative constant. Suppose there exist $K_1, r, A_0, r > 0, p > n$ such that for all $x \in M$, $f \in C_0^\infty(B_{g_0}(x, 4r))$,*

$$(5.2)\quad \begin{cases} H_{g_0} \leq \kappa_0 < 0; \\ \fint_{B_{g_0}(x,r)} |Rm(g_0)|^p \, d\mu_{g_0} \leq K_1; \\ \left(\fint_{B_{g_0}(x,4r)} |f|^{\frac{2n}{n-1}} \, d\mu_{g_0}\right)^{\frac{n-1}{n}} \leq A_0 r^2 \fint_{B_{g_0}(x,4r)} |\nabla f|^2 \, d\mu_{g_0}. \end{cases}$$

Then M supports a unique Kähler-Einstein metric with negative scalar curvature.

Proof. By [24], there is a complete short time solution $g(t)$ to the Ricci flow with $g(0) = g_0$ such that $|\mathrm{Rm}(g(t))| \leq Ct^{-a}$ for some $0 < a < \frac{1}{2}$ and hence $|\nabla \mathrm{Rm}(g(t))| \leq Ct^{-a-\frac{1}{2}}$ by [15]. On the other hand for fixed $g(t)$, there is an exhaustion function ρ with uniformly bounded gradient and uniformly bounded Hessian. Since $g(t)$ is uniformly equivalent to g_0 and $\nabla_{g(t)} g_0$ is bounded, ρ is also an exhaustion function ρ with uniformly bounded gradient and uniformly bounded Hessian with respect to g_0. By Corollary 5.1, the result follows. □

We also want to discuss metrics which are uniformly equivalent to g_0 as in the previous corollaries. The following is an immediate consequence of Theorem 4.2 and Theorem 5.1.

Corollary 5.3. *Let (M^n, g_0) be a complete Kähler manifold and h is a fixed complete Hermitian metric on M such that g_0, h satisfy the assumptions in Theorem 4.2 with $k > 0$. Then M supports a unique Kähler-Einstein metric with negative scalar curvature.*

Let us prove Theorem 5.1. Here we do not assume existence of a good exhaustion function for h. However, the distance function $d(x, t)$ in a Ricci flow behaves well. First using the idea by Chen [2], we have the following:

Lemma 5.1. *Let $(M^m, g(t))$ be a complete noncompact solution to the Ricci flow on $M \times [0,T]$ with $0 < T < \infty$, where $m \geq 2$ is the real dimension of M. Let Q be a smooth function so that*

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq -\alpha Q^2 + \beta$$

for some $\alpha, \beta > 0$ at the point where $Q > 0$. Then

$$tQ(x,t) \leq \frac{1 + \sqrt{1 + 4\alpha\beta T^2}}{2\alpha}.$$

on $M \times (0,T]$.

Proof. Let $x_0 \in M$, and let $r_0 > 0$ be small enough so that:

$$\mathrm{Ric}(x,t) \leq (m-1)r_0^{-2}$$

for $x \in B_t(x_0, r_0)$, $t \in [0,T]$. By [11] (see also [2]), we then have

$$\left(\frac{\partial}{\partial t} - \Delta\right) d_t(x, x_0) \geq -\frac{5(m-1)}{3} r_0^{-1} \tag{5.3}$$

whenever $d_t(x,x_0) \geq r_0$ in the sense of barrier, where $d_t(x,x_0)$ is the distance function from x_0 with respect to $g(t)$. In the following, argue as in [7], we may assume that $d_t(x,x_0)$ to be smooth when applying maximum principle. We consider the function

$$u(x,t) = t\varphi\left(\frac{1}{Ar_0}\left[d_t(x,x_0) + \frac{5(m-1)t}{3r_0}\right]\right) Q(x,t),$$

where A is sufficiently large so that $Ar_0 >> \frac{5(m-1)T}{3r_0}$, and φ is a fixed smooth nonnegative non-increasing function such that $\varphi \equiv 1$ on $(-\infty, \frac{1}{2}]$, vanishes outside $[0,1]$ and satisfies $|2\frac{(\varphi')^2}{\varphi}| + |\varphi''| \leq c_1$ for some absolute constant. Note that u also depends on A. However,

$$u(x_0,t) = tQ(x_0,t),$$

if $Ar_0 \geq \frac{10(m-1)T}{3r_0}$.

If $u \leq 0$, then we are done. Suppose the function $u > 0$ somewhere, then there exists (x_1,t_1) with $0 < t_1 \leq T$ so that u attains its maximum at (x_1,t_1).

At (x_1, t_1) we have

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta\right) u; \quad \nabla Q = -\frac{\nabla \phi}{\phi} Q.$$

Suppose $d_{t_1}(x_1, x_0) < r_0$, then $u(x,t) = tQ(x,t)$ near (x_1, t_1) provided Ar_0 is large enough. Then at (x_1, t_1), we have

$$\begin{aligned} 0 \leq & \left(\frac{\partial}{\partial t} - \Delta\right) u \\ =& t_1 \left(\frac{\partial}{\partial t} - \Delta\right) Q + Q \\ \leq & -\alpha t_1 Q^2 + \beta t_1 + Q \end{aligned}$$

and so

$$0 \leq -\alpha u^2 + u + \beta T^2$$

which implies

$$u(x_0, t) \leq u(x_1, t_1) \leq \frac{1 + \sqrt{1 + 4\alpha\beta T^2}}{2\alpha}. \tag{5.4}$$

for $t \in [0, T]$.

Suppose $d_{t_1}(x_1, x_0) \geq r_0$, then at (x_1, t_1),

$$\begin{aligned} 0 \leq & \left(\frac{\partial}{\partial t} - \Delta\right) u \\ =& Qt \left(\frac{\partial}{\partial t} - \Delta\right) \varphi + \varphi \left(\frac{\partial}{\partial t} - \Delta\right) (Qt) - 2t\langle \nabla \varphi, \nabla Q \rangle \\ \leq & Qt\varphi' \frac{1}{Ar_0} \left[\left(\frac{\partial}{\partial t} - \Delta\right) d_t(x,p) + \frac{5}{3}(m-1) r_0^{-1}\right] \\ & + |\varphi''| \frac{1}{(Ar_0)^2} tQ + \varphi\left(-\alpha t Q^2 + \beta t + Q\right) + 2tQ \frac{1}{(Ar_0)^2} \cdot \frac{(\phi')^2}{\phi} \\ \leq & -\alpha t \varphi Q^2 + \varphi Q + \beta t \varphi + c_1 Qt \frac{1}{(Ar_0)^2}. \end{aligned}$$

Multiply both the inequality by $t\varphi = t_1\varphi$, we have

$$0 \leq -\alpha u^2 + \left(1 + \frac{c_1 T}{A^2 r_0^2}\right) u + \beta T^2.$$

Hence we have

$$(5.5) \qquad u(x_0,t) \le u(x_1,t_1) \le \frac{1+\frac{c_1T}{A^2r_0^2}+\sqrt{\left(1+\frac{c_1T}{A^2r_0^2}\right)^2+4\alpha\beta T^2}}{2\alpha}$$

for $0 < t \le T$. Let $A \to \infty$ together with (5.5) and the fact that x_0 is any point in M, we conclude the lemma is true. □

As an application of the lemma, we can recover the following well-known result on uniqueness of complete Kähler-Einstein metric. Here we do not assume the curvature is bounded, see also [21]. Note that the result can also be obtained by using elliptic theory.

Proposition 5.1. *Suppose ω_1 and ω_2 are complete noncompact Kähler Einstein metrics on M with* $\mathrm{Ric}(\omega_i) = -\omega_i$ *for $i = 1, 2$. Then $\omega_1 = \omega_2$ on M.*

Proof. Let Let $\widetilde{\omega}_1(t) = (t+1)\omega_1$ and $\widetilde{\omega}_2(t) = (t+1)\omega_2$. Then both $\widetilde{\omega}_1, \widetilde{\omega}_2$ are solutions to the Kähler-Ricci flow on $M \times [0, +\infty)$. Define $F(x,t) = F(x)$ to be the function

$$F(x,t) = \log\left[\frac{\widetilde{\omega}_2^n}{\widetilde{\omega}_1^n}\right]^{\frac{1}{n}} = \log\left[\frac{\omega_2^n}{\omega_1^n}\right]^{\frac{1}{n}}$$

which is independent of t. The function F is independent of $t > 0$ but we treat it as a function over the Kähler-Ricci flow. Let Δ be the Laplacian of $\widetilde{\omega}_1$. Then it satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right) F &= \frac{1}{t+1}\left(1 - \frac{1}{n}\operatorname{tr}_{\omega_1}\omega_2\right)\\ &\le \frac{1}{t+1}\left(1 - e^F\right)\\ &\le -\frac{1}{4}F^2\end{aligned}$$

whenever $F > 0$ on $M \times [0,1]$.

Apply Lemma 5.1 on $M \times [0,1]$, $tF \le 4$. In particular, $F(x)$ is bounded from above uniformly on M. By interchanging ω_1 and ω_2, we conclude that F is a bounded function on M. Let Δ_1 be the Laplacian of ω_1, we have as above

$$-\Delta_1 F \le 1 - e^F.$$

By the generalized maximum principle [3], we conclude that $F \le 0$. Interchanging the roles of ω_1 and ω_2, we can prove similarly that $F \ge 0$. Hence $F = 0$. So $\partial\bar{\partial}F = 0$ and $\omega_1 = \omega_2$ because they are Kähler-Einstein. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $g(t)$ and h be as in the Theorem. Let $\Lambda = \operatorname{tr}_g h$. By Corollary 3.1, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Lambda \le -c_1 k\Lambda^2,$$

for some c_1 depending only on n. By Lemma 5.1 with $\beta = 0$, we conclude that

$$\Lambda(x,t) \le \frac{1}{c_1 kt} \tag{5.6}$$

on $M \times (0,\infty)$.

On the other hand, let $R(x,t)$ be the scalar curvature of $g(t)$ at x and let $R_-(x,t)$ be its negative part. For any $\epsilon > 0$, let $f = \frac{1}{2}\left((R^2+\epsilon^2)^{\frac{1}{2}} - R\right)$. Note that if $\epsilon \to 0$, then $f \to R_-$. Using the fact that

$$\left(\frac{\partial}{\partial t} - \Delta\right) R \ge \frac{1}{n}R^2,$$

direct computations show that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \le -\frac{1}{n} f(f-2c_1\epsilon) \le -\frac{1}{n}(f-c_1\epsilon)^2 + c_2\epsilon$$

for some absolute constant $c_1 > 0$ and $c_2 > 0$ depending only on n. By Lemma 5.1, we conclude that

$$t(f-c_1\epsilon) \le \frac{n}{2}\left(1+\sqrt{1+\frac{4c_2\epsilon}{n}}\right)$$

on $M \times (0,\infty)$. Let $\epsilon \to 0$, we conclude that

$$tR(x,t) \ge -n. \tag{5.7}$$

Since

$$\frac{\partial}{\partial t}\log\left(\frac{\det g(t)}{\det h}\right) = -R \le \frac{n}{t},$$

we conclude for any bounded open set Ω, there is a constant C_1 depending only on $\Omega, g(1), h, n$ such that

$$\frac{\det g(t)}{\det h} \le C_1 t^n$$

on $\Omega \times [1, \infty)$. Combining this with (5.6), we conclude that

$$C_2^{-1} th \le g \le C_2 th$$

on $\Omega \times [1, \infty)$ for some constant $C_2 > 0$ depending only on $\Omega, g(1), h, k, n$.

Consider the normalized metric

$$\widetilde{g}(x, s) = e^{-s} g(x, e^s).$$

Then we have

$$\frac{\partial}{\partial s}\widetilde{g} = -\mathrm{Ric}(\widetilde{g}) - \widetilde{g} \tag{5.8}$$

on $M \times [0, \infty)$, and

$$C_2^{-1} h \le \widetilde{g}(s) \le C_2 h \tag{5.9}$$

on $\Omega \times [0, \infty)$.

In the following, let $\omega(t)$, $\widetilde{\omega}(s)$ be the Kähler forms of $g(t)$, $\widetilde{g}(s)$ respectively. By [16, Theorem 2.17], we conclude that for any bounded open set in M and $\ell \ge 0$, there is a constant C_3 depending only on $\Omega, g(1), h, k, n$ and ℓ such that

$$||\widetilde{\omega}(s)||_{C^\ell(\Omega, \widetilde{g}_0)} \le C_3. \tag{5.10}$$

On the other hand, let

$$\widetilde{\phi}(x, s) = e^{-s} \int_0^s e^\tau \log\left(\frac{(\widetilde{\omega}(\tau))^n}{(\widetilde{\omega}(0))^n}\right) d\tau.$$

Then

$$\widetilde{\omega}(s) = e^{-s}\widetilde{\omega}(0) - (1 - e^{-s})\mathrm{Ric}(\widetilde{\omega}(0)) + \sqrt{-1}\partial\bar{\partial}\widetilde{\phi}(s). \tag{5.11}$$

Moreover,

$$\begin{cases} \frac{\partial}{\partial s}\widetilde{\phi} = & \log\left(\frac{\widetilde{\omega}^n}{(\widetilde{\omega}(0))^n}\right) - \widetilde{\phi} \ \text{ in } M \times [0, \infty); \\ \widetilde{\phi}(0) = & 0 \ \text{ on } M. \end{cases} \tag{5.12}$$

Denote $\partial_s\widetilde{\phi}$ by $\widetilde{\phi}'$ etc., and let $\widetilde{R}$ be the scalar curvature of $\widetilde{g}$, then

$$\begin{aligned}\widetilde{\phi}''+\widetilde{\phi}' &= -\widetilde{R}-n\\ &= -e^sR(g(e^s))-n\\ &= -e^s\left(R(g(e^s))+e^{-s}n\right)\\ &\leq 0\end{aligned}\tag{5.13}$$

by (5.7). Hence $\widetilde{\phi}'+\widetilde{\phi}$ is non-increasing and $\widetilde{\phi}'+\widetilde{\phi}\leq 0$ because $\widetilde{\phi}'+\widetilde{\phi}=0$ at $s=0$. On the other hand, by (5.10) and (5.12), we conclude that for any bounded open set Ω, there exists $s_i\to\infty$ such that

$$(\widetilde{\phi}'+\widetilde{\phi})(s_i)$$

converges uniformly in C^∞ norm in Ω. By the monotonicity of $\widetilde{\phi}'+\widetilde{\phi}$, we conclude that $\widetilde{\phi}'+\widetilde{\phi}$ converges in C^∞ norm in Ω to some function.

By (5.13), we have

$$(e^s\widetilde{\phi}')'\leq 0,$$

and so $\widetilde{\phi}'\leq 0$ because $\widetilde{\phi}'=0$ at $s=0$. Combine this with (5.10) and (5.11), we conclude that $\widetilde{\phi}$ also converges in C^∞ norm to some function $\widetilde{\phi}_\infty$. Hence $\widetilde{\phi}'$ also converge in C^∞ norm to some function. However, by (5.9) we conclude that ϕ is bounded from below. This implies that $\widetilde{\phi}'\to 0$ as $s\to\infty$. Moreover, $\widetilde{\omega}(s)\to\widetilde{\omega}_\infty$ in C^∞ norm in Ω as $s\to\infty$ with

$$\widetilde{\omega}_\infty=-\mathrm{Ric}(\widetilde{\omega}(0))+\sqrt{-1}\partial\bar{\partial}\widetilde{\phi}_\infty.$$

Note that $\widetilde{\omega}_\infty$ is a Kähler form of a Kähler metric by (5.9). Moreover,

$$\widetilde{\phi}_\infty=\log\left(\frac{\widetilde{\omega}^n}{(\widetilde{\omega}(0))^n}\right).$$

Taking $\partial\bar{\partial}$ to both sides, we conclude that

$$\mathrm{Ric}(\widetilde{\omega}_\infty)=-\widetilde{\omega}_\infty.$$

Suppose $\bar{\omega}$ is a complete Ricci flat metric compatible with the same complex structure of h. Then $\omega(t)=\bar{\omega}$ is a steady solution of the Kähler-Ricci flow. By the convergence of normalized Kähler-Ricci flow, $t^{-1}\omega(t)$ converges to a Kähler Einstein metric on M which is impossible since $t^{-1}\omega(t)\equiv t^{-1}\bar{\omega}$ converges to a zero tensor on M. This completes the proof. □

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