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Stochastically complete submanifolds with parallel mean curvature vector field in a Riemannian space form

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In this paper, we deal with stochastically complete submanifolds M^n immersed with nonzero parallel mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$. In this setting, we use the weak Omori-Yau maximum principle jointly with a suitable Simons type formula in order to show that either such a submanifold M^n must be totally umbilical or it holds a sharp estimate for the norm of its total umbilicity tensor, with equality if and only if the submanifold is isometric to an open piece of a hyperbolic cylinder $\mathbb{H}^1\left(-\sqrt{1+r^2}\right) \times \mathbb{S}^{n-1}(r)$, when $c = -1$, a circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$, when $c = 0$, and a Clifford torus $\mathbb{S}^1\left(\sqrt{1-r^2}\right) \times \mathbb{S}^{n-1}(r)$, when $c = 1$.

1. Introduction

The problem of characterizing hypersurfaces immersed with constant mean curvature in a Riemannian space form constitutes a classical thematic into the theory of isometric immersions. In this branch, Klotz and Osserman [10] characterized totally umbilical spheres and circular cylinders as the only complete surfaces immersed into the Euclidean 3-space $\mathbb{R}^3$ with nonzero constant mean curvature and whose Gaussian curvature does not change sign. Afterwards, Hoffman [8] and Tribuzy [21] gave an extension of that result to the case of surfaces with constant mean curvature in the Euclidean 3-sphere $\mathbb{S}^3$ and in the hyperbolic 3-space $\mathbb{H}^3$, respectively. Later on, Alencar and do Carmo [1] showed that a constant mean curvature compact hypersurface of the $(n+1)$-sphere $\mathbb{S}^{n+1}$ must be either totally umbilical or isometric to certain Clifford torus, provided that the traceless part of the corresponding second fundamental form satisfies an appropriate boundedness which amounts to a previous one due to Simons [19] related to the case of compact minimal hypersurfaces in $\mathbb{S}^{n+1}$.

More recently, Alías and García-Martínez [2] extended the results of [1, 8, 10, 21] considering the so-called stochastically complete hypersurfaces immersed with constant mean curvature in a Riemannian space form. We recall that a Riemannian manifold M^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x,t) \in M^n \times (0,+\infty)$, the heat kernel $p(x,y,t)$ of the Laplace-Beltrami operator Δ satisfies the conservation property

$$\int_M p(x,y,t)d\mu(y) = 1. \tag{1.1}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. Furthermore, for the Brownian motion on a manifold, the conservation property (1.1) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [4–6, 20]).

Proceeding with the picture described above, our purpose in this paper is to revisit the results of [2] in the higher codimensional case. More precisely, we will deal with stochastically complete submanifolds immersed with nonzero parallel mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{-1,0,1\}$. We recall that a submanifold has nonzero parallel mean curvature vector field when its mean curvature is a positive constant and its mean curvature vector field is parallel as a section of the normal bundle (for more details, see Section 3).

In this setting, we apply a crucial result due to Pigola, Rigoli and Setti which asserts that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle (cf. Theorem 1.1 of [16] or Theorem 3.1 of [17]) jointly with a suitable Simons type formula (cf. Proposition 1) in order to establish the following characterization result:

Theorem 1. *Let M^n be a stochastically complete submanifold immersed with nonzero parallel mean curvature vector field in a Riemannian space form $\mathbb{Q}_c^{n+p}$ of constant sectional curvature $c \in \{-1,0,1\}$, with $n \geq 4$ and such that its normalized scalar curvature satisfies $R \geq c$. When $c=-1$, assume in addition that $H > 1$. Then*

(i) *either $\sup_M |\Phi|^2 = 0$ and M^n is a totally umbilical submanifold,*

(ii) *or*

$$\sup_M |\Phi|^2 \geq \frac{n}{4(n-1)} \left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)H\right)^2 > 0. \tag{1.2}$$

Moreover, the equality holds in (1.2) *and this supremum is attained at some point of* M^n *if, and only if,* M^n *is isometric to an open piece of a*

(a) *hyperbolic cylinder* $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$, *with* $r > 0$, *when* $c = -1$.
(b) *circular cylinder* $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$, *with* $r > 0$, *when* $c = 0$.
(c) *Clifford torus* $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, *with* $0 < r < \sqrt{(n-1)/n}$, *when* $c = 1$.

Here, Φ stands for the traceless part of the second fundamental form of the submanifold M^n. When M^n is complete (which happens, for instance, when M^n is properly immersed), we obtain the following consequence of Theorem 1:

Corollary 1. *Let* M^n *be a complete submanifold immersed with nonzero parallel mean curvature vector field in a Riemannian space form* $\mathbb{Q}_c^{n+p}$ *of constant sectional curvature* $c = -1, 0, 1$, *with* $n \geq 4$ *and such that its normalized scalar curvature satisfies* $R \geq c$. *When* $c = -1$, *assume in addition that* $H > 1$. *Then*

(i) *either* $\sup_M |\Phi|^2 = 0$ *and* M^n *is a totally umbilical submanifold,*

(ii) *or the inequality* (1.2) *is satisfied. Moreover, the equality holds in* (1.2) *and this supremum is attained at some point of* M^n *if, and only if,* M^n *is isometric to a*
  (a) *hyperbolic cylinder* $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$, *with* $r > 0$, *when* $c = -1$.
  (b) *circular cylinder* $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$, *with* $r > 0$, *when* $c = 0$.
  (c) *Clifford torus* $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, *with* $0 < r < \sqrt{(n-1)/n}$, *when* $c = 1$.

Recall that a Riemannian manifold M^n is said to be *parabolic* if the constant functions are the only subharmonic functions on M^n which are bounded from above; that is, for a function $u \in \mathcal{C}^2(M)$

$$\Delta u \geq 0 \quad \text{and} \quad u \leq u^* < +\infty \quad \text{implies} \quad u = \text{constant}.$$

In this setting, we obtain the following consequence of Theorem 1 related to complete parabolic submanifolds of $\mathbb{Q}_c^{n+p}$:

Corollary 2. *Let* M^n *be a complete parabolic submanifold immersed with nonzero parallel mean curvature vector field in a Riemannian space form*

$\mathbb{Q}_c^{n+p}$ *of constant sectional curvature* $c = -1, 0, 1$, *with* $n \geq 4$ *and such that its normalized scalar curvature satisfies* $R \geq c$. *When* $c = -1$, *assume in addition that* $H > 1$. *Then*

(i) *either* $\sup_M |\Phi|^2 = 0$ *and* M^n *is a totally umbilical submanifold,*

(ii) *or the inequality* (1.2) *is satisfied. Moreover, the equality holds in* (1.2) *if, and only if,* M^n *is isometric to a*
  (a) *hyperbolic cylinder* $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$, *with* $r > 0$, *when* $c = -1$.
  (b) *circular cylinder* $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$, *with* $r > 0$, *when* $c = 0$.
  (c) *Clifford torus* $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$, *with* $0 < r < \sqrt{(n-1)/n}$, *when* $c = 1$.

The proofs of Theorem 1 and Corollaries 1 and 2 in Section 4.

We close this section pointing out that, compared with recent rigidity results concerning closed submanifolds with parallel mean curvature vector field in $\mathbb{S}^{n+p}$ which have as hypothesis a previous control of the square length of the second fundamental form of the submanifold through the mean curvature and the second largest eigenvalue of the fundamental matrix (cf. [11, 14]), our results offer the advantage that they do not suppose such a control. Furthermore, we also note that our constraint on the normalized scalar curvature already appears in several papers of the current literature (cf. [3, 7, 12, 13]).

2. Preliminaries

Let M^n be an n-dimensional connected submanifold immersed in a space form $\mathbb{Q}_c^{n+p}$, with constant sectional curvature c. We choose a local field of orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in $\mathbb{Q}_c^{n+p}$, with dual coframe $\{\omega_1, \dots, \omega_{n+p}\}$, such that, at each point of M^n, $e_1, \dots, e_n$ are tangent to M^n and $e_{n+1}, \dots, e_{n+p}$ are normal to M^n. We will use the following convection for indices

$$1 \leq A, B, C, \dots \leq n+p, \quad 1 \leq i, j, k, \dots \leq n \quad \text{and} \quad n+1 \leq \alpha, \beta, \gamma, \dots n+p.$$

When restricting on M^n, the second fundamental form A, the curvature tensor R and the normal curvature tensor $R^{\perp}$ of M^n are given by

$$\omega_{i\alpha} = \sum_j h_{ij}^{\alpha} \omega_j, \quad A = \sum_{i,j,\alpha} h_{ij}^{\alpha} \omega_i \otimes \omega_j e_\alpha,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2}\sum_{k,l} R^\perp_{\alpha\beta kl}\omega_k \wedge \omega_l.$$

Moreover, the components h^α_{ijk} of the covariant derivative ∇A satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ki}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.1}$$

The Gauss equation is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.2}$$

In particular, the components of the Ricci tensor R_{ik} and the normalized scalar curvature R are given, respectively, by

$$R_{ik} = (n-1)\delta_{ik} + n\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk} \tag{2.3}$$

where $H^\alpha = \frac{1}{n}\sum_j h^\alpha_{jj}$, and

$$R = \frac{1}{(n-1)}\sum_i R_{ii}. \tag{2.4}$$

From (2.3) and (2.4), we get the following relation

$$n(n-1)R = n(n-1)c + n^2H^2 - S, \tag{2.5}$$

where $S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the norm square of the second fundamental form and, being $h = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n}\sum_\alpha \left(\sum_k h^\alpha_{kk}\right) e_\alpha$ the mean curvature vector field, $H = |h|$ is the mean curvature function of M^n.

By exterior differentiation of (2.1), we have the following Ricci identity

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R^\perp_{\beta\alpha kl}. \tag{2.6}$$

The Codazzi equation and the Ricci equation are given by

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik} \tag{2.7}$$

and

$$R^{\perp}_{\alpha\beta ij} = \sum_{k}(h^{\alpha}_{ik}h^{\beta}_{kj} - h^{\alpha}_{jk}h^{\beta}_{ki}). \tag{2.8}$$

3. A Simons type formula

From now on, we will deal with submanifolds M^n of $\mathbb{Q}^{n+p}_c$ having *nonzero parallel mean curvature vector field*, which means that the mean curvature function H is, in fact, a positive constant and that the corresponding mean curvature vector field h is parallel as a section of the normal bundle.

In this context, we can choose a local orthonormal frame $\{e_1, \ldots, e_{n+p}\}$ such that $e_{n+1} = \dfrac{h}{H}$. Thus,

$$H^{n+1} = \frac{1}{n}\mathrm{tr}(h^{n+1}) = H \quad \text{and} \quad H^{\alpha} = \frac{1}{n}\mathrm{tr}(h^{\alpha}) = 0,\ \alpha \geq n+2. \tag{3.1}$$

We will also consider the following symmetric tensor

$$\Phi = \sum_{\alpha,i,j}\Phi^{\alpha}_{ij}\omega_i \otimes \omega_j e_{\alpha}, \tag{3.2}$$

where $\Phi^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}$. Consequently, we have that

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^{\alpha}_{ij} = h^{\alpha}_{ij}, \quad n+2 \leq \alpha \leq n+p. \tag{3.3}$$

Let $|\Phi|^2 = \sum_{\alpha,i,j}(\Phi^{\alpha}_{ij})^2$ be the square of the length of Φ. From (2.5), it is not difficult to verify that Φ is traceless with

$$|\Phi|^2 = S - nH^2 = n(n-1)(c + H^2 - R). \tag{3.4}$$

Extending the ideas of [7], we obtain the following Simons type formula

Proposition 1. *Let M^n be an n-dimensional $(n \geq 2)$ submanifold immersed with nonzero parallel mean curvature vector field in a Riemannian space form $\mathbb{Q}^{n+p}_c$. Then, we have*

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + cn|\Phi|^2 + n\sum_{\beta,i,j,k}Hh^{n+1}_{ij}h^{\beta}_{jk}h^{\beta}_{ki} - \sum_{i,j,k,l}\left(\sum_{\alpha}h^{\alpha}_{ij}h^{\alpha}_{kl}\right)^2 - \sum_{i,j,\alpha,\beta}(R^{\perp}_{\alpha\beta ij})^2.$$

Proof. Taking into account that

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} + \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2, \tag{3.5}$$

where the Laplacian Δh^α_{ij} of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$, using Codazzi equation (2.7) into (3.5) we have

$$\frac{1}{2}\Delta S = \sum_{\alpha,i,j} h^\alpha_{ij}\left(\sum_k h^\alpha_{ijkk}\right) + \sum_{\alpha,i,j,k}(h^\alpha_{ijk})^2 = |\nabla A|^2 + \sum_{\alpha,i,j,k} h^\alpha_{ij}h^\alpha_{kijk}. \tag{3.6}$$

Thus, since (3.2) and (3.5) imply that $|\nabla A|^2 = |\nabla\Phi|^2$ and $\Delta|\Phi|^2 = \Delta S$, from (2.6) and (3.6) we conclude that

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 &+ \sum_{\alpha,i,j,k}(h^\alpha_{ij}h^\alpha_{kki})_j - \sum_{\alpha,i,j,k} h^\alpha_{ijj}h^\alpha_{kki} + \sum_{\alpha,i,j,m} h^\alpha_{ij}h^\alpha_{mi}R_{mj}\\ &+ \sum_{\alpha,i,j,k,m} h^\alpha_{ij}h^\alpha_{km}R_{mijk} + \sum_{\beta,\alpha,i,j,k} h^\alpha_{ij}h^\beta_{ki}R^\perp_{\beta\alpha jk}.\end{aligned} \tag{3.7}$$

Since H is constant, from Codazzi equation (2.7) we obtain that

$$\sum_{\alpha,i,j,k}(h^\alpha_{ij}h^\alpha_{kki})_j - \sum_{\alpha,i,j,k} h^\alpha_{ijj}h^\alpha_{kki} = \sum_{i,j,\alpha} nH^\alpha_{ij}h^\alpha_{ij} = 0. \tag{3.8}$$

From (2.2) and (2.3) we also conclude that

$$\begin{aligned}&\sum_{\alpha,i,j,k,m} h^\alpha_{ij}h^\alpha_{mk}R_{mijk} + \sum_{\alpha,i,j,m} h^\alpha_{ij}h^\alpha_{im}R_{mj} + \sum_{\beta,\alpha,i,j,k} h^\alpha_{ji}h^\beta_{ik}R^\perp_{\beta\alpha jk}\\ &\quad = c|\Phi|^2 - \sum_{\alpha,\beta,i,j,k,m} h^\alpha_{ij}h^\beta_{ij}h^\alpha_{mk}h^\beta_{mk} + n\sum_{\alpha,\beta,i,j,m} H^\beta h^\beta_{mj}h^\alpha_{ij}h^\alpha_{im}\\ &\qquad - \sum_{\alpha,\beta,i,j,m,l} h^\alpha_{ij}h^\alpha_{im}h^\beta_{ml}h^\beta_{ij} + \sum_{\alpha,\beta,i,j,k,m} h^\alpha_{ij}h^\alpha_{km}h^\beta_{jm}h^\beta_{ik} + \sum_{\alpha,\beta,i,j,k} h^\alpha_{ji}h^\beta_{ik}R^\perp_{\beta\alpha jk}.\end{aligned} \tag{3.9}$$

On the other hand, from (3.1) we get

$$\sum_{\alpha,\beta,i,j,m} H^\beta h^\beta_{mj}h^\alpha_{ij}h^\alpha_{im} = \sum_{\beta,i,j,k} Hh^{n+1}_{ij}h^\beta_{jk}h^\beta_{ki}, \tag{3.10}$$

and

$$\sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha} h_{ij}^{\beta} h_{mk}^{\alpha} h_{mk}^{\beta} = \sum_{i,j,k,l} \left(\sum_{\alpha,\beta} h_{ij}^{\alpha} h_{kl}^{\alpha} h_{ij}^{\beta} h_{kl}^{\beta} \right) = \sum_{i,j,k,l} \left(\sum_{\alpha} h_{ij}^{\alpha} h_{kl}^{\alpha} \right)^2 . \tag{3.11}$$

Using (2.8) we also have that

$$\begin{aligned} \sum_{\alpha,\beta,j,k} (R_{\alpha\beta jk}^{\perp})^2 &= \sum_{\alpha,\beta,j,k} \left[\sum_i (h_{ji}^{\beta} h_{ik}^{\alpha} - h_{ji}^{\alpha} h_{ik}^{\beta}) \right] R_{\beta\alpha jk}^{\perp} \\ &= \sum_{\alpha,\beta,i,j,k} h_{ji}^{\beta} h_{ik}^{\alpha} R_{\beta\alpha jk}^{\perp} - \sum_{\alpha,\beta,i,j,k} h_{ji}^{\alpha} h_{ik}^{\beta} R_{\beta\alpha jk}^{\perp} \\ &= \sum_{\alpha,\beta,i,j,m,l} h_{ij}^{\alpha} h_{im}^{\alpha} h_{ml}^{\beta} h_{lj}^{\beta} - \sum_{\alpha,\beta,i,j,k,m} h_{ij}^{\alpha} h_{km}^{\alpha} h_{jm}^{\beta} h_{ik}^{\beta} \\ &\quad - \sum_{\alpha,\beta,i,j,k} h_{ji}^{\alpha} h_{ik}^{\beta} R_{\beta\alpha jk}^{\perp}. \end{aligned} \tag{3.12}$$

Therefore, considering (3.8), (3.9), (3.10), (3.11) and (3.12) in (3.7), we conclude the proof. □

4. Proofs of Theorem 1 and Corollaries 1 and 2

In order to prove Theorem 1 we will also need of two algebraic lemmas. The proofs of them can be found in [18] and [12], respectively.

Lemma 1. *Let $B, C : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be symmetric linear maps that $BC - CB = 0$ and $trB = trC = 0$, then*

$$-\frac{n-2}{\sqrt{n(n-1)}} |B|^2 |C| \leq \mathrm{tr}(B^2 C) \leq \frac{n-2}{\sqrt{n(n-1)}} |B|^2 |C|.$$

Lemma 2. *Let $B^1, B^2, \cdots, B^n$ be symmetric $(n \times n)$-matrices. Set $S_{\alpha\beta} = tr(B^{\alpha} B^{\beta})$, $S_{\alpha} = S_{\alpha\alpha}$, $S = \sum_{\alpha} S_{\alpha}$, then*

$$\sum_{\alpha,\beta} |B^{\alpha} B^{\beta} - B^{\beta} B^{\alpha}|^2 + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2} \left(\sum_{\alpha} S_{\alpha} \right)^2 .$$

Now we are in position to proceed with the proof of Theorem 1.

Proof of Theorem 1. From Proposition 1 we have that

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + cn|\Phi|^2 + n\sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta} - \sum_{i,j,k,l}\left(\sum_{\alpha} h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^2 - \sum_{i,j,\alpha,\beta}(R^{\perp}_{\alpha\beta ij})^2. \tag{4.1}$$

From (3.1) and (3.3) we get

$$\begin{aligned}\sum_{i,j,k,\beta} & Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta} \\ &= \sum_{i,j,k} Hh_{ij}^{n+1}h_{jk}^{n+1}h_{ki}^{n+1} + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} Hh_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta} \\ &= H\mathrm{tr}(\Phi^{n+1}+HI)^3 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta} \\ &\quad + \sum_{\beta=n+2}^{n+p} H^2|\Phi^{\beta}|^2 \\ &= H\mathrm{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^{\beta}|^2 \\ &\quad + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta}.\end{aligned} \tag{4.2}$$

On the other hand, taking into account $R^{\perp}_{(n+1)\beta ij} = 0$ for every β, i, j, from Ricci equation (2.8) we get that $\Phi^{n+1}\Phi^{\beta} - \Phi^{\beta}\Phi^{n+1} = 0$, for every β. Thus, since $\mathrm{tr}\,\Phi^{\beta} = 0$ for every β, we can apply Lemma 1 to obtain

$$H\mathrm{tr}(\Phi^{n+1})^3 + 3H^2|\Phi^{n+1}|^2 + nH^4 + \sum_{\beta=n+2}^{n+p} H^2|\Phi^{\beta}|^2 + \sum_{\beta=n+2}^{n+p}\sum_{i,j,k} H\Phi_{ij}^{n+1}\Phi_{jk}^{\beta}\Phi_{ki}^{\beta} \tag{4.3}$$

$$\geq -\frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}|^3 + 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4$$
$$-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\beta=n+2}^{n+p} H|\Phi^{n+1}||\Phi^\beta|^2$$
$$= 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2.$$

Hence, from (4.2) and (4.3) we have

$$\sum_{\beta,i,j,k} Hh_{ij}^{n+1}h_{jk}^{\beta}h_{ki}^{\beta} \geq 2H^2|\Phi^{n+1}|^2 + H^2|\Phi|^2 + nH^4 - \frac{n-2}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2. \tag{4.4}$$

From Ricci equation (2.3) we get

$$\begin{aligned}\sum_{i,j,k,l}\left(\sum_\alpha h_{ij}^\alpha h_{kl}^\alpha\right)^2 &+ \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2 \\ &= \sum_{\alpha,\beta}(\mathrm{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha\neq n+1,\beta\neq n+1,i,j}(R^\perp_{\alpha\beta ij})^2 \\ &= [\mathrm{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(A^{n+1}A^\beta)]^2 \\ &\quad + \sum_{\alpha\neq n+1,\beta\neq n+1}(\mathrm{tr}(A^\alpha A^\beta))^2 + \sum_{\alpha\neq n+1,\beta\neq n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2.\end{aligned} \tag{4.5}$$

But, using (3.3) and Lemma 2 we obtain

$$\begin{aligned}\sum_{\alpha\neq n+1,\beta\neq n+1}[\mathrm{tr}(A^\alpha A^\beta)]^2 &+ \sum_{\alpha\neq n+1,\beta\neq n+1}|A^\alpha A^\beta - A^\beta A^\alpha|^2 \\ &\leq \frac{3}{2}\left(\sum_{\beta\neq n+1} tr(A^\beta A^\alpha)\right)^2 \leq \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^\beta|\right)^2.\end{aligned} \tag{4.6}$$

Hence, from (4.5) and (4.6) we have

(4.7)

$$\begin{aligned}
&\sum_{i,j,k,l}\left(\sum_{\alpha} h_{ij}^{\alpha}h_{kl}^{\alpha}\right)^2 + \sum_{\alpha,\beta,i,j}(R^{\perp}_{\alpha\beta ij})^2 \\
&\quad \le [\mathrm{tr}(A^{n+1}A^{n+1})]^2 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(A^{n+1}A^{\beta})]^2 + \frac{3}{2}\left(\sum_{\beta\neq n+1}|\Phi^{\beta}|^2\right)^2 \\
&\quad = |\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2\sum_{\beta\neq n+1}[\mathrm{tr}(\Phi^{n+1}\Phi^{\beta})]^2 \\
&\qquad + \frac{3}{2}(|\Phi|^2 - |\Phi^{n+1}|^2)^2 \\
&\quad \le \frac{5}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 + 2|\Phi^{n+1}|^2(|\Phi|^2 - |\Phi^{n+1}|^2) \\
&\qquad + \frac{3}{2}|\Phi|^4 - 3|\Phi|^2|\Phi^{n+1}|^2 \\
&\quad = \frac{1}{2}|\Phi^{n+1}|^4 + 2nH^2|\Phi^{n+1}|^2 + n^2H^4 - |\Phi|^2|\Phi^{n+1}|^2 + \frac{3}{2}|\Phi|^4.
\end{aligned}$$

Therefore, from (4.1), (4.4) and (4.7) we get

(4.8)

$$\begin{aligned}
\frac{1}{2}\Delta|\Phi|^2 &\ge cn|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi^{n+1}||\Phi|^2 + nH^2|\Phi|^2 \\
&\quad - \frac{1}{2}|\Phi^{n+1}|^4 + |\Phi|^2|\Phi^{n+1}|^2 - \frac{3}{2}|\Phi|^4 \\
&= |\Phi|^2\left(-|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(H^2+c)\right) \\
&\quad + (|\Phi| - |\Phi^{n+1}|) \\
&\qquad \times \left(\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2\right).
\end{aligned}$$

On the other hand, we note that holds the following algebraic inequality (3.5) of [7]

(4.9)
$$(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \le \frac{32}{27}|\Phi|^3.$$

Moreover, since that $R \geq c$ and using (2.5), we also have

$$n^2H^2 = S + n(n-1)(R-c) \geq S = |\Phi|^2 + nH^2,$$

which give us

$$H \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|. \tag{4.10}$$

Thus, from (4.9) and (4.10) we conclude that

$$\begin{aligned} &\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^2 - \frac{1}{2}(|\Phi| - |\Phi^{n+1}|)(|\Phi| + |\Phi^{n+1}|)^2 \\ &\quad \geq \left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3. \end{aligned} \tag{4.11}$$

But, taking into account our assumption that $n \geq 4$, we have that

$$\frac{n-2}{n-1} - \frac{16}{27} > 0. \tag{4.12}$$

Consequently, from (4.3), (4.11) and (4.12) we get that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geq -|\Phi|^2 P_{H,c}(|\Phi|) + (|\Phi| - |\Phi^{n+1}|)\left(\frac{n-2}{n-1} - \frac{16}{27}\right)|\Phi|^3 \\ &\geq -|\Phi|^2 P_{H,c}(|\Phi|), \end{aligned} \tag{4.13}$$

where

$$P_{H,c}(x) = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(H^2 + c).$$

If $\sup_M |\Phi|^2 = 0$, then M^n is totally umbilical and, hence, item (i) holds. If $\sup_M |\Phi|^2 = +\infty$, then (ii) is trivially satisfied. So, let us suppose that $0 < \sup_M |\Phi|^2 < +\infty$, then by applying Theorem 1.1 of [16] (see also Theorem 3.1 of [17]) to the function $|\Phi|^2$ we obtain a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^n such that, for every $k \in \mathbb{N}$,

$$\lim_{k\to+\infty} |\Phi|^2(p_k) = \sup_M |\Phi|^2 \quad \text{and} \quad \Delta|\Phi|^2(p_k) < \frac{1}{k}. \tag{4.14}$$

Hence, from (4.13) and (4.14), we get

$$\frac{1}{k} > \Delta|\Phi|^2(p_k) \geq -2|\Phi|^2(p_k)P_{H,c}(|\Phi|^2(p_k)). \tag{4.15}$$

Taking into (4.15) the limit when $k \to +\infty$, by continuity, we have

$$\left(\sup_M |\Phi|\right)^2 P_{H,c}\left(\sup_M |\Phi|\right) \geq 0.$$

Since $\sup_M |\Phi| > 0$, we obtain that

$$P_{H,c}\left(\sup_M |\Phi|\right) \geq 0. \tag{4.16}$$

Observe that, since $H^2 + c > 0$, the polynomial $P_H(x)$ has a unique positive root given by

$$x_0 = \frac{\sqrt{n}}{2\sqrt{(n-1)}}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)H\right).$$

Therefore, (4.16) implies

$$\sup_M |\Phi|^2 \geq x_0^2 = \frac{n}{4(n-1)}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)H\right)^2.$$

This proves inequality (1.2).

Moreover, suppose that the equality holds in (1.2) or, equivalently, $\sup_M |\Phi|^2 = x_0^2$. Thus, in this case, $P_{H,c}(|\Phi|) \leq 0$ on M^n, which jointly with (4.13) implies that $\Delta|\Phi|^2 \geq 0$ on M^n. Hence, if there exists a point $p_0 \in M^n$ such that $|\Phi(p_0)| = \sup_M |\Phi|$, from the maximum principle the function $|\Phi|^2$ must be constant and, consequently, $|\Phi| \equiv x_0$. Thus,

$$0 = \frac{1}{2}\Delta|\Phi|^2 = -|\Phi|^2 P_{H,c}(|\Phi|).$$

Thus, all the inequalities along the proof this prove must be equalities. In particular,

$$|\nabla\Phi|^2 = |\nabla A|^2 = 0.$$

So, it follows that λ_i is constant for every $i = 1, \ldots, n$, that is, M^n is an isoparametric submanifold. Now, suppose that M^n is not totally umbilical, which means that $|\Phi|$ a positive constant. In this case, taking into account (4.12), from (4.13) we conclude that $|\Phi| = |\Phi^{n+1}|$ and, consequently, $\Phi^\alpha = 0$, for all $n+2 \leq \alpha \leq n+p$. Thus, since e_{n+1} is parallel in the normal bundle of M^n, we are in position to apply Theorem 1 of [22] to conclude that M^n is, in fact, isometrically immersed in a $(n+1)$-dimensional totally geodesic submanifold $\mathbb{Q}_c^{n+1}$ of $\mathbb{Q}_c^{n+p}$. Therefore, we can use Theorem 5 of [2] to finish our proof. □

We proceed with the proof of Corollary 1.

Proof of Corollary 1. From (2.2) we obtain

$$R_{ijij} = c + \sum_{\alpha} h^{\alpha}_{ii} h^{\alpha}_{jj} - \sum_{\alpha} (h^{\alpha}_{ij})^2. \tag{4.17}$$

Since $S \leq (nH)^2$, we have that

$$(h^{\alpha}_{ij})^2 \leq S \leq (nH)^2,$$

for every α, i, j and, hence,

$$|h^{\alpha}_{ii} h^{\alpha}_{jj}| = |h^{\alpha}_{ii}||h^{\alpha}_{jj}| \leq (nH)^2. \tag{4.18}$$

Thus, since we are supposing that H is constant on M^n, it follows from (4.17) and (4.18) that the sectional curvatures of M^n are bounded from below. Therefore, we can apply the classical maximum principle of Omori [15] and the result follows directly from Theorem 1. □

We close our paper proving Corollary 2.

Proof of Corollary 2. First all recall that the weak Omori-Yau maximum principle holds on every parabolic Riemannian manifold. Then, if $\sup_M |\Phi|^2 = +\infty$ then there is nothing to prove. On the other hand, in the case that $0 < \sup_M |\Phi|^2 < +\infty$, reasoning as in the first part of the proof of Theorem 1, we guarantee that $\sup_M |\Phi|^2 \geq x_0$. Moreover, if equality holds in (1.2), then we have $P_{H,c}(|\Phi|) \leq 0$ and, consequently, the function $|\Phi|^2$ is a subharmonic on M^n. Therefore, from the parabolicity of M^n we conclude that the function $|\Phi|^2$ must be constant and equal to x_0. At this point, taking into account that the circular cylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$, the Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$ and the hyperbolic cylinder $\mathbb{H}^1(-\sqrt{1+r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$ are parabolic (cf. Section 2.1 of [9]), we can reason as in the proof of Theorem 1. □

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