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Rectifiability and Minkowski bounds for the zero loci of $\mathbb{Z}/2$ harmonic spinors in dimension 4

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This article proves that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 4 dimensional manifold is 2-rectifiable and has locally finite Minkowski content.

1. Introduction

1.1. Background

The notion of $\mathbb{Z}/2$ harmonic spinors was first introduced by Taubes [12, 14] to describe the behaviour of certain non-convergent sequences of flat $PSL_2(\mathbb{C})$ connections on a three manifold. It also appears in the compactifications of the moduli spaces of solutions to Kapustin-Witten equations [13], Vafa-Witten equations [16], and Seiberg-Witten equations with multiple spinors [8, 15]. These equations may have important topological applications. For example, Witten [17] has conjectured that the space of solutions to the Kapustin-Witten equations can be used to compute the Jones polynomials and the Khovanov homology for knots. Haydys [7] conjectured a relation between the multiple spinor Seiberg-Witten monopoles, Fueter sections, and $G2$ instantons. More recently, Doan and Walpuski [6] conjectured a relation between generalized Seiberg-Witten equations and counting of associative manifolds on $G2$ manifolds.

All of these applications require a better understanding of the compactifications for the relevant moduli spaces. The zero locus of $\mathbb{Z}/2$ harmonic spinor plays a crucial role in the description of the boundaries of the compactifications. It is the set of points where the sequence of solutions blow up after normalizations. Takahashi [10, 11] studied the moduli spaces of $\mathbb{Z}/2$ harmonic spinors with additional regularity assumptions on the zero locus, where the zero locus was assumed to be a union of embedded circles in the case of dimension 3, and an embedded surface in the case of dimension 4. In general, the zero locus may not have this regularity. Taubes [14] proved that the zero locus must have Hausdorff codimension at least 2. This article improves the regularity result by proving that the zero locus is rectifiable and has locally finite Minkowski content. The arguments are inspired by [4], where a similar problem was studied for Dir-minimizing Q-valued functions. The proof relies on a general method developed recently by Naber and Valtorta [9].

1.2. Statement of results

Let X be a 4-dimensional Riemannian manifold. Let $\mathcal{V}$ be a Clifford bundle over X. That is, $\mathcal{V}$ is a unitary vector bundle equipped with an extra structure $\rho \in \mathrm{Hom}(TX, \mathrm{Hom}(\mathcal{V}, \mathcal{V}))$, such that $\rho(e)^2 = -\|e\|^2 \cdot \mathrm{id}$ and $\|\rho(e)(u)\| = \|e\| \cdot \|u\|$ for every $e \in T_pX$ and $u \in \mathcal{V}|_p$. Let ∇ be a connection on V that is compatible with $(X, \mathcal{V}, \rho)$. Namely, for every pair of smooth vector fields e, e', and every smooth section u of $\mathcal{V}$, one has

$$\nabla_e(\rho(e')\cdot u) = \rho(\nabla_e e')\cdot u + \rho(e')\cdot \nabla_e(u).$$

The Dirac operator on $\mathcal{V}$ is defined by

$$D(u) = \sum_{i=1}^{4} \rho(e_i)\nabla_{e_i} u,$$

where $\{e_i\}$ is a local orthonormal frame for TX.

Let Q be a positive integer. For a vector space E, define $\mathcal{A}_Q(E)$ to be the set of unordered Q-tuples of points in E. If $P_1, P_2, \cdots, P_Q$ are Q points in E, use $\sum_{i=1}^{Q} [\![P_i]\!] \in \mathcal{A}_Q(E)$ to denote the Q-tuple given by the collection of P_i's. If E is endowed with a Euclidean metric, one can define a metric on $\mathcal{A}_Q(E)$ by

$$\operatorname{dist}\Big(\sum_i [\![P_i]\!], \sum_i [\![S_i]\!]\Big) = \min_{\sigma\in\mathcal{P}_Q} \sqrt{\sum_i |P_i - S_{\sigma(i)}|^2},$$

where $\mathcal{P}_Q$ is the permutation group of $\{1, 2, \cdots, Q\}$. If $T \in \mathcal{A}_Q(E)$, define $|T| = \operatorname{dist}(T, Q[\![0]\!])$.

A map from X is called a Q-valued section of $\mathcal{V}$ if it maps every $x \in X$ to an element of $\mathcal{A}_Q(\mathcal{V}|_x)$. A Q-valued section is called continuous if it is continuous under local trivializations of $\mathcal{V}$.

Definition 1.1. *Let U be a continuous 2-valued section of $\mathcal{V}$. Then U is called a $\mathbb{Z}/2$ harmonic spinor if the following conditions hold.*

1) *U is not identically $2[\![0]\!]$.*
2) *Let Z be the set of U where $U = 2[\![0]\!]$. For every $x \in X - Z$, there exists a neighborhood of x, such that on this neighborhood U can be written as $U = [\![u]\!] + [\![-u]\!]$, where u is a smooth section of $\mathcal{V}$ satisfying $D(u) = 0$.*
3) *Near a point $x \in X - Z$, write U as $[\![u]\!] + [\![-u]\!]$, then the function $|\nabla u|$ is a well defined smooth function on $X - Z$. The section U satisfies*

$$\int_{X-Z} |\nabla u|^2 < \infty.$$

This definition is equivalent to the definition of $\mathbb{Z}/2$ harmonic spinors given in [14].

For $x \in X$ and $r > 0$, let $B_x(r)$ be the set of points on X whose distance to x is less than or equal to r. As in (1.5) of [14], we make the following additional assumption on U.

Assumption 1.2. *There exits a constant $\epsilon > 0$ such that the following holds. For every $x \in X$ with $U(x) = 2[\![0]\!]$, there exist constants $C, r_0 > 0$, depending on x, such that*

$$\int_{B_x(r)} |U(y)|^2 \, dy < C \cdot r^{4+\epsilon}, \quad \textit{for every } r \in (0, r_0).$$

Assumption 1.2 is necessary for the integration-by-parts arguments in the proof of [14, Lemma 2.3], which is essential for most of the estimates developed in this article. In all the known cases [8, 13, 15, 16], the $\mathbb{Z}/2$ harmonic spinors that arised from the study of gauge-theoretic equations satisfy this assumption.

Assume U is a $\mathbb{Z}/2$ harmonic spinor, and let Z be the set of U where $U = 2[\![0]\!]$. Taubes [14] proved the following theorem.

Theorem 1.3 (Taubes [14]). *If U satisfies Assumption 1.2, then the Hausdorff dimension of Z is at most 2.*

This article improves Theorem 1.3 to the following result.

Theorem 1.4. *If U satisfies Assumption 1.2, then Z is a 2-rectifiable set. Moreover, for every compact subset $A \subset X$, there exist constants C and r_0 depending on A and Z, such that for every $r < r_0$,*

$$\textit{Vol}\,(\{x : \textit{dist}(x, A \cap Z) < r\}) < C \cdot r^2.$$

In other words, Z is a 2-rectifiable set with locally finite 2 dimensional Minkowski content. Since the Minkowski content controls the Hausdorff measure, Theorem 1.4 implies that Z has locally finite 2 dimensional Hausdorff measure.

Taubes [14] also defined and studied the zero loci of $\mathbb{Z}/2$ harmonic spinors on three and two dimensional manifolds. Since every $\mathbb{Z}/2$ harmonic spinor on a 3-manifold Y with zero locus Z induces an $\mathbb{R}$–invariant $\mathbb{Z}/2$ harmonic spinor on $\mathbb{R} \times Y$ with zero locus $\mathbb{R} \times Z$, Theorem 1.4 implies that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 3-manifold is 1-rectifiable and has locally finite Minkowski content. Similarly, a further dimension reduction argument implies that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 2-manifold is a locally finite set of points, which is already proved in [14, Section 5(a)].

Acknowledgments

This project originated from an idea of Clifford Taubes that one should be able to apply the techniques for Dir-minimizing Q-valued functions to the study of $\mathbb{Z}/2$ harmonic spinors. I would like to express my most sincere gratitude for his insightful guidance and encouragement. I also want to thank Thomas Walpuski and Aaron Naber for many helpful discussions and correspondences.

2. $\mathbb{Z}/2$ harmonic spinors as Sobolev sections

Almgren [2] developed a Sobolev theory for Q-valued functions on $\mathbb{R}^m$. For a quicker introduction, one can see for example [5]. For an open set $\Omega \subset \mathbb{R}^m$, the space $W^{1,2}(\Omega, \mathcal{A}_Q)$ is defined to be the space of Q valued functions T on Ω, such that $|T| \in L^2(\Omega)$, and that T has distributional derivatives which are also in $L^2(\Omega)$. The Sobolev theory extends to Q-valued sections of vector bundles without any difficulty. This section proves the following lemma.

Lemma 2.1. *If U is a $\mathbb{Z}/2$ harmonic spinor, then U is in $W^{1,2}(X, \mathcal{A}_2)$. Moreover, $D(U) = 0$ in the distributional sense.*

This lemma allows us to study the compactness properties of $\mathbb{Z}/2$ harmonic spinors by the Sobolev theory for Q-valued functions.

We start with the following definition.

Definition 2.2. *Let T be a Q-valued section of $\mathcal{V}$. It is called a smooth Q-valued section, if for every $x \in X$, there exists a neighborhood of x on which T can be written as*

$$T = \sum_{i=1}^{Q} [\![f_i]\!],$$

where f_i's are smooth sections of $\mathcal{V}$.

If T is a smooth Q-valued section and is locally written as $\sum_i [\![f_i]\!]$, then the function $\sum_i |f_i|^2 + \sum_i |\nabla f_i|^2$ is well defined on X. In this case, the $W^{1,2}$ norm of T is given by $(\int_X \sum_i |f_i|^2 + \sum_i |\nabla f_i|^2)^{1/2}$.

Proof of Lemma 2.1. The proof is essentially the same as Lemma 2.4 of [14].

Let χ be a smooth non-increasing function on $\mathbb{R}$, such that $\chi(t) = 1$ when $t \le 1$, and $\chi(t) = 0$ when $t \ge 2$. For $s > 0$, let $\tau_s = \chi(\ln |U| / \ln s)$. Then $\tau_s(x) = 0$ when $|U(x)| \le s^2$, and $\tau_s(x) = 1$ when $|U(x)| \ge s$.

The section $\tau_s U$ is a 2-valued smooth section of $\mathcal{V}$. Recall that on $X - Z$, the $\mathbb{Z}/2$ harmonic spinor U can be locally written as $U = [\![u]\!] + [\![-u]\!]$. Although u is only defined up to a sign, the functions $|u|$ and $|\tau_s \nabla u + \nabla \tau_s \cdot u|$ are well defined on $X - Z$. Thus the $W^{1,2}$ norm of $\tau_s U$ is given by

$$\|\tau_s U\|_{W^{1,2}} = \sqrt{2} \int_X (|\tau_s|^2 |u|^2 + |\tau_s \nabla u + \nabla \tau_s \cdot u|^2).$$

Notice that

$$|\nabla \tau_s| \cdot |u| \leq \frac{1}{|\ln s|} (\sup |\chi'|) \cdot |\nabla u|,$$

hence its L^2 norm converges to zero as $s \to 0$. Therefore,

$$\lim_{s \to 0} \|\tau_s U\|_{W^{1,2}} = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2). \tag{1}$$

In particular, $\tau_s U$ is bounded in $W^{1,2}$ as $s \to 0$, thus a subsequence of it weakly converges in $W^{1,2}$ to an element $U' \in W^{1,2}$. Since $\tau_s U$ also uniformly converges to U, one must have $U' = U$. Therefore $U \in W^{1,2}$.

Since D is a smooth first-order differential operator, $D(U) \in L^2_{loc}(X)$. By the definition of $\mathbb{Z}/2$ harmonic spinors, $D(U) = 0$ on $X - Z$. By section 2.2.1 of [5], the derivatives of U are zero at the Lebesgue points of Z, hence $D(U) = 0$ on those points. That proves $D(U) = 0$ in the distributional sense. □

The argument of Lemma 2.1 also shows that U can be $W^{1,2}$ approximated by smooth sections. We write it as a separate lemma for later reference.

Lemma 2.3. *Let U be a $\mathbb{Z}/2$ harmonic spinor. Then there exits a sequence of smooth sections U_i, such that $U_i = -U_i$, and*

$$\lim_{i \to \infty} U_i = U \text{ in } W^{1,2}.$$

Proof. Since $|U|$ and $|\nabla U|$ are zero on the Lebesgue points of Z, one has

$$\|U\|_{W^{1,2}} = \int_{X-Z} (|U|^2 + |\nabla U|^2) = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2).$$

Define τ_s as in the proof of Lemma 2.1. It was proved previously that there is a sequence $s_i \to 0$, such that $\tau_{s_i} U$ converges weakly to U in $W^{1,2}$.

As a consequence,

$$\liminf_{i\to\infty} \|\tau_{s_i} U\|_{W^{1,2}} \geq \|U\|_{W^{1,2}}$$

On the other hand, by (1),

$$\lim_{i\to\infty} \|\tau_{s_i} U\|_{W^{1,2}} = \sqrt{2}\int_{X-Z} (|u|^2 + |\nabla u|^2) = \|U\|_{W^{1,2}}.$$

Therefore $\tau_{s_i} U$ converges strongly to U in $W^{1,2}$. □

3. Frequency functions

The frequency functions were first introduced by Amgren [1] to study the singular set of elliptic partial differential equations, and they were adapted by Taubes [14] to study the zero loci of $\mathbb{Z}/2$ harmonic spinors. This section recalls some results about the frequency functions from [14].

Let U be a $\mathbb{Z}/2$ harmonic spinor. On $X - Z$ the section U can be locally written as $U = [\![u]\!] + [\![-u]\!]$. As before, we will use notations like $|u|$ and $|\nabla u|$ to denote the corresponding functions on $X - Z$ if they can be globally defined. The functions $|u|$ and $|\nabla u|$ extend to X by defining them to be zero on Z.

The following C^0 estimate was established in [14].

Lemma 3.1 ([14], Lemma 2.3). *Let $A \subset B$ be two open subsets of X, and assume the closure of A is compact and contained in B. Then there exists a constant K, depending on A, B and the norms of the curvatures of X and $\mathcal{V}$, such that*

$$\sup_{x\in A} |u(x)|^2 \leq K \int_B |u(x)|^2\, dx.$$

Now introduce some notations. Fix a point $x_0 \in X$. Take $R > 0$ such that the exponential map of X at x_0 is well-defined on the closed ball with radius $1500R$, and that the injectivity radius of X is greater than $1000R$ for every point in $B_{x_0}(500R)$.

Later on we will need to work on both the Euclidean space and the manifold X, so we need to differentiate the notations. We will use $B_x(r)$ to denote the geodesic ball on X with center $x \in X$ and radius $r > 0$. Use $\bar{B}_x(r)$ to denote the Euclidean ball with center x in the Euclidean space and radius $r > 0$. When the center is the origin, $\bar{B}_0(r)$ is also denoted by $\bar{B}(r)$. Use $d(x, y)$ to denote the distance function on X, and use $|x - y|$ to denote the distance function on $\mathbb{R}^4$.

For every $x \in B_{x_0}(500R)$, use the normal coordinate centered at x to identify $B_x(500R)$ with the ball $\bar{B}(500R) \subset \mathbb{R}^4$. Let g_x be the function of metric matrices on $\bar{B}(500R)$ corresponding to $B_x(500R)$. For each $z \in \bar{B}(500R)$, let $K_x(z), \kappa_x(z)$ be the largest and smallest eigenvalue of $g_x(z)$. Assume that R is sufficiently small so that for every $x \in B_{x_0}(500R)$, $z \in \bar{B}(500R)$,

$$(2) \qquad \left(\frac{11}{12}\right)^2 \le \kappa_x(z) \le K_x(z) \le \left(\frac{12}{11}\right)^2$$

In order to prove Theorem 1.4, one only needs to study the rectifiability and the Minkowski content of $Z \cap B_{x_0}(R/2)$.

For $x \in B_{x_0}(500R)$, $r \in (0, 500R]$, define the height function

$$H(x,r) = \int_{\partial B_x(r)} |u|^2,$$

then $H(x,r)$ is always positive [14, Lemma 3.1]. Define

$$D(x,r) = \int_{B_x(r)} |\nabla u|^2,$$

and define the frequency function

$$N(x,r) = \frac{rD(x,r)}{H(x,r)}.$$

Section 3(a) of [14] proved the following monotonicity properties for N and H:

Lemma 3.2 ([14], (3.6) and Lemma 3.2). *The functions N and H are absolutely continuous with respect to r, and there exist constants $\kappa > 0$ and $r_0 > 0$, depending only on the norms of curvatures of X and $\mathcal{V}$ on $B_{x_0}(1000R)$, such that when $r \le r_0$,*

$$(3) \qquad \frac{\partial}{\partial r} H \ge \frac{3}{r} H - \kappa r H,$$

$$(4) \qquad \frac{\partial}{\partial r} N \ge -\kappa r(1+N).$$

$$(5) \qquad \left(\frac{N}{r} + \kappa r\right) \frac{H}{r^3} \ge \frac{\partial}{\partial r}\left(\frac{H}{r^3}\right) \ge \left(\frac{N}{r} - \kappa r\right) \frac{H}{r^3}$$

By shrinking the size of R, we assume without loss of generality that $r_0 = 500R$, hence inequalities (3), (4), and (5) hold for all $x \in B_{x_0}(500R)$ and $r \le 500R$.

Inequality (3) gives the following lemma

Lemma 3.3 ([14], Lemma 3.1). *There exists a constant $\kappa > 0$, such that when $s < r < 500R$,*

$$H(x,r) \geq \big(\frac{r}{s}\big)^3 \cdot e^{-\kappa(r^2-s^2)} \cdot H(x,s).$$

Inequality (4) gives

Lemma 3.4. *There exists a constant $\kappa > 0$, such that when $s < r < 500R$,*

$$N(x,r) \geq e^{-\kappa(r^2-s^2)} N(x,s) - \kappa(r^2 - s^2).$$

Since $N(x, 500R)$ is continuous with respect to x, Lemma 3.4 implies that $N(x,r)$ is bounded for all $x \in B_{x_0}(500R)$, $r \leq 500R$. Let Λ be an upper bound for N. From now on Λ will be treated as a constant. For the rest of this article, unless otherwise stated, C, C_1, C_2, $\cdots$ will denote positive constants that depend on Λ, R, and the norms of the curvatures of X and $\mathcal{V}$, but independent of U. The values of C, C_1, C_2, $\cdots$ may be different in different appearances.

If $|g| \leq C \cdot f$ for some constant C, we write $g = O(f)$.

Inequality (5) then implies that there exists a constant C such that

$$\Big|\frac{\partial}{\partial r}\big(\ln(\frac{H}{r^3})\big)\Big| = O(\frac{1}{r}). \tag{6}$$

Inequality (4) implies that there exists $C > 0$, such that whenever $r \geq s$,

$$N(x,r) \geq N(x,s) - C(r^2 - s^2).$$

4. Smoothed frequency functions

We need to use a modified version of frequency functions. Let ϕ be a non-increasing smooth function on $\mathbb{R}$ such that $\phi(t) = 1$ when $t \leq 3/4$, and $\phi(t) = 0$ when $t \geq 1$. From now on ϕ will be fixed, hence the values of ϕ and its derivatives are considered as universal constants. Following [4], we define the smoothed frequency functions as follows.

Definition 4.1. *For $x \in X$, let ν_x be the gradient vector field of the distance function $d(x, \cdot)$. For $x \in B_{x_0}(500R)$, $r \leq 500R$, introduce the following* *functions*

$$D_\phi(x,r) = \int |\nabla u(y)|^2 \phi\Big(\frac{d(x,y)}{r}\Big)\,dy,$$
$$H_\phi(x,r) = -\int |u(y)|^2 d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy,$$
$$N_\phi(x,r) = \frac{rD_\phi(x,r)}{H_\phi(x,r)},$$
$$E_\phi(x,r) = -\int |\nabla_{\nu_x} u(y)|^2 d(x,y)\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy.$$

Inequality (6) has the following useful corollary.

Lemma 4.2. *There exists a constant C with the following property. Let $r \in (0, 32R]$. Assume $s_1 \le 10r$, $s_2 \ge r/10$. Then for any two points x, y with $d(x,y) \le r$, one has*

$$H_\phi(x,s_1) \le C(H_\phi(y,s_2)).$$

Proof. Since the constant K in Lemma 3.1 only depends on the norms of the curvatures and the sets A, B, a rescaling argument gives

$$|u(z)|^2 \le \frac{C_1}{r^4}\int_{B_z(r)} |u|^2, \quad \forall B_z(r) \subset B_{x_0}(500R).$$

Therefore for every $z \in \partial B_x(s_1)$,

$$|u(z)|^2 \le \frac{C_2}{r^4}\int_{B_y(12r)} |u|^2.$$

On the other hand, inequality (6) and Lemma 3.3 gives

$$\frac{1}{r^4}\int_{B_y(12r)} |u|^2 \le \frac{C_3}{r^3} H(y,s_2).$$

Therefore

$$H(x,s_1) = O(H(y,s_2)).$$

Apply (6) again, one obtains

$$H(y,s_2) = O(H_\phi(y,s_2)),$$
$$H_\phi(x,s_1) = O(H(x,s_1)),$$

hence the lemma is proved. □

Lemma 4.3. *For $x \in B_{x_0}(32R)$, $r \le 32R$, one has*

$$\int_{B_x(r)} |u(y)|^2 dy = O(rH_\phi(x,r)),$$

$$\int_{B_x(r)} |u(y)||\nabla u(y)| dy = O(H_\phi(x,r)),$$

$$\int_{B_x(r)} |\nabla u(y)|^2 dy = O(\frac{1}{r} H_\phi(x,r)).$$

Proof. The first equation follows from inequality (6) and Lemma 3.3. For the third,

$$\begin{aligned}\int_{B_x(r)} |\nabla u(y)|^2 dy &\le D_\phi(x,2r) \\ &= \frac{1}{2r} N_\phi(x,2r) H_\phi(x,2r) \\ &= O(\frac{1}{r} H_\phi(x,r)).\end{aligned}$$

The second equation then follows from Cauchy's inequality. □

The main result of this section is the following proposition.

Proposition 4.4. *The functions D_ϕ, H_ϕ, N_ϕ, and E_ϕ are smooth in both variables. Assume $x \in B_{x_0}(32R)$, $r \le 32R$, and $v \in T_x(X)$. Consider the normal coordinate centered at x with radius r, extend the vector v to a vector field on $B_x(r)$ by requiring that the coordinate functions of v are constants. Then the following equations hold*

$$D_\phi(x,r) = -\frac{1}{r} \int \phi'\Big(\frac{d(x,y)}{r}\Big) \nabla_{\nu_x} u(y) \cdot u(y)\, dy + O(rH_\phi(x,r)), \tag{7}$$

$$\partial_r D_\phi(x,r) = \frac{2}{r} D_\phi(x,r) + \frac{2}{r^2} E_\phi(x,r) + O(H_\phi(x,r)), \tag{8}$$

$$\partial_v D_\phi(x,r) = -\frac{2}{r} \int \phi'\Big(\frac{d(x,y)}{r}\Big) \nabla_{\nu_x} u(y) \cdot \nabla_v u(y)\, dy + O(H_\phi(x,r)), \tag{9}$$

$$\partial_r H_\phi(x,r) = \frac{3}{r} H_\phi(x,r) + 2D_\phi(x,r) + O(rH_\phi(x,r)), \tag{10}$$

$$\begin{aligned}\partial_v H_\phi(x,r) = &-2 \int u(y) \cdot \nabla_v u(y)\, d(x,y)^{-1} \phi'\Big(\frac{d(x,y)}{r}\Big)\, dy \\ &+ O(rH_\phi(x,r)).\end{aligned} \tag{11}$$

The smoothness of the functions follows from the fact that ϕ is smooth and $|u|$, $|\nabla u|$ are both in L^2.

Proof of (7). It was proved in [14, Section 2(c)] that

$$\int_{\partial B_x(s)} \nabla_{\nu_x} u(y)\cdot u(y)\,dy = \int_{B_x(s)} |\nabla u(y)|^2\,dy + \int_{B_x(s)} \langle u(y), \mathcal{R}u(y)\rangle\,dy, \tag{12}$$

where $\mathcal{R}$ is a bounded curvature term from the Weitzenböck formula.

Therefore, by Lemma 4.3,

$$\begin{aligned} D_\phi(x,r) &= -\frac{1}{r}\int_0^r \phi'\Big(\frac{s}{r}\Big)\int_{B_x(s)} |\nabla u(y)|^2\,dy\,ds \\ &= -\frac{1}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x} u(y)\cdot u(y)\,dy \\ &\quad + \frac{1}{r}\int_0^r \phi'\Big(\frac{s}{r}\Big)\int_{B_x(s)} \langle u, \mathcal{R}u\rangle\,dy\,ds \\ &= -\frac{1}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x} u(y)\cdot u(y)\,dy + O(rH_\phi(x,r)). \end{aligned}$$

□

Proof of (8).

$$\begin{aligned} \partial_r D_\phi(x,r) &= -\frac{1}{r^2}\int |\nabla u(y)|^2 \phi'\Big(\frac{d(x,y)}{r}\Big)\cdot d(x,y)\,dy \\ &= -\frac{1}{r^2}\int_0^r \phi'\Big(\frac{s}{r}\Big)\cdot s\int_{\partial B_x(s)} |\nabla u(y)|^2\,dy\,ds \end{aligned} \tag{13}$$

It was proved in [14, Section 2(d)] that

$$\begin{aligned} \int_{\partial B_x(s)} |\nabla u(y)|^2\,dy &= 2\int_{\partial B_x(s)} |\nabla_{\nu_x} u(y)|^2\,dy + \frac{2}{s}\int_{B_x(s)} |\nabla u(y)|^2\,dy \\ &\quad + \frac{2}{s}\int_{B_x(s)} \langle u(y), \mathcal{R}u(y)\rangle\,dy - \int_{\partial B_x(s)} \langle \mathcal{R}_1 u(y), \nabla u(y)\rangle\,dy \\ &\quad + \int_{\partial B_x(s)} \langle u(y), \mathcal{R}_2 u(y)\rangle\,dy, \end{aligned}$$

where $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_2$ are smooth tensors, $\mathcal{R}$ and $\mathcal{R}_2$ are bounded, the norm of $\mathcal{R}_1$ is bounded by $C_1\cdot r$.

Notice that

$$-\int_0^r \phi'\Big(\frac{s}{r}\Big)\cdot s\int_{\partial B_x(s)} |\nabla_{\nu_x} u(y)|^2\,dy\,ds = E_\phi(x,r),$$
$$-\frac{1}{r}\int_0^r \phi'\Big(\frac{s}{r}\Big)\int_{B_x(s)} |\nabla u(y)|^2\,dy\,ds = D_\phi(x,r).$$

Plug into equation (13), we have

$$\begin{aligned}\partial_r D_\phi(x,r) &= \frac{2}{r}D_\phi(x,r) + \frac{2}{r^2}E_\phi(x,r)\\ &\quad - \frac{1}{r^2}\int_0^r \phi'\Big(\frac{s}{r}\Big)\cdot s\cdot\Big[\frac{2}{s}\int_{B_x(s)} \langle u(y),\mathcal{R}u(y)\rangle\,dy\\ &\quad - \int_{\partial B_x(s)} \langle \mathcal{R}_1 u(y),\nabla u(y)\rangle\,dy + \int_{\partial B_x(s)}\langle u(y),\mathcal{R}_2 u(y)\rangle\,dy\Big]\,ds.\end{aligned}$$

Lemma 4.3 implies

$$\begin{aligned}-\frac{1}{r^2}\int_0^r \phi'\Big(\frac{s}{r}\Big)\cdot s\cdot\Big[\frac{2}{s}\int_{B_x(s)} \langle u(y),\mathcal{R}u(y)\rangle\,dy&\\ + \int_{\partial B_x(s)}\langle u(y),\mathcal{R}_2 u(y)\rangle\,dy\Big]\,ds&\\ = O(H_\phi(x,r)).&\end{aligned}$$

On the other hand,

$$\begin{aligned}&\Big| -\frac{1}{r^2}\int_0^r \phi'\Big(\frac{s}{r}\Big)\cdot s\cdot\Big[-\int_{\partial B_x(s)} \langle \mathcal{R}_1 u(y),\nabla u(y)\rangle\,dy\Big]\,ds\Big|\\ &\quad\le C_2\cdot\int_0^r \Big|\phi'\Big(\frac{s}{r}\Big)\Big|\int_{\partial B_x(s)} |u(y)||\nabla u(y)|\,dy\,ds\\ &\quad\le C_3\int_{B_x(r)} |u(y)||\nabla u(y)|dy = O(H_\phi(x,r)).\end{aligned}$$

Hence the result is proved. □

Proof of (9). For a function $G(x,y)$ defined on $X\times X$ and a vector field w, use $\frac{\partial x}{\partial w}G$ to denote the directional derivative of G with respect to x, use $\frac{\partial y}{\partial w}G$ to denote the directional derivative with respect to y.

The first variation formula of geodesic lengths gives

$$\frac{\partial x}{\partial v} d(x,y) + \frac{\partial y}{\partial v} d(x,y) = O(d(x,y)^2).$$

We have

$$\begin{aligned}\frac{\partial x}{\partial v} D_\phi(x,r) &= \frac{1}{r}\int |\nabla u(y)|^2 \phi'\Big(\frac{d(x,y)}{r}\Big)\cdot \frac{\partial x}{\partial v} d(x,y)\,dy \\ &= -\frac{1}{r}\int |\nabla u(y)|^2 \phi'\Big(\frac{d(x,y)}{r}\Big)\cdot \frac{\partial y}{\partial v} d(x,y)\,dy \\ &\quad + O(r)\int_{B_x(r)} |\nabla u(y)|^2 \\ &= -\int |\nabla u(y)|^2 \cdot \frac{\partial y}{\partial v}\phi\Big(\frac{d(x,y)}{r}\Big)\,dy + O(H_\phi(x,r)). \end{aligned} \tag{14}$$

One needs to establish the following lemma.

Lemma 4.5. *Let F be the curvature of $\mathcal{V}$, and $\{e_i\}$ be an orthonormal basis of TX. Let φ be a smooth function with $supp\,\varphi \subset B_x(r)$. Then*

$$\begin{aligned}\int |\nabla u|^2 \partial_v \varphi = 2\int \langle d\varphi \otimes \nabla_v u, \nabla u\rangle - 2\int \sum_i \varphi \langle F(v,e_i)u, \nabla_{e_i} u\rangle \\ - 2\int \sum_i \varphi\langle \nabla_{[v,e_i]} u, \nabla_{e_i} u\rangle - \int |\nabla u|^2 \varphi\, div(v) \\ + 2\int \sum_i \varphi \langle \nabla_v u, \nabla_{\nabla_{e_i} e_i} u\rangle + 2\int \sum_i \varphi\langle \nabla_v u, \nabla_{e_i} u\rangle\, div(e_i) \\ + 2\int \varphi\langle \nabla_v u, \mathcal{R}_0 u\rangle, \end{aligned}$$

where $\mathcal{R}_0$ is the curvature term in the Weitzenböck formula.

Proof of Lemma 4.5. By Lemma 2.3, there exists a sequence of smooth 2-valued section U_i, such that $U_i = -U_i$ and $U_i \to U$ in $W^{1,2}$. By partitions of unity, integration by parts works for U_i. For any U_i, locally write it as $[\![w]\!] + [\![-w]\!]$ where w is a smooth section of $\mathcal{V}$, then

$$\begin{aligned}\int |\nabla w|^2 \partial_v \varphi &= -\int \sum_i \varphi \nabla_v \langle \nabla_{e_i} w, \nabla_{e_i} w\rangle - \int |\nabla w|^2 \varphi\, \mathrm{div}(v) \\ &= -2\int \sum_i \varphi\langle \nabla_{e_i}\nabla_v w, \nabla_{e_i} w\rangle - 2\int \sum_i \varphi\langle F(v,e_i)w, \nabla_{e_i} w\rangle \end{aligned}$$

$$- 2\int \sum_i \varphi\langle \nabla_{[v,e_i]} w, \nabla_{e_i} w\rangle - \int |\nabla w|^2 \varphi \operatorname{div}(v)$$

Here F denotes the curvature of $\mathcal{V}$. For the first term in the formula above,

$$\begin{aligned}
&\int \sum_i \varphi\langle \nabla_{e_i}\nabla_v w, \nabla_{e_i} w\rangle \\
&= -\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle - \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i}\nabla_{e_i} w\rangle \\
&\quad - \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i} w\rangle \operatorname{div}(e_i) \\
&= -\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle + \int \sum_i \varphi\langle \nabla_v w, \nabla^\dagger \nabla w\rangle \\
&\quad - \int \sum_i \varphi\langle \nabla_v w, \nabla_{\nabla_{e_i} e_i} w\rangle - \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i} w\rangle \operatorname{div}(e_i)
\end{aligned}$$

For the second term in the formula above, let $\mathcal{R}_0$ be the curvature term in the Weitzenböck formula, then

$$\begin{aligned}
\int \sum_i \varphi\langle \nabla_v w, \nabla^\dagger \nabla w\rangle &= \int \langle \varphi \nabla_v w, D^2 w - \mathcal{R}_0 w\rangle \\
&= -\int \varphi\langle \nabla_v w, \mathcal{R}_0 w\rangle + \int \langle \rho(\nabla\varphi)\nabla_v w, Dw\rangle \\
&\quad - \int \langle \varphi\langle [\nabla_v, D] w, Dw\rangle + \int \varphi\langle \nabla_v(Dw), Dw\rangle \\
&= -\int \varphi\langle \nabla_v w, \mathcal{R}_0 w\rangle + \int \langle \rho(\nabla\varphi)\nabla_v w, Dw\rangle \\
&\quad - \int \langle \varphi\langle [\nabla_v, D] w, Dw\rangle - \frac{1}{2}\int \partial_v \varphi |Dw|^2 \\
&\quad - \frac{1}{2}\int \varphi |Dw|^2 \operatorname{div}(v)
\end{aligned}$$

Therefore

$$\begin{aligned}
\int |\nabla w|^2 \partial_v \varphi = &-2\int \sum_i \varphi\langle F(v,e_i) w, \nabla_{e_i} w\rangle - 2\int \sum_i \varphi\langle \nabla_{[v,e_i]} w, \nabla_{e_i} w\rangle \\
&- \int |\nabla w|^2 \varphi \operatorname{div}(v) + 2\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle
\end{aligned}$$

$$
\begin{aligned}
&+2\int\sum_i\varphi\langle\nabla_v w,\nabla_{\nabla_{e_i}e_i}w\rangle+2\int\sum_i\varphi\langle\nabla_v w,\nabla_{e_i}w\rangle\operatorname{div}(e_i)\\
&+2\int\varphi\langle\nabla_v w,\mathcal{R}_0 w\rangle-2\int\langle\rho(\nabla\varphi)\nabla_v w,Dw\rangle\\
&+2\int\langle\varphi\langle[\nabla_v,D]w,Dw\rangle+\int\partial_v\varphi|Dw|^2-\int\varphi|Dw|^2\operatorname{div}(v)
\end{aligned}
$$

Take limit $U_i\to U$, one has

$$
\begin{aligned}
\int|\nabla u|^2\partial_v\varphi=&-2\int\sum_i\varphi\langle F(v,e_i)u,\nabla_{e_i}u\rangle-2\int\sum_i\varphi\langle\nabla_{[v,e_i]}u,\nabla_{e_i}u\rangle\\
&-\int|\nabla u|^2\varphi\operatorname{div}(v)+2\int\sum_i(\nabla_{e_i}\varphi)\langle\nabla_v u,\nabla_{e_i}u\rangle\\
&+2\int\sum_i\varphi\langle\nabla_v u,\nabla_{\nabla_{e_i}e_i}u\rangle+2\int\sum_i\varphi\langle\nabla_v u,\nabla_{e_i}u\rangle\operatorname{div}(e_i)\\
&+2\int\varphi\langle\nabla_v u,\mathcal{R}_0 u\rangle-2\int\langle\rho(\nabla\varphi)\nabla_v u,Du\rangle\\
&+2\int\langle\varphi\langle[\nabla_v,D]u,Du\rangle+\int\partial_v\varphi|Du|^2-\int\varphi|Du|^2\operatorname{div}(v)\\
=&-2\int\sum_i\varphi\langle F(v,e_i)u,\nabla_{e_i}u\rangle-2\int\sum_i\varphi\langle\nabla_{[v,e_i]}u,\nabla_{e_i}u\rangle\\
&-\int|\nabla u|^2\varphi\operatorname{div}(v)+2\int\sum_i(\nabla_{e_i}\varphi)\langle\nabla_v u,\nabla_{e_i}u\rangle\\
&+2\int\sum_i\varphi\langle\nabla_v u,\nabla_{\nabla_{e_i}e_i}u\rangle+2\int\sum_i\varphi\langle\nabla_v u,\nabla_{e_i}u\rangle\operatorname{div}(e_i)\\
&+2\int\varphi\langle\nabla_v u,\mathcal{R}_0 u\rangle
\end{aligned}
$$

Notice that

$$
\sum_i(\nabla_{e_i}\varphi)\langle\nabla_v u,\nabla_{e_i}u\rangle=\langle d\varphi\otimes\nabla_v u,\nabla u\rangle,
$$

therefore the lemma is proved. $\square$

Back to the proof of equation (9). Take $\varphi(y)=\phi(d(x,y)/r)$. By Lemma 4.3,

$$
-2\int\sum_i\varphi\langle F(v,e_i)u,\nabla_{e_i}u\rangle+2\int\varphi\langle\nabla_v u,\mathcal{R}_0 u\rangle=O(H_\phi(x,r)).
$$

On the other hand, $|\mathrm{div}(v)| = O(r)$, and one can choose $\{e_i\}$ such that $|[v, e_i]| = O(r)$, $|\mathrm{div}(e_i)| = O(r)$, and $|\nabla_{e_i} e_i| = O(r)$. Thus by Lemma 4.3,

$$\begin{aligned} &-2\int \sum_i \varphi\langle \nabla_{[v,e_i]}u, \nabla_{e_i}u\rangle - \int |\nabla u|^2 \varphi \,\mathrm{div}(v) + 2\int \sum_i \varphi\langle \nabla_v u, \nabla_{\nabla_{e_i}e_i} u\rangle \\ &\quad + 2\int \sum_i \varphi \langle \nabla_v u, \nabla_{e_i} u\rangle \,\mathrm{div}(e_i) = O(H_\phi(x,r)). \end{aligned}$$

Equation (9) then follows immediately from equation (14) and Lemma 4.5. □

Proof of (10). By [14, Equation (2.11)],

$$\partial_s H(x,s) = \frac{3}{s} H(x,s) + 2D(x,s) + \int_{B_x(s)} \langle u, \mathcal{R}u\rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2, \tag{15}$$

where $\mathcal{R}$ is a curvature term from the Weitzenböck formula, and $\mathfrak{t}$ comes from the mean curvature of $\partial B_x(s)$. The function $\mathfrak{t}$ satisfies $|\mathfrak{t}(y)| = O(d(x,y))$. Notice that

$$H_\phi(x,r) = \int_0^r -\phi'(s/r)\cdot \frac{1}{s}\cdot H(s)\,ds = \int_0^1 -\phi'(\lambda)\frac{1}{\lambda}\cdot H(\lambda r)\,d\lambda.$$

Therefore

$$\begin{aligned} \partial_r H_\phi(x,r) &= \int_0^1 -\phi'(\lambda)\cdot(\partial_r H)(\lambda r)\,d\lambda \\ &= \int_0^1 -\phi'(\lambda)\Big[\frac{3}{\lambda r} H(x,\lambda r) + 2D(x,\lambda r) + \int_{B_x(\lambda r)} \langle u, \mathcal{R}u\rangle + \int_{\partial B_x(\lambda r)} \mathfrak{t}|u|^2\Big]\,d\lambda \\ &= -\frac{1}{r}\int_0^r \phi'(s/r)\Big[\frac{3}{s}H(x,s) + 2D(x,s) + \int_{B_x(s)} \langle u, \mathcal{R}u\rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2\Big]\,ds \\ &= \frac{3}{r}H_\phi(x,r) + 2D_\phi(x,r) - \frac{1}{r}\int_0^r \phi'(s/r)\Big[\int_{B_x(s)} \langle u, \mathcal{R}u\rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2\Big]\,ds \\ &= \frac{3}{r}H_\phi(x,r) + 2D_\phi(x,r) + O(rH_\phi(x,r)). \end{aligned}$$

□

Proof of (11). As in the proof of (9), for a function $G(x,y)$, use $\frac{\partial x}{\partial v}G$ to denote the directional derivative of G with respect to x, and use $\frac{\partial y}{\partial v}G$ to denote the directional derivative with respect to y. Recall that we have

$$\frac{\partial x}{\partial v}d(x,y)+\frac{\partial y}{\partial v}d(x,y)=O(d(x,y)^2),$$

therefore

$$(\frac{\partial x}{\partial v}+\frac{\partial y}{\partial v})\Big[d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\Big]=O(1).$$

We have

$$\begin{aligned}\partial_v H(x,r) &= -\int |u(y)|^2\frac{\partial x}{\partial v}\Big[d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\Big]dy\\ &= \int |u(y)|^2\frac{\partial y}{\partial v}\Big[d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\Big]dy + O(\int_{B_x(r)}|u|^2)\\ &= -\int \frac{\partial}{\partial v}|u(y)|^2 d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)dy\\ &\quad -\int |u(y)|^2 d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,\mathrm{div}(v)dy + O(rH_\phi(x,r))\\ &= -2\int u(y)\cdot\nabla_v u(y)\,d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy + O(rH_\phi(x,r))\end{aligned}$$

The last equality follows from $|\mathrm{div}(v)|=O(r)$ and $\int_{B_x(r)}|u|^2=O(rH_\phi(x,r))$. □

Remark 4.6. When both X and $\mathcal{V}$ are flat, all the curvature terms in the computations above are zero. Therefore, Proposition 4.4 becomes

$$\begin{aligned}D_\phi(x,r) &= -\frac{1}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x}u(y)\cdot u(y)\,dy,\\ \partial_r D_\phi(x,r) &= \frac{2}{r}D_\phi(x,r)+\frac{2}{r^2}E_\phi(x,r)\\ \partial_v D_\phi(x,r) &= -\frac{2}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x}u(y)\cdot\nabla_v u(y)\,dy\\ \partial_r H_\phi(x,r) &= \frac{3}{r}H_\phi(x,r)+2D_\phi(x,r)\\ \partial_v H_\phi(x,r) &= -2\int u(y)\cdot\nabla_v u(y)\,d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy\end{aligned}$$

Corollary 4.7. *Let $\eta_x(y)=d(x,y)\cdot\nu_x(y)$. Under the assumptions of Proposition 4.4, one has*

$$(16)\quad \partial_v N_\phi(x,r) = \frac{2}{H_\phi(x,r)} \int -\frac{1}{d(x,y)}\phi'\Big(\frac{d(x,y)}{r}\Big)\cdot \\ (\nabla_{\eta_x} u(y) - N_\phi(x,r)u(y)) \cdot \nabla_v u(y)\, dy + O(r).$$

$$(17)\quad \partial_r N_\phi(x,r) = \frac{2}{rH_\phi(x,r)} \int -\phi'\Big(\frac{d(x,y)}{r}\Big)\cdot \\ d(x,y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x,r)u(y)|^2\, dy + O(r),$$

As a consequence, there exists a constant C, such that $\Big(N_\phi(x,r) + Cr^2\Big)$ is increasing in r.

Proof. The first equation follows immediately from Proposition 4.4 by combining equations (9) and (11). For the first one, Lemma 4.4 gives

$$\partial_r N_\phi(x,r) = \frac{2}{rH_\phi(x,r)}\Big(E_\phi(x,r) - \frac{r^2 D_\phi(x,r)^2}{H_\phi(x,r)}\Big) + O(r),$$

and we have

$$\begin{aligned} &E_\phi(x,r) - \frac{r^2 D_\phi(x,r)^2}{H_\phi(x,r)} \\ &= E_\phi(x,r) - 2rD_\phi(x,r)N_\phi(x,r) + N_\phi(x,r)^2 H_\phi(x,r) \\ &= \int -\phi'\Big(\frac{d(x,y)}{r}\Big) d(x,y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x,r)u(y)|^2\, dy + O(r^2 H_\phi(x,r)) \end{aligned}$$

Hence the second equation is verified.

□

5. Compactness

This section proves a compactness result for $\mathbb{Z}/2$ harmonic spinors.

Consider the ball $\bar{B}(5) \subset \mathbb{R}^4$ centered at the origin. Let $\mathcal{V}$ be a fixed trivial vector bundle on Ω. Assume g_n is a sequence of Riemannian metrics on $\bar{B}(5)$, A_n is a sequence of connection forms on $\mathcal{V}$, and ρ_n is a sequence of Clifford bundle structures of $\mathcal{V}$. Assume that (g_n, A_n, ρ_n) are compatible, and assume that (g_n, A_n, ρ_n) converge to (g_{encl}, A, ρ) in C^∞, where g_{eucl} is the Euclidean metric on $\bar{B}(5)$. Then for sufficiently large n, the injectivity radius at each point in $\bar{B}(2)$ is at least 2.5. Without loss of generality, assume that this property holds for every n.

Fix $\epsilon, \Lambda > 0$. For every n, assume U_n is a 2-valued section of $\mathcal{V}$ defined on $\bar{B}(5)$, with the following properties:

1) The section U_n is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5)$ with respect to (g_n, A_n, ρ_n).
2) U_n satisfies Assumption 1.2 with respect to ϵ.
3) Let $N_\phi^{(n)}$ be the smoothed frequency function for the extended U_n. Then whenever $N_\phi(x, r)$ is defined,
$$N_\phi^{(n)}(x, r) \leq \Lambda.$$
4) Let $H_\phi^{(n)}$ be the smoothed height function of U_n, then $H_\phi^{(n)}(0, 1) = 1$.

The main result of this section is the following proposition.

Proposition 5.1. *Let U_n be given as above. Then there exits a subsequence of $\{U_n\}$, such that the sequence converges strongly in $W^{1,2}(\bar{B}(2))$ to a section U. The section U is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(2)$ with respect to (g_{eucl}, A, ρ), and U satisfies Assumption 1.2 for a possibly smaller value of ϵ. Moreover, U_n converges to U uniformly on $\bar{B}(2)$.*

Proof. Fix a trivialization of $\mathcal{V}$, and fix $s \in (0, 0.5)$. The bound on $N_\phi^{(n)}$ and the assumption that $H_\phi^{(n)}(0, 1) = 1$ implies that $\|U\|_{L^2(\bar{B}(2+s))} \leq C_1$ for some constant C_1. The upper bound on N_ϕ then implies $\|\nabla_{A_n} U\|_{L^2(\bar{B}(2+s/2))} \leq C_2$. Since $A_n \to A$ in C^∞, this implies that U_n is bounded in $W^{1,2}(\bar{B}(2+s/2))$. Therefore, there is a subsequence of $\{U_n\}$ which converges weakly in $W^{1,2}(\bar{B}(2+s/2))$ and converges strongly in $L^2(\bar{B}(2+s/2))$. To avoid complicated notations, the subsequence is still denoted by $\{U_n\}$. Denote the limit of $\{U_n\}$ on $\bar{B}(2+s/2)$ by U. Let $H_\phi^{(n)}$, $D_\phi^{(n)}$, $N_\phi^{(n)}$ be the smoothed frequency functions for U_n, let H_ϕ, D_ϕ, N_ϕ be the corresponding functions for U. Since $U_n \to U$ strongly in L^2, one has $H_\phi(0, 1) = 1$, thus U is not identically $2[\![0]\!]$.

By [14, Section 3(e)], there exists constants $K > 0$ and $\alpha \in (0, 1)$, depending on ϵ, Λ, R and the C^1 norms of the curvatures of $\{g_n\}$ and A_n, such that
$$\|U_n\|_{C^\alpha(\bar{B}(2+s/2))} \leq K.$$
By the Arzela-Ascoli theorem, there exists a further subsequence of $\{U_n\}$ which converges uniformly to U on $\bar{B}(2+s/2)$. Still denote this subsequence by $\{U_n\}$. Since solutions to the Dirac equation are closed under C^0 limits,

U is a $\mathbb{Z}/2$ harmonic spinor. U is also Hölder continuous, so it satisfies Assumption 1.2.

Locally write U_n as $[\![u_n]\!] + [\![-u_n]\!]$, and write U as $[\![u]\!] + [\![-u]\!]$. The weak convergence of U_n to U implies

$$\liminf_{n\to\infty} \int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 \geq \int_{\bar{B}(2)} |\nabla_A u|^2.$$

We want to prove that

$$\lim_{n\to\infty} \int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 = \int_{\bar{B}(2)} |\nabla_A u|^2.$$

Assume the contrary, then there exists a subsequence of n such that

$$\int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 \geq \int_{\bar{B}(2)} |\nabla_A u|^2 + \delta$$

for some $\delta > 0$. Since $\int_{\bar{B}(r)} |\nabla_A u|^2$ is continuous in r, and $\int_{\bar{B}(r)} |\nabla_{A_n} u_n|^2$ is non-decreasing in r for every n, there exists $r \in (2, 2+s/2)$ and $\sigma \in (1, (2+s/2)/r)$, such that for every $t \in [2, r]$,

$$\int_{\bar{B}(t)} |\nabla_{A_n} u_n|^2 \geq \int_{\bar{B}(\sigma t)} |\nabla_A u|^2 + \delta/2 \tag{18}$$

Use $B_n(t)$ to denote the geodesic ball of center 0 and radius t with metric g_n. Since $g_n \to g_{eucl}$, we have $\bar{B}(t) \subset B_n(\sigma t)$ for sufficiently large n. Equation (18) then gives

$$\int_{B_n(\sigma t)} |\nabla_{A_n} u_n|^2 \geq \int_{\bar{B}(\sigma t)} |\nabla_A u|^2 + \delta/2, \quad \text{for } t \in [2, r] \tag{19}$$

when n is sufficiently large.

By equation (15), for every t,

$$\partial_t H^{(n)}(0,t) = \frac{3}{t} H^{(n)}(0,t) + 2D^{(n)}(0,t) + \int_{B_n(t)} \langle u, \mathcal{R}^{(n)} u\rangle + \int_{\partial B_n(t)} \mathfrak{t}^{(n)} |u|^2,$$

$$\partial_t H(0,t) = \frac{3}{t} H(0,t) + 2D(0,t) + \int_{\bar{B}(t)} \langle u, \mathcal{R} u\rangle + \int_{\partial \bar{B}(t)} \mathfrak{t} |u|^2,$$

where $\mathcal{R}^{(n)}$ and $\mathfrak{t}^{(n)}$ are bounded terms that are uniformly convergent to $\mathcal{R}$ and $\mathfrak{t}$ as n goes to infinity. The uniform convergence of $|u_n|$ and g_n then imply

$$\lim_{s\to\infty}\int_{2\sigma}^{\sigma r} D^{(n)}(0,t)\,dt = \int_{2\sigma}^{\sigma r} D(0,t)\,dt,$$

which contradicts (19). In conclusion,

$$\lim_{n\to\infty}\int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 = \int_{\bar{B}(2)} |\nabla_A u|^2.$$

Since $(A_n, g_n) \to (A, g_{eucl})$ in C^∞, this implies

$$\lim_{n\to\infty} \|U_i\|_{W^{1,2}}(\bar{B}(2)) = \|U\|_{W^{1,2}}(\bar{B}(2)),$$

therefore U_i convergence strongly to U in $W^{1,2}(\bar{B}(2))$. □

Corollary 5.2. *Let $\sigma > 1$. Let g_* be a metric on $\mathbb{R}^4$ given by a constant metric matrix, such that all eigenvalues of the matrix are in the interval $[\sigma^{-2}, \sigma^2]$.*

Assume $\{(g_n, A_n, \rho_n)\}_{n\geq 1}$ is a sequence of geometric data on $\bar{B}(5\sigma^2)$, and assume (g_n, A_n, ρ_n) converge to (g_*, A, ρ) in C^∞. Let U_n be a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5\sigma^2)$ with respect to (g_n, A_n, ρ_n), such that the sequence U_n satisfies conditions (2) to (4) listed before Proposition 5.1. Then a subsequence of U_n converges to a $\mathbb{Z}/2$ harmonic spinor in $W^{1,2}(\bar{B}(2))$ with respect to (g_*, A, ρ). The limit U satisfies Assumption 1.2, and the sequence U_n converges to U uniformly.

Proof. Take a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T^*(g_*)$ is the Euclidean metric. Then $(T^*g_n, T^*A_n, T^*\rho_n, T^*U_n)$ gives a sequence of $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5\sigma)$. Since T^*g_n converges to the Euclidean metric, one can apply Lemma 5.1 and find a convergent subsequence on $\bar{B}(2\sigma)$. Now pull back by T^{-1}, one obtains a convergent subseqence of U_n on $\bar{B}(2)$. □

6. Frequency pinching estimates

For $x \in B_{x_0}(32R)$ and $0 < s < r \leq 32R$, define

$$W_s^r(x) = N_\phi(x,r) - N_\phi(x,s).$$

This section proves the following estimate

Proposition 6.1. *There exists a constant C with the following property. Let $r \in (0, 8R]$. Assume $x_1, x_2 \in B_{x_0}(32R)$, such that $d(x_1,x_2) \leq r/4$. Let x*

be a point on the short geodesic γ bounded by x_1 and x_2. Let v be a unit tangent vector of γ at x. Then

$$d(x_1,x_2)\cdot|\partial_v N_\phi(x,r)| \le C\Big[\sqrt{|W^{4r}_{r/4}(x_1)|}+\sqrt{|W^{4r}_{r/4}(x_2)|}+r\Big].$$

The proof is adapted from the arguments of [4, Section 4]. First, one needs to prove the following lemma.

Lemma 6.2. *There exists a constant C, such that for every $x\in B_{x_0}(32R)$ and $r\le 8R$, one has*

$$\int_{B_x(3r)-B_x(r/3)} |\nabla_{\eta_x}u(y)-N_\phi(x,d(x,y))u(y)|^2dy \\ \le CrH_\phi(x,r)(W^{4r}_{r/4}(x)+Cr^2).$$

Proof. By equation (17),

$$\begin{aligned}
&\int_{r/4}^{4r}\partial_s N_\phi(x,s)ds+O(r^2)\\
&=\int_{r/4}^{4r}\frac{2}{sH_\phi(x,s)}\int -\phi'\Big(\frac{d(x,y)}{s}\Big)d(x,y)^{-1}|\nabla_{\eta_x}u(y)-N_\phi(x,s)u(y)|^2\,dyds\\
&\ge \frac{1}{C_1rH_\phi(x,r)}\int_{r/4}^{4r}\int -\phi'\Big(\frac{d(x,y)}{s}\Big)d(x,y)^{-1}|\nabla_{\eta_x}u(y)-N_\phi(x,s)u(y)|^2\,dyds\\
&\ge \frac{1}{C_1rH_\phi(x,r)}\int_{r/3}^{4r}\int -\phi'\Big(\frac{d(x,y)}{s}\Big)d(x,y)^{-1}|\nabla_{\eta_x}u(y)-N_\phi(x,s)u(y)|^2\,dyds\\
&=:(A)
\end{aligned}$$

For every pair (y,s) in the support of the integration in (A), one has $d(x,y)\in[r/4,4r]$, hence

$$|N_\phi(x,s)-N_\phi(x,d(x,y))|\le W^{4r}_{r/4}(x)+C_2r^2.$$

Therefore,

$$(A)\ge\frac{1}{C_1rH_\phi(x,r)}\\ \times\underbrace{\int_{r/3}^{4r}\int -\phi'\Big(\frac{d(x,y)}{s}\Big)d(x,y)^{-1}|\nabla_{\eta_x}u(y)-N_\phi(x,d(x,y))u(y)|^2\,dyds}_{=:I}$$

$$- \frac{C_3(W^{4r}_{r/4}(x) + C_2 r^2)}{rH_\phi(x,r)}$$
$$\times \underbrace{\int_{r/3}^{4r} \int -\phi'\Big(\frac{d(x,y)}{s}\Big) d(x,y)^{-1} \Big[|\nabla u(y)||u(y)|d(x,y) + |u(y)|^2\Big] dyds}_{=:II}.$$

By Lemma 4.3, $II = O(rH_\phi(x,4r)) = O((rH_\phi(x,r))$. By Fubini's theorem,

$$I = \int_{B_x(4r)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y))u(y)|^2 \int_{r/3}^{4r} -\phi'\Big(\frac{d(x,y)}{s}\Big) d(x,y)^{-1}\, dsdy$$

Notice that

$$\inf_{\{y|d(x,y)\in[r/3,3r]\}} \int_{r/3}^{4r} -\phi'\Big(\frac{d(x,y)}{s}\Big) d(x,y)^{-1}\, ds > 0,$$

Therefore

$$I \geq \frac{1}{C_4} \int_{B_x(3r)-B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y))u(y)|^2\, dy,$$

In conclusion,

$$(A) \geq \frac{1}{C_5 rH_\phi(x,r)} \int_{B_x(3r)-B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y))u(y)|^2\, dy$$
$$- C_6(W^{4r}_{r/4}(x) + C_2 r^2),$$

hence

$$C_7 rH_\phi(x,r)(W^{4r}_{r/4}(x) + C_8 r^2)$$
$$\geq \int_{B_x(3r)-B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y))u(y)|^2 dy.$$

□

One also needs the following technical lemma.

Lemma 6.3. *Assume M is a compact manifold, possibly with boundary. Let $\varphi^\zeta : \Omega \subset \overline{B_{x_0}(64R)} \to \mathbb{R}^4$ be a smooth family of smooth embeddings,* *parametrized by $\zeta \in M$. For every $\zeta \in M$ and $x \in B_{x_0}(64R)$, one can define a vector field η_x^ζ on $B_{x_0}(64R)$ as follows. For every $y \in B_{x_0}(64R)$, let*

$$\eta_x^\zeta(y) = [(\varphi^\zeta)_*(y)]^{-1}(\varphi^\zeta(y) - \varphi^\zeta(x)).$$

Then there exists a constant $\Theta > 0$, depending on φ, such that

$$|\eta_x^\zeta(y) - \eta_x(y)| \leq \Theta \cdot d(x,y)^2.$$

Proof. Fix x, compute the covariant derivates of η_x^ζ and η_x at x. Since both vector fields are zero at x, their covariant derivatives at x are independent of the connections. Let $e \in T_xX$. Taking derivate in the Euclidean coordinates φ^ζ, one obtains $\nabla_e(\eta_x^\zeta)(x) = e$. Taking derivative in the normal coordinates centered at x, one obtains $\nabla_e(\eta_x)(x) = e$. Therefore, η_x^ζ and η_x have the same derivatives at x. Since we are working on compact manifolds, $|\eta_x^\zeta(y) - \eta_x(y)| \leq \Theta \cdot d(x,y)^2$ for some constant Θ independent of x. □

Proof of Proposition 6.1. Assume that v points from x_1 towards x_2. Extend v to a vector field on $B_x(r)$, such that the coordinates of v are constant under the normal coordinate centered at x. Now apply Lemma 6.3. Let $M = \overline{B_{x_0}(32R)}$. For every $\zeta \in \overline{B_{x_0}(32R)}$, let φ^ζ be the exponential map centered at ζ. Then for every $z \in B_x(r)$,

$$v(z) = \frac{\eta_{x_1}^x(z) - \eta_{x_2}^x(z)}{|\varphi^x(x_1) - \varphi^x(x_2)|}. \tag{20}$$

By Lemma 6.3,

$$|\eta_{x_1}^x(z) - \eta_{x_1}(z)| = O(r^2), \quad |\eta_{x_2}^x(z) - \eta_{x_2}(z)| = O(r^2) \tag{21}$$

Notice that since φ^x is the exponential map centered at x,

$$|\varphi^x(x_1) - \varphi^x(x_2)| = d(x_1, x_2). \tag{22}$$

Combine (20), (21) and (22) together, one obtains

$$\left| v(z) - \frac{\eta_{x_1}(z) - \eta_{x_2}(z)}{d(x_1,x_2)} \right| = O(r^2/d(x_1,x_2)).$$

Define

$$\mathcal{E}_l(z) = \nabla_{\eta_{x_l}} u(z) - N_\phi(x_l, d(z,x_l))u(z) \qquad \text{for } l = 1,2.$$

Then

$$\begin{aligned} d(x_1,x_2)\nabla_v u(z) =& \nabla_{\eta_{x_1}} u(z) - \nabla_{\eta_{x_2}} u(z) + O(r^2|\nabla u|) \\ =& \underbrace{\big(N_\phi(x_1, d(z,x_1)) - N_\phi(x_2, d(z,x_2))\big)}_{=:\mathcal{E}_3(z)} u(z) \\ &+ \mathcal{E}_1(z) - \mathcal{E}_2(z) + O(r^2|\nabla u|). \end{aligned}$$

To simplify notations, define the measure

$$d\mu_x = -d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\, dy.$$

Using (16), one can write

$$\begin{aligned} & d(x_1,x_2)\cdot \partial_v N_\phi(x,r) \\ &= O(r^2) + \frac{2}{H_\phi(x,r)}\int \nabla_{\eta_x} u(y)\cdot(\mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3 u + O(r^2|\nabla u|))d\mu_x \\ &\quad - \frac{2}{H_\phi(x,r)}\int uN_\phi(x,r)\cdot(\mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3 u + O(r^2|\nabla u|))d\mu_x \\ &= \underbrace{\frac{2}{H_\phi(x,r)}\int \nabla_{\eta_x} u(y)\cdot(\mathcal{E}_1 - \mathcal{E}_2)d\mu_x}_{=:(A)} - \underbrace{\frac{2N_\phi(x,r)}{H_\phi(x,r)}\int u\cdot(\mathcal{E}_1 - \mathcal{E}_2)d\mu_x}_{=:(B)} \\ &\quad + \underbrace{\frac{2}{H_\phi(x,r)}\int \mathcal{E}_3 u(\nabla_{\eta_x} u - N_\phi(x,r)u)\, d\mu_x}_{=:(C)} + O(r) \end{aligned}$$

To bound (C), notice that

$$\begin{aligned} \mathcal{E}_3(z) = & \underbrace{N_\phi(x_1,r) - N_\phi(x_2,r)}_{=:\mathcal{E}} + \underbrace{[N_\phi(x_1,d(z,x_1)) - N_\phi(x_1,r)]}_{=:\mathcal{E}_4(z)} \\ & - \underbrace{[N_\phi(x_2,d(z,x_2)) - N_\phi(x_2,r)]}_{=:\mathcal{E}_5(z)}. \end{aligned}$$

By (7),

$$\begin{aligned} \int u\cdot\nabla_{\eta_x} u\, d\mu_x &= rD_\phi(x,r) + O(r^2 H_\phi(x,r)) \\ &= N_\phi(x,r)H_\phi(x,r) + O(r^2 H_\phi(x,r)) \\ &= N_\phi(x,r)\int |u|^2\, d\mu_x + O(r^2 H_\phi(x,r)). \end{aligned}$$

Hence

$$\int u \cdot (\nabla_{\eta_x} u - N_\phi(x,r)u) d\mu_x = O(r^2 H_\phi(x,r)),$$

therefore

$$\int \mathcal{E} u \cdot (\nabla_{\eta_x} u - N_\phi(x,r)u) d\mu_x = O(r^2 H_\phi(x,r)).$$

By Lemma 4.3,

$$2\int |u|(|\nabla_{\eta_x} u| + |N_\phi(x,r)||u|)\, d\mu_x = O(H_\phi(x,r)).$$

In addition, notice that

$$\sup_{z \in \text{ supp } \mu_x} |\mathcal{E}_4(z)| + |\mathcal{E}_5(z)| \leq W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_1 r^2.$$

Therefore,

$$\begin{aligned}\int (|\mathcal{E}_4| + |\mathcal{E}_5|) \cdot \left| u(\nabla_{\eta_x} u - N_\phi(x,r)u)\right| d\mu_x \\ \leq C_2 H_\phi(x,r)(W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_1 r^2).\end{aligned}$$

As a result,

$$(C) \leq C_3(W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_4 r^2).$$

To bound (A), use Cauchy's inequality to obtain

$$\begin{aligned}(A) &\leq \frac{C_5}{H_\phi(x,r)}\Big(\int_{B_x(r)} |\nabla u|^2 d\mu_x\Big)^{1/2}\Big(\int_{B_x(r)-B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) d\mu_x\Big)^{1/2} \\ &\leq \frac{C_6}{r^{1/2}\, H_\phi(x,r)^{1/2}}\Big(\int_{B_x(r)-B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) d\mu_x\Big)^{1/2}.\end{aligned}$$

Now apply Lemma 6.2,

$$\begin{aligned}\int_{B_x(r)-B_x(3r/4)} \mathcal{E}_1^2\, \mu_x &\leq \int_{B_{x_1}(5r/4)-B_{x_1}(r/2)} \mathcal{E}_1^2\, \mu_x \\ &\leq C_7 r H_\phi(x_1,r)(W^{4r}_{r/4}(x_1) + C_7 r^2)\end{aligned}$$

A similar estimate works for the integral of $\mathcal{E}_2$. Therefore

$$(A) \le C_8\Big[\sqrt{|W^{4r}_{r/4}(x_1)|} + \sqrt{|W^{4r}_{r/4}(x_2)|} + r\Big].$$

Similarly, applying Cauchy's inequality on (B) leads to

$$\begin{aligned}(B) &\le \frac{C_9}{rH_\phi(x,r)}\Big(\int_{B_x(r)} |u|^2 d\mu_x\Big)^{1/2}\Big(\int_{B_x(r)-B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) d\mu_x\Big)^{1/2} \\ &\le \frac{C_{10}}{r^{1/2}}\Big(\int_{B_x(r)-B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) d\mu_x\Big)^{1/2}\end{aligned}$$

Lemma 6.2 then gives

$$(B) \le C_{11}\Big[\sqrt{|W^{4r}_{r/4}(x_1)|} + \sqrt{|W^{4r}_{r/4}(x_2)|} + r\Big],$$

and the proposition is proved. □

Corollary 6.4. *Assume $x_1, x_2 \in B_{x_0}(32R)$, assume $r \in (0, 8R]$. If $d(x_1, x_2) \le r/4$, then*

$$|N_\phi(x_1, r) - N_\phi(x_2, r)| \le C\Big[\sqrt{|W^{4r}_{r/4}(x_1)|} + \sqrt{|W^{4r}_{r/4}(x_2)|} + r\Big].$$

□

7. L^2 approximation by planes

This section establishes a distortion bound in the spirit of [9]. Assume U satisfies Assumption 1.2 with respect to $\epsilon > 0$. In this section, the constants C, C_1, C_2, $\cdots$ will denote constants that depend on Λ, R, the C^1 norms of the curvatures, as well as ϵ. The techniques in this section were developed by [9], and the presentation here is adapted from Section 5 of [4].

Definition 7.1. *Suppose μ is a Radon measure on $\mathbb{R}^4$. For $x \in \mathbb{R}^4$, $r > 0$, define*

$$D^2_\mu(x, r) = \inf_L r^{-4} \int_{\bar{B}_x(r)} dist(y, L)^2 \, d\mu(y),$$

where L is taken among the set of 2-dimensional affine subspaces.

Remark. In the literature $D^2_\mu(x, r)$ usually called the Jone's β-number, and is denoted by $\beta^2_{\mu,2}(x, r)$. The notation $D^2_\mu(x, r)$ follows from [4].

For a measure μ supported in Z, we wish to bound the value of $D^2_\mu(x,r)$ in terms of the frequency functions. However, we have to be careful, since X is a Riemannian manifold, but $D^2_\mu(x,r)$ is only defined for Euclidean spaces. We identify $B_{x_0}(32R)$ with $\bar{B}(32R)$ using the exponential map centered at x_0. From now on, we will only work with the Euclidean metric induced by this identification.

The main result of this section is the following

Proposition 7.2. *There exists a positive constant $R_0 \leq R$ and a constant C with the following property. Let μ be a Radon measure supported in Z. For $x \in \bar{B}(R)$ and $r \leq R_0$, one has*

$$D^2_\mu(x, r/8) \leq \frac{C}{r^2} \int_{\bar{B}_x(r/8)} (W^{4r}_{r/4}(z) + Cr^2) d\mu(z).$$

First, observe that the function $D^2_\mu(x,r)$ has the following geometric interpretation. Assume $\mu(\bar{B}_x(r)) > 0$, let

$$\bar{z} = \frac{1}{\mu(\bar{B}_x(r))} \int_{\bar{B}_r(x)} z \, d\mu(z),$$

Define a non-negative bilinear form b on $\mathbb{R}^4$ as

$$b(v,w) = \int_{\bar{B}_x(r)} \big((z-\bar{z})\cdot v\big)\big((z-\bar{z})\cdot w\big)\, d\mu(z).$$

Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_4$ be the eigenvalues of b, then

$$D^2_\mu(x,r) = r^{-4}(\lambda_1 + \lambda_2).$$

Let v_i be an eigenvector with eigenvalue λ_i, a straightforward argument of linear algebra shows that

$$\int_{B_x(r)} \big((z-\bar{z})\cdot v_i\big) z \, d\mu(z) = \lambda_i \, v_i. \tag{23}$$

The following lemma can be understood as a version of Poincaré inequality for $\mathbb{Z}/2$ harmonic spinors.

Lemma 7.3. *There exist constants $C, R_0 > 0$ with the following property. Let v_1, v_2, v_3 be orthonormal vectors in $\mathbb{R}^4$. Let $x \in \bar{B}(R)$, $r \leq R_0$. Assume*

$Z \cap \bar{B}_x(r/8) \neq \emptyset$, *then*

$$\int_{\bar{B}_x(5r/4)-\bar{B}_x(3r/4)} \sum_{j=1}^{3} |\nabla_{v_j} u(z)|^2 \, dz \geq \frac{H_\phi(x,r)}{Cr}.$$

Proof. Assume such constants do not exist. Then there exists a sequence

$$\{(x_n, r_n, U_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})\}_{n\geq 1},$$

such that $r_n \leq \frac{1}{n}$, the vectors $v_1^{(n)}, v_2^{(n)}, v_3^{(n)}$ are orthonormal in $\mathbb{R}^4$,

$$\int_{\bar{B}_{x_n}(5r_n/4)-\bar{B}_{x_n}(3r_n/4)} \sum_{j=1}^{3} |\nabla_{v_j^{(n)}} u(z)|^2 \, dz \leq \frac{H_\phi(x_n,r_n)}{nr_n}, \tag{24}$$

and $Z \cap \bar{B}_{x_n}(r_n/8) \neq \emptyset$.

Let $\sigma = (12/11)^2$. Rescale the ball $\bar{B}_{x_n}(5\sigma^2 r_n)$ to $\bar{B}(5\sigma^2)$, and normalize the restriction of U. By Assumption (2), the pull back metrics g_n are given by matrix-valued functions on $\bar{B}(5\sigma^2)$ with eigenvalues bounded by $1/\sigma^2$ and σ^2. There is a subsequence of the pull backs of $(g_n, A_n, \rho_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})$ that converges to some data set $(g, A, \rho, v_1, v_2, v_3)$ in C^∞, and since $r_n \to 0$, the limit data set (g, A, ρ) is invariant under translations. By corollary 5.2, after taking a subsequence, the rescaled U_n converges to a $\mathbb{Z}/2$ harmonic spinor U^* on $\bar{B}(2)$ with respect to (g, A, ρ), which satisfies Assumption 1.2.

The assumption that $Z \cap \bar{B}_{x_n}(r_n/8) \neq \emptyset$ implies that U^* has at least one zero point in $\bar{B}(1/8)$. Inequality (24) gives

$$\int_{\bar{B}(5/4)-\bar{B}(3/4)} \sum_{j=1}^{3} |\nabla_{v_j} u^*(z)|^2 \, dz = 0$$

Theorem 1.3 implies that U^* is not identically zero on $\bar{B}(5/4) - \bar{B}(3/4)$. It also follows from Theorem 1.3 that the complement of the zero locus of a $\mathbb{Z}/2$ harmonic spinor is always open and connected, hence the unique continuation property still holds for $\mathbb{Z}/2$ harmonic spionors. Since U^* solves the Dirac equation on the complement of its zero locus, the unique continuation property implies that $|U|$ is constant in 3 linearly independent directions in $\bar{B}(5/4) - \bar{B}(3/4)$, hence Theorem 1.3 implies that U is everywhere non-zero in $\bar{B}(5/4)$, and that is a contradiction. $\square$

Now one can give the proof of Proposition 7.2. The proof is adapted from the proof of Proposition 5.3 in [4].

Proof of Proposition 7.2. Let R_0 be given by Lemma 7.3, and assume $r \leq R_0$. Without loss of generality, assume that $D_\mu^2(x, r/8) > 0$. In particular, $\mu(\bar{B}_x(r/8)) > 0$, thus $Z \cap \bar{B}_x(r/8) \neq \emptyset$. Let

$$\bar{z} = \frac{1}{\mu(\bar{B}_x(r/8))} \int_{\bar{B}_x(r/8)} z d\mu(z).$$

Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_4$ be the corresponding eigenvalues, then $D_\mu^2(x, r/8) > 0$ implies $\lambda_2 > 0$. Let v_i be the unit eigenvector with eigenvalue λ_i. Let $\operatorname{grad} u(z)$ be the vector in $T_z\mathbb{R}^4 \otimes \mathcal{V}$, such that for every $v \in T_z\mathbb{R}^4$,

$$\langle v, \operatorname{grad} u(z) \rangle_{\mathbb{R}^4} = \nabla_v u(z).$$

By Theorem 1.3, $\operatorname{grad} u$ is well-defined almost everwhere. By (2), $\|\operatorname{grad} u(z)\|_{\mathbb{R}^4} \leq (\frac{12}{11})\|\nabla u\|_X$. Equation (23) gives

$$-\lambda_i v_i \cdot \operatorname{grad} u(y) = \int_{\bar{B}_x(r/8)} \big((z - \bar{z}) \cdot v_i\big)\big((y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big) d\mu(z)$$

for any constant α. By Cauchy's inequality

$$\begin{aligned} &\lambda_i^2 |v_i \cdot \operatorname{grad} u(y)|^2 \\ &\leq \int_{\bar{B}_x(r/8)} \big|(z - \bar{z}) \cdot v_i\big|^2 d\mu(z) \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(z) \\ &= \lambda_i \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(y) \end{aligned}$$

Therefore, when $\lambda_i \neq 0$,

$$\lambda_i |v_i \cdot \operatorname{grad} u(y)|^2 \leq \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(z).$$

Integrate with respect to y on $\bar{B}_x(5r/4) - \bar{B}_x(3r/4)$, and sum up $i = 2, 3, 4$,

$$\begin{aligned} (25) \quad &\int_{\bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \sum_{i=2}^4 \lambda_i |v_i \cdot \operatorname{grad} u(y)|^2 \, dy \\ &\leq 3 \int_{y \in \bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \int_{z \in \bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(z) dy \\ &\leq 3 \int_{z \in \bar{B}_x(r/8)} \int_{y \in \bar{B}_z(11r/8) - \bar{B}_z(5r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 \, dy d\mu(z). \end{aligned}$$

On the other hand,

$$r^2 D_\mu^2(x,r)\sum_{i=2}^4 |v_i\cdot \operatorname{grad} u(y)|^2 = r^{-2}(\lambda_1+\lambda_2)\sum_{i=2}^4 |v_i\cdot \operatorname{grad} u(y)|^2 \leq \frac{2}{r^2}\sum_{i=2}^4 \lambda_i |v_i\cdot \operatorname{grad} u(y)|^2$$

Therefore

$$r^2 D_\mu^2(x,r)\int_{\bar B_x(5r/4)-\bar B_x(3r/4)}\sum_{i=2}^4 |v_i\cdot \operatorname{grad} u(y)|^2\,dy \leq \frac{2}{r^2}\int_{\bar B_x(5r/4)-\bar B_x(3r/4)}\sum_{i=2}^4 \lambda_i |v_i\cdot \operatorname{grad} u(y)|^2\,dy$$

By Lemma 7.3, this implies

$$r^2 H_\phi(x,r) D_\mu^2(x,r)\leq \frac{C_1}{r}\int_{\bar B_x(5r/4)-\bar B_x(3r/4)}\sum_{i=2}^4 \lambda_i |v_i\cdot \operatorname{grad} u(y)|^2\,dy$$

Therefore inequality (25) gives

$$\begin{aligned}(26)\quad & r^2 H_\phi(x,r) D_\mu^2(x,r)\\ &\leq \frac{3C_1}{r}\int_{\bar B_x(r/8)}\underbrace{\int_{\bar B_z(11r/8)-\bar B_z(5r/8)}\big|(y-z)\cdot \operatorname{grad} u(y)-\alpha u(y)\big|^2\,dy}_{=:A(z,r)}\,d\mu(z).\end{aligned}$$

where the constant C_1 is independent of α.

Notice that

$$\begin{aligned}A(z,r)\leq 3\Big(&\underbrace{\int_{\bar B_z(11r/8)-\bar B_z(5r/8)}\big|\eta_z(y)\cdot \operatorname{grad} u(y)-N_\phi(z,d(z,y))u(y)\big|^2\,dy}_{=:A_1(z,r)}\\ &+\underbrace{\int_{\bar B_z(11r/8)-\bar B_z(5r/8)}|(y-z)-\eta_z(y)|^2|\operatorname{grad} u(y)|^2 dy}_{=:A_2(z,r)}\\ &+\underbrace{\int_{\bar B_z(11r/8)-\bar B_z(5r/8)}\big(N_\phi(z,d(z,y))-\alpha\big)^2|u(y)|^2 dy}_{=:A_3(z,r)}\Big)\end{aligned}$$

Notice that by (2), we have $\bar{B}_z(11r/8) - \bar{B}_z(5r/8) \subset B_z(3r/2) - B_z(r/2)$. Therefore, by Lemma 6.2,

$$A_1(z,r) \leq C_2 r H_\phi(z,r)(W^{4r}_{r/4}(z) + C_2 r^2).$$

By Lemma 6.3 and Lemma 4.3,

$$A_2(z,r) = O(r^4 \int_{B_z(3r/2)} |\nabla u|^2) = O(r^3 H_\phi(x,r)).$$

To bound $A_3(z,r)$, first break it into two parts

$$\begin{aligned} A_3(z,r) \leq C_3 &\underbrace{\int_{B_z(3r/2)-B_z(r/2)} \big(N_\phi(z,d(z,y)) - N_\phi(z,r)\big)^2 |u(y)|^2 dy}_{=:A_4(z,r)} \\ &+ C_4 \underbrace{\int_{B_z(3r/8)-B_z(r/2)} \big(N_\phi(z,r) - \alpha\big)^2 |u(y)|^2 dy}_{=:A_5(z,r)} \end{aligned}$$

Here the balls $B_z(3r/2)$ and $B_z(r/2)$ are the geodesic balls on X, and the measure dy is the volume form of X. The monotonicity of N_ϕ implies that

$$\begin{aligned} A_4(z,r) &\leq (W^{4r}_{r/4}(z) + C_5 r^2) \int_{B_z(3r/2)} |u(y)|^2 dy \\ &\leq C_6 r H_\phi(x,r)(W^{4r}_{r/4}(z) + C_5 r^2). \end{aligned}$$

Now take $p \in B_x(r/8)$, such that

$$|W^{4r}_{r/4}(p)| = \inf_{q \in B_x(r/8)} |W^{4r}_{r/4}(q)|,$$

and take $\alpha = N_\phi(p,r)$. Then by Corollary 6.4, for $z \in B_x(r/8)$,

$$\begin{aligned} A_5(z,r) &\leq \int_{B_z(3r/2)-B_z(r/2)} \Big(C_7\big(\sqrt{|W^{4r}_{r/4}(z)|} + \sqrt{|W^{4r}_{r/4}(p)| + r}\big)\Big)^2 |u(y)|^2 dy \\ &\leq C_8\big(W^{4r}_{r/4}(z) + C_8 r^2\big) \int_{B_z(3r/2)-B_z(r/2)} |u(y)|^2 dy \\ &\leq C_9 r H_\phi(x,r)\big(W^{4r}_{r/4}(z) + C_8 r^2\big) \end{aligned}$$

In conclusion,

$$A(z,r) \le C_{10} r H_\phi(x,r)\big(W^{4r}_{r/4}(z) + C_{11} r^2\big).$$

Therefore Proposition 7.2 follows from inequality (26). □

8. Approximate spines

Definition 8.1. *Given a set of points $\{p_i\}_{i=0}^k \subset \mathbb{R}^4$ and a number $\beta > 0$, one says that $\{p_i\}_{i=0}^k$ is β-linearly independent, if for every $j \in \{0, 1, \cdots, k\}$, the distance between p_j and the affine subspace spanned by $\{p_i\}_{i=0}^k \backslash \{p_j\}$ is at least β.*

Given a set $F \subset \mathbb{R}^4$, one says that F β-spans a k-dimsensional affine subspace, if there exit $(k+1)$ points in F that are β-linearly independent.

Remark. β-linear independence is only defined for subsets of the Euclidean space with respect to the Euclidean metric.

Lemma 8.2. *If F is a bounded set that does not β-span a k-dimensional affine space, then there exists a $(k-1)$-dimensional affine space V, such that F is contained in the 2β-neighborhood of V.*

Proof. For k points $\{q_1, \cdots, q_k\}$ in $\mathbb{R}^4$, let $V(q_1, \cdots, q_k)$ be the volume of the $(k-1)$ dimensional simplex spanned by these points. Let $\{p_1, \cdots, p_k\} \subset F$ be k points in F such that

$$V(p_1, \cdots, p_k) \ge \frac{1}{2} \sup_{q_1, \cdots, q_k \in F} V(q_1, \cdots, q_k). \tag{27}$$

If the volume $V(p_1, \cdots, p_k)$ is zero, then F is contained in a $(k-1)$-dimensional affine subspace, and the statement is trivial. If the volume is positive, then the set $\{p_1, \cdots, p_k\}$ spans a $k-1$ dimensional affine space V. If F is contained in the 2β neighborhood of V, then the statement is verified. Otherwise, there exists a point $p_{k+1} \in F$, such that the distance of p_{k+1} and V is greater than 2β. Let d_j be the distance between p_j and the affine subspace spanned by $\{p_i\}_{i=0}^{k+1} \backslash \{p_j\}$, then $d_{k+1} \ge 2\beta$. By (27), $2d_j \ge d_{k+1}$ for every j. Therefore $\{p_1, \cdots, p_{k+1}\}$ is β-linearly independent, and that contradicts the assumption on F. □

As in Section 7, use the normal coordinate centered at x_0 to identify $B_{x_0}(32R)$ with the ball $\bar{B}(32R)$ in $\mathbb{R}^4$. Recall that by the assumption (2),

$$\Big(\frac{11}{12}\Big)^2 \leq \kappa_{x_0}(z) \leq K_{x_0}(z) \leq \Big(\frac{12}{11}\Big)^2,$$

where $\kappa_{x_0}(z)$ and $K_{x_0}(z)$ are the upper and lower bound of the eigenvalues of the metric matrix at $z \in \bar{B}_x(32R)$.

The compactness property of $\mathbb{Z}/2$ harmonic spinors leads to the following lemma.

Lemma 8.3. *Let $\beta, \bar{\beta}, \tilde{\beta} \in (0,1)$ be given. Then there exits $\delta > 0$, depending on $\beta, \bar{\beta}, \tilde{\beta}$, the upper bound Λ of the frequency function, the value of R, the curvatures of X and $\mathcal{V}$, and the constant ϵ in Assumption 1.2, such that the following holds. If $x \in \bar{B}(R)$, $r \leq \delta$, and $\{p_1, p_2, p_3\}$ is a set of $\bar{\beta}r$-linearly independent points in $\bar{B}_x(r)$, such that*

$$N_\phi(p_i, 2r) - N_\phi(p_i, \tilde{\beta}r) < \delta \quad i = 1, 2, 3.$$

Let V be the affine space spanned by p_1, p_2, p_3. Then the set $Z \cap \bar{B}_x(r)$ is contained in the βr neighborhood of $V \cap \bar{B}_x(r)$.

Proof. Assume such δ does not exist. Then there exist sequences $\{p_i^{(n)}\}_{i=1}^3$, x_n, and r_n, such that $r_n \to 0$, the points $\{p_i^{(n)}\}_{i=1}^3$ are contained in $\bar{B}_{x_n}(r_n)$ and are $\bar{\beta}r_n$-linearly independent, and

$$N_\phi(p_i^{(n)}, 2r_n) - N_\phi(p_i^{(n)}, \tilde{\beta}r_n) < \frac{1}{n} \quad i = 1, 2, 3,$$

and there exists $y_n \in Z$ such that the distance from y_n to the affine space spanned by $\{p_i^{(n)}\}_{i=1}^3$ is greater than βr_n.

Let $\sigma = 12/11$. Rescale the balls $\bar{B}_{x_n}(10\sigma^2 r_n)$ to radius $10\sigma^2$, and normalize the section U. Corollary 5.2 then gives a limit section U^* which satisfies the following properties:

1) U^* is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation invariant Clifford multiplication. U^* satisfies Assumption 1.2.
2) There exist points $p_1^*, p_2^*, p_3^* \in \bar{B}(1)$, such that they are $\bar{\beta}$-linearly independent, and

$$N_\phi(p_i^*, 2) - N_\phi(p_i^*, \tilde{\beta}) = 0 \quad i = 1, 2, 3, \tag{28}$$

3) Let V^* be the affine space spanned by $\{p_i^*\}_{i=1}^3$. There exits a point $q \in \bar{B}(1)$ in the zero set of U^*, such that the distance from q to $V^* \cap \bar{B}(1)$ is at least β.

Since U^* is defined on a flat manifold with flat bundle, remark 4.6 indicates that for U^*,

$$\partial_r N_\phi(x,r) = \frac{2}{rH_\phi(x,r)} \times \int -\phi'\Big(\frac{d(x,y)}{r}\Big) d(x,y)^{-1} |\nabla_{\eta_x} u(y) - N_\phi(x,r)u(y)|^2 \, dy.$$

Therefore equation (28) implies that for $i \in \{1,2,3\}$, the section U^* is homogeneous on $\bar{B}_{p_i^*}(2) - \bar{B}_{p_i^*}(\tilde{\beta})$ with respect to the center p_i^*. The unique continuation property for solutions to the Dirac equation implies that U^* is homogeneous on $\bar{B}(2)$ with respect to p_i^*. An elementary argument (see for example [4, Lemma 6.8]) then shows that the section U^* is zero on the affine space V^*, and that U^* is invariant in the directions parallel to V^*. Therefore, property (3) of U^* implies that U^* is zero on a 3-dimensional affine subspace, which contradicts Theorem 1.3. □

Similarly, one has

Lemma 8.4. *Let $\beta, \bar{\beta}, \tilde{\beta} \in (0,1)$ and $\tau > 0$ be given. Then there exits $\delta > 0$, depending on $\beta, \bar{\beta}, \tilde{\beta}, \tau$, the upper bound Λ of the frequency function, the value of R, the curvatures of X and $\mathcal{V}$, and the constant ϵ in Assumption 1.2, such that the following holds. Assume $x \in \bar{B}(R)$, and $r \leq \delta$, and $\{p_1, p_2, p_3\}$ is a set of points in $\bar{B}_x(r)$ that is $\bar{\beta}r$-linearly independent, such that*

$$N_\phi(p_i, 2r) - N_\phi(p_i, \tilde{\beta}r) < \delta \quad i = 1,2,3.$$

Let V be the affine space spanned by $\{p_i\}$. Then for all $y, y' \in \bar{B}_x(r) \cap Z$, one has

$$|N_\phi(y, \beta r) - N_\phi(y', \beta r)| < \tau.$$

Proof. Assume such δ does not exist, then arguing as before, one obtains a 2-valued section U^* on $\bar{B}(4)$ with the following properties:

1) U^* is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation invariant Clifford multiplication. U^* satisfies Assumption 1.2.

2) There exist points $p_1^*, p_2^*, p_3^* \in \bar{B}(1)$, such that they are $\bar{\beta}$-linearly independent, and

$$N_\phi(p_i^*, 2) - N_\phi(p_i^*, \tilde{\beta}) = 0 \quad i = 1, 2, 3, \tag{29}$$

3) Let Z^* be the zero set of U^*. There exist $y, y' \in \bar{B}(1) \cap Z^*$, such that

$$|N_\phi(y, \beta) - N_\phi(y', \beta)| \geq \tau.$$

However, as in the proof of the previous lemma, the first two properties imply that U^* is invariant in the directions parallel to the plane V^* spanned by p_1^*, p_2^*, p_3^*, and $Z^* \subset V^*$, which contradicts property (3). □

9. Rectifiability and the Minkowski bound

This section only will work in the Euclidean metric. To simplify the notation, for the rest of this section, we will use $B_x(r)$ and $B(r)$ to denote the Euclidean balls.

Definition 9.1. *Let $\mathcal{Z}$ be a Borel subset of $B(R) \subset \mathbb{R}^4$. A function $\mathcal{I}(x, r)$ defined for $x \in \mathcal{Z}$ and $r \leq 128R$ is called a taming function for $\mathcal{Z}$, if the following conditions hold.*

1) *$\mathcal{I}(x, r)$ is non-negative, bounded, continuous, and non-decreasing in r.*
2) *Given $\beta, \bar{\beta} \in (0, 1)$ and $\tau > 0$, there exists $\epsilon(\beta, \bar{\beta}, \tau) > 0$, depending on $\beta, \bar{\beta}, \tau$, such that the following holds. Assume $x \in B(R)$, $r \leq R$, and $\{p_1, p_2, p_3\}$ is a set of points in $B_x(r)$ that is $\bar{\beta}r$-linearly independent, such that*

$$\mathcal{I}(p_i, 2r) - \mathcal{I}(p_i, \beta r/2) < \epsilon(\beta, \bar{\beta}, \tau) \quad i = 1, 2, 3.$$

Then for all $y, y' \in B_x(r) \cap \mathcal{Z}$, one has

$$|\mathcal{I}(y, \beta r/2) - \mathcal{I}(y', \beta r/2)| < \tau.$$

3) *There exists a constant C, such that for every Radon measure μ supported in $\mathcal{Z}$, the following inequality holds for every $x \in B(2R)$ and $r \leq 2R$:*

$$D_\mu^2(x, r) \leq \frac{C}{r^2} \int_{\bar{B}_x(r)} [\mathcal{I}(z, 32r) - \mathcal{I}(z, 2r)]\, d\mu(z).$$

The following result follows almost verbatim from [4], and a large part of the argument is originated from [9]. We will give a sketch of the proof for the reader's convenience. For more details, the reader may refer to Sections 7 and 8 of [4].

Theorem 9.2 ([9], [4]). *Assume $\mathcal{Z}$ is a Borel subset of $B(R)$ and there exists a taming function $\mathcal{I}(x,r)$ for $\mathcal{Z}$. Then the set $\mathcal{Z}\cap B(R/2)$ is 2-rectifiable and has finite 2-dimensional Minkowski content.*

The proof of Theorem 9.2 makes use of the following Reifenberg-type theorem. We state the theorem for the cases of dimension 4 and codimension 2.

Theorem 9.3 ([9], Theorem 3.4). *There exist universal constants $K_0>0$ and $\delta_0>0$ such that the following holds. Assume $\{B_{x_i}(r_i)\}$ is a collection of balls in $B(2R)$, such that $\{B_{x_i}(r_i/4)\}$ are disjoint. Define a measure $\mu=\sum_i r_i^2\delta_{x_i}$. Suppose*

$$\int_{B_x(r)}\int_0^r \frac{D_\mu^2(z,s)}{s}\,ds d\mu(z) < \delta_0 r^2$$

for every $B_x(r)\subset B(2R)$, then $\mu(B(R))\le K_0R^2$.

Proof of Theorem 9.2. Assume $B_x(r)\subset B(R)$. If one rescales $B_x(r)$ to $B(R)$, then the function $\mathcal{I}'(y,s)=\mathcal{I}(x+(ry)/R, sr/R)$ is a taming function for $[(\mathcal{Z}-x)\cdot(R/r)]\cap B(R)$ with the same function $\epsilon(\beta,\bar\beta,\tau)$ and constant C. Therefore Definition 9.1 is invariant under rescaling, thus one only needs to consider the case for $R=2$.

Let $\beta=1/10$, let $\bar\beta\le 1/100$ be a positive universal constant, let $\tau>0$ be a constant that depends on $\bar\beta$ and C, and let $\delta>0$ be a constant that depends on $\bar\beta,\tau$, and ϵ and C from Definition 9.1. The exact values of $\bar\beta,\tau$ and δ will be determined later in the proof.

Let Λ be an upper bound of $\mathcal{I}$, namly $\Lambda\ge\sup_{x\in\mathcal{Z},r\le 128R}\mathcal{I}(x,r)=\sup_{x\in\mathcal{Z}}\mathcal{I}(x,256)$.

Define

$$D_\delta(r)=B(R/2)\cap\{x\in\mathcal{Z}\,|\,\mathcal{I}(x,\beta r/2)\ge\Lambda-\delta\}. \tag{30}$$

Define

$$W_{r_1}^{r_2}(x)=\mathcal{I}(x,r_1)-\mathcal{I}(x,r_2).$$

If $\{B_{x_i}(r_i)\}$ is a family of balls, we call the sum $\sum_i r_i^2$ its 2-dimensional volume.

Step 1. First, require that $\delta < \epsilon(\beta, \bar{\beta}, \tau)$. For $B_x(r) \subset B(2)$, and a set $A \subset \mathcal{Z}$, define an operator $\mathcal{F}_A$, which turns $B_x(r)$ into a finite set of balls. It has the property that either $\mathcal{F}_A(B_x(r)) = \{B_x(r)\}$, or $\mathcal{F}_A(B_x(r))$ is a family of balls with radius βr. In either case, the balls in $\mathcal{F}(B_x(r))$ will cover the set $A \cap \mathcal{Z}$. The operator $\mathcal{F}_A$ is defined as follows. If $A \cap D_\delta(r)$ does not $\bar{\beta}r$-span a 2-dimensional affine space, then it is called "bad". Otherwise, it is called "good". In the bad case, define $\mathcal{F}_A(B_x(r)) = \{B_x(r)\}$. In the good case, cover $A \cap \mathcal{Z}$ by a family of balls $\{B_{x_i}(\beta r)\}$ with the following properties

1) The distance between x_i and x_j is at least $\beta r/2$ for $\forall i \neq j$,
2) Each x_i is an element of $A \cap \mathcal{Z}$.

Define $\mathcal{F}_A(B_x(r))$ to be the family $\{B_{x_i}(\beta r)\}$.

Obviouly the descriptions above do not uniquely specify the operator $\mathcal{F}_A$. When there are more than one possibilities, choose one arbitrarily.

If $B_x(r)$ is a good ball, let $p_1, p_2, p_3 \in D_\delta(r) \cap B_x(r)$ be three points that $\bar{\beta}r$ span a plane, let $\mathcal{F}_A(B_x(r)) = \{B_{x_i}(\beta r)\}$. By Condition (2) of Definition 9.1,

$$|\mathcal{I}(x_i, \beta r/2) - \mathcal{I}(p_i, \beta r/2)| \leq \tau.$$

Therefore by (30),

$$\mathcal{I}(x_i, \beta r/2) \geq \Lambda - \delta - \tau. \tag{31}$$

The operator $\mathcal{F}_A$ can be extended to act on a collection of balls. Assume $\{B_{x_i}(r)\}_{i=1}^n$ is a collection of balls with the same radius. Let $A \subset \bigcup B_{x_i}(r) \cap \mathcal{Z}$. Assume $\{B_{x_i}(r)\}_{i=1}^k$ are the good balls, and $\{B_{x_i}(r)\}_{i=k+1}^n$ are the bad balls. Then there exists a collection of balls $\{B_{y_j}(\beta r)\}$, such that

1) $\{B_{y_j}(\beta r)\}$ covers $\bigcup_{i=1}^k (A \cap B_{x_i}(r))$.
2) $|y_j - y_l| \geq \beta r/2$, for $\forall j \neq l$.
3) $y_j \in \bigcup_{i=1}^k A \cap B_{x_i}(r)$, for $\forall j$.

Inequality (31) still holds when x_i is replaced by y_j. Define $\mathcal{F}_A\{B_{x_i}(r)\}$ to be the union of $\{B_{y_j}(\beta r)\}$ and $\{B_{x_i}(r)\}_{i=k+1}^n$.

Step 2. Let $N > 0$ be a positive integer. Let $A_0(x, r) = \mathcal{Z} \cap B_x(r)$. Apply the operator $\mathcal{F}_{A_0}$ to $B_x(r)$ to obain a set of balls, which we denote by

$\mathcal{S}_1(x,r)$. Assume $\mathcal{S}_1(x,r)$ splits to two sets $\mathcal{S}_1(x,r) = \mathcal{S}_{1,g}(x,r) \bigcup \mathcal{S}_{1,b}(x,r)$, where $\mathcal{S}_{1,g}(x,r)$ is the collection of good balls and $\mathcal{S}_{1,b}(x,r)$ is the collection of bad balls. Let

$$A_1(x,r) = A_0(x,r) - \bigcup_{B_{x_i}(r_i)\in\mathcal{S}_{1,b}(x,r)} B_{x_i}(r_i).$$

Apply $\mathcal{F}_{A_1(x,r)}$ to $\mathcal{S}_{1,g}(x,r)$, we obtain a new set of balls

$$\mathcal{S}_2(x,r) = \mathcal{F}_{A_1(x,r)}(\mathcal{S}_{1,g}(x,r)) \bigcup \mathcal{S}_{1,b}(x,r).$$

Similarly, write $\mathcal{S}_2(x,r) = \mathcal{S}_{2,g}(x,r) \bigcup \mathcal{S}_{2,b}(x,r)$, where $\mathcal{S}_{2,g}(x,r)$ is the collection of good balls and $\mathcal{S}_{2,b}(x,r)$ is the collection of bad balls, and define

$$A_2(x,r) = A_1(x,r) - \bigcup_{B_{x_i}(r_i)\in\mathcal{S}_{2,b}(x,r)} B_{x_i}(r_i),$$

and $\mathcal{S}_3 = \mathcal{F}_{A_2}(\mathcal{S}_{2,g}) \bigcup \mathcal{S}_{2,b}$. Repeat the procedure N times to obtain a set of balls $\mathcal{S}_N(x,r)$.

The family $\mathcal{S}_N(x,r)$ has the following property. If $B_{x_1}(r_1)$ and $B_{x_2}(r_2)$ are two distinct elements of $\mathcal{S}_N(x,r)$, then

$$|x_1 - x_2| \geq (r_1 + r_2)/4. \tag{32}$$

Inequality (32) can be proved by induction. For $N = 1$, it follows from the definition of $\mathcal{F}_A$. Assume (32) holds for $N-1$, and write $\mathcal{S}_N = \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g}) \bigcup \mathcal{S}_{N-1,b}$. Let $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{S}_N$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g})$, then (32) follows from the definition of $\mathcal{F}$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{S}_{N-1,b}$, then (32) follows from the induction hypothesis. If $B_{x_1}(r_1) \in \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g})$, $B_{x_2}(r_2) \in \mathcal{S}_{N-1,b}$, then $x_1 \notin B_{x_2}(r_2)$. By the construction of $\mathcal{F}$, one has $r_1 \leq \beta r_2$. Since $\beta = 1/10$, one has $|x_1 - x_2| \geq r_2 \geq (r_1 + r_2)/2$.

By (31), either $\mathcal{S}_N = \{B_x(r)\}$, or

$$\mathcal{I}(x_i, r_i/2) \geq \Lambda - \delta - \tau, \quad \forall\, B_{x_i}(r_i) \in \mathcal{S}_N. \tag{33}$$

Step 3. We claim that there exists a universal constant $K_1 > 1$, such that for τ and δ sufficiently small, we have

$$\sum_{B_{x_i}(r_i)\in\mathcal{S}_N(x,r)} r_i^2 < K_1\, r^2. \tag{34}$$

Without loss of generality, assume $\mathcal{S}_N(x,r) \neq \{B_x(r)\}$. Let $r_j = \beta^{N-j}\, r$. Define Radon measures

$$\mu = \sum_{B_y(s)\in \mathcal{S}_N(x,r)} s^2 \delta_y,$$

$$\mu_j = \sum_{B_y(s)\in \mathcal{S}_N(x,r), s\le r_j} s^2 \delta_y.$$

Notice that by (32), there exists a universal constant K_2 such that

$$\mu_0(B_x(r_0)) \le K_2\, r_0^2, \quad \forall x. \tag{35}$$

Let K_0 be the constant given by Theorem 9.3, let $K_3 = \max\{K_0, K_2\}$. We prove that if τ, δ are sufficiently small, then for every $j = 0, 1, \cdots, N-3$, and every $B_y(r_j) \subset B_x(2r)$, one has

$$\mu_j(B_y(r_j)) \le K_3\, r_j^2. \tag{36}$$

The claim is proved by induction on j. The case for $j = 0$ follows from (35). Assume that the claim is proved for $0, 1, \cdots, j$, and $j < N-3$. Then there exists a universal constant $M > 1$, such that for every $y \in B_x(2r)$, $k \le j+1$, and $s \in [r_k/2, 2r_k]$,

$$\mu_{k+3}(B_y(s)) \le M\,(K_3+1)\, s^2 \tag{37}$$

We want to use Theorem 9.3 and (37) to prove

$$\mu_{j+1}(B_y(r_{j+1})) \le K_3\, r_{j+1}^2, \text{ for } \forall B_y(r_{j+1}) \subset B_x(2r).$$

If $\mu_{j+1}(B_y(r_{j+1})) = 0$, the inequality is trivial. From now on assume $\mu(B_y(r_{j+1})) > 0$. Since $r_{j+1} \le r_{N-3} = r/8$, and $\mathrm{supp}\mu \subset B_x(r)$, we have $B_y(4r_{j+1}) \subset B_x(2r)$.

Notice that for $B_{x_i}(s_i) \in \mathcal{S}_N$, if $t < \min_k |x_i - x_k|$, then

$$D_\mu^2(x_i, t) = 0.$$

Define

$$\overline{W}_{2t}^{32t}(x_i) = \begin{cases} 0 & \text{if } t < s_i/4, \\ W_{2t}^{32t}(x_i) & \text{if } t \ge s_i/4. \end{cases}$$

Inequality (32) and Condition (3) of Definition 9.1 gives

$$D_\mu^2(q,t) \le C \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3}\, d\mu(p) \tag{38}$$

for every (q,t).

For $B_z(s) \subset B_y(2r_{j+1})$, assume $s \in [r_k/2, 2r_k]$ for $k \le j+1$. Inequality (38) gives

$$\begin{aligned} &\int_{B_z(s)} \int_0^s \frac{D^2_{\mu_{j+1}}(q,t)}{t}\, dt\, d\mu_{j+1}(q) \\ &\le C \int_{B_z(s)} \int_0^s \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3}\, d\mu_{j+1}(p)\, dt\, d\mu_{j+1}(q) \\ &\le C \int_{B_z(s)} \int_0^s \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3}\, d\mu_{k+3}(p)\, dt\, d\mu_{k+3}(q) \qquad (39) \\ &\le C \int_{B_z(2s)} \int_0^s \int_{B_p(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3}\, d\mu_{k+3}(q)\, ds\, d\mu_{k+3}(p) \\ &\le CM(K_3+1) \int_{B_z(2s)} \int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t}\, dt\, d\mu_{k+3}(p), \qquad (40) \end{aligned}$$

where inequality (39) follows from (32). For $p \in \operatorname{supp}\mu_{j+1}$, let s_p be the radius of the ball in $\mathcal{S}_N$ with center p. If $s \ge s_p/4$, then

$$\begin{aligned} \int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t}\, dti &= \int_{s_p/4}^s \frac{W_{2t}^{32t}(p)}{t}\, dt = \int_{2s}^{32s} \mathcal{I}(p,t)\, dt - \int_{s_p/a}^{16s_p} \mathcal{I}(p,t)\, dt \\ &\le W_{s_p/2}^{32s}(p) \int_2^{32} \frac{1}{t}\, dt \le \ln(16)\,(\delta+\tau). \qquad (41) \end{aligned}$$

The last inequality above follows from (33). Therefore, the right hand side of (40) is bounded by

$$\begin{aligned} &CM(K_3+1) \int_{B_z(2s)} \int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t}\, dt\, d\mu_{k+3}(p) \\ &\le CM(K_3+1)\, \mu_{k+3}(B_z(2s))\, \ln(16)\,(\tau+\delta) \\ &\le 4CM^2(K_3+1)^2 \ln(16)(\tau+\delta)\, s^2 \end{aligned}$$

Let δ_0 be the constant given by Theorem 9.3. Take

$$\tau < \frac{\delta_0}{8CM^2(K_3+1)^2\ln(16)},$$

and

$$\delta < \frac{\delta_0}{8CM^2(K_3+1)^2\ln(16)},$$

then the conditions of Theorem 9.3 are satisfied, therefore $\mu_{j+1}(B_y(r_{j+1})) \leq K_0\, r_{j+1}^2$. By induction, (36) is proved. Inequality (34) then follows from (36) by the the case of $j = N-3$.

Step 4. By Lemma 8.2, the result obtained from the previous steps can be summarized as follows. For any integer $N > 0$, and any ball $B_x(r)$, there is a covering of $\mathcal{Z} \cap B_x(r)$ by a family of balls $\mathcal{S}_N(x,r) = \{B_{x_i}(r_i)\}_i$, a splitting of $\mathcal{Z}$ into $\mathcal{Z} = \bigcup_i \mathcal{E}_i$, such that the following properties hold:

1) $\mathcal{E}_i \subset B_{x_i}(r_i)$.
2) The radius of each ball is at least $\beta^N r$.
3) For a all $B_{x_i}(r_i) \in \mathcal{S}_N$, either $r_i = \beta^N r$, or $r_i = \beta^j r$ for some integer $j < N$, and $\mathcal{E}_i \cap D_\delta(r_i)$ is contained in the $2\bar{\beta} r_i$ neighborhood of a line.
4) $\sum_i r_i^2 \leq K_1\, r^2$.

As a consequence,

Lemma 9.4. *There exists a universal constant $K_1 > 1$, and a constant δ, such that the following property holds. For any $B_x(r) \subset B(2)$, and $s \in (0,r)$, there exists a covering of $\mathcal{Z} \cap B_x(r)$ by balls $\mathcal{S} = \{B_{x_i}(r_i)\}_i$, such that*

1) *The radius of each ball is at least βs.*
2) *For each ball $B_{x_i}(r_i) \in \mathcal{S}$, either $r_i \leq s$, or $B_{x_i}(r_i) \cap D_\delta(r_i)$ is contained in the $2\bar{\beta} r_i$ neighborhood of a line.*
3) $\sum_i r_i^2 \leq K_1\, r^2$.

Step 5. We prove the following lemma

Lemma 9.5. *There exists a universal constant K_4, and a constant δ, such that the following property holds. For any $B_x(r) \subset B(2)$, and $s \in (0,r)$, there exists a splitting of $\mathcal{Z}$ into $\mathcal{Z} = \bigcup_i \mathcal{E}_i$, and a family of balls $\mathcal{S} = \{B_{x_i}(r_i)\}_i$, such that*

1) *$\mathcal{E}_i \subset B_{x_i}(r_i)$.*

2) *The radius of each ball is at least $4\bar{\beta}s$.*

3) *For every ball $B_{x_i}(r_i) \in \mathcal{S}$, either $r_i \in [4\bar{\beta}s, s]$, or $\mathcal{E}_i \cap D_\delta(r_i) = \emptyset$*

4) *$\sum_i r_i^2 \le K_4\, r^2$.*

Proof of Lemma 9.5. Notice that by the assumptions on β and $\bar{\beta}$, we have $4\bar{\beta} < \beta$.

If $\{B_{x_i}(r_i)\}_i$ is a covering of $\mathcal{Z} \cap B_x(r)$, such that there exists a splitting $\mathcal{Z} = \cup \mathcal{E}_i$, which satisfies the four properties given by Lemma 9.4 with respect to s, we say that $\{B_{x_i}(r_i)\}_i$ is an s-admissible covering of $B_x(r) \cap \mathcal{Z}$. Fix $s > 0$, by Lemma 9.4, s-admissible coverings of $B_x(r) \cap \mathcal{Z}$ exist.

Let $\{B_{x_i}(r_i)\}$ be an s-admissible covering of $B_x(r) \cap \mathcal{Z}$ with respect to $\{\mathcal{E}_i\}$. Then the family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ satisfies Conditions (1), (2) of Lemma 9.5, and $\sum_i r_i^2 \le K_1\, r^2$. However, it may not satisfy Condition (3). In the following, we will give a procedure to adjust the family, such that at each step the covering still satisfies property (2) of s-admissibility, and after finitely many steps of adjustments, the family will satisfy property (3) of Lemma 9.5. At the same time, $\sum_i r_i^2$ is being contorlled throughout the adjustments.

Assume $\{B_{x_i}(r_i)\}$ is an s-admissible covering of $B_x(r) \cap \mathcal{Z}$, and $\mathcal{E}_i \subset B_{x_i}(r_i)$, $B_x(r) \cap \mathcal{Z} = \bigcup \mathcal{E}_i$. Assume $(\mathcal{E}_0, B_{x_0}(r_0))$ does not satisfy property (3) of Lemma 9.5. Then $r_0 > s$.

By property (2) of s-admissibility, $B_{x_0}(r_0) \cap D_\delta(r_0)$ is contained in the $2\bar{\beta}r_0$ neighborhood of a line. Thus one can cover $B_{x_0}(r_0) \cap D_\delta(r_0)$ by a family of no more than $[10/\bar{\beta}]$ balls with radius $4\bar{\beta}r_0$. Let $\{B_{y_j}(t_j)\}$ be this family. If $4\bar{\beta}r_0 > s$, apply Lemma 9.4 again to each ball $B_{y_j}(t_j)$ and replace it with an s-admissible covering of $B_{y_j}(t_j) \cap D_\delta(r_0)$. Otherwise keep the family $\{B_{y_j}(t_j)\}$ as it is. Let $\{B_{z_j}(l_j)\}$ be the result of this procedure. Then $\{B_{z_j}(l_j)\}$ covers $B_{x_0}(r_0) \cap D_\delta(r_0)$, and it has the following properties

1) $4\bar{\beta}s \le l_j \le 4\bar{\beta}r_0$ for each j,

2) $\sum_j l_j^2 \le [10/\bar{\beta}] \cdot K_1\, (4\bar{\beta}r_0)^2$.

Take $\bar{\beta} \le 1/(320K_1)$, then $\sum_j l_j^2 \le \frac{1}{2} r_0^2$.

The adjustment of the family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ is defined as follows. First, remove $(\mathcal{E}_0, B_{x_0}(r_0))$ from the family, and add $(\mathcal{E}_0 \backslash D_\delta(r_0), B_{x_0}(r_0))$ into the family. Next, add the family $\{(\mathcal{E}_0 \cap B_{z_j}(l_j), B_{z_j}(l_j))\}$ constructed from the previous paragraph into this family.

This adjustment replaces an element $(\mathcal{E}_0, B_{x_0}(r_0))$ which does not satisfy property (3) of Lemma 9.5 by a family of balls, such that the biggest ball in this family has the same radius r_0 and satisfies property (3). The rest of the balls have radius in the interval $[4\bar{\beta}s, 4\bar{\beta}r_0]$ and their 2-dimensional volume is bounded by $\frac{1}{2}r_0^2$. Moreover, the new family still satisfies property (2) of Lemma 9.4. Therefore, after finitely many times of adjustments, we will obtain a family that satisfies conditions (1), (2), (3), with 2-dimensional volume

$$\sum_i r_i^2 \leq 2K_1 r^2,$$

hence the lemma is proved. □

Step 6. Given $s \in (0,1)$, we use Lemma 9.5 to construct a covering of $\mathcal{Z} \cap B(1)$ by a family of balls $\{B_{x_i}(r_i)\}$ with radius $r_i \in [4\bar{\beta}s, s]$, such that the 2-dimensional volume of the covering is bounded.

We call a family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ a split-covering of a set A, if $\mathcal{E}_i \subset B_{x_i}(r_i)$, and $A = \bigcup \mathcal{E}_i$.

If a split-covering of $\mathcal{Z} \cap B_x(r)$ satisfies the properties given by Lemma 9.5, we say that it is strongly s-admissible.

Let $\mathcal{S}$ be a strongly s-admissible split-covering of $\mathcal{Z} \cap B(1)$. For every $B_{x_i}(r_i) \in \mathcal{S}$, if $r_i \leq s$, we say it is of type I. Otherwise, we say it is of type II. Assume $B_{x_i}(r_i)$ is a ball of type II, then the function $\mathcal{I}(x,r)$ is at most $\Lambda - \delta$ for $x \in \mathcal{E}_i$, $r_i \leq \beta r_i/2$. There exists a universal constant L such that $\mathcal{E}_i$ can be covered by L balls $B_{y_j}(\beta r_i/512)$ with radius $(\beta r_i/512)$. Therefore, for each ball, the set $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$ has a strongly s-admissible split-covering, with Λ replaced by $\Lambda - \delta$.

Change $(B_{x_i}(r_i), \mathcal{E}_i)$ to the union of the L strongly s-admissible split-coverings of $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$, we obtain a split-covering of $\mathcal{E}_i$ with 2-dimensional volume at most $LK_4(\beta r_i/512)^2$. Define an operation $\mathcal{G}$ on $\mathcal{S}$, such that $\mathcal{G}(\mathcal{S})$ is constructed from $\mathcal{S}$ by replacing every type II element in $\mathcal{S}$ with the union of the L split-coverings described above.

Notice that for the balls $B_{y_j}(\beta r_i/512)$, the upper bound Λ is replaced by $\Lambda - \delta$. Therefore, this procedure can only be carried for at most $N = \lceil \frac{\Lambda}{\delta} \rceil$ times. After that, every ball in $\mathcal{G}^{(N)}(\mathcal{S})$ is of type I. Namely, every ball in $\mathcal{G}^{(N)}(\mathcal{S})$ has radius in the interval $[4\bar{\beta}s, s]$.

Let V_n be the 2 dimensional volume of $\mathcal{G}^{(n)}(\mathcal{S})$, then we have

$$V_{n+1} \leq (1 + LK_4(\beta/512)^2)V_n.$$

Therefore the total 2-dimensional volume of $\mathcal{G}^{(n)}(\mathcal{S})$ is bounded by

$$V_n \le (1 + LK_4(\beta/512)^2)^N K_4.$$

Since s can be taken to be arbitrarily small, the Minkowski content of $\mathcal{Z} \cap B(1)$ is bounded by a contant K depending on Λ, ϵ and C.

By rescaling, we conclude that the Minkowski content of $\mathcal{Z} \cap B_x(r)$ is bounded by $K\, r^2$. Since the Minkowski content bounds the Hausdorff measure, there exists a constant K' depending on Λ, ϵ and C, such that

$$\mathcal{H}_2(\mathcal{Z} \cap B_x(r)) \le K'\, r^2. \tag{42}$$

Step 7. So far we have been using Theorem 9.3 to prove an upper bound for the Minkowski content of $\mathcal{Z}$. It turns out that a more careful look at the proof of Theorem 9.3 also gives a rectifiable map for $\mathcal{Z}$, hence it concludes the proof of Theorem 9.2.

Another way to show the rectifiability of $\mathcal{Z}$ is to cite the following theorem of Azzam and Tolsa. This method takes an unnecessary detour, but it allows us to finish the proof without citing implicit statements from [9].

Theorem 9.6 ([3], Corollary 1.3). *Assume $S \subset B(2)$ is a $\mathcal{H}_2$-measurable set and has finite Hausdorff measure, let λ be the restriction of $\mathcal{H}_2$ to S. Assume that for λ-a.e. z,*

$$\int_0^1 \frac{D_\lambda^2(z,s)}{s}\, ds < +\infty,$$

then S is 2-rectifiable.

Now invoke Theorem 9.6 and let S be the set $\mathcal{Z}$. By (42),

$$\begin{aligned}
\int_{B(1)} \int_0^1 \frac{D_\lambda^2(z,s)}{s}\, ds\, d\lambda(z) &\le C \int_{B(1)} \int_0^1 \int_{B_z(s)} \frac{W_{2s}^{32s}(p)}{s^3}\, d\lambda(p)\, ds\, d\lambda(z) \\
&\le C \int_{B(2)} \int_0^1 \int_{B_p(s)} \frac{W_{2s}^{32s}(p)}{s^3}\, d\lambda(z)\, ds\, d\lambda(p) \\
&\le CK' \int_{B(2)} \int_0^1 \frac{W_{2s}^{32s}(p)}{s}\, ds\, d\lambda(p)
\end{aligned}$$

The same estimate as (41) gives

$$\int_0^1 \frac{W_{2s}^{32s}(p)}{s}\, ds \leq \ln(16)\Lambda.$$

Thus

$$CK' \int_{B(2)} \int_0^1 \frac{W_{2s}^{32s}(p)}{s}\, ds\, d\lambda(p) \leq 4C(K')^2 \ln(16)\Lambda < \infty.$$

Therefore, the conditions of Theorem 9.6 are satisfied for $\mathcal{Z} \cap B(1)$, hence $\mathcal{Z} \cap B(1)$ is a rectifiable set, and the result is proved. □

Proof of Theorem 1.4. Let R_0 be the constant given by Proposition 7.2. Cover $B_{x_0}(R)$ by finitely many Euclidean balls of radius $R_0/32$. Let $B_{x_i}(R_0/32)$ be such a ball, we claim that there exists a constant C such that

$$\mathcal{I}(x,r) = N_\phi(x,r) + Cr^2$$

is a taming function for $Z \cap B_{x_i}(R_0/16)$ on the ball $B_{x_i}(R_0/16)$.

In fact, it follows from the definition that $N_\phi(x,r)$ is non-negative and continuous. By equation (17), there exists $C_1 > 0$ such that $\mathcal{I}_1(x,r) = N_\phi(x,r) + C_1 r^2$ is increasing in r. By Proposition 7.2, there exists C_2, such that for $\mathcal{I}_2(x,r) = \mathcal{I}_1(x,r) + C_2 r^2$, one has

$$D_\mu^2(x,r) \leq \frac{C_1}{r^2} \int_{B_x(r)} [\mathcal{I}_2(32r) - \mathcal{I}_2(2r)] d\mu(x)$$

for every Radon measure supported in $Z \cap B_{x_i}(R_0)$ and $r \leq 8R_0$, thus $\mathcal{I}_2$ satisfies Condition (3) of Definition 9.1.

Notice that since $\mathcal{I}_1(x,r)$ is increasing in r, for $\tilde{\beta} > 0$, the inequality

$$\mathcal{I}_2(x,2r) - \mathcal{I}_2(x,\tilde{\beta}r) < \delta$$

implies that $r < \sqrt{\delta/(4C_2)}$. Therefore, Lemma 8.4 implies $\mathcal{I}_2$ satisfies Condition (2) of Definition 9.1.

In conclusion, $\mathcal{I}_2(x,r)$ is a taming function for Z on $B_{x_i}(R_0/16)$, therefore Theorem 1.4 follows from Theorem 9.2. □

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