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New surfaces with canonical map of high degree

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We give an algorithm that, for a given value of the geometric genus p_g, computes all regular product-quotient surfaces with abelian group that have at most canonical singularities and have canonical system with at most isolated base points. We use it to show that there are exactly two families of such surfaces with canonical map of degree 32. We also construct a surface with $q = 1$ and canonical map of degree 24. These are regular surfaces with $p_g = 3$ and base point free canonical system. We discuss the case of regular surfaces with $p_g = 4$ and base point free canonical system.

1. Introduction

Let S be a smooth surface of general type with irregularity q and geometric genus $p_g \geq 3$. Denote by ϕ the canonical map of S and let $d := \deg(\phi)$. It is known since Beauville [6] that if the canonical image $\phi(S)$ is a surface, then

$$d \leq 36 - 9q \text{ if } q \leq 3, \quad d \leq 8 \text{ if } q \geq 4.$$

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$. Despite being a classical problem, for $d > 8$ the number of known examples is scarce. Tan's example [16, §5] with $d = 9, q = 0$ and Persson's example [12] with $d = 16$, $q = 0$ are well known. Du and Gao [10] show that if the canonical map is an abelian cover of $\mathbb{P}^2$, then these are the only possibilities for $d > 8$. More recently the third author has given examples with $d = 16$, $q = 2$ [15] and $d = 24$, $q = 0$ [14]. There is a paper [17] claiming the existence of the case $d = 36$, but, to our knowledge, the proof is not correct.

In this paper we consider the problem of finding product-quotient surfaces $(A \times B)/G$ with at most canonical singularities having canonical map of maximum degree. For these surfaces $K^2 \leq 8\chi$ (see [1]), equality holding if and only if the *quotient model* $(A \times B)/G$ is smooth, *i.e.* the action of G

is free. Here Beauville's argument gives

$$d \leq 32 - 8q \quad \text{if} \quad q \leq 3,$$

equality holding if and only if G acts freely, $p_g = 3$ and the canonical system is base point free. In order to be able to understand this system, we restrict our study to abelian groups G. Such surfaces are then abelian coverings of the product $(A/G) \times (B/G)$, and we can use Pardini's [11] formulas to understand their canonical curves.

We give an algorithm that, for a given value of the geometric genus p_g and some $n \in \mathbb{N}$, computes all regular product-quotient surfaces with abelian group G that have at most canonical singularities and have canonical system with at most n base points. Applying it to the case $K^2 = 32$, we get exactly two families of surfaces with $p_g = 3$, $q = 0$ and canonical map of degree $K^2 = 32$ onto $\mathbb{P}^2$. We describe these surfaces as $(\mathbb{Z}/2)^4$-coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ in Section 3.

We have also found a family of product-quotient surfaces with $p_g = 3$, $q = 1$ and canonical map of degree $K^2 = 24$ onto $\mathbb{P}^2$. We give the construction as a $(\mathbb{Z}/2)^3$-covering of $E \times \mathbb{P}^1$ in Section 4, where E is an elliptic curve. One can show that this is the unique such family with group $G = (\mathbb{Z}/2)^3$, we give the idea for the proof of this fact in Remark 4.1.

For product-quotient surfaces with $p_g \geq 4$ and $q \leq 3$, Beauville's proof gives the inequality

$$d \leq 8\left(1 + \frac{3-q}{p_g - 2}\right) \leq 20.$$

Strangely enough the output of our algorithm for $p_g = 4$ does not contain any quotient $(A \times B)/G$ with G acting freely, and therefore there exists no product-quotient surface $(A \times B)/G$ with G abelian and canonical map of degree 20. We show that the maximum degree for regular such surfaces is 12. The value $p_g = 4$ is a surprising gap. Indeed Catanese constructed in [9] regular product-quotient surfaces with $p_g = 5$ and 6 of the form $(A \times B)/G$ with G abelian acting freely and canonical system without base points. Catanese's examples, having canonical map of degree 1 and $\phi(S)$ of very high degree, have been an important source of inspiration for this paper.

To keep the paper as simple as possible, for the convenience of the readers, we describe our examples directly as abelian covers of $(A/G) \times (B/G)$ instead of as quotients $(A \times B)/G$. We refer the interested reader to [2, 3] and the references therein for the theory of product-quotient surfaces in the general case of arbitrary singularities.

An implementation of our algorithm as MAGMA script may be downloaded at `http://www.science.unitn.it/~pignatel/papers/CanonicalMapProg.magma`

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2. $(\mathbb{Z}/2)^r$-coverings and canonical systems

The following result is taken from [8, Proposition 6.6] (see also [11]).

Proposition 2.1. *A normal finite $G \cong (\mathbb{Z}/2)^r$-covering $Y \to X$ of a smooth variety X is completely determined by the datum of*

1) *reduced effective divisors D_σ, $\forall \sigma \in G$, with no common components;*
2) *divisor linear equivalence classes $L_1, \dots, L_r$, for $\chi_1, \dots, \chi_r$ a basis of the dual group of characters $G^\vee$, such that*

$$2L_i \equiv \sum_{\chi_i(\sigma)=1} D_\sigma.$$

Conversely, given (1) and (2), one obtains a normal scheme Y with a finite $G \cong (\mathbb{Z}/2)^r$-covering $Y \to X$, with branch curves the divisors D_σ.

The covering $\psi\colon Y \to X$ is embedded in the total space of the direct sum of the line bundles whose sheaves of sections are the $\mathcal{O}_X(L_i)$, and is there defined by equations

$$u_{\chi_i} u_{\chi_j} = u_{\chi_i+\chi_j} \prod_{\chi_i(\sigma)=\chi_j(\sigma)=1} x_\sigma,$$

where x_σ is a section such that $\operatorname{div}(x_\sigma) = D_\sigma$.

The scheme Y is irreducible if $\{\sigma | D_\sigma > 0\}$ generates G.

If the branch locus of ψ is simple normal crossing, then the surface Y is smooth and its invariants are

$$\chi(\mathcal{O}_Y) = 2^r \chi(\mathcal{O}_X) + \frac{1}{2} \sum_{\chi \in G^{\vee *}} \left(L_\chi^2 + K_X L_\chi \right),$$

$$p_g(Y) = p_g(X) + \sum_{\chi \in G^{\vee *}} h^0(X, \mathcal{O}_X(K_X + L_\chi)).$$

For each $\sigma \in G$, denote by $R_\sigma \subset Y$ the reduced divisor supported on $\psi^*(D_\sigma)$. We get from [11, Proposition 4.1, c)] and [4, Proposition 2.1)] that, if X is Gorenstein, for any χ,

$$(\psi_* \omega_Y)^{(\chi)} \cong \omega_X(L_\chi).$$

Combining with the Hurwitz formula,

$$K_Y = \psi^*(K_X) + \sum_{\sigma \in G^*} R_\sigma,$$

we obtain that the canonical linear system of Y is generated by

$$\psi^* |K_X + L_i| \sum_{\chi_i(\sigma)=0} R_\sigma, \quad i \in J, \tag{1}$$

where $J := \{j : |K_X + L_j| \neq \emptyset\}$.

3. The families with $\deg(\phi) = 32$

Let f, g be the rational fibrations of $X := \mathbb{P}^1 \times \mathbb{P}^1$, and let $F_1, \ldots, F_6$ be distinct fibres of f and $E_1, \ldots, E_6$ be distinct fibres of g. Denote by $e_1, \ldots, e_4$ the generators of $(\mathbb{Z}/2)^4$. We set $e_{i_1 \cdots i_r} := e_{i_1} + \cdots + e_{i_r}$.

3.1. Building data $2 \times (1, 1, 1, 1, 1, 1)$

Consider the $(\mathbb{Z}/2)^4$-covering

$$\psi : Y \to X$$

given by

$$D_{e_1} := F_1,\ D_{e_2} := F_2,\ D_{e_3} := F_3,\ D_{e_4} := F_4,\ D_{e_{13}} := F_5,\ D_{e_{24}} := F_6,$$

$$D_{e_{234}} := E_1,\ D_{e_{134}} := E_2,\ D_{e_{124}} := E_3,\ D_{e_{123}} := E_4,\ D_{e_{14}} := E_5,\ D_{e_{23}} := E_6.$$

For $i, j, k, l \in \mathbb{Z}/2$, let χ_{ijkl} denote the character which takes the value i, j, k, l on e_1, e_2, e_3, e_4, respectively. There exist divisors L_{ijkl} such that

$$2L_{ijkl} \equiv \sum_{\chi_{ijkl}(\sigma)=1} D_\sigma,$$

thus the covering ψ is well defined. Since there is no 2-torsion in the Picard group of X, then ψ is uniquely determined. The surface Y is smooth because the curves $D_{e_1}, \ldots, D_{e_{234}}$ are smooth with pairwise transverse intersections only.

We have

$$L_{1100} \equiv L_{0011} \equiv L_{1111} \equiv 2F + 2E,$$

where F is a fibre of f and E is a fibre of g. For the remaining cases we have

$$L_{ijkl} \equiv F + 2E \quad \text{or} \quad 2F + E.$$

This implies that

$$\chi(\mathcal{O}_Y) = 16 + \frac{1}{2}\sum \left(L_{ijkl}^2 + K_X L_{ijkl}\right) = 4$$

and

$$p_g(Y) = 0 + \sum h^0(X, \mathcal{O}_X(K_X + L_{ijkl})) = 3.$$

We get from (1) that K_Y is generated by the following divisors, respectively associated to the characters χ_{0011}, χ_{1100} and χ_{1111}:

$$\widehat{F}_1 + \widehat{F}_2 + \widehat{E}_1 + \widehat{E}_2,\ \widehat{F}_3 + \widehat{F}_4 + \widehat{E}_3 + \widehat{E}_4,\ \widehat{F}_5 + \widehat{F}_6 + \widehat{E}_5 + \widehat{E}_6,$$

where $\widehat{F}_i := \frac{1}{2}\psi^*(F_i)$ and $\widehat{E}_i := \frac{1}{2}\psi^*(E_i)$.

The fact $\widehat{F}_i\widehat{E}_j = 4$ implies $K_Y^2 = 32$.

By looking to their images on X, one verifies that the above three divisors have no common intersection. Thus $|K_Y|$ is base-point free and then $K_Y^2 > 0$ implies that the canonical map of Y is not composed with a pencil. Hence its image is $\mathbb{P}^2$, a surface of degree 1, therefore the degree formula implies that the canonical map of Y is of degree $K_Y^2 = 32$.

3.2. Building data $2 \times (2,1,1,1,1)$

Here we only give the building data of the covering, the verifications are analogous to the ones in the previous section.

$$D_{e_1} := F_1,\ D_{e_{134}} := F_2,\ D_{e_{123}} := F_3 + F_4,\ D_{e_{13}} := F_5,\ D_{e_{14}} := F_6,$$

$$D_{e_2} := E_1,\ D_{e_{234}} := E_2,\ D_{e_{124}} := E_3 + E_4,\ D_{e_{23}} := E_5,\ D_{e_{24}} := E_6.$$

As in the previous case, setting $\widehat{F}_i := \frac{1}{2}\psi^*(F_i)$ and $\widehat{E}_i := \frac{1}{2}\psi^*(E_i)$, K_Y is generated by the divisors $\widehat{F}_1 + \widehat{F}_2 + \widehat{E}_1 + \widehat{E}_2$, $\widehat{F}_3 + \widehat{F}_4 + \widehat{E}_3 + \widehat{E}_4$, $\widehat{F}_5 + \widehat{F}_6 + \widehat{E}_5 + \widehat{E}_6$, respectively associated to the characters χ_{0011}, χ_{1100} and χ_{1111}.

4. A family with $\deg(\phi) = 24$, $q = 1$

Let

$$X := E \times F,$$

with $F \cong \mathbb{P}^1$ and E a smooth elliptic curve. Let $E_1, \ldots, E_6 \subset X$ be distinct elliptic fibres and $F_1, F_2, F_3 \subset X$ be distinct rational fibres. Since the sum of two points in an elliptic curve is divisible by 2 in the Picard Group, there are fibres F_{ij} such that $2F_{ij} \equiv F_i + F_j$, $i, j \in \{1, 2, 3\}$.

Let e_1, e_2, e_3 be the generators of $(\mathbb{Z}/2)^3$, set $e_{i_1 \cdots i_r} := e_{i_1} + \cdots + e_{i_r}$ and consider the divisors

$$D_{e_1} := E_1 + E_2,\ D_{e_2} := E_3 + E_4,\ D_{e_3} := E_5 + E_6,$$

$$D_{e_{23}} := F_1,\ D_{e_{13}} := F_2,\ D_{e_{12}} := F_3,$$

$$L_{100} := E + F_{23},\ L_{010} := E + F_{13},\ L_{001} := E + F_{12}.$$

For $i, j, k \in \mathbb{Z}/2$, let χ_{ijk} denote the character which takes the value i, j, k on e_1, e_2, e_3, respectively. The above data satisfies

$$2L_{ijk} \equiv \sum_{\chi_{ijk}(\sigma)=1} D_\sigma,$$

thus from Proposition 2.1 it defines a $(\mathbb{Z}/2)^3$-covering

$$\psi : Y \longrightarrow X.$$

Note that there are four different possible choices for each F_{ij}: a different choice produces a different Y. The surface Y is smooth because the curves $D_{e_1}, \dots, D_{e_{23}}$ are smooth with pairwise transverse intersections only.

The fact

$$L_{\chi+\eta} \equiv L_\chi + L_\eta - \sum_{\chi(\sigma)=\eta(\sigma)=1} D_\sigma$$

implies that

$$L_{111} \equiv 3E + T,$$

$$L_{110} \equiv 2E + F'_{12},\ L_{101} \equiv 2E + F'_{13},\ L_{011} \equiv 2E + F'_{23},$$

where

$$T := F_{12} + F_{13} + F_{23} - F_1 - F_2 - F_3$$

and F'_{ij} is a fibre linearly equivalent to $F_{ij} + T$.

We notice that the divisor class $2T$ is trivial. We choose the F_{ij} so that the divisor class T is not trivial, so T is a 2-torsion element of the Picard group.

Since $K_X \equiv -2E$, we have that

$$\chi(\mathcal{O}_Y) = 0 + \frac{1}{2}\sum \left(L_{ijk}^2 + K_X L_{ijk}\right) = 3,$$

$$p_g(Y) = 0 + \sum h^0(X, \mathcal{O}_X(K_X + L_{ijk})) = 3,$$

and then $q(Y) = 1$.

We get from (1) that K_Y is generated by the following divisors:

$$\widehat{E}_5 + \widehat{E}_6 + \widehat{F}_3 + \widetilde{F}'_{12},\ \widehat{E}_3 + \widehat{E}_4 + \widehat{F}_2 + \widetilde{F}'_{13},\ \widehat{E}_1 + \widehat{E}_2 + \widehat{F}_1 + \widetilde{F}'_{23},$$

corresponding respectively to the characters χ_{110}, χ_{101} and χ_{011}, where $\widehat{E}_i := \frac{1}{2}\psi^*(E_i)$, $\widehat{F}_i := \frac{1}{2}\psi^*(F_i)$, $\widetilde{F}_{ij} := \psi^*(F_{ij})$ and $\widetilde{F}'_{ij} := \psi^*(F'_{ij})$.

The facts $\widehat{E}_i\widehat{F}_j = 2$ and $\widehat{E}_i\widetilde{F}_{ij} = 4$ imply $K_Y^2 = 24$.

The fibres E_i, F_j, F'_{kl} are distinct with the only possible exceptions $F'_{ij} = 2F_k$, $\{i,j,k\} = \{1,2,3\}$. Then the above three divisors have no common intersection since their images on X have no common intersection. Thus $|K_Y|$ is base-point free and then, arguing as in Section 3, the canonical map of Y is of degree $K_Y^2 = 24$.

Remark 4.1. We have a proof that these are the only irregular product-quotient surfaces of the form $(A \times B)/(\mathbb{Z}/2)^3$ with canonical map of degree 24. We quickly sketch here the main point of the proof.

Such surfaces S are $(\mathbb{Z}/2)^3-$covers of $E \times F$ (E elliptic, F rational) branched on an union of elliptic fibres E_i and rational fibres F_j. Since the action of $(\mathbb{Z}/2)^3$ on $A \times B$ is free, each D_σ is either of the form $\sum E_i$ or of the form $\sum F_j$.

By (1) the canonical system is generated by three divisors corresponding to three characters. If these characters are linearly independent we can assume w.l.o.g. that they are χ_{100}, χ_{010} and χ_{001}. Then $h^0(E \times F, K_{E\times F} + L_{100}) = 1$. It is easy to prove that the class of $K_{E\times F} + L_{100}$ can't be trivial, so it is the class of a rational fibre F_1, and analogous statement holds for $K_{E\times F} + L_{010}$ and $K_{E\times F} + L_{001}$. Then all the three divisors contain the pull-back of a rational fibre F_i. Then $\forall i \in \{1, 2, 3\}$, D_{e_i} cannot contain any elliptic fibre E_j or there would be a base point of K_S on $\widehat{E}_j$. By Hurwitz formula one deduces $D_{e_{ij}} \equiv E$, $D_{e_{123}} \equiv 2E$. Since there is at least a rational fibre F_0 in the branch locus, w.l.o.g. $F_0 \le D_{e_1}$ and one finds a base point of K_S on $F_0 \cap D_{e_{23}}$, a contradiction.

So, the three characters are linearly dependent. The rest of the proof uses similar arguments.

5. The algorithm

In this section we describe our algorithm, producing all regular product-quotient surfaces whose quotient model $Y := (A \times B)/G$ has G abelian, at most rational double points as singularities, and canonical system with at most isolated base points.

By [1, Remark 2.5] every singular point $y \in Y$ is then of type A_{n_y}, $n_y \in \mathbb{N}$.

Lemma 5.1. *Let $Y := (A \times B)/G$ be the quotient model of a product-quotient surface with only canonical singularities such that G is abelian. Set $g(A), g(B)$ for the genus of the curve A, B, respectively, and assume w.l.o.g. $g(A) \ge g(B)$. Set also $\chi := \chi(\mathcal{O}(Y))$. Then*

$$g(B) \le 1 + 2\chi + 2\sqrt{\chi^2 + 2\chi}\ , \qquad g(A) \le 4\chi\frac{g(B)+1}{g(B)-1} + 1\ .$$

Proof. According to [13, Proposition 3.10],

$$\chi = \frac{(g(A)-1)(g(B)-1)}{|G|} + \frac{1}{12}\sum_{y\in \operatorname{Sing} Y}\frac{n_y^2-1}{n_y} \ge \frac{(g(A)-1)(g(B)-1)}{|G|}\ .$$

Since G is abelian, we have $|G| \leq 4g(B)+4$ by [7, Corollary 9.6], which implies

$$\chi(4g(B)+4) \geq \chi|G| \geq (g(A)-1)(g(B)-1) \geq (g(B)-1)^2 \ .$$

In particular

$$g(B)^2 - (4\chi+2)g(B) + 1 - 4\chi \leq 0.$$

□

We assume Y regular, then $E := A/G \cong F := B/G \cong \mathbb{P}^1$. Since G is abelian, then the finite map $\psi\colon Y \to E \times F \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a Galois cover with Galois group G. The branching locus of ψ is the union of the lines $E_i := E \times q_i$, $F_j := p_j \times F$, where p_j are the branching points of $A \to E$ and q_i are the branching points of $B \to F$. The cover $A \to E$ associates naturally to each point p_j its *local monodromy*, an element g_j of G, that is also the local monodromy of F_j for ψ. The element g_j is the image of a small loop around p_j for the map in [1, page 1002], not depending on the choice of the loop since G is abelian. In the notation of [11], F_j is a component of the divisor $D_{H,\eta}$ with $H = \langle g_j \rangle$ and $\eta \in H^*$ defined by $\eta(g_j) = e^{\frac{2\pi i}{m_j}}$ where m_j is the order of g_j in G.

By the Riemann Existence Theorem, the local monodromies give a bijection among the Galois covers of $\mathbb{P}^1$ and the maps $\{p_j\} \to G$ such that p_j is a finite subset of $\mathbb{P}^1$ and the image is a set of generators of G that is *spherical*, *i.e.* such that the sum of the images of the p_j is zero. So we produce regular product-quotient surfaces by producing two sets of spherical generators of G and then choosing freely the points p_j, q_i.

The *type* of the set of generators is the *multiset* (a set whose elements are allowed to have a multiplicity in $\mathbb{N}$) of the orders m_j of the local monodromies of the p_j. See [1] for details.

Fix now $p_g(Y) \in \mathbb{N}$. The algorithm is the following:

1st Step: Since $\chi = p_g(Y)+1$, Lemma 5.1 determines finitely many possible pairs of genera $(g(A), g(B))$ and so finitely many possible orders of the group $|G| \leq 4g(B)+4$. The inequalities in [7, Theorem 9.1, Corollary 9.6] and [13, Proposition 3.6] limit the types T_1 and T_2 of the coverings $A \to E$ and $B \to F$ to finitely many possibilities. Our program lists first all possible $5-$tuples $[g(A), g(B), |G|, T_1, T_2]$.

2nd Step: For each resulting $5-$tuple $[g(A), g(B), |G|, T_1, T_2]$, the program searches among all groups of order $|G|$ for pairs of systems of spherical generators of respective types T_1 and T_2. For each such pairs it computes the singularities of the resulting surface $(A \times B)/G$ using [2, Proposition 1.17] (or the equivalent [11, Proposition 3.3]) and discards all pairs giving singularities not canonical.

3^{rd} Step: Finally the program discards, among the surfaces produced by Step 2, those whose canonical system has a 1-dimensional base component, as follows. Since the singularities of Y are Gorenstein, we can use Pardini's formula ([11, Proposition 4.1, c)] and [4, Proposition 2.1)]) for splitting its canonical system as in (1). More precisely we obtain subsystems of the form $\psi^*|M_\chi| + \Phi_\chi$, $\chi \in G^*$, generating the canonical system, where $|M_\chi|$ is a (possibly empty) complete linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ and Φ_χ is an effective divisor supported on the union of the E_i and the F_j. Since every complete linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ is base point free, then the canonical system of the product-quotient surface has at most isolated base points if and only if the divisors Φ_χ such that $|M_\chi| \neq \emptyset$ meet only at a finite number of points.

The program returns: the group G, the types T_i, a pair of generating vectors, the systems M_χ that are not empty, the singularities of Y, and the number of base points of the canonical system.

Remark 5.2. By Beauville's argument,

$$\deg \phi \leq \frac{8(p_g+1) - b - \frac{2}{3}\sum_{y \in \operatorname{Sing} Y} \frac{n_y^2-1}{n_y}}{p_g - 2},$$

where b is the number of base points. The equality holds if and only if $\phi(S) \subset \mathbb{P}^{p_g-1}$ is a surface of minimal degree $p_g - 2$.

Running the program for $p_g = 3$, we obtain the following result.

Proposition 5.3. *There are exactly* 2 *families of regular product-quotient surfaces*
$(A \times B)/G$ with G abelian acting freely, $p_g = 3$ and canonical system base point free, the families described in Section 3.

There are further 17 *families of regular product-quotient surfaces $(A \times B)/G$ with G abelian, $p_g = 3$, canonical system base point free whose quotient model has only canonical singularities.*

We notice that the two families are distinct even as families in the Gieseker moduli space of surfaces of general type. Indeed by [5, Theorem 1.3] a surface in one family is not even deformation equivalent to any surface in the other family.

The degrees of the canonical maps of the surfaces in the 17 further families form the set $\{2, 4, 6, 8, 16\}$.

6. The case $p_g = 4$

Running the program for $p_g = 4$ we get:

Proposition 6.1. *There are no regular product-quotient surfaces $(A \times B)/G$ with G abelian acting freely, $p_g = 4$ and canonical system base point free.*

There are 60 families of regular product-quotient surfaces $(A \times B)/G$ with G abelian, $p_g = 4$, canonical system base point free whose quotient model has only canonical singularities.

The highest degree realized by the 60 families in Proposition 6.1 is 12, realized by a family with group $G = (\mathbb{Z}/3)^2$. The branching divisor is the union of 8 lines, 4 for each ruling: $E_1 + E_2 + E_3 + E_4 + F_1 + F_2 + F_3 + F_4$, with local monodromies

$$\begin{array}{llll} E_1 \mapsto (1,1,2) & E_2 \mapsto (2,2,0) & E_3 \mapsto (1,2,1) & E_4 \mapsto (2,1,0) \\ F_1 \mapsto (2,1,1) & F_2 \mapsto (0,1,2) & F_3 \mapsto (1,2,1) & F_4 \mapsto (0,2,2) \end{array}$$

The surface has 9 singular points of type A_2 and $K^2 = 24$. There are 4 characters χ with $|M_\chi| \neq \emptyset$, here are the respective Φ_χ:

$$\begin{array}{ll} \Phi_{(0,1,0)} = \widehat{E}_1 + \widehat{E}_4 + \widehat{F}_1 + \widehat{F}_2 & \Phi_{(1,0,1)} = 2\widehat{E}_1 + 2\widehat{F}_1 \\ \Phi_{(0,2,0)} = \widehat{E}_2 + \widehat{E}_3 + \widehat{F}_3 + \widehat{F}_4 & \Phi_{(2,2,2)} = 2\widehat{E}_4 + 2\widehat{F}_2 \end{array}$$

with $\widehat{E}_i = \frac{1}{3}\psi^*(E_i)$, $\widehat{F}_i = \frac{1}{3}\psi^*(F_i)$.

Recall that since the canonical system is base point free and has positive self-intersection, the canonical map is not composed with a pencil. Since $2\Phi_{(0,1,0)} = \Phi_{(1,0,1)} + \Phi_{(0,2,0)}$, the image of the canonical map is contained in a quadric cone and therefore is the quadric cone. More precisely, one can choose sections x_0, x_1, x_2, x_3 of $H^0(S, K_S)$ with respective divisors $\Phi_{(0,1,0)}$, $\Phi_{(1,0,1)}$, $\Phi_{(0,2,0)}$, $\Phi_{(2,2,2)}$ such that the canonical image is the quadric $x_0^2 = x_1 x_2$.

There are three more families of surfaces in our list of 60 with $K_S^2 \geq 24$: one with $K^2 = 36$ and two with $K^2 = 32$. We can show that their canonical image is not contained in a quadric, and therefore the maximal canonical degree we can reach for $p_g = 4$ is 12.

References

[1] I. Bauer, F. Catanese, F. Grunewald, and R. Pignatelli, *Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups*, Amer. J. Math. **134** (2012), no. 4, 993–1049.

[2] I. Bauer and R. Pignatelli, *The classification of minimal product-quotient surfaces with $p_g = 0$*, Math. Comp. **81** (2012), no. 280, 2389–2418.

[3] ———, *Product-quotient surfaces: new invariants and algorithms*, Groups Geom. Dyn. **10** (2016), no. 1, 319–363.

[4] ———, *Rigid but not infinitesimally rigid compact complex manifolds*, Duke Math. J. **170** (2021), no. 8, 1757–1780.

[5] I. C. Bauer and F. Catanese, *Some new surfaces with $p_g = q = 0$*, in The Fano Conference, 123–142, Univ. Torino, Turin (2004).

[6] A. Beauville, *L'application canonique pour les surfaces de type général*, Invent. Math. **55** (1979), no. 2, 121–140.

[7] T. Breuer, Characters and automorphism groups of compact Riemann surfaces, Vol. 280 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge (2000), ISBN 0-521-79809-4.

[8] F. Catanese, *Differentiable and deformation type of algebraic surfaces, real and symplectic structures*, in Symplectic 4-manifolds and algebraic surfaces, Vol. 1938 of *Lecture Notes in Math.*, 55–167, Springer, Berlin (2008).

[9] ———, *On the canonical map of some surfaces isogenous to a product*, in Local and global methods in algebraic geometry, Vol. 712 of *Contemp. Math.*, 33–57, Amer. Math. Soc., Providence, RI (2018).

[10] R. Du and Y. Gao, *Canonical maps of surfaces defined by abelian covers*, Asian J. Math. **18** (2014), no. 2, 219–228.

[11] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991) 191–213.

[12] U. Persson, *Double coverings and surfaces of general type*, in Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977), Vol. 687 of *Lecture Notes in Math.*, 168–195, Springer, Berlin (1978).

[13] F. Polizzi, *Standard isotrivial fibrations with $p_g = q = 1$*, J. Algebra **321** (2009), no. 6, 1600–1631.

[14] C. Rito, *A surface with canonical map of degree 24*, Internat. J. Math. **28** (2017), no. 6, 1750041, 10.

[15] ———, *A surface with $q = 2$ and canonical map of degree 16*, Michigan Math. J. **66** (2017), no. 1, 99–105.

[16] S.-L. Tan, *Surfaces whose canonical maps are of odd degrees*, Math. Ann. **292** (1992), no. 1, 13–29.

[17] S.-K. Yeung, *A surface of maximal canonical degree*, Math. Ann. **368** (2017), no. 3-4, 1171–1189.

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