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Positivity preserving along a flow over projective bundles

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In this paper, we introduce a flow over the projective bundle $p : P(E^*) \to M$, a natural generalization of both Hermitian-Yang-Mills flow and Kähler-Ricci flow. We prove that the semi-positivity of curvature of the hyperplane line bundle $\mathcal{O}_{P(E^*)}(1)$ is preserved along this flow under the null eigenvector assumption. As applications, we prove that the semi-positivity is preserved along the flow if the base manifold M is a curve, which implies that the Griffiths semi-positivity is preserved along the Hermitian-Yang-Mills flow over a curve. And we also reprove that the nonnegativity of holomorphic bisectional curvature is preserved under Kähler-Ricci flow.

Introduction **1541**

1 Preliminaries **1546**

2 A flow over projective bundles **1555**

3 Applications **1568**

References **1571**

Introduction

In the celebrated paper [29], Siu and Yau presented a differential geometric proof of the famous Frankel conjecture in Kähler geometry states that a

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compact Kähler manifold M^n with positive holomorphic bisectional curvature must be biholomorphic to a complex projective space $\mathbb{P}^n$. For a compact Kähler manifold with nonnegative holomorphic bisectional curvature, Mok [24] proved a generalized Frankel conjecture and obtained a uniformization theorem, and H. Gu [17] gave a new proof of Mok's uniformization theorem. They both used the Kähler-Ricci flow and considered the variation of holomorphic bisectional curvature along this flow. Especially, the nonnegativity of holomorphic bisectional curvature is preserved under Kähler-Ricci flow [24, Proposition 1.1]. Later on, there are many references about studying and generalizing the Frankel conjecture by using Kähler-Ricci flow, including [8, 9, 18, 28].

In 1979, Mori [25] proved the famous Hartshorne's conjecture. If the ground field is $\mathbb{C}$, a compact complex manifold M was proved biholomorphic to $\mathbb{P}^n$ if its tangent bundle is ample, which implies the Frankel conjecture. A holomorphic vector bundle E is ample in the sense of Hartshorne if and only if the hyperplane line bundle $\mathcal{O}_{P(E^*)}(1)$ is a positive line bundle over $P(E^*)$ (see [19, Proposition 3.2]), i.e., there is a positive curvature metric on $\mathcal{O}_{P(E^*)}(1)$. If (E, h) is a Hermitian vector bundle with Griffiths positive curvature (see Definition 0.3), then E is an ample vector bundle. Griffiths conjectured its converse in [16] holds that E can admit a Hermitian metric with Griffiths positive curvature if E is ample. Both Mori's theorem and Griffiths conjecture would be proved if one can deform the given positive curvature metric to another "better" metric with positive curvature, for example, the Kähler metric with positive holomorphic bisectional curvature for Mori's theorem, and Hermitian metric with Griffiths positive curvature for Griffiths conjecture. Naturally, one wants to define a certain flow over the projective bundle $P(E^*)$ such that the positivity of the curvature of $\mathcal{O}_{P(E^*)}(1)$ is preserved under this flow. This is also the motivation for the author to study the positivity preserving along the flow (0.1) over the projective bundle.

Let M be a compact complex manifold of dimension n, and $\pi : E \to M$ be a holomorphic vector bundle of rank r over M. Let E^* denote the dual bundle of E, and $p : P(E^*) := (E^*)^o/\mathbb{C}^* \to M$ denote the projective bundle, where $(E^*)^o$ denotes the set of all the nonzero elements in E^*. For any strongly pseudoconvex complex Finsler metric G on E^* (see Definition 1.1), there exists the following canonical decomposition (see Remark 1.5 (4))

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_{FS}(G),$$

which is a curvature form of $\mathcal{O}_{P(E^*)}(1)$ and represents the first Chern class $2\pi c_1(\mathcal{O}_{P(E^*)}(1))$. Here Ψ is called the Kobayashi curvature (see [12, Definition 1.2]), which is given in (1.4), and $\omega_{FS}(G)$ is a positive $(1,1)$-form along each fiber of $p : P(E^*) \to M$, which is defined by (1.10). According to this decomposition, Kobayashi [20] characterized ample vector bundle, i.e., E is ample if and only if a strongly pseudoconvex complex Finsler metric exists on E^* such that $\Psi < 0$.

Let $\omega(G) = p^*\omega$, where $\omega = \sqrt{-1}g_{\alpha\bar{\beta}}(G)dz^\alpha \wedge d\bar{z}^\beta$ is a Kähler metric on M depending smoothly on a Finsler metric G. One can define a Hermitian metric on $P(E^*)$ by

$$\Omega := \omega(G) + \omega_{FS}(G),$$

Let G_0 be a strongly pseudoconvex complex Finsler metric on E^*, one has $\omega_{FS}(G_0) > 0$ (means positive along fibers). Now we consider the following flow over the projective bundle $P(E^*)$:

$$\text{(0.1)} \qquad \begin{cases} \frac{\partial}{\partial t}\log G = \Delta_\Omega \log G, \\ \omega_{FS}(G) > 0, \\ G(0) = G_0. \end{cases}$$

Here $\Delta_\Omega := \sqrt{-1}\Lambda\partial\bar{\partial}$, Λ is the adjoint operator of $\Omega \wedge \bullet$.

One can also define a horizontal and real $(1,1)$-form T on $P(E^*)$ as follow,

$$\text{(0.2)} \qquad (-\sqrt{-1})T(u,\overline{u}) := \langle\sqrt{-1}R^g(u,\overline{u}), -\Psi\rangle_\Omega - \left|i_u\partial^V\Psi\right|^2_\Omega$$

for any horizontal vector $u = u^\alpha \frac{\delta}{\delta z^\alpha}$, where $\langle\sqrt{-1}R^g(u,\overline{u}), -\Psi\rangle_\Omega := (-\Psi)_{\alpha\bar{\delta}}g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}R^g_{\gamma\bar{\beta}\sigma\bar{\tau}}u^\sigma\bar{u}^\tau$ and $\left|i_u\partial^V\Psi\right|^2_\Omega := (\log G)^{a\bar{b}}\partial_a\Psi_{\alpha\bar{\beta}}\overline{\partial_b\Psi_{\gamma\bar{\tau}}}u^\alpha\overline{u}^\gamma g^{\tau\bar{\beta}}$, R^g denotes the Chern curvature of ω. Now we assume that T satisfies the *the null eigenvector assumption* (see Theorem 1.8), namely $(-\sqrt{-1})T(U,\overline{U}) \geq 0$ whenever $\partial\bar{\partial}\log G(U,\overline{U}) \geq 0$ and $i_U(\partial\bar{\partial}\log G) = 0$ for a $(1,0)$-type vector U of $TP(E^*)$. By using the maximum principle for real $(1,1)$-forms, we obtain

Theorem 0.1 (=Theorem 2.4). *Let $\pi : (E^*, G_0) \to M$ be a holomorphic Finsler vector bundle over the compact Kähler manifold M with $\sqrt{-1}\partial\bar{\partial}\log G_0 \geq 0$. If the horizontal $(1,1)$-form T satisfies the null eigenvector assumption, then $\sqrt{-1}\partial\bar{\partial}\log G(t) \geq 0$ along the flow (0.1) for all $t \geq 0$ such that the solution exists.*

In this paper, we will give two applications of Theorem 0.1.

For the first application, we consider the case of the curve, i.e., $\dim M = 1$. In this case, any Hermitian metric on M is Kähler automatically, and one can prove that the $(1,1)$-form T satisfies the null eigenvector assumption. By Theorem 0.1, we obtain

Proposition 0.2 (=Proposition 3.1). *If M is a curve, then the semi-positivity of the curvature of $\mathcal{O}_{P(E^*)}(1)$ is preserved along the flow (0.1).*

In particular, if $G_0 = h_0^{i\bar{j}} v_i \bar{v}_j$ comes from a Hermitian metric $(h_0^{i\bar{j}})$ of E^* and

$$\Omega = p^*\omega + \omega_{FS}(G) \tag{0.3}$$

for a fixed Hermitian metric ω, by Remark 2.1 (1), (0.1) is equivalent to the following Hermitian-Yang-Mills flow:

$$\begin{cases} h^{-1} \cdot \frac{\partial h}{\partial t} + \Lambda R^h + (r-1)I = 0, \\ (h_{i\bar{j}}(t)) > 0, \\ h_{i\bar{j}}(0) = (h_0)_{i\bar{j}}. \end{cases} \tag{0.4}$$

Recall that

Definition 0.3 ([16]). The Chern curvature of the metric $(h_{i\bar{j}})$ is called *Griffiths positive* (resp. *Griffiths semi-positive*) if

$$R_{i\bar{j}\alpha\bar{\beta}} X^i \overline{X^j} Y^\alpha \overline{Y^\beta} > 0 \quad (\text{resp.} \geq 0)$$

for any two nonzero vectors $X = X^i e_i \in E$ and $Y = Y^\alpha \frac{\partial}{\partial z^\alpha} \in TM$. Here $R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} + h^{\bar{l}k} \frac{\partial h_{i\bar{l}}}{\partial z^\alpha} \frac{\partial h_{k\bar{j}}}{\partial \bar{z}^\beta}$ denotes the Chern curvature of $(h_{i\bar{j}})$. For the case of $E = TM$, the Hermitian metric $(h_{i\bar{j}})$ is called *has positive (nonnegative) holomorphic bisectional curvature* if its Chern curvature is Griffiths positive (semi-positive).

By Proposition 0.2, we have

Corollary 0.4 (=Corollary 3.2). *If M is a curve, the Griffiths semi-positivity is preserved along the Hermitian-Yang-Mills flow (0.4).*

For the second application, we assume that $E = TM$ and take

$$\omega(G) = \sqrt{-1} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta},$$

where $(g_{\alpha\bar{\beta}})$ denotes the inverse of the matrix $\left(\frac{\partial^2 G}{\partial v_\alpha \partial \bar{v}_\beta}\right)$ (see Section 1 for the definition of $\frac{\partial^2 G}{\partial v_\alpha \partial \bar{v}_\beta}$). Let $G_0 = g_0^{\alpha\bar{\beta}} v_\alpha \bar{v}_\beta$ be the strongly pseudoconvex complex Finsler metric on T^*M induced by the following Kähler metric

$$\omega_0 = \sqrt{-1} (g_0)_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}.$$

Then the flow (0.1) is equivalent to the following Kähler-Ricci flow

$$\begin{cases} \frac{\partial \omega}{\partial t} + \mathrm{Ric}(\omega) + (n-1)\omega = 0, \\ \omega > 0, \\ \omega(0) = \omega_0. \end{cases} \tag{0.5}$$

The solution of (0.1) is induced from the Kähler metric $\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}$. In this case, the $(1,1)$-form T can also be proved to satisfy the null eigenvector assumption (see [5, Page 254, Claim 2.2]). From Theorem 0.1 and Definition 0.3, we can reprove the following Mok's proposition, which is contained in [24, Proposition 1.1] (see also [5, Theorem 5.2.10]).

Proposition 0.5 ([24, Proposition 1.1]). *If (M, ω_0) is a compact Kähler manifold with nonnegative holomorphic bisectional curvature, then the nonnegativity is preserved along the Kähler-Ricci flow (0.5).*

Remark 0.6. By (0.4) and (0.5), the flow (0.1) is a natural generalization of both Hermitian-Yang-Mills flow and Kähler-Ricci flow. And there are some other flows, which are also the generalizations of the Kähler-Ricci flow. For example, Gill [15] introduced the *Chern-Ricci flow* on Hermitian manifolds, and many properties of the flow were established in [33, 34]. Especially, Yang [38] proved the nonnegativity of the holomorphic bisectional curvature is not necessarily preserved under the Chern-Ricci flow. In [30], Streets and Tian introduced the *Hermitian curvature flow*, proved short time existence for this flow, and derived basic long-time blowup and regularity results, and in [31, 32] they introduced a parabolic flow of pluriclosed metrics and obtained some regularity results for solutions to this equation. For a particular version of the Hermitian curvature flow, Ustinovskiy [35] proved that the property of Griffiths positive (nonnegative) Chern curvature is preserved along this flow.

This article is organized as follows. In Section 1, we shall fix the notation and recall some basic definitions and facts on complex Finsler vector bundles, Griffiths positivity (semi-positivity), and maximal principle for real $(1,1)$-forms. In Section 2, we will define a flow over projective bundle $P(E^*)$ and study the positivity along this flow; Theorem 0.1 will be proved in this section. In Section 3, we will give two applications of Theorem 0.1, and we will prove Proposition 0.2, Corollary 0.4 and Proposition 0.5.

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1. Preliminaries

In this section, we shall fix the notation and recall some basic definitions and facts on complex Finsler vector bundles, Griffiths positivity (semi-positivity), and maximal principle for real $(1,1)$-forms. For more details we refer to [1, 2, 7, 10, 12–14, 16, 20, 23, 26, 36].

1.1. Complex Finsler vector bundle

Let M be a compact complex manifold of dimension n, and let $\pi : E \to M$ be a holomorphic vector bundle of rank r over M. Let $z = (z^1, \cdots, z^n)$ be a local coordinate system in M, and $\{e_i\}_{1\leq i\leq r}$ be a local holomorphic frame of E. With respect to the local frame of E, an element of E can be written as

$$v = v^i e_i \in E,$$

where we adopt the summation convention of Einstein. In this way, one gets a local coordinate system of the complex manifold E:

$$(z; v) = (z^1, \cdots, z^n; v^1, \cdots, v^r). \tag{1.1}$$

Definition 1.1 ([20]). A Finsler metric G on the holomorphic vector bundle E is a continuous function $G : E \to \mathbb{R}$ satisfying the following conditions:

F1): G is smooth on $E^o = E \setminus O$, where O denotes the zero section of E;

F2): $G(z, v) \geq 0$ for all $(z, v) \in E$ with $z \in M$ and $v \in \pi^{-1}(z)$, and $G(z, v) = 0$ if and only if $v = 0$;

F3): $G(z,\lambda v)=|\lambda|^2G(z,v)$ for all $\lambda\in\mathbb{C}$.

Moreover, G is called strongly pseudoconvex if

F4): the Levi form $\sqrt{-1}\partial\bar\partial G$ on E^o is positive-definite along each fiber $E_z=\pi^{-1}(z)$ for $z\in M$.

For any Hermitian metric on E, one can associate it with a strongly pseudoconvex complex Finsler metric on E.

We write

$$G_i=\partial G/\partial v^i,\quad G_{\bar j}=\partial G/\partial\bar v^j,\quad G_{i\bar j}=\partial^2G/\partial v^i\partial\bar v^j,$$
$$G_{i\alpha}=\partial^2G/\partial v^i\partial z^\alpha,\quad G_{i\bar j\bar\beta}=\partial^3G/\partial v^i\partial\bar v^j\partial\bar z^\beta,\quad etc.,$$

to denote the differentiation with respect to $v^i,\bar v^j$ $(1\le i,j\le r)$, $z^\alpha,\bar z^\beta$ $(1\le\alpha,\beta\le n)$. In the following lemma, we collect useful identities related to a Finsler metric G.

Lemma 1.2 ([7, 20]). *The following identities hold for any $(z,v)\in E^o$, $\lambda\in\mathbb{C}$:*

$$G_i(z,\lambda v)=\bar\lambda G_i(z,v),\quad G_{i\bar j}(z,\lambda v)=G_{i\bar j}(z,v)=\overline{G_{j\bar i}(z,v)};$$
$$G_i(z,v)v^i=G_{\bar j}(z,v)\bar v^j=G_{i\bar j}(z,v)v^i\bar v^j=G(z,v);$$
$$G_{ij}(z,v)v^i=G_{i\bar jk}(z,v)v^i=G_{i\bar j\bar k}(z,v)\bar v^j=0.$$

Suppose G is a strongly pseudoconvex complex Finsler metric on M. In that case, there is a canonical h-v decomposition of the holomorphic tangent bundle TE^o of E^o (see [7, §5] or [12, §1]).

$$TE^o=\mathcal{H}\oplus\mathcal{V}.$$

In terms of local coordinates,

$$\mathcal{H}=\mathrm{span}_{\mathbb{C}}\left\{\frac{\delta}{\delta z^\alpha}=\frac{\partial}{\partial z^\alpha}-G_{\alpha\bar j}G^{\bar jk}\frac{\partial}{\partial v^k},1\le\alpha\le n\right\},$$
$$\mathcal{V}=\mathrm{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial v^i},1\le i\le r\right\}.$$

The dual bundle T^*E^o also has a smooth h-v decomposition $T^*E^o = \mathcal{H}^* \oplus \mathcal{V}^*$:

$$\begin{aligned} \mathcal{H}^* &= \mathrm{span}_{\mathbb{C}}\{dz^\alpha, 1 \le \alpha \le n\}, \\ \mathcal{V}^* &= \mathrm{span}_{\mathbb{C}}\{\delta v^i = dv^i + G^{\bar{j}i} G_{\alpha\bar{j}} dz^\alpha, \quad 1 \le i \le r\}. \end{aligned} \tag{1.2}$$

Moreover, the differential operators

$$\partial^V = \frac{\partial}{\partial v^i} \otimes \delta v^i, \quad \partial^H = \frac{\delta}{\delta z^\alpha} \otimes dz^\alpha. \tag{1.3}$$

are well-defined.

With respect to the h-v decomposition (1.2), the $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}\log G$ has the following decomposition. For readers' convenience, we give proof of the following lemma due to Kobayashi and Aikou (cf. [2, 20]).

Lemma 1.3 ([2, 20]). *Let G be a strongly pseudoconvex complex Finsler metric on E. One has*

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_V,$$

where Ψ is called the Kobayashi curvature (see [12, Definition 1.2]),

$$\Psi = \sqrt{-1} R_{i\bar{j}\alpha\bar{\beta}} \frac{v^i \bar{v}^j}{G} dz^\alpha \wedge d\bar{z}^\beta, \quad \omega_V = \sqrt{-1} \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j, \tag{1.4}$$

with

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 G_{i\bar{j}}}{\partial z^\alpha \partial \bar{z}^\beta} + G^{\bar{l}k} \frac{\partial G_{i\bar{l}}}{\partial z^\alpha} \frac{\partial G_{k\bar{j}}}{\partial \bar{z}^\beta}.$$

Proof. By (1.2), we have

$$\begin{aligned} \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j &= \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} (dv^i + G_{\alpha\bar{l}} G^{\bar{l}i} dz^\alpha) \wedge (d\bar{v}^j + G_{\bar{\beta}k} G^{\bar{j}k} d\bar{z}^\beta) \\ &= \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} dv^i \wedge d\bar{v}^j + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\bar{\beta}k} G^{\bar{j}k} dv^i \wedge d\bar{z}^\beta \\ &\quad + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha\bar{l}} G^{\bar{l}i} dz^\alpha \wedge d\bar{v}^j \\ &\quad + \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha\bar{l}} G^{\bar{l}i} G_{k\bar{\beta}} G^{\bar{j}k} dz^\alpha \wedge d\bar{z}^\beta. \end{aligned} \tag{1.5}$$

For the last three terms in the RHS of (1.5), one has from Lemma 1.2

$$\frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\bar{\beta}k} G^{\bar{j}k} dv^i \wedge d\bar{z}^\beta = \frac{G G_{i\bar{j}} - G_i G_{\bar{j}}}{G^2} G_{\bar{\beta}k} G^{\bar{j}k} dv^i \wedge d\bar{z}^\beta = \frac{\partial^2 \log G}{\partial v^i \partial \bar{z}^\beta} dv^i \wedge d\bar{z}^\beta, \tag{1.6}$$

$$\frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha\bar{l}} G^{\bar{l}i} dz^\alpha \wedge d\bar{v}^j = \frac{G G_{i\bar{j}} - G_i G_{\bar{j}}}{G^2} G_{\alpha\bar{l}} G^{\bar{l}i} dz^\alpha \wedge d\bar{v}^j = \frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{v}^j} dz^\alpha \wedge d\bar{v}^j \tag{1.7}$$

and

$$\begin{aligned} &\frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} G_{\alpha\bar{l}} G^{\bar{l}i} G_{k\bar{\beta}} G^{\bar{j}k} dz^\alpha \wedge d\bar{z}^\beta \\ &\quad = \frac{G G_{i\bar{j}} - G_i G_{\bar{j}}}{G^2} G_{\alpha\bar{l}} G^{\bar{l}i} G_{k\bar{\beta}} G^{\bar{j}k} dz^\alpha \wedge d\bar{z}^\beta \\ &\quad = \frac{1}{G^2} (G G^{\bar{l}k} G_{\alpha\bar{l}} G_{k\bar{\beta}} - G_\alpha G_{\bar{\beta}}) dz^\alpha \wedge d\bar{z}^\beta. \end{aligned} \tag{1.8}$$

Submitting (1.6), (1.7) and (1.8) into (1.5), we obtain

$$\begin{aligned} \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} \delta v^i \wedge \delta \bar{v}^j &= \frac{\partial^2 \log G}{\partial v^i \partial \bar{v}^j} dv^i \wedge d\bar{v}^j + \frac{\partial^2 \log G}{\partial v^i \partial \bar{z}^\beta} dv^i \wedge d\bar{z}^\beta \\ &\quad + \frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{v}^j} dz^\alpha \wedge d\bar{v}^j \\ &\quad + \frac{1}{G^2} (G G^{\bar{l}k} G_{\alpha\bar{l}} G_{k\bar{\beta}} - G_\alpha G_{\bar{\beta}}) dz^\alpha \wedge d\bar{z}^\beta \\ &= \partial\bar{\partial} \log G + \frac{1}{G} (G^{\bar{l}k} G_{\alpha\bar{l}} G_{k\bar{\beta}} - G_{\alpha\bar{\beta}}) dz^\alpha \wedge d\bar{z}^\beta \\ &= \partial\bar{\partial} \log G + (G^{\bar{l}k} G_{\alpha i\bar{l}} G_{k\bar{j}\bar{\beta}} - G_{i\bar{j}\alpha\bar{\beta}}) \frac{v^i \bar{v}^j}{G} dz^\alpha \wedge d\bar{z}^\beta \\ &= \partial\bar{\partial} \log G - \sqrt{-1}\Psi, \end{aligned}$$

which completes the proof. □

Let q denote the natural projection

$$q : E^o \to P(E) := E^o/\mathbb{C}^* \quad (z; v) \mapsto (z; [v]) := (z^1, \cdots, z^n; [v^1, \cdots, v^r]),$$

which gives a local coordinate system of $P(E)$ by

$$(z;w) := (z^1,\cdots,z^n;w^1,\cdots,w^{r-1}) = \left(z^1,\cdots,z^n;\frac{v^1}{v^k},\cdots,\frac{v^{k-1}}{v^k},\frac{v^{k+1}}{v^k},\cdots,\frac{v^r}{v^k}\right) \tag{1.9}$$

on $U_k := \{(z,[v]) \in P(E), v^k \neq 0\}$.

Denote by $((\log G)^{\bar{b}a})_{1\leq a,b\leq r-1}$ the inverse of the matrix

$$\left((\log G)_{a\bar{b}} := \frac{\partial^2 \log G}{\partial w^a \bar{w}^b}\right)_{1\leq a,b\leq r-1}$$

and set

$$\delta w^a = dw^a + (\log G)_{\alpha\bar{b}}(\log G)^{\bar{b}a} dz^\alpha.$$

One can define a vertical $(1,1)$-form on $P(E)$ by

$$\omega_{FS}(G) := \sqrt{-1}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b}\delta w^a \wedge \delta \bar{w}^b, \tag{1.10}$$

which is well-defined (see e.g. [14, Section 1]). Moreover,

Lemma 1.4. *By the pullback $q^* : A^{1,1}(P(E)) \to A^{1,1}(E^o)$, one has*

$$q^*\omega_{FS}(G) = \omega_V. \tag{1.11}$$

Proof. We only need to prove (1.11) at one point $(z,[v]) \in U_k$. Without loss of generality, we assume that $k = r$. For any point $(z,[v]) \in U_r$, one has from (1.9)

$$q_*\left(\frac{\partial}{\partial v^r}\right) = -\sum_{a=1}^{r-1}\frac{v^a}{(v^r)^2}\frac{\partial}{\partial w^a}, \qquad q_*\left(\frac{\partial}{\partial v^b}\right) = \frac{1}{v^r}\frac{\partial}{\partial w^b}, \quad 1 \leq b \leq r-1. \tag{1.12}$$

By (1.12), one has

$$\begin{aligned}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b} &= (\partial\bar{\partial}\log G)(\frac{\partial}{\partial w^a}, \frac{\partial}{\partial \bar{w}^b}) \\ &= (\partial\bar{\partial}\log G)(q_*(v^r\frac{\partial}{\partial v^a}), q_*(\bar{v}^r\frac{\partial}{\partial \bar{v}^b})) \\ &= q^*(\partial\bar{\partial}\log G)(v^r\frac{\partial}{\partial v^a}, \bar{v}^r\frac{\partial}{\partial \bar{v}^b}) \\ &= |v^r|^2\frac{\partial^2 \log G}{\partial v^a \partial \bar{v}^b}.\end{aligned} \tag{1.13}$$

Similarly,

$$\frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{w}^b} = \bar{v}^r\frac{\partial^2 \log G}{\partial z^\alpha \partial \bar{v}^b}, \quad \frac{\partial^2 \log G}{\partial w^a \partial \bar{z}^\beta} = v^r\frac{\partial^2 \log G}{\partial v^a \partial \bar{z}^\beta} \tag{1.14}$$

and

$$\begin{aligned}\frac{\partial^2 \log G}{\partial v^a \partial \bar{v}^r} &= -\frac{1}{|v^r|^2}\frac{\bar{v}^b}{\bar{v}^r}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b}, \\ \frac{\partial^2 \log G}{\partial v^r \partial \bar{v}^b} &= -\frac{1}{|v^r|^2}\frac{v^a}{v^r}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b}, \\ \frac{\partial^2 \log G}{\partial v^r \partial \bar{v}^r} &= \frac{v^a\bar{v}^b}{|v^r|^4}\frac{\partial^2 \log G}{\partial w^a \partial \bar{w}^b}.\end{aligned} \tag{1.15}$$

By direct checking, one has

$$(\log G)^{\bar{b}a} = \frac{G}{|v^r|^2}\left(-\frac{v^a}{v^r}G^{\bar{b}r} + G^{\bar{b}a} + \frac{\bar{v}^b v^a}{|v^r|^2}G^{\bar{r}r} - \frac{\bar{v}^b}{\bar{v}^r}G^{\bar{r}a}\right). \tag{1.16}$$

By (1.14) and (1.16), we have

$$\begin{aligned}(\log G)_{a\bar{\beta}}(\log G)^{\bar{b}a} &= \left(\frac{1}{\bar{v}^r}G^{\bar{b}i} - \frac{\bar{v}^b}{(\bar{v}^r)^2}G^{\bar{r}i}\right)(G_{i\bar{\beta}} - \frac{1}{G}G_{\bar{\beta}}G_i) \\ &= \frac{1}{\bar{v}^r}G^{\bar{b}i}G_{i\bar{\beta}} - \frac{\bar{v}^b}{(\bar{v}^r)^2}G^{\bar{r}i}G_{i\bar{\beta}}.\end{aligned}$$

So

$$
\begin{aligned}
q^*(\delta\bar{w}^b) &= q^*(d\bar{w}^b + (\log G)_{a\bar{\beta}}(\log G)^{\bar{b}a}d\bar{z}^\beta) \\
&= \frac{1}{\bar{v}^r}(d\bar{v}^b + G^{\bar{b}i}G_{i\bar{\beta}}d\bar{z}^\beta) - \frac{\bar{v}^b}{(\bar{v}^r)^2}(d\bar{v}^r + G^{\bar{r}i}G_{i\bar{\beta}}d\bar{z}^\beta) \\
&= \frac{1}{\bar{v}^r}\delta\bar{v}^b - \frac{\bar{v}^b}{(\bar{v}^r)^2}\delta\bar{v}^r.
\end{aligned} \tag{1.17}
$$

From (1.13), (1.15) and (1.17), we obtain

$$
\begin{aligned}
q^*\omega_{FS}(G) &= q^*\left(\sqrt{-1}\frac{\partial^2\log G}{\partial w^a\partial\bar{w}^b}\delta w^a\wedge\delta\bar{w}^b\right) \\
&= \sqrt{-1}|v^r|^2\frac{\partial^2\log G}{\partial v^a\partial\bar{v}^b}\left(\frac{1}{v^r}\delta v^a - \frac{v^a}{(v^r)^2}\delta v^r\right)\left(\frac{1}{\bar{v}^r}\delta\bar{v}^b - \frac{\bar{v}^b}{(\bar{v}^r)^2}\delta\bar{v}^r\right) \\
&= \sqrt{-1}\frac{\partial^2\log G}{\partial v^a\partial\bar{v}^b}\bigg(\delta v^a\wedge\delta\bar{v}^b - \frac{\bar{v}^b}{\bar{v}^r}\delta v^a\wedge\delta\bar{v}^r \\
&\qquad - \frac{v^a}{v^r}\delta v^r\wedge\delta\bar{v}^b + \frac{v^a\bar{v}^b}{|\bar{v}^r|^2}\delta v^r\wedge\delta\bar{v}^r\bigg) \\
&= \sqrt{-1}\sum_{i,j=1}^{r}\frac{\partial^2\log G}{\partial v^i\partial\bar{v}^j}\delta v^i\wedge\delta\bar{v}^j = \omega_V.
\end{aligned}
$$

□

Remark 1.5. (1) By (1.12) and (1.16), we get

$$
\begin{aligned}
q_*(\frac{\delta}{\delta z^\alpha}) &= q_*(\frac{\partial}{\partial z^\alpha} - G_{\alpha\bar{l}}G^{\bar{l}k}\frac{\partial}{\partial v^k}) \\
&= \frac{\partial}{\partial z^\alpha} - (\frac{1}{v^r}G_{\alpha\bar{l}}G^{\bar{l}a} - \frac{v^a}{(v^r)^2}G_{\alpha\bar{l}}G^{\bar{l}r})\frac{\partial}{\partial w^a} \\
&= \frac{\partial}{\partial z^\alpha} - (\log G)_{\alpha\bar{b}}(\log G)^{\bar{b}a}\frac{\partial}{\partial w^a}.
\end{aligned}
$$

For convenience, we will identify $\frac{\delta}{\delta z^\alpha}$ with $q_*(\frac{\delta}{\delta z^\alpha})$, and denote $N_\alpha^a := (\log G)_{\alpha\bar{b}}(\log G)^{\bar{b}a}$, so

$$
\frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - N_\alpha^a\frac{\partial}{\partial w^a}. \tag{1.18}
$$

(2) For any smooth function $f\in C^\infty(P(E))$, the vertical Laplacian is defined by

$$
\Delta^V f := (\log G)^{\bar{b}a}\frac{\partial^2}{\partial w^a\partial\bar{w}^b}f.
$$

By identifying f with q^*f, one has from (1.12) and (1.16)

$$\Delta^V f = (\log G)^{\bar{b}a} \frac{\partial^2}{\partial w^a \partial \bar{w}^b} f = G G^{i\bar{j}} \frac{\partial^2}{\partial v^i \partial \bar{v}^j} f.$$

(3) For the Hermitian metric

$$\Omega = \omega(G) + \omega_{FS}(G)$$

on $P(E)$, where $\omega(G) = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, then one can define the *horizontal laplacian* by

$$\Delta^H f := g^{\alpha\bar{\beta}} (\partial\bar{\partial} f)(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}) \tag{1.19}$$

for any smooth function $f \in C^\infty(P(E))$.

(4) Note that $\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log G$ is a $(1,1)$-form on $P(E)$, which represents the first Chern class $c_1(\mathcal{O}_{P(E)}(1))$. And Ψ is also a $(1,1)$-form on $P(E)$, combining Lemma 1.3 with Lemma 1.4, one has when restricting to $P(E)$

$$\sqrt{-1}\partial\bar{\partial} \log G = -\Psi + \omega_{FS}(G). \tag{1.20}$$

Proposition 1.6. *A Finsler metric G is strongly pseudoconvex if and only if $\omega_{FS}(G)$ is positive along each fiber of $p : P(E) \to M$.*

Proof. By Definition 1.1, G is strongly pseudoconvex if $(G_{i\bar{j}})$ is a positive definite matrix, which gives an inner product $\langle \cdot, \cdot \rangle$ on the vertical subbundle $\mathcal{V}$.

Denote $T = v^i \frac{\partial}{\partial v^i}$. If G is strongly pseudoconvex, for any $X = X^i \frac{\partial}{\partial v^i}$, then

$$\begin{aligned}(-\sqrt{-1})\omega_V(X, \bar{X}) &= \frac{1}{G^2}(G G_{i\bar{j}} - G_i G_{\bar{j}}) X^i \bar{X}^j \\ &= \frac{1}{G^2}(\|X\|^2 \|T\|^2 - |\langle X, T\rangle|^2) \geq 0,\end{aligned}$$

the equality holds if and only if $X = \lambda T$ for some constant $\lambda \in \mathbb{C}$. So ω_V has $r-1$ positive eigenvalues and one zero eigenvalue. Since $\omega_V(T, \overline{T}) = 0$ and $q_*(T) = 0$, by Lemma 1.4, $\omega_{FS}(G)$ is positive along each fiber of $p : P(E) \to M$.

Conversely, if $\omega_{FS}(G)$ is positive along each fiber, then $\omega_V = q^*\omega_{FS}(G)$ has $r-1$ positive eigenvalues and one zero eigenvalue, and $\omega_V(T, \overline{T}) = 0$.

So

$$G_{i\bar{j}}X^i\bar{X}^j = \frac{1}{G}|G_iX^i|^2 + G(-\sqrt{-1})\omega_V(X,\overline{X}) \geq 0.$$

Moreover, $G_{i\bar{j}}X^i\bar{X}^j = 0$ if and only if $X = \lambda T$ and $G_iX^i = 0$, if and only if $\lambda = 0$. So $(G_{i\bar{j}})$ is a positive definite matrix. □

Let $(h_{i\bar{j}})$ be a Hermitian metric on E with respect to a local holomorphic frame $\{e_i\}_{1\leq i\leq r}$. The Hermitian metric $(h_{i\bar{j}})$ induces a strongly pseudoconvex complex Finsler metric on E^* by

$$G := h^{i\bar{j}}v_i\bar{v}_j.$$

By Remark 1.5, we have the following decomposition

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_{FS}(G),$$

where

$$-\Psi = -R^{i\bar{j}}{}_{\alpha\bar{\beta}}\frac{v_i\bar{v}_j}{G}\sqrt{-1}dz^\alpha\wedge d\bar{z}^\beta = R_{k\bar{l}\alpha\bar{\beta}}\frac{G^{\bar{l}i}v_iG^{\bar{j}k}\bar{v}_j}{G}\sqrt{-1}dz^\alpha\wedge d\bar{z}^\beta.$$

From Proposition 1.6, $\omega_{FS}(G)$ is positive along fibers. So

Proposition 1.7. *The Chern curvature of $(h_{i\bar{j}})$ is Griffiths positive (resp. semi-positive)if and only if $\sqrt{-1}\partial\bar{\partial}\log G$ is a positive (resp. semi-positive) $(1,1)$-form on $P(E^*)$.*

1.2. Maximum principle for real (1, 1)-forms

This subsection recalls the maximum principle for real $(1,1)$-forms. For more details, one can refer to [4, 10, 24, 27]. The following version maximum principle is the same as the tensor maximum principle [10, Theorem 4.6], so we omit its proof.

Theorem 1.8 ([10, Theorem 4.6]). *Let $\omega(t) = \sqrt{-1}g_{\alpha\bar{\beta}}(t)dz^\alpha\wedge d\bar{z}^\beta$ be a smooth 1-parameter family of Hermitian metrics on a compact complex manifold M. Let $\eta(t) = \sqrt{-1}\eta_{\alpha\bar{\beta}}(t)dz^\alpha\wedge d\bar{z}^\beta$ be a real $(1,1)$-form satisfying*

$$\frac{\partial}{\partial t}\eta \geq \Delta_{\omega(t)}\eta + \sigma,$$

where $\Delta_{\omega(t)} := g^{\alpha\bar{\beta}}\nabla_{\frac{\partial}{\partial z^\alpha}}\nabla_{\frac{\partial}{\partial\bar{z}^\beta}}$, ∇ denotes the Chern connection of $\omega(t)$, $\sigma(\omega,t)$ is a real $(1,1)$-form which is locally Lipschitz in all its arguments

and satisfies the **null eigenvector assumption** *that*

$$(-\sqrt{-1})\sigma(V,\overline{V})(z,t) = (\sigma_{\alpha\bar{\beta}}V^{\alpha}\overline{V^{\beta}})(z,t) \geq 0$$

whenever $V(z,t) = V^{\alpha}\frac{\partial}{\partial z^{\alpha}}$ *is a null eigenvector of* $\eta(t)$*, that is whenever*

$$(\eta_{\alpha\bar{\beta}}V^{\alpha})(z,t) = 0.$$

If $\eta(0) \geq 0$*, then* $\eta(t) \geq 0$ *for all* $t \geq 0$ *such that the solution exists.*

2. A flow over projective bundles

2.1. Definition of the flow

Let M be a compact complex manifold of dimension n. Let $\pi : E \to M$ be a holomorphic vector bundle of rank r over M. Let E^* denote the dual bundle of E, $\{e^i\}_{i=1}^r$ be a local holomorphic frame of E^*. Then

$$(z,v) = (z^1,\cdots,z^n;v_1,\cdots,v_r)$$

gives a local coordinate system of the complex manifold E^*, which represents the point $v^i e_i \in E^*$. For any strongly pseudoconvex complex Finsler metric G on E^*, by Remark 1.5, we have the following decomposition

$$\sqrt{-1}\partial\bar{\partial}\log G = -\Psi + \omega_{FS}(G).$$

By Proposition 1.6, $\omega_{FS}(G)$ is positive along each fiber of $p : P(E^*) \to M$.

Let

$$\omega(G) = \sqrt{-1}g_{\alpha\bar{\beta}}(G)dz^{\alpha}\wedge d\bar{z}^{\beta}$$

be a horizontal $(1,1)$-form on $P(E^*)$ depending smoothly on the Finsler metric G, which is positive on horizontal directions, namely $(g_{\alpha\bar{\beta}}(G))$ is a positive definite matrix.

Then one can define a Hermitian metric on $P(E^*)$ by

$$\Omega := \omega(G) + \omega_{FS}(G).$$

Let G_0 be a strongly pseudoconvex complex Finsler metric on E^*. We consider the following flow:

$$\begin{cases} \frac{\partial}{\partial t}\log G = \Delta_\Omega \log G, \\ \omega_{FS}(G) > 0, \\ G(0) = G_0. \end{cases} \tag{2.1}$$

Here $\Delta_\Omega := \sqrt{-1}\Lambda\partial\bar{\partial}$, Λ is the adjoint operator of $\Omega \wedge \bullet$.

Note that

$$\Delta_\Omega \log G = \Lambda\sqrt{-1}\partial\bar{\partial}\log G = \Lambda_{\omega(G)}(-\Psi) + (r-1) \tag{2.2}$$

is a smooth function on $P(E^*)$, then

$$G(t) = e^{\int_0^t \Delta_\Omega \log G dt} G_0,$$

By Definition 1.1, one sees that $G(t)$ is always a complex Finsler metric on E^*. Moreover, G_0 is strongly pseudoconvex, i.e. $\omega_{FS}(G_0) > 0$, so $G(t)$ is strongly pseudoconvex automatically as t small.

Remark 2.1. 1) For the case

$$\omega(G) = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$$

is a fixed Kähler metric on M, and $G_0 = h_0^{i\bar{j}} v_i \bar{v}_j$ comes from a Hermitian metric $(h_0^{i\bar{j}})$ of E^*, then

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\log G - \Delta_\Omega \log G \\ &= \frac{1}{G}\frac{\partial G}{\partial t} + \Lambda_\omega \Psi - (r-1) \\ &= \frac{v_i \overline{v_j}}{G}\left(\frac{\partial h^{i\bar{j}}(t)}{\partial t} + g^{\alpha\bar{\beta}} R^{i\bar{j}}{}_{\alpha\bar{\beta}} - (r-1)h^{\bar{j}i}(t)\right), \end{aligned} \tag{2.3}$$

where $h^{i\bar{j}}(t) := \frac{\partial^2 G(t)}{\partial v_i \partial \bar{v}_j}$. From the above equation, one has

$$\frac{\partial h^{i\bar{j}}(t)}{\partial t} + g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial v_i \partial \bar{v}_j}(R^{i\bar{j}}{}_{\alpha\bar{\beta}} v_i \bar{v}_j) - (r-1)h^{\bar{j}i}(t) = 0.$$

Since $h^{i\bar{j}}(0) = h_0^{i\bar{j}}$ is a Hermitian metric, which is independent of the vertical coordinates $\{v_i, 1 \le i \le r\}$, so $h^{i\bar{j}}(t)$ is also a Hermitian metric.

In fact, by induction, we assume that $(\frac{\partial^k h^{i\bar{j}}}{\partial t^k})|_{t=0}$ is independent of fibers, then

$$\begin{aligned}\frac{\partial^{k+1} h^{i\bar{j}}(t)}{\partial t^{k+1}}|_{t=0} &= \frac{\partial^k}{\partial t^k}\left(-g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial v_i \partial \bar{v}_j}(R^{i\bar{j}}{}_{\alpha\bar{\beta}} v_i \bar{v}_j) + (r-1)h^{\bar{j}i}(t)\right)|_{t=0} \\ &= -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial v_i \partial \bar{v}_j}\left((\frac{\partial^k}{\partial t^k}R^{i\bar{j}}{}_{\alpha\bar{\beta}})|_{t=0} v_i \bar{v}_j\right) + (r-1)(\frac{\partial^k}{\partial t^k}h^{\bar{j}i})|_{t=0} \\ &= -g^{\alpha\bar{\beta}}(\frac{\partial^k}{\partial t^k}R^{i\bar{j}}{}_{\alpha\bar{\beta}})|_{t=0} + (r-1)(\frac{\partial^k}{\partial t^k}h^{\bar{j}i})|_{t=0},\end{aligned}$$

which is also independent of fibers, because $\frac{\partial^k}{\partial t^k}R^{i\bar{j}}{}_{\alpha\bar{\beta}}$ is the combination by $h^{\bar{j}i}$ and $\frac{\partial^l}{\partial t^l}h^{\bar{j}i}, 1 \leq l \leq k$. It follows that

$$h^{\bar{j}i}(t) = h^{\bar{j}i}(0) + \frac{\partial h^{i\bar{j}}(t)}{\partial t}|_{t=0}t + \cdots + \frac{\partial^k h^{i\bar{j}}(t)}{\partial t^k}|_{t=0}t^k + \cdots$$

is independent of fibers for small t. By Proposition 1.6 and $\omega_{FS}(G) > 0$, $h^{\bar{j}i}(t)$ is a positive definite matrix, so $(h^{\bar{j}i}(t))$ is a Hermitian metric on E^*.

Therefore, (2.3) is equivalent to

$$\frac{\partial h^{i\bar{j}}(t)}{\partial t} + g^{\alpha\bar{\beta}}(R^{h^{-1}})^{i\bar{j}}{}_{\alpha\bar{\beta}} - (r-1)h^{\bar{j}i}(t) = 0. \tag{2.4}$$

Multiplying by $h_{k\bar{j}}$ to both sides of (2.4), one has

$$h^{-1}\cdot\frac{\partial h}{\partial t} + \Lambda R^h + (r-1)I := h^{\bar{j}i}\frac{\partial h_{k\bar{j}}}{\partial t} + g^{\alpha\bar{\beta}}(R^h)^i_{k\alpha\bar{\beta}} + (r-1)\delta^i_k = 0,$$

which is exactly the Hermitian-Yang-Mills flow [3, 11] (see also [21]).

2) For the case of $E = TM$. Let

$$\omega(G) = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta,$$

where $(g_{\alpha\bar{\beta}})$ denotes the inverse of the matrix $\left(\frac{\partial^2 G}{\partial v_\alpha \partial \bar{v}_\beta}\right)$. Let $G_0 = g_0^{\alpha\bar{\beta}}v_\alpha \bar{v}_\beta$ be a strongly pseudoconvex complex Finsler metric on T^*M

induced by the Hermitian metric

$$\omega_0 = \sqrt{-1}(g_0)_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Similar to the above case, the flow (2.1) is equivalent to the following *Hermitian curvature flow*

$$\frac{\partial g_{\gamma\bar{\delta}}}{\partial t} + g^{\alpha\bar{\beta}} R_{\gamma\bar{\delta}\alpha\bar{\beta}} + (r-1) g_{\gamma\bar{\delta}} = 0, \tag{2.5}$$

which was first given in [30, (3)], and many results were obtained there on this kind of flow with an arbitrary quadratic tensor Q in the torsion. This flow (2.5) was also appeared in [22, (7.11)], and by [22, Theorem 7.1] (or [22, Remark 7.2]), if the initial metric ω_0 is a Kähler metric, then this flow is reduced to the usual Kähler-Ricci flow (see [6]).

2.2. Positivity preserving along the flow

In this subsection, we shall discuss the positivity preserving along the flow (2.1). We assume that the initial metric $G(0) = G_0$ satisfies $\sqrt{-1}\partial\bar{\partial}\log G_0 \geq 0$, and will prove

Proposition 2.2. *Let $(z_0, [v_0], t_0)$ be a point and time such that $\sqrt{-1}\partial\bar{\partial}\log G \geq 0$ for all $0 \leq t < t_0$, and there is a $(1,0)$-type vector u such that $i_u(\partial\bar{\partial}\log G)(z_0, [v_0], t_0) = 0$. Then*

$$\frac{\partial}{\partial t}\left(\partial\bar{\partial}\log G(U, \overline{U})\right) = \Delta^H_\Omega(\partial\bar{\partial}\log G(U, \overline{U})) + \langle R^g(u, \overline{u}), -\Psi\rangle_\Omega - \left|i_u \partial^V \Psi\right|^2_\Omega.$$

at this point $(z_0, [v_0], t_0)$, where U is defined by (2.30), which is a locally extended vector field of u, and

$$\langle\sqrt{-1}R^g(u, \overline{u}), -\Psi\rangle_\Omega := (-\Psi)_{\alpha\bar{\delta}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R^g_{\gamma\bar{\beta}\sigma\bar{\tau}} u^\sigma \bar{u}^\tau,$$
$$\left|i_u \partial^V \Psi\right|^2_\Omega := (\log G)^{a\bar{b}} \partial_a \Psi_{\alpha\bar{\beta}} \overline{\partial_b \Psi_{\gamma\bar{\tau}}} u^\alpha \overline{u}^\gamma g^{\tau\bar{\beta}}.$$

Here $\Omega = \omega(G) + \omega_{FS}(G)$, $\omega(G) = p^*\omega$, ω is a Kähler metric on M depending on the Finsler metric G.

Firstly, let $\epsilon > 0$ small enough such that

$$\begin{aligned}(2.6)\qquad \Omega_\epsilon &= \omega(G) + \epsilon\sqrt{-1}\partial\bar\partial \log G \\ &= \sqrt{-1}(g_{\alpha\bar\beta} - \epsilon\Psi_{\alpha\bar\beta})dz^\alpha \wedge d\bar z^\beta + \epsilon\sqrt{-1}\frac{\partial^2 \log G}{\partial w^a \partial \bar w^b}\delta w^a \wedge \delta \bar w^b\end{aligned}$$

is a Hermitian metric on $P(E^*)$, where $\Psi_{\alpha\bar\beta}$ is given by $\Psi = \sqrt{-1}\Psi_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$. Denote by ∇^ϵ the Chern connection of the Hermitian metric Ω_ϵ, which is the unique connection preserving the holomorphic structure and the metric Ω_ϵ. For any two $(1,0)$-type vector fields X, Y of $P(E^*)$, then

$$\begin{aligned}(2.7)\qquad \nabla^\epsilon_X Y &= \langle \nabla^\epsilon_X Y, \frac{\delta}{\delta z^\beta}\rangle_\epsilon g_\epsilon^{\bar\beta\alpha}\frac{\delta}{\delta z^\alpha} + \langle \nabla^\epsilon_X Y, \frac{\partial}{\partial w^b}\rangle_\epsilon \frac{1}{\epsilon}(\log G)^{\bar b a}\frac{\partial}{\partial w^a} \\ &= X\langle Y, \frac{\delta}{\delta z^\beta}\rangle_\epsilon g_\epsilon^{\bar\beta\alpha}\frac{\delta}{\delta z^\alpha} - \langle Y, \overline{X}(\frac{\delta}{\delta z^\beta})\rangle_\epsilon g_\epsilon^{\bar\beta\alpha}\frac{\delta}{\delta z^\alpha} \\ &\quad + X\langle Y^V, \frac{\partial}{\partial w^b}\rangle_\epsilon \frac{1}{\epsilon}(\log G)^{\bar b a}\frac{\partial}{\partial w^a} \\ &= X\langle Y, \frac{\delta}{\delta z^\beta}\rangle_\epsilon g_\epsilon^{\bar\beta\alpha}\frac{\delta}{\delta z^\alpha} + \langle Y^V, \overline{X}(N^b_\beta)\frac{\partial}{\partial w^b}\rangle_\epsilon g_\epsilon^{\bar\beta\alpha}\frac{\delta}{\delta z^\alpha} \\ &\quad + X\langle Y^V, \frac{\partial}{\partial w^b}\rangle_\epsilon \frac{1}{\epsilon}(\log G)^{\bar b a}\frac{\partial}{\partial w^a},\end{aligned}$$

where Y^V denotes the vertical part of Y, $(g^{\bar\beta\alpha})_\epsilon$ denotes the inverse of $g_{\epsilon\alpha\bar\beta} := g_{\alpha\bar\beta} - \epsilon\Psi_{\alpha\bar\beta}$, $\langle\cdot,\cdot\rangle_\epsilon$ is the inner product defined by Ω_ϵ.

Denote by Z^A the coordinates z^α or w^a, $1 \le A \le n+r-1$. We rewrite Ω_ϵ as the following form:

$$(2.8)\qquad \Omega_\epsilon = \sqrt{-1}\Omega_{\epsilon A\bar B}dZ^A \wedge d\bar Z^B.$$

In this form, the Chern connection is given by

$$(2.9)\qquad \nabla^\epsilon_{\frac{\partial}{\partial Z^A}}\frac{\partial}{\partial Z^B} = \Gamma^C_{AB}\frac{\partial}{\partial Z^C}, \quad \Gamma^C_{AB} = \frac{\partial \Omega_{\epsilon B\bar D}}{\partial Z^A}\Omega_\epsilon^{\bar D C}.$$

Here $(\Omega_\epsilon^{\bar D C})$ denotes the inverse of the matrix $(\Omega_{\epsilon C\bar D})$. The Chern curvature tensor of Ω_ϵ is defined by

$$\begin{aligned}(2.10)\quad R^\epsilon_{A\bar B C\bar D} &:= R(\frac{\partial}{\partial Z^A}, \frac{\partial}{\partial \bar Z^B}, \frac{\partial}{\partial Z^C}, \frac{\partial}{\partial \bar Z^D}) \\ &:= \langle \frac{\partial}{\partial Z^A}, (\nabla^\epsilon_{\frac{\partial}{\partial Z^D}}\nabla^\epsilon_{\frac{\partial}{\partial \bar Z^C}} - \nabla^\epsilon_{\frac{\partial}{\partial \bar Z^C}}\nabla^\epsilon_{\frac{\partial}{\partial Z^D}} - \nabla^\epsilon_{[\frac{\partial}{\partial Z^D}, \frac{\partial}{\partial \bar Z^C}]})\frac{\partial}{\partial Z^B}\rangle_\epsilon \\ &= -\Omega_{\epsilon A\bar E}\frac{\partial \overline{\Gamma^E_{DB}}}{\partial Z^C}.\end{aligned}$$

The Chern connection ∇^ϵ induces a natural connection on the cotangent bundle $T^*P(E^*)$ (resp. $\overline{T^*P(E^*)}$) by

$$\begin{aligned}(2.11)\qquad &\nabla^\epsilon_A(f_C dZ^C) = (\partial_A f_C - \Gamma^B_{AC} f_B)dZ^C,\\ &(\text{resp. } \nabla^\epsilon_{\bar B}(f_{\bar D} d\bar Z^D) = (\partial_{\bar B} f_{\bar D} - \overline{\Gamma^E_{BD}} f_{\bar E})dZ^C)\end{aligned}$$

for any smooth $(1,0)$-form $f_C dZ^C$ (resp. $(0,1)$-form $f_{\bar D} d\bar Z^D$), where $\nabla^\epsilon_A := \nabla^\epsilon_{\frac{\partial}{\partial Z^A}}$. For convenience, we denote

$$(2.12)\qquad \nabla^\epsilon_A f_C := \partial_A f_C - \Gamma^B_{AC} f_B, \quad \nabla^\epsilon_{\bar B} f_{\bar D} := \partial_{\bar B} f_{\bar D} - \overline{\Gamma^E_{BD}} f_{\bar E}$$

and

$$\begin{aligned}(2.13)\qquad &(\nabla^\epsilon_A f_{C\bar D})dZ^C \wedge d\bar Z^D := \nabla^\epsilon_{\frac{\partial}{\partial Z^A}}(f_{C\bar D} dZ^C \wedge d\bar Z^D),\\ &\nabla^\epsilon_A f_{C\bar D} = \partial_A f_{C\bar D} - \Gamma^B_{AC} f_{B\bar D}.\end{aligned}$$

By using the above notations, we have

$$(2.14)\qquad \nabla^\epsilon_A \Omega_{\epsilon C\bar D} = \partial_A \Omega_{\epsilon C\bar D} - \Gamma^B_{AC}\Omega_{\epsilon B\bar D} = 0.$$

By taking $\partial\bar\partial$ to the both sides of the first equation of (2.1), one has

$$(2.15)\qquad \frac{\partial}{\partial t}\partial\bar\partial \log G = \partial\bar\partial \Delta_\Omega \log G = \partial\bar\partial(tr_{\omega(G)}(-\Psi)),$$

where the last equality follows from (2.2). Since $\lim_{\epsilon\to 0}(\omega(G) - \epsilon\Psi) = \omega(G)$, so

$$\begin{aligned}(2.16)\qquad \frac{\partial}{\partial t}\partial\bar\partial \log G &= \lim_{\epsilon\to 0}\partial\bar\partial(tr_{\omega(G)-\epsilon\Psi}(-\Psi))\\ &= \lim_{\epsilon\to 0}\partial\bar\partial \Delta_{\Omega_\epsilon}\log G\\ &= \lim_{\epsilon\to 0}\partial_C\partial_{\bar D}(\Omega_\epsilon^{A\bar B}\partial_A\partial_{\bar B}\log G)dZ^C\wedge d\bar Z^D.\end{aligned}$$

Denote $f := \log G$ and

$$f_{A\bar B} := \partial_A\partial_{\bar B} f, \quad f_{A\bar B C} := \partial_A\partial_{\bar B}\partial_C f, \quad f_{A\bar B C\bar D} := \partial_A\partial_{\bar B}\partial_C\partial_{\bar D} f, \quad etc..$$

By (2.13) and (2.14), one has

$$\begin{aligned}(2.17)\quad \partial_C\partial_{\bar D}(\Omega_\epsilon^{A\bar B}\partial_A\partial_{\bar B}\log G) &= \partial_C\partial_{\bar D}(\Omega_\epsilon^{A\bar B}f_{A\bar B}) = \nabla_C^\epsilon\nabla_{\bar D}^\epsilon(\Omega_\epsilon^{AB}f_{A\bar B})\\ &= \Omega_\epsilon^{A\bar B}\nabla_C^\epsilon\nabla_{\bar D}^\epsilon f_{A\bar B}\\ &= \Omega_\epsilon^{A\bar B}\nabla_C^\epsilon(f_{A\bar B\bar D} - \overline{\Gamma_{DB}^E}f_{A\bar E})\\ &= \Omega_\epsilon^{A\bar B}(f_{A\bar BC\bar D} - \Gamma_{CA}^F f_{F\bar B\bar D} - \partial_C\overline{\Gamma_{DB}^E}f_{A\bar E}\\ &\quad - \overline{\Gamma_{DB}^E}(f_{AC\bar E} - \Gamma_{CA}^F f_{F\bar E})).\end{aligned}$$

Similarly,

$$\begin{aligned}(2.18)\qquad \Omega_\epsilon^{A\bar B}(\nabla_A^\epsilon\nabla_{\bar B}^\epsilon f_{C\bar D}) &= \Omega_\epsilon^{A\bar B}(f_{A\bar BC\bar D} - \Gamma_{AC}^F f_{F\bar B\bar D}\\ &\quad - \partial_A\overline{\Gamma_{BD}^E}f_{C\overline{E}} - \overline{\Gamma_{BD}^E}(f_{AC\bar E} - \Gamma_{AC}^F f_{F\bar E})).\end{aligned}$$

Combining (2.17) with (2.18), we obtain

$$\begin{aligned}(2.19)\qquad &\partial_C\partial_{\bar D}(\Omega_\epsilon^{A\bar B}f_{A\bar B}) - \Omega_\epsilon^{A\bar B}(\nabla_A^\epsilon\nabla_{\bar B}^\epsilon f_{C\bar D})\\ &\quad= \Omega_\epsilon^{A\bar B}(\Gamma_{AC}^F - \Gamma_{CA}^F)f_{F\bar D\bar B} + \Omega_\epsilon^{A\bar B}(\overline{\Gamma_{BD}^E} - \overline{\Gamma_{DB}^E})f_{AC\bar E}\\ &\qquad+ \Omega_\epsilon^{A\bar B}(\overline{\Gamma_{DB}^E}\Gamma_{CA}^F - \overline{\Gamma_{BD}^E}\Gamma_{AC}^F)f_{F\bar E}\\ &\qquad+ \Omega_\epsilon^{A\bar B}\partial_A\overline{\Gamma_{BD}^E}f_{C\bar E} - \Omega_\epsilon^{A\bar B}\partial_C\overline{\Gamma_{DB}^E}f_{A\bar E}\\ &\quad= \Omega_\epsilon^{A\bar B}(\Gamma_{AC}^F - \Gamma_{CA}^F)\nabla_{\bar D}f_{F\bar B} + \Omega_\epsilon^{A\bar B}(\overline{\Gamma_{BD}^E} - \overline{\Gamma_{DB}^E})\nabla_A f_{C\bar E}\\ &\qquad- \Omega_\epsilon^{A\bar B}\Omega_\epsilon^{F\bar E}R_{F\bar DA\bar B}^\epsilon f_{C\bar E} + \Omega_\epsilon^{A\bar B}\Omega_\epsilon^{F\bar E}R_{F\bar BC\bar D}^\epsilon f_{A\bar E}.\end{aligned}$$

Substituting (2.19) into (2.16), we have

$$\begin{aligned}(2.20)\quad \frac{\partial}{\partial t}\partial\bar\partial\log G &= \lim_{\epsilon\to 0}\Big(\Omega_\epsilon^{A\bar B}\nabla_A^\epsilon\nabla_{\bar B}^\epsilon(\partial\bar\partial\log G)\\ &\quad+\Big(\Omega_\epsilon^{A\bar B}(\Gamma_{AC}^F - \Gamma_{CA}^F)\nabla_{\bar D}f_{F\bar B} + \Omega_\epsilon^{A\bar B}(\overline{\Gamma_{BD}^E} - \overline{\Gamma_{DB}^E})\nabla_A f_{C\bar E}\\ &\qquad-\Omega_\epsilon^{A\bar B}\Omega_\epsilon^{F\bar E}R_{F\bar DA\bar B}^\epsilon f_{C\bar E} + \Omega_\epsilon^{A\bar B}\Omega_\epsilon^{F\bar E}R_{F\bar BC\bar D}^\epsilon f_{A\bar E}\Big)\,dZ^C\wedge d\bar Z^D\Big).\end{aligned}$$

As in the proof of Theorem 1.8, we assume that $\sqrt{-1}\partial\bar\partial\log G \geq 0$ for all $0 \leq t < t_0$, and $(z_0, [v_0], t_0)$ is a point and time, and $u = u^A\frac{\partial}{\partial Z^A}$ is a vector such that

$$(2.21)\qquad f_{C\bar D}u^C(z_0,[v_0],t_0) = (\partial_C\partial_{\bar D}\log G)u^C(z_0,[v_0],t_0) = 0$$

and

$$(\partial\bar\partial \log G(W,\overline{W}))(z,[v],t) = (\partial_C\partial_{\bar D}\log G)W^C\overline{W^D}(z,[v],t) \geq 0 \tag{2.22}$$

for all $(z,[v]) \in P(E^*)$, $t \in [0,t_0]$, and tangent vectors $W \in T_{(z,[v])}P(E^*)$. This implies that

$$u = u^\alpha \frac{\delta}{\delta z^\alpha} \in q_*\mathcal{H}. \tag{2.23}$$

Indeed, one may assume that $u = u_1 + u_2$, where $u_1 = u^\alpha \frac{\delta}{\delta z^\alpha}$, $u_2 = u^a \frac{\partial}{\partial w^a}$ are the horizontal and vertical parts of u respectively. By (1.20) and $\omega_{FS}(G) > 0$, one has

$$\begin{aligned} 0 &= (\sqrt{-1}\partial\bar\partial \log G)(u,\bar u) \\ &= (\sqrt{-1}\partial\bar\partial \log G)(u_1+u_2, \overline{u_1}+\overline{u_2}) \\ &= (\sqrt{-1}\partial\bar\partial \log G)(u_1,\overline{u_1}) + \omega_{FS}(G)(u_2,\overline{u_2}) \\ &\geq \omega_{FS}(G)(u_2,\overline{u_2}) \geq 0, \end{aligned} \tag{2.24}$$

and all equalities hold if and only if $u_2 = 0$, namely $u = u_1$. From Lemma 1.3 and (2.23), (2.21) is equivalent to

$$(i_u\Psi)(z_0,[v_0],t_0) = 0. \tag{2.25}$$

For any $\epsilon > 0$, by parallel translation, one can extend u to a vector field $U_\epsilon = U_\epsilon^A \frac{\partial}{\partial Z^A}$ defined in a neighborhood of $(z_0,[v_0],t_0)$ such that $U_\epsilon(z_0,[v_0],t_0) = u$ and

$$\frac{\partial U_\epsilon}{\partial t}(z_0,[v_0],t_0) = 0, \quad (\nabla^\epsilon U_\epsilon)(z_0,[v_0],t_0) = 0. \tag{2.26}$$

This can be done by parallel translating u along radial rays with respect to the connection ∇^ϵ, and then by extending to be independent of time t.

We assume that

$$U_\epsilon(z,[v],t_0) = U_\epsilon^\alpha \frac{\delta}{\delta z^\alpha} + U_\epsilon^a \frac{\partial}{\partial w^a}.$$

By (2.7), one has

$$\begin{aligned} \nabla^\epsilon U_\epsilon = \bar\partial U_\epsilon &+ \Big(\partial(U_\epsilon^\alpha g_{\epsilon\alpha\bar\beta}) + U_\epsilon^a \partial(\overline{N_\beta^b})\epsilon(\log G)_{a\bar b}\Big) g_\epsilon^{\bar\beta\gamma}\frac{\delta}{\delta z^\gamma} \\ &+ \partial(U_\epsilon^a(\log G)_{a\bar b})(\log G)^{\bar b c}\frac{\partial}{\partial w^c}. \end{aligned} \tag{2.27}$$

So the second equation of (2.26) is equivalent to

$$\begin{cases} \bar\partial U^\alpha_\epsilon = 0, \\ \bar\partial(-U^\alpha_\epsilon N^a_\alpha + U^a_\epsilon) = 0, \\ \partial(U^\alpha_\epsilon g_{\epsilon\alpha\bar\beta}) + U^a_\epsilon \partial(\overline{N^b_\beta})\epsilon(\log G)_{a\bar b} = 0, \\ \partial(U^a_\epsilon (\log G)_{a\bar b})(\log G)^{\bar b c} = 0, \end{cases} \tag{2.28}$$

at the point $(z_0, [v_0], t_0)$. By (2.23), $U^a_\epsilon = 0$ at the point $(z_0, [v_0], t_0)$, so (2.28) is equivalent to

$$\begin{cases} \bar\partial U^\alpha_\epsilon = 0, \\ \bar\partial U^a_\epsilon = u^\alpha \bar\partial N^a_\alpha, \\ \partial U^\alpha_\epsilon + g^{\bar\beta\alpha}_\epsilon \partial g_{\epsilon\gamma\bar\beta} u^\gamma = 0, \\ \partial U^a_\epsilon = 0. \end{cases} \tag{2.29}$$

Since $\lim_{\epsilon\to 0} g_{\epsilon\alpha\bar\beta} = g_{\alpha\bar\beta}$, so $\lim_{\epsilon\to 0} U_\epsilon = U$ which satisfies the following equations:

$$\begin{cases} \bar\partial U^\alpha = 0, \\ \bar\partial U^a = u^\alpha \bar\partial N^a_\alpha, \\ \partial U^\alpha + g^{\bar\beta\alpha} \partial g_{\gamma\bar\beta} u^\gamma = 0, \\ \partial U^a = 0 \end{cases} \tag{2.30}$$

and $\frac{\partial U}{\partial t} = 0$ at the point $(z_0, [v_0], t_0)$. By (2.20), (2.21) and (2.26), one has at the point $(z_0, [v_0], t_0)$,

$$\begin{aligned} \frac{\partial}{\partial t}\left(\partial\bar\partial \log G(U, \overline{U})\right) &= \lim_{\epsilon\to 0}\Big(\Omega^{A\bar B}_\epsilon \nabla^\epsilon_A \nabla^\epsilon_{\bar B}(\partial\bar\partial \log G)(u, \overline{u}) \\ &\quad + \Big(\Omega^{A\bar B}_\epsilon(\Gamma^F_{AC} - \Gamma^F_{CA})\nabla_{\bar D} f_{F\bar B} + \Omega^{A\bar B}_\epsilon(\overline{\Gamma^E_{BD}} - \overline{\Gamma^E_{DB}})\nabla_A f_{C\bar E} \\ &\quad - \Omega^{A\bar B}_\epsilon \Omega^{F\bar E}_\epsilon R_{F\bar D A\bar B} f_{C\bar E} + \Omega^{A\bar B}_\epsilon \Omega^{F\bar E}_\epsilon R^\epsilon_{F\bar B C\bar D} f_{A\bar E}\Big) u^C \bar u^D\Big) \\ &= \lim_{\epsilon\to 0}\Big(\Omega^{A\bar B}_\epsilon \partial_A \partial_{\bar B}(\partial\bar\partial \log G(U_\epsilon, \overline{U}_\epsilon)) \\ &\quad + \Big(\Omega^{A\bar B}_\epsilon(\Gamma^F_{AC} - \Gamma^F_{CA})\nabla_{\bar D} f_{F\bar B} + \Omega^{A\bar B}_\epsilon(\overline{\Gamma^E_{BD}} - \overline{\Gamma^E_{DB}})\nabla_A f_{C\bar E} \\ &\quad + \Omega^{A\bar B}_\epsilon \Omega^{F\bar E}_\epsilon R^\epsilon_{F\bar B C\bar D} f_{A\bar E}\Big) u^C \bar u^D\Big). \end{aligned} \tag{2.31}$$

In order to deal with (2.31), we assume that $\omega(G) = p^*\omega$ for some Kähler metric ω on M, so

$$\Omega_\epsilon = \omega(G) + \epsilon\sqrt{-1}\partial\bar\partial \log G \tag{2.32}$$

is a Kähler metric on $P(E^*)$ for $\epsilon > 0$ small enough. Thus, (2.31) is reduced to

$$\begin{aligned}\frac{\partial}{\partial t}\left(\partial\bar\partial \log G(U,\overline{U})\right)&\\ = \lim_{\epsilon\to 0}\left(\Omega_\epsilon^{A\bar B}\partial_A\partial_{\bar B}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon)) + \Omega_\epsilon^{A\bar B}\Omega_\epsilon^{F\bar E}R^\epsilon_{F\bar BC\bar D}f_{A\bar E}u^C\bar u^D\right).&\end{aligned} \tag{2.33}$$

For the first term in the RHS of (2.33), we have

$$\begin{aligned}\Omega_\epsilon^{A\bar B}\partial_A\partial_{\bar B}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon))&\\ = \sqrt{-1}\Lambda\partial\bar\partial(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon))&\\ = \Delta^H_{\Omega_\epsilon}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon)) + \Delta^V_{\Omega_\epsilon}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon)).&\end{aligned} \tag{2.34}$$

Here $\Delta^V_{\Omega_\epsilon} = \frac{1}{\epsilon}(\log G)^{\bar b a}\frac{\partial^2}{\partial w^a\partial\bar w^b}$ is the vertical Laplacian, while $\Delta^H_{\Omega_\epsilon}f = g_\epsilon^{\alpha\bar\beta}(\partial\bar\partial f)(\frac{\delta}{\delta z^\alpha},\frac{\delta}{\delta\bar z^\beta})$ is the horizontal Laplacian (see Remark 1.5). Since $\lim_{\epsilon\to o} g_{\epsilon\alpha\bar\beta} = g_{\alpha\bar\beta}$ and $\lim_{\epsilon\to 0}U_\epsilon = U$, so

$$\lim_{\epsilon\to 0}\Delta^H_{\Omega_\epsilon}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon)) = \Delta^H_\Omega(\partial\bar\partial\log G(U,\overline{U})). \tag{2.35}$$

By Remark 1.5 (1), one has

$$\begin{aligned}\Delta^V_{\Omega_\epsilon}(\partial\bar\partial\log G(U_\epsilon,\overline{U}_\epsilon)) &= \Delta^V_{\Omega_\epsilon}(f_{c\bar d}U^c_\epsilon\overline{U}^d_\epsilon) + \Delta^V_{\Omega_\epsilon}((-\Psi)_{\alpha\bar\beta}U^\alpha_\epsilon\overline{U}^\beta_\epsilon)\\ &= \frac{1}{\epsilon}f^{a\bar b}\frac{\partial^2}{\partial w^a\partial\bar w^b}(f_{c\bar d}U^c_\epsilon\overline{U}^d_\epsilon)\\ &\quad + \frac{1}{\epsilon}f^{a\bar b}\frac{\partial^2}{\partial w^a\partial\bar w^b}((-\Psi)_{\alpha\bar\beta}U^\alpha_\epsilon\overline{U}^\beta_\epsilon).\end{aligned} \tag{2.36}$$

For the first term in the RHS of (2.36), by (2.29) and $U^c = 0$ at the point $(z_0,[v_0],t_0)$, we have

$$\begin{aligned}\frac{1}{\epsilon}f^{a\bar b}\frac{\partial^2}{\partial w^a\partial\bar w^b}(f_{c\bar d}U^c_\epsilon\overline{U}^d_\epsilon) &= \frac{1}{\epsilon}f^{a\bar b}f_{c\bar d}\partial_{\bar b}U^c_\epsilon\partial_a\overline{U}^d_\epsilon\\ &= \frac{1}{\epsilon}f^{a\bar b}f_{c\bar d}u^\alpha\overline{u}^\beta\partial_{\bar b}N^c_\alpha\partial_a\overline{N}^d_\beta.\end{aligned} \tag{2.37}$$

The following lemma is proved in [37, (3.46)]. For readers' convenience, we give a proof here.

Lemma 2.3.

$$f^{\bar{b}a}\frac{\partial^2}{\partial w^a\partial\bar{w}^b}(-\Psi)_{\alpha\bar{\beta}} = \partial\bar{\partial}\log\det(f_{a\bar{b}})(\frac{\delta}{\delta z^\alpha},\frac{\delta}{\delta\bar{z}^\beta}) - \langle\bar{\partial}^V\frac{\delta}{\delta z^\alpha},\bar{\partial}^V\frac{\delta}{\delta z^\beta}\rangle, \tag{2.38}$$

where

$$\bar{\partial}^V\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial\bar{w}^d}(-f_{\alpha\bar{b}}f^{\bar{b}c})\delta\bar{w}^d\otimes\frac{\partial}{\partial w^c}$$

and

$$\langle\bar{\partial}^V\frac{\delta}{\delta z^\alpha},\bar{\partial}^V\frac{\delta}{\delta z^\beta}\rangle := f^{\bar{b}a}\partial_{\bar{b}}N^c_\alpha\partial_a\overline{N}^d_\beta f_{c\bar{d}}.$$

Proof. Let $(-\Psi)_{\alpha\bar{\beta}}$ denote the coefficient of $-\Psi$, i.e. $-\Psi = \sqrt{-1}(-\Psi)_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$, then

$$(-\Psi)_{\alpha\bar{\beta}} = f_{\alpha\bar{\beta}} - f_{\alpha\bar{d}}f^{\bar{d}c}f_{c\bar{\beta}}. \tag{2.39}$$

In fact, by the decomposition (1.20), one has

$$\begin{aligned}(-\Psi)_{\alpha\bar{\beta}} &= (-\sqrt{-1})(-\Psi)(\frac{\delta}{\delta z^\alpha},\frac{\delta}{\delta\bar{z}^\beta})\\ &= (\partial\bar{\partial}f)(\frac{\partial}{\partial z^\alpha}-f_{\alpha\bar{b}}f^{\bar{b}a}\frac{\partial}{\partial w^a},\frac{\partial}{\partial\bar{z}^\beta}-f_{\bar{\beta}a}f^{a\bar{b}}\frac{\partial}{\partial\bar{w}^b})\\ &= f_{\alpha\bar{\beta}}-f_{\alpha\bar{d}}f^{\bar{d}c}f_{c\bar{\beta}},\end{aligned}$$

which proves (2.39).

For any fixed point $(z,[v])\in P(E^*)|_z$, $z\in M$, we take normal coordinates near $(z,[v])$ such that $f_{a\bar{b}}(z,[v])=\delta_{ab}$, $f_{a\bar{b}c}(z,[v])=0$. Evaluating at $(z,[v])$ we see that

$$\begin{aligned}&f^{\bar{b}a}\frac{\partial^2}{\partial w^a\partial\bar{w}^b}(-\Psi)_{\alpha\bar{\beta}}\\ &= f^{\bar{b}a}\frac{\partial^2}{\partial w^a\partial\bar{w}^b}(f_{\alpha\bar{\beta}}-f_{\alpha\bar{d}}f^{\bar{d}c}f_{c\bar{\beta}})\\ &= f^{\bar{b}a}\left(f_{a\bar{b}\alpha\bar{\beta}}-f_{\alpha\bar{c}a\bar{b}}f_{c\bar{\beta}}-f_{\alpha\bar{c}}f_{c\bar{\beta}a\bar{b}}+f_{\alpha\bar{d}}f_{a\bar{b}d\bar{c}}f_{c\bar{\beta}}-f_{\alpha\bar{c}a}f_{c\bar{\beta}\bar{b}}-f_{\alpha\bar{c}\bar{b}}f_{c\bar{\beta}a}\right)\\ &= f^{\bar{b}a}(-\bar{\partial}(\partial f_{a\bar{d}}f^{\bar{d}c})f_{c\bar{b}})(\frac{\delta}{\delta z^\alpha},\frac{\delta}{\delta\bar{z}^\beta})-f^{\bar{b}a}\partial_{\bar{b}}(-f_{\alpha\bar{c}}f^{\bar{c}d})\partial_a(-f_{k\bar{\beta}}f^{k\bar{l}})f_{d\bar{l}}\\ &= \partial\bar{\partial}\log\det(f_{a\bar{b}})(\frac{\delta}{\delta z^\alpha},\frac{\delta}{\delta\bar{z}^\beta})-\langle\bar{\partial}^V\frac{\delta}{\delta z^\alpha},\bar{\partial}^V\frac{\delta}{\delta z^\beta}\rangle.\end{aligned}$$

which completes the proof. □

By Lemma 2.3 and (2.29), one has

$$
\begin{aligned}
(2.40)\qquad & \frac{1}{\epsilon}f^{a\bar{b}}\frac{\partial^2}{\partial w^a\partial\bar{w}^b}((-\Psi)_{\alpha\bar{\beta}}U_\epsilon^\alpha\overline{U}_\epsilon^\beta)\\
&=\frac{1}{\epsilon}f^{a\bar{b}}\Big(\partial_a\partial_{\bar{b}}(-\Psi)_{\alpha\bar{\beta}}u^\alpha\bar{u}^\beta+\partial_a(-\Psi)_{\alpha\bar{\beta}}\partial_{\bar{b}}\overline{U}_\epsilon^\beta u^\alpha\\
&\qquad+\partial_{\bar{b}}(-\Psi)_{\alpha\bar{\beta}}\partial_a U_\epsilon^\alpha\overline{u}^\beta+\partial_a U_\epsilon^\alpha\overline{\partial_b U_\epsilon^b}(-\Psi)_{\alpha\bar{\beta}}\Big)\\
&=\frac{1}{\epsilon}\partial\bar{\partial}\log\det(f_{a\bar{b}})(u,\bar{u})-\frac{1}{\epsilon}f^{a\bar{b}}f_{c\bar{d}}u^\alpha\overline{u}^\beta\partial_{\bar{b}}N_\alpha^c\partial_a\overline{N}_\beta^d\\
&\quad-2f^{a\bar{b}}\partial_a\Psi_{\alpha\bar{\beta}}\overline{\partial_b\Psi_{\gamma\bar{\tau}}}u^\alpha\overline{u}^\gamma g^{\tau\bar{\beta}}+O(\epsilon),
\end{aligned}
$$

where the last equality follows from

$$
\partial_a U_\epsilon^\alpha=-g_\epsilon^{\bar{\beta}\alpha}\partial_a g_{\epsilon\gamma\bar{\beta}}u^\gamma=\epsilon g_\epsilon^{\bar{\beta}\alpha}\partial_a\Psi_{\gamma\bar{\beta}}u^\gamma=O(\epsilon).
$$

Substituting (2.37) and (2.40) into (2.36), we obtain

$$
\begin{aligned}
(2.41)\qquad & \Delta_{\Omega_\epsilon}^V(\partial\bar{\partial}\log G(U_\epsilon,\overline{U}_\epsilon))\\
&=\frac{1}{\epsilon}\partial\bar{\partial}\log\det(f_{a\bar{b}})(u,\bar{u})-2\left|i_u\partial^V\Psi\right|_\Omega^2+O(\epsilon).
\end{aligned}
$$

Here we denote $\left|i_u\partial^V\Psi\right|_\Omega^2:=f^{a\bar{b}}\partial_a\Psi_{\alpha\bar{\beta}}\overline{\partial_b\Psi_{\gamma\bar{\tau}}}u^\alpha\overline{u}^\gamma g^{\tau\bar{\beta}}$.

For the second term in the RHS of (2.33), we have

$$
\begin{aligned}
(2.42)\qquad & \Omega_\epsilon^{A\bar{B}}\Omega_\epsilon^{F\bar{E}}R^\epsilon_{F\bar{B}C\bar{D}}f_{A\bar{E}}u^C\bar{u}^D\\
&\qquad=(-\Psi)_{\alpha\bar{\delta}}g_\epsilon^{\alpha\bar{\beta}}g_\epsilon^{\gamma\bar{\delta}}R^\epsilon_{\gamma\bar{\beta}\sigma\bar{\tau}}u^\sigma\bar{u}^\tau+\frac{1}{\epsilon^2}f^{a\bar{b}}R^\epsilon_{a\bar{b}\sigma\bar{\tau}}u^\sigma\bar{u}^\tau,
\end{aligned}
$$

where $R^\epsilon_{\gamma\bar{\beta}\sigma\bar{\tau}}=R^\epsilon(\frac{\delta}{\delta z^\gamma},\frac{\delta}{\delta\bar{z}^\beta},\frac{\delta}{\delta z^\sigma}\frac{\delta}{\delta\bar{z}^\tau})$ and $R^\epsilon_{a\bar{b}\sigma\bar{\tau}}=R^\epsilon(\frac{\partial}{\partial w^a},\frac{\partial}{\partial\bar{w}^b},\frac{\delta}{\delta z^\sigma},\frac{\delta}{\delta\bar{z}^\tau})$. By (2.7) and (2.10), one has

$$
\begin{aligned}
(2.43)\qquad R^\epsilon_{\gamma\bar{\beta}\sigma\bar{\tau}}&=R^\epsilon(\frac{\delta}{\delta z^\gamma},\frac{\delta}{\delta\bar{z}^\beta},\frac{\delta}{\delta z^\sigma},\frac{\delta}{\delta\bar{z}^\tau})\\
&=\langle\frac{\delta}{\delta z^\gamma},(\nabla^\epsilon_{\frac{\delta}{\delta z^\tau}}\nabla^\epsilon_{\frac{\delta}{\delta\bar{z}^\sigma}}-\nabla^\epsilon_{\frac{\delta}{\delta\bar{z}^\sigma}}\nabla^\epsilon_{\frac{\delta}{\delta z^\tau}}-\nabla^\epsilon_{[\frac{\delta}{\delta z^\tau},\frac{\delta}{\delta\bar{z}^\sigma}]})\frac{\delta}{\delta z^\beta}\rangle_\epsilon\\
&=\Big(\bar{\partial}(\partial g_{\epsilon\gamma\bar{\delta}}\cdot g_\epsilon^{\bar{\delta}\alpha})g_{\epsilon\alpha\bar{\beta}}\Big)(\frac{\delta}{\delta z^\sigma},\frac{\delta}{\delta\bar{z}^\tau})-\epsilon f_{c\bar{d}}\frac{\delta}{\delta z^\sigma}\overline{N}_\beta^d\frac{\delta}{\delta\bar{z}^\tau}N_\gamma^c\\
&=R^g_{\gamma\bar{\beta}\sigma\bar{\tau}}+O(\epsilon),
\end{aligned}
$$

where $R^g_{\gamma\bar\beta\sigma\bar\tau} := -\frac{\partial^2 g_{\gamma\bar\beta}}{\partial z^\sigma \partial \bar z^\tau} + g^{\bar\tau\alpha}\frac{\partial g_{\alpha\bar\beta}}{\partial \bar z^\tau}\frac{\partial g_{\gamma\bar\delta}}{\partial z^\sigma}$ denotes the Chern curvature of the Kähler metric $\omega = \sqrt{-1}g_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta$. And

$$
\begin{aligned}
(2.44)\quad \frac{1}{\epsilon^2}f^{a\bar b}R^\epsilon_{a\bar b\sigma\bar\tau}u^\sigma\overline{u}^\tau &= \frac{1}{\epsilon^2}f^{a\bar b}R^\epsilon(\frac{\partial}{\partial w^a},\frac{\partial}{\partial \bar w^b},\frac{\delta}{\delta z^\sigma},\frac{\delta}{\delta \bar z^\tau})u^\sigma\overline{u}^\tau\\
&= \frac{1}{\epsilon^2}f^{a\bar b}\langle\frac{\partial}{\partial w^a},(\nabla^\epsilon_{\frac{\delta}{\delta z^\tau}}\nabla^\epsilon_{\frac{\delta}{\delta \bar z^\sigma}} - \nabla^\epsilon_{\frac{\delta}{\delta \bar z^\sigma}}\nabla^\epsilon_{\frac{\delta}{\delta z^\tau}} - \nabla^\epsilon_{[\frac{\delta}{\delta z^\tau},\frac{\delta}{\delta \bar z^\sigma}]})\frac{\partial}{\partial w^b}\rangle_\epsilon u^\sigma\overline{u}^\tau\\
&= \frac{1}{\epsilon}f^{a\bar b}\left(\bar\partial(\partial f_{a\bar d}\cdot f^{\bar d c})f_{c\bar b}\right)(u,\overline{u}) + \frac{\delta}{\delta z^\sigma}\overline{N}^b_\beta\frac{\delta}{\delta \bar z^\tau}N^a_\alpha f_{a\bar b}g^{\bar\beta\alpha}_\epsilon u^\sigma\overline{u}^\tau\\
&= -\frac{1}{\epsilon}\partial\bar\partial\log\det(f_{a\bar b})(u,\overline{u}) + \left|i_u\partial^V\Psi\right|^2_\Omega + O(\epsilon),
\end{aligned}
$$

where the last equality follows from the following equalities:

$$
\begin{aligned}
\frac{\delta}{\delta\bar z^\gamma}N^a_\alpha f_{a\bar b} &= (\partial_{\bar\gamma} - f_{c\bar\gamma}f^{c\bar e}\partial_{\bar e})(f_{\alpha\bar d}f^{\bar d a})f_{a\bar b}\\
&= f_{a\bar b\bar\gamma} - f_{\alpha\bar d}f^{\bar d a}f_{a\bar b\bar\gamma} - f_{c\bar\gamma}f^{c\bar e}f_{\bar e\alpha\bar b} + f_{c\bar\gamma}f^{c\bar e}f_{\alpha\bar d}f^{\bar d a}f_{a\bar b\bar e}\\
&= \partial_{\bar b}(f_{\alpha\bar\gamma} - f_{\alpha\bar b}f^{\bar b a}f_{a\bar\gamma})\\
&= \partial_{\bar b}(-\Psi)_{\alpha\bar\gamma}.
\end{aligned}
$$

Substituting (2.43) and (2.44) into (2.42), we have

$$
\begin{aligned}
(2.45)\quad \Omega^{A\bar B}_\epsilon\Omega^{F\bar E}_\epsilon R^\epsilon_{F\bar BC\bar D}f_{A\bar E}u^C\overline{u}^D &= -\frac{1}{\epsilon}\partial\bar\partial\log\det(f_{a\bar b})(u,\overline{u})\\
&\quad + \left|i_u\partial^V\Psi\right|^2_\Omega + \langle R^g(u,\overline{u}),-\Psi\rangle_\Omega + O(\epsilon).
\end{aligned}
$$

Here we denote $\langle\sqrt{-1}R^g(u,\overline{u}),-\Psi\rangle_\Omega = (-\Psi)_{\alpha\bar\delta}g^{\alpha\bar\beta}g^{\gamma\bar\delta}R^g_{\gamma\bar\beta\sigma\bar\tau}u^\sigma\overline{u}^\tau$.

Substituting (2.35), (2.41) and (2.45) into (2.33), we obtain

$$
\begin{aligned}
(2.46)\quad \frac{\partial}{\partial t}\left(\partial\bar\partial\log G(U,\overline{U})\right) &= \Delta^H_\Omega(\partial\bar\partial\log G(U,\overline{U}))\\
&\quad + \langle\sqrt{-1}R^g(u,\overline{u}),-\Psi\rangle_\Omega - \left|i_u\partial^V\Psi\right|^2_\Omega.
\end{aligned}
$$

at the point $(z_0,[v_0],t_0)$, which completes the proof of Proposition 2.2.

Now we define a horizontal and real $(1,1)$-form T as follow,

$$
(2.47)\qquad (-\sqrt{-1})T(X,\overline{X}) := \langle\sqrt{-1}R^g(X,\overline{X}),-\Psi\rangle_\Omega - \left|i_X\partial^V\Psi\right|^2_\Omega
$$

for any horizontal vector $X = X^\alpha\frac{\delta}{\delta z^\alpha}$. And we assume that T satisfies the null eigenvector assumption (see Theorem 1.8), by Theorem 1.8, we obtain

Theorem 2.4. *Let $\pi : (E^*, G_0) \to M$ be a holomorphic Finsler vector bundle over M with $\sqrt{-1}\partial\bar{\partial}\log G_0 \geq 0$. Consider the following flow over the projective bundle $p : P(E^*) \to M$:*

$$\begin{cases} \frac{\partial}{\partial t}\log G = \Delta_\Omega \log G, \\ \omega_{FS}(G) > 0, \\ G(0) = G_0, \end{cases} \tag{2.48}$$

where $\Omega = \omega(G) + \omega_{FS}(G)$, $\omega(G) = p^*\omega$, ω is a Kähler metric on M depending on the Finsler metric G. If the horizontal $(1,1)$-form T satisfies the null eigenvector assumption, then

$$\sqrt{-1}\partial\bar{\partial}\log G(t) \geq 0$$

for all $t \geq 0$ such that the solution exists.

3. Applications

This section will give two applications of Theorem 2.4.

3.1. The case of curve

In this subsection, we consider the case of $\dim M = 1$, i.e., M is a curve. In this case, any Hermitian metric

$$\omega = \sqrt{-1}g dz \wedge d\bar{z}$$

on M is Kähler automatically. The Gaussian curvature is then given by

$$K = -\frac{1}{g}\frac{\partial^2}{\partial z \partial \bar{z}}\log g = \frac{1}{g^2}R^g(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}) =: \frac{1}{g^2}R^g_{z\bar{z}z\bar{z}}. \tag{3.1}$$

Now we assume that

$$\Omega = p^*\omega + \omega_{FS}(G), \tag{3.2}$$

where $\omega = \omega(G)$ is a metric on M depending smoothly on the Finsler metric G. Then at the point $(z_0, [v_0], t_0)$, by (2.30), one has

(3.3)
$$\begin{aligned}(-\sqrt{-1})T(u,\overline{u}) &= \langle\sqrt{-1}R^g(u,\overline{u}), -\Psi\rangle_\Omega - \left|i_u\partial^V\Psi\right|^2_\Omega \\ &= (-\Psi)_{z\bar{z}}g^{-2}R^g_{z\bar{z}z\bar{z}}|u|^2 - f^{a\bar{b}}\partial_a\Psi_{z\bar{z}}\overline{\partial_b\Psi_{z\bar{z}}}|u|^2g^{-1} \\ &= K\cdot\sqrt{-1}\Psi(u,\overline{u}) \\ &\quad - f^{a\bar{b}}\left(\partial_a(-\sqrt{-1}\Psi(U,\overline{U})) + \sqrt{-1}\Psi(\partial_aU,u)\right)\overline{\partial_b\Psi_{z\bar{z}}}g^{-1} \\ &= 0,\end{aligned}$$

since $i_u\Psi = 0$ and $\sqrt{-1}\Psi(U,\overline{U})$ attains its local minimal value at the point $(z_0, [v_0], t_0)$. Therefore, we prove

Proposition 3.1. *If M is a curve, then the semi-positivity of the curvature of $\mathcal{O}_{P(E^*)}(1)$ is preserved along the flow (2.48).*

In particular, if $G_0 = h_0^{i\bar{j}}v_i\bar{v}_j$ comes from a Hermitian metric $(h_0^{i\bar{j}})$ of E^* and

$$\Omega = p^*\omega + \omega_{FS}(G) \tag{3.4}$$

for a fixed Hermitian metric ω, by Remark 2.1 (1), (2.48) is equivalent to the following Hermitian-Yang-Mills flow:

$$\begin{cases} h^{-1}\cdot\frac{\partial h}{\partial t} + \Lambda R^h + (r-1)I = 0 \\ (h_{i\bar{j}}(t)) > 0, \\ h_{i\bar{j}}(0) = (h_0)_{i\bar{j}}. \end{cases} \tag{3.5}$$

By Proposition 1.7 and Proposition 3.1, we have

Corollary 3.2. *If M is a curve, the Griffiths semi-positivity is preserved along the Hermitian-Yang-Mills flow (3.5).*

3.2. Kähler-Ricci flow

In this section, we assume that $E = TM$. As the discussion in Remark 2.1 (2), if we take

$$\omega(G) = \sqrt{-1}g_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta,$$

where $(g_{\alpha\bar{\beta}})$ denotes the inverse of the matrix $\left(\frac{\partial^2 G}{\partial v_\alpha \partial \overline{v}_\beta}\right)$. And $G_0 = g_0^{\alpha\bar{\beta}} v_\alpha \bar{v}_\beta$ is a strongly pseudoconvex complex Finsler metric on T^*M induced by the following Kähler metric

$$\omega_0 = \sqrt{-1}(g_0)_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

By (2.5), the flow (2.48) is equivalent to the following Kähler-Ricci flow

$$\text{(3.6)} \qquad \begin{cases} \frac{\partial \omega}{\partial t} + \mathrm{Ric}(\omega) + (n-1)\omega = 0, \\ \omega > 0, \\ \omega(0) = \omega_0. \end{cases}$$

The solution of (2.48) is induced from the Kähler metric $\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$. In this case,

$$\begin{aligned} \text{(3.7)} \qquad \langle \sqrt{-1} R^g(u,\overline{u}), -\Psi\rangle_\Omega &= (-\Psi)_{\alpha\bar{\delta}} g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R^g_{\gamma\bar{\beta}\sigma\bar{\tau}} u^\sigma \bar{u}^\tau \\ &= \frac{1}{G} R^g_{\mu\bar{\nu}\alpha\bar{\delta}} g^{\mu\bar{\sigma}} g^{\bar{\nu}\tau} \overline{v_\sigma} v_\tau g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R^g_{\gamma\bar{\beta}\sigma\bar{\tau}} u^\sigma \bar{u}^\tau \\ &= \sum_{\alpha,\beta=1}^{\dim M} R^g(V,\overline{V},e_\alpha,\overline{e_\beta}) R^g(e_\beta,\overline{e_\alpha},u,\overline{u}), \end{aligned}$$

where $V = \frac{1}{\sqrt{G}} g^{\alpha\bar{\sigma}} \overline{v_\sigma} \frac{\partial}{\partial z^\alpha}$ and $\{e_\alpha\}$ is a local orthonormal basis of (TM,ω). On the other hand, by (2.25), one has at the point $(z_0,[v_0],t_0)$,

$$\begin{aligned} \text{(3.8)} \qquad \left|i_u \partial^V \Psi\right|^2_\Omega &= f^{a\bar{b}} \partial_a \Psi_{\alpha\bar{\beta}} \overline{\partial_b \Psi_{\gamma\bar{\tau}}} u^\alpha \overline{u}^\gamma g^{\tau\bar{\beta}} \\ &= \frac{1}{G} (R^g)^{\mu\bar{\sigma}}{}_{\alpha\bar{\beta}} \overline{v_\sigma} u^\alpha (R^g)^{\delta\bar{\nu}}{}_{\tau\bar{\gamma}} v_\delta \overline{u}^\gamma g_{\mu\bar{\nu}} g^{\tau\bar{\beta}} \\ &= \sum_{\alpha,\beta=1}^{\dim M} \left|R^g(V,\overline{e_\alpha},u,\overline{e_\beta})\right|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{(3.9)} \quad & (-\sqrt{-1}) T(u,\overline{u}) \\ & = \sum_{\alpha,\beta=1}^{\dim M} \left(R^g(V,\overline{V},e_\alpha,\overline{e_\beta}) R^g(e_\beta,\overline{e_\alpha},u,\overline{u}) - \left|R^g(V,\overline{e_\alpha},u,\overline{e_\beta})\right|^2 \right) \geq 0 \end{aligned}$$

by [5, Page 254, Claim 2.2]. From Proposition 1.7 and Theorem 2.4, we can reprove the following Mok's proposition, which is contained in [24, Proposition 1.1] (see also [5, Theorem 5.2.10]).

Proposition 3.3 ([24, Proposition 1.1]). *If (M, ω_0) is a compact Kähler manifold with nonnegative holomorphic bisectional curvature, then the nonnegativity is preserved along the Kähler-Ricci flow (3.6).*

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