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Homogeneous metrics with prescribed Ricci curvature on spaces with non-maximal isotropy

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Consider a compact Lie group G and a closed subgroup $H < G$. Suppose $\mathcal{M}$ is the set of G-invariant Riemannian metrics on the homogeneous space $M = G/H$. We obtain a sufficient condition for the existence of $g \in \mathcal{M}$ and $c > 0$ such that the Ricci curvature of g equals cT for a given $T \in \mathcal{M}$. This condition is also necessary if the isotropy representation of M splits into two inequivalent irreducible summands.

1. Introduction

Consider a smooth manifold M and a symmetric (0,2)-tensor field T on M. The prescribed Ricci curvature problem consists in finding a Riemannian

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metric g such that

$$\mathrm{Ric}\, g = T, \tag{1.1}$$

where $\mathrm{Ric}\, g$ denotes the Ricci curvature of g. The investigation of this problem is an important segment of geometric analysis with strong ties to flows and relativity. While many mathematicians have made significant contributions to the study of (1.1), a particularly large amount of work was done by D. DeTurck and his collaborators. The reader will find surveys in [8, Chapter 5] and [7, Section 6.5]. For more recent results, see [12, 13, 23, 24] and references therein.

Suppose the manifold M is closed and the tensor field T is positive-definite. It is possible for equation (1.1) to have no solutions. Moreover, in a number of settings, a metric g such that

$$\mathrm{Ric}\, g = cT \tag{1.2}$$

only exists for one value of $c \in \mathbb{R}$; see, e.g., [17, 24]. This observation suggests a change of paradigm in the study of the prescribed Ricci curvature problem. Namely, instead of trying to solve (1.1), one should search for a metric g and a constant $c > 0$ satisfying (1.2). The idea of shifting focus from (1.1) to (1.2) dates back to R. Hamilton's work [17] and D. DeTurck's work [14]. Note that such a shift may be unreasonable on an open manifold or a manifold with non-empty boundary.

In the paper [24], the second-named author initiated the investigation of equation (1.2) on homogeneous spaces. More precisely, consider a compact connected Lie group G and a closed connected subgroup $H < G$. Let M be the homogeneous space G/H. We denote by $\mathcal{M}$ the set of G-invariant Riemannian metrics on M and suppose the tensor field T lies in $\mathcal{M}$. The main theorem of [24] states that a metric $g \in \mathcal{M}$ and a constant $c > 0$ satisfying (1.2) can be found if H is a maximal connected Lie subgroup of G (the dimension of M is assumed to be at least 3). Further results in [24] address the prescribed Ricci curvature problem on M in the case where the isotropy representation of M splits into two inequivalent irreducible summands. The reader will find a classification of homogeneous spaces possessing this property in [15, 18]. Several authors have studied their geometry in detail; see, e.g., [5, 11, 25].

The main result of the present paper, Theorem 2.9, provides a sufficient condition for the existence of $g \in \mathcal{M}$ and $c > 0$ satisfying (1.2) in the case where the maximality assumption on H does not hold. This condition is, in fact, necessary when the isotropy representation of M splits into two inequivalent irreducible summands. To describe the result further, assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H. As before, we demand that T lie in $\mathcal{M}$. Imposing natural requirements on the Lie subalgebras of $\mathfrak{g}$ that contain $\mathfrak{h}$, we show that the existence of $g \in \mathcal{M}$ and $c > 0$ satisfying (1.2) is guaranteed by an array of simple inequalities for T.

Theorem 2.9 applies on a broad class of homogeneous spaces. For instance, its assumptions hold if M is a generalised flag manifold. Previous literature provides little information concerning the solvability of (1.2) on such manifolds. However, several other aspects of their geometry have been investigated thoroughly; see the survey [3].

It is interesting to place our analysis of (1.2) into the context of the theory of homogeneous Einstein metrics. We refer to [8, Chapter 7] for an introduction to this theory and some foundational results. The surveys [3, 21, 26, 27] contain overviews of more recent work. According to [28, Theorem (2.2)], a metric $g \in \mathcal{M}$ satisfying the Einstein equation

$$\operatorname{Ric} g = \lambda g \tag{1.3}$$

for some $\lambda \in \mathbb{R}$ exists if H is a maximal connected Lie subgroup of G. Whether such $g \in \mathcal{M}$ can be found when this assumption does not hold is a long-standing open question. The papers [9, 10] offer several sufficient conditions for the answer to be positive, while [28, §3] discusses a situation in which the answer is negative.

One observes a number of similarities *and* differences between the analytical properties of (1.2) and those of (1.3) on homogeneous spaces. As shown in [24], a metric $g \in \mathcal{M}$ satisfies (1.2) for some $c \in \mathbb{R}$ if and only if it is (up to scaling) a critical point of the scalar curvature functional S on the set

$$\mathcal{M}_T = \{g \in \mathcal{M} \mid \operatorname{tr}_g T = 1\}, \tag{1.4}$$

where $\operatorname{tr}_g T$ denotes the trace of T with respect to g. Under the assumptions of Theorem 2.9, S has a global maximum on $\mathcal{M}_T$. Correspondingly, it is well-known that $g \in \mathcal{M}$ satisfies (1.3) if and only if it is (up to scaling) a critical point of S on the set

$$\mathcal{M}_1 = \{g \in \mathcal{M} \mid M \text{ has volume 1 with respect to } g\}. \tag{1.5}$$

This fact underlies the proofs of the main results of [9, 10, 28]. However, according to [28, Theorem (2.4)] and [9, Theorem 1.2], it is only in very special situations that S can have a global maximum on $\mathcal{M}_1$.

The paper is organised as follows. In Section 2, we state and prove our main result, Theorem 2.9. We also present a corollary. Section 3 explores equation (1.2) on homogeneous spaces with two inequivalent irreducible isotropy summands. We demonstrate, by appealing to [24, Proposition 3.1], that Theorem 2.9 is optimal in this setting. Section 4 discusses the application of our results on generalised flag manifolds. As a specific example, we consider the space $G_2/U(2)$ with $U(2)$ corresponding to the long root of G_2. This space has three pairwise inequivalent irreducible summands in its isotropy representation.

Most of the results of the present paper, including Theorem 2.9, are announced in [16].

2. The existence of metrics with prescribed Ricci curvature

As in Section 1, we consider a compact connected Lie group G and a closed connected subgroup $H < G$. Assume the homogeneous space $M = G/H$ has dimension 3 or higher, i.e.,

$$\dim M = n \geq 3. \tag{2.1}$$

Choose a scalar product Q on $\mathfrak{g}$ induced by a bi-invariant Riemannian metric on G. If $\mathfrak{u}$ and $\mathfrak{v}$ are subspaces of $\mathfrak{g}$ such that $\mathfrak{u} \subset \mathfrak{v}$, we use the notation $\mathfrak{v} \ominus \mathfrak{u}$ for the Q-orthogonal complement of $\mathfrak{u}$ in $\mathfrak{v}$. Define

$$\mathfrak{m} = \mathfrak{g} \ominus \mathfrak{h}.$$

It is clear that $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. The representation $\mathrm{Ad}(H)|_{\mathfrak{m}}$ is equivalent to the isotropy representation of G/H. We standardly identify $\mathfrak{m}$ with the tangent space $T_H M$.

2.1. Preliminaries

The space $\mathcal{M}$ of G-invariant Riemannian metrics on M carries a natural smooth manifold structure; see, e.g., [21, pages 6318–6319]. The properties of this space are discussed in [9, Subsection 4.1] in great detail. In what follows, we implicitly identify $g \in \mathcal{M}$ with the bilinear form induced by g on $\mathfrak{m}$ via the identification of $T_H M$ and $\mathfrak{m}$. The scalar curvature $S(g)$ of a metric $g \in \mathcal{M}$ is constant on M. Therefore, we may interpret $S(g)$ as the result of applying a functional $S : \mathcal{M} \to \mathbb{R}$ to $g \in \mathcal{M}$. Standard formulas for the scalar curvature (see, e.g., [8, Corollary 7.39]) imply that S is differentiable on $\mathcal{M}$. Given $T \in \mathcal{M}$, the space $\mathcal{M}_T$ defined by (1.4) has a smooth manifold structure inherited from $\mathcal{M}$.

The following result is a special case of [24, Lemma 2.1]. It provides a variational interpretation of the prescribed Ricci curvature equation (1.2) on homogeneous spaces.

Lemma 2.1. *Given $T \in \mathcal{M}$ and $g \in \mathcal{M}_T$, formula (1.2) holds for some $c \in \mathbb{R}$ if and only if g is a critical point of the restriction of the functional S to $\mathcal{M}_T$.*

We will use this lemma in the proof of our main result, Theorem 2.9.

Remark 2.2. The restriction of S to $\mathcal{M}_T$ is bounded above for every $T \in \mathcal{M}$. This is a consequence of [28, Equation (1.3)] and the definition of $\mathcal{M}_T$; cf. Lemma 2.24 below. If the homogeneous space M is effective and T lies in $\mathcal{M}$, then the following statements are equivalent:

1) The restriction of S to $\mathcal{M}_T$ is bounded below.
2) The universal cover of M is the product of several isotropy irreducible homogeneous spaces and a Euclidean space.

One can prove this equivalence by repeating the argument from [28, Proof of Theorem (2.1)] with minor modifications. If the two statements above hold, then all the metrics in $\mathcal{M}$ have the same Ricci curvature; see [25, Lemma 3.2]. In this case, the analysis of (1.2) is easy.

Given a bilinear form R on $\mathfrak{m}$ and a nonzero subspace $\mathfrak{u} \subset \mathfrak{m}$, we write $R|_\mathfrak{u}$ for the restriction of R to $\mathfrak{u}$. Let $\operatorname{tr}_Q R|_\mathfrak{u}$ be the trace of $R|_\mathfrak{u}$ with respect to $Q|_\mathfrak{u}$. If R' is a bilinear form on $\mathfrak{u}$, denote

$$\begin{aligned} &\lambda_-(R') = \inf\{R'(X,X) \,|\, X \in \mathfrak{u} \text{ and } Q(X,X) = 1\}, \\ &\lambda_+(R') = \sup\{R'(X,X) \,|\, X \in \mathfrak{u} \text{ and } Q(X,X) = 1\}. \end{aligned} \tag{2.2}$$

Thus, $\lambda_-(R')$ and $\lambda_+(R')$ are the smallest and the largest eigenvalue of the matrix of R' in a $Q|_\mathfrak{u}$-orthonormal basis of $\mathfrak{u}$. We will use the notation

$$\omega(\mathfrak{u}) = \min\{\dim \mathfrak{v} \,|\, \mathfrak{v} \text{ is a nonzero } \operatorname{Ad}(H)\text{-invariant subspace of } \mathfrak{u}\}.$$

It is clear that $\omega(\mathfrak{u})$ always lies between 1 and $\dim \mathfrak{u}$. In fact, $\omega(\mathfrak{u})$ equals $\dim \mathfrak{u}$ if $\operatorname{Ad}(H)|_\mathfrak{u}$ is irreducible.

Given $\mathrm{Ad}(H)$-invariant subspaces $\mathfrak{u} \subset \mathfrak{m}$, $\mathfrak{v} \subset \mathfrak{m}$ and $\mathfrak{w} \subset \mathfrak{m}$, define a tensor $\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) \in \mathfrak{u} \otimes \mathfrak{v}^* \otimes \mathfrak{w}^*$ by the formula

$$\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})(X, Y) = \pi_{\mathfrak{u}}[X, Y], \qquad X \in \mathfrak{v},\ Y \in \mathfrak{w}. \tag{2.3}$$

Here and in what follows, $\pi_{\mathfrak{u}}$ stands for the Q-orthogonal projection onto $\mathfrak{u}$. Let $\langle \mathfrak{u}\mathfrak{v}\mathfrak{w} \rangle$ be the squared norm of $\Delta(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ with respect to the scalar product on $\mathfrak{u} \otimes \mathfrak{v}^* \otimes \mathfrak{w}^*$ induced by $Q|_{\mathfrak{u}}$, $Q|_{\mathfrak{v}}$ and $Q|_{\mathfrak{w}}$. The fact that Q comes from a bi-invariant metric on G implies

$$\langle \mathfrak{u}\mathfrak{v}\mathfrak{w} \rangle = \langle \mathfrak{w}\mathfrak{u}\mathfrak{v} \rangle = \langle \mathfrak{v}\mathfrak{w}\mathfrak{u} \rangle = \langle \mathfrak{v}\mathfrak{u}\mathfrak{w} \rangle = \langle \mathfrak{u}\mathfrak{w}\mathfrak{v} \rangle = \langle \mathfrak{w}\mathfrak{v}\mathfrak{u} \rangle.$$

It is easy to compute $\langle \mathfrak{u}\mathfrak{v}\mathfrak{w} \rangle$ in terms of the structure constants of the homogeneous space M; see formula (2.18) below.

2.2. The sufficient condition

Our main result, Theorem 2.9, requires the following hypothesis. The class of homogeneous spaces for which this hypothesis holds is very broad. We discuss examples in Sections 3 and 4.

Hypothesis 2.3. *Every Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{s}$ and $\mathfrak{h} \neq \mathfrak{s}$ meets the following requirements:*

1) *The representations $\mathrm{Ad}(H)|_{\mathfrak{u}}$ and $\mathrm{Ad}(H)|_{\mathfrak{v}}$ are inequivalent for every pair of nonzero $\mathrm{Ad}(H)$-invariant spaces $\mathfrak{u} \subset \mathfrak{s} \ominus \mathfrak{h}$ and $\mathfrak{v} \subset \mathfrak{g} \ominus \mathfrak{s}$.*
2) *The commutator $[\mathfrak{r}, \mathfrak{s}]$ is nonzero for every $\mathrm{Ad}(H)$-invariant 1-dimensional subspace $\mathfrak{r}$ of $\mathfrak{g} \ominus \mathfrak{s}$.*

Remark 2.4. One can show that requirement 1 of Hypothesis 2.3 holds for every $\mathfrak{s}$ if the isotropy representation of M splits into pairwise inequivalent irreducible summands; cf. the proof of Proposition 4.1 below. However, this requirement may be satisfied (at least, for some $\mathfrak{s}$) even if M does not possess this property. To give an example, suppose $H = SO(k-2)$ with $k \geq 4$ embedded naturally into $G = SO(k)$. Then M is the Stiefel manifold $V_2\mathbb{R}^k$. Let $\mathfrak{s}$ be the direct sum of $\mathfrak{so}_2$ and $\mathfrak{h} = \mathfrak{so}_{k-2}$ embedded naturally into $\mathfrak{g} = \mathfrak{so}_k$. Then the representation $\mathrm{Ad}(H)|_{\mathfrak{s}\ominus\mathfrak{h}}$ is trivial, while the representation $\mathrm{Ad}(H)|_{\mathfrak{g}\ominus\mathfrak{s}}$ splits into two equivalent $(k-2)$-dimensional irreducible summands; see [19, Section 4].

Remark 2.5. In a sense, requirement 2 of Hypothesis 2.3 is necessary for Theorem 2.9 to hold. We explain this after the proof of Lemma 2.15.

Remark 2.6. Suppose $\mathfrak{r}$ is an $\mathrm{Ad}(H)$-invariant 1-dimensional subspace of $\mathfrak{g} \ominus \mathfrak{s}$. If the commutator $[\mathfrak{r}, \mathfrak{s}]$ equals $\{0\}$, then the direct sum of $\mathfrak{r}$ and $\mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to the direct sum of $\mathbb{R}$ and $\mathfrak{s}$. It is obvious that requirement 2 of Hypothesis 2.3 holds for $\mathfrak{s}$ if no such subalgebra exists.

Remark 2.7. In Section 4, we will encounter cases where $\mathfrak{g} \ominus \mathfrak{s}$ does not have any $\mathrm{Ad}(H)$-invariant 1-dimensional subspaces. In these cases, requirement 2 of Hypothesis 2.3 is automatically satisfied for $\mathfrak{s}$.

Suppose $\mathfrak{k}$ and $\mathfrak{k}'$ are Lie subalgebras of $\mathfrak{g}$ such that

$$\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{k}' \supset \mathfrak{h}. \tag{2.4}$$

In order to state our main result, we need to introduce some terminology and notation.

Definition 2.8. We call (2.4) a *simple chain* if $\mathfrak{k}'$ is a maximal Lie subalgebra of $\mathfrak{k}$ and $\mathfrak{h} \neq \mathfrak{k}'$.

Let us emphasise that Definition 2.8 allows the equality $\mathfrak{k} = \mathfrak{g}$ but not $\mathfrak{k}' = \mathfrak{k}$. We denote

$$\mathfrak{j} = \mathfrak{g} \ominus \mathfrak{k}, \qquad \mathfrak{j}' = \mathfrak{g} \ominus \mathfrak{k}', \qquad \mathfrak{l} = \mathfrak{k} \ominus \mathfrak{k}', \qquad \mathfrak{n} = \mathfrak{k}' \ominus \mathfrak{h}. \tag{2.5}$$

It is obvious that

$$\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{h} = \mathfrak{j}' \oplus \mathfrak{n} \oplus \mathfrak{h}, \quad \mathfrak{j}' = \mathfrak{j} \oplus \mathfrak{l}, \quad \mathfrak{k} = \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{h}, \quad \mathfrak{k}' = \mathfrak{n} \oplus \mathfrak{h}.$$

Here and in what follows, the symbol $\oplus$ stands for the Q-orthogonal sum.

Suppose (2.4) is a simple chain. In order to state our main result, we need to associate a number, denoted $\eta(\mathfrak{k}, \mathfrak{k}')$, to this simple chain. Let B be the Killing form of the Lie algebra $\mathfrak{g}$. Define $\eta(\mathfrak{k}, \mathfrak{k}')$ by the formula

$$\eta(\mathfrak{k}, \mathfrak{k}') = \frac{2 \operatorname{tr}_Q B|_{\mathfrak{n}} + 2\langle \mathfrak{n}\mathfrak{j}'\mathfrak{j}' \rangle + \langle \mathfrak{n}\mathfrak{n}\mathfrak{n} \rangle}{\omega(\mathfrak{n})(2 \operatorname{tr}_Q B|_{\mathfrak{l}} + \langle \mathfrak{l}\mathfrak{l}\mathfrak{l} \rangle + 2\langle \mathfrak{l}\mathfrak{j}\mathfrak{j} \rangle)}. \tag{2.6}$$

Lemma 2.15 below shows, when Hypothesis 2.3 is satisfied, that the denominator in (2.6) can never equal 0 and that $\eta(\mathfrak{k}, \mathfrak{k}') \geq 0$. We are now ready to formulate the main result of the present paper. We prove it in Subsections 2.3–2.7.

Theorem 2.9. *Suppose Hypothesis 2.3 is satisfied for the homogeneous space M. Consider a tensor field $T \in \mathcal{M}$. If the inequality*

$$\frac{\lambda_-(T|_{\mathfrak{n}})}{\operatorname{tr}_Q T|_{\mathfrak{l}}} > \eta(\mathfrak{k}, \mathfrak{k}') \tag{2.7}$$

holds for every simple chain of the form (2.4), then there exists a Riemannian metric $g \in \mathcal{M}_T$ such that $S(g) \geq S(h)$ for all $h \in \mathcal{M}_T$. The Ricci curvature of g coincides with cT for some $c > 0$.

Subsection 2.3 contains simple and "practical" formulas for the quantities appearing in (2.7). Specifically, the eigenvalue $\lambda_-(T|_{\mathfrak{n}})$ and the trace $\operatorname{tr}_Q T|_{\mathfrak{l}}$ are given by (2.14), while the computation of $\eta(\mathfrak{k}, \mathfrak{k}')$ on concrete homogeneous spaces is likely to involve (2.13), (2.16) and (2.18). One can also find $\eta(\mathfrak{k}, \mathfrak{k}')$ with the aid of Lemma 2.15.

In Sections 3 and 4, we discuss several classes of examples that illustrate the use of Theorem 2.9. As part of this discussion, we compute the numbers $\eta(\mathfrak{k}, \mathfrak{k}')$ explicitly for all simple chains on certain generalised flag manifolds. In Subsection 2.8, we state two corollaries of Theorem 2.9. One of them provides an alternative to (2.7), and the other deals with the case where (2.7) holds for all $T \in \mathcal{M}$.

Remark 2.10. Theorem 2.9 assumes that the tensor field T is positive-definite. Let us make a few comments related to this assumption. If T is degenerate, then the restriction of S to $\mathcal{M}_T$ may be unbounded above. This is possible even if M satisfies Hypothesis 2.3; see [24, Remark 3.2] for a class of examples. If T has mixed signature, the techniques used in our proof of Theorem 2.9 appear to be ineffective. Particularly, the estimates in Lemmas 2.22, 2.24 and 2.31 seem to break down. Finally, if T is negative-definite, a Riemannian metric $g \in \mathcal{M}_T$ with Ricci curvature cT does not exist for any $c > 0$. This is a consequence of Bochner's theorem; see [8, Theorem 1.84].

Remark 2.11. Given $T \in \mathcal{M}$, if $\mathfrak{h}$ is not a maximal Lie subalgebra of $\mathfrak{g}$, Hypothesis 2.3 is satisfied, and (2.7) holds for every simple chain of the form (2.4), then the restriction of S to $\mathcal{M}_T$ cannot be proper. This observation follows from Remark 2.34 and Lemma 2.36 below. In a sense, it is an analogue of the "only if" part of [28, Theorem (2.2)], a result concerning the restriction of S to the set $\mathcal{M}_1$ given by (1.5).

2.3. Some background and preparatory lemmas

The background material in this subsection is mostly standard. It is presented in greater detail in, for example, [21, 28]. However, to the best of the authors' knowledge, Lemmas 2.12, 2.14 and 2.15, as well as Proposition 2.18, are new.

Throughout Subections 2.3–2.7, we assume Hypothesis 2.3 holds. Some of our lemmas can actually be proven under milder conditions than those imposed. This is explained in Remark 2.37. As above, throughout Subsections 2.3–2.4, we suppose $\mathfrak{k}$ and $\mathfrak{k}'$ are distinct Lie subalgebras of $\mathfrak{g}$ satisfying the inclusions $\mathfrak{h} \subset \mathfrak{k}' \subset \mathfrak{k}$. However, unless stated otherwise, we do not require (2.4) to be a simple chain. The spaces $\mathfrak{j}$, $\mathfrak{j}'$, $\mathfrak{l}$ and $\mathfrak{n}$ are defined by (2.5).

Consider a Q-orthogonal $\mathrm{Ad}(H)$-invariant decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{2.8}$$

such that $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is irreducible for each $i = 1, \ldots, s$. Let d_i denote the dimension of $\mathfrak{m}_i$. Generally speaking, the space $\mathfrak{m}$ admits more than one decomposition of the form (2.8). However, the number s and the multiset $\{d_1, \ldots, d_s\}$ must be the same for all such decompositions.

The summands $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ are determined uniquely up to order if $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is inequivalent to $\mathrm{Ad}(H)|_{\mathfrak{m}_j}$ whenever $i \neq j$. This fact can be derived from Schur's lemma; see, e.g., [25, Subsection 2.1].

Our analysis will rely heavily on the following consequence of Hypothesis 2.3.

Lemma 2.12. *There exists a set $J_{\mathfrak{k}} \subset \{1, \ldots, s\}$ satisfying the equality*

$$\mathfrak{k} \ominus \mathfrak{h} = \bigoplus_{j \in J_{\mathfrak{k}}} \mathfrak{m}_j. \tag{2.9}$$

Evidently, such a set is unique.

Throughout the paper, we assume

$$\bigoplus_{j \in \emptyset} \mathfrak{m}_j = \{0\}.$$

Proof of Lemma 2.12. Fix a Q-orthogonal $\mathrm{Ad}(H)$-invariant decomposition

$$\mathfrak{m} = \mathfrak{m}'_1 \oplus \cdots \oplus \mathfrak{m}'_s$$

such that $\mathrm{Ad}(H)|_{\mathfrak{m}'_j}$ is irreducible for each $j = 1, \ldots, s$ and

$$\mathfrak{k} \ominus \mathfrak{h} = \mathfrak{m}'_1 \oplus \cdots \oplus \mathfrak{m}'_p$$

for some $p = 1, \ldots, s$. One can easily verify that such a decomposition exists. Consider the map $\pi_{jk} : \mathfrak{m}_j \to \mathfrak{m}'_k$ sending a vector in $\mathfrak{m}_j$ to its Q-orthogonal projection onto $\mathfrak{m}'_k$. Clearly, this map is $\mathrm{Ad}(H)$-invariant for all $j, k = 1, \ldots, s$. It is, therefore, an isomorphism or zero by Schur's lemma. Define

$$J_{\mathfrak{k}} = \{j \in [1, s] \cap \mathbb{N} \,|\, \pi_{jk} \text{ is an isomorphism for some } k \in [1, p] \cap \mathbb{N}\}.$$

We claim that (2.9) holds. To prove this, we first fix $k \leq p$ and show that

$$\mathfrak{m}'_k \subset \bigoplus_{j \in J_{\mathfrak{k}}} \mathfrak{m}_j. \tag{2.10}$$

Consider the map $\pi'_{kl} : \mathfrak{m}'_k \to \mathfrak{m}_l$ sending a vector in $\mathfrak{m}'_k$ to its Q-orthogonal projection onto $\mathfrak{m}_l$. Choose $X \in \mathfrak{m}'_k$. The equality

$$X = \pi'_{k1} X + \cdots + \pi'_{ks} X$$

holds true. To prove formula (2.10), it suffices to show that $l \in J_{\mathfrak{k}}$ whenever $\pi'_{kl} X \neq 0$. Clearly, $\mathfrak{m}'_k$ is not orthogonal to $\mathfrak{m}_l$ if $\pi'_{kl} X \neq 0$. Therefore, $\pi_{lk} \neq 0$ if this inequality holds. Schur's lemma then implies that π_{lk} must be an isomorphism. Therefore, l lies in $J_{\mathfrak{k}}$, formula (2.10) holds, and $\mathfrak{k} \ominus \mathfrak{h}$ is a subset of $\bigoplus_{j \in J_{\mathfrak{k}}} \mathfrak{m}_j$.

We now fix $k > p$ and $l \in J_{\mathfrak{k}}$. Our next step is to prove that $Q(\mathfrak{m}'_k, \mathfrak{m}_l) = \{0\}$. This equality implies that the Q-orthogonal complement of $\bigoplus_{j \in J_{\mathfrak{k}}} \mathfrak{m}_j$ contains the Q-orthogonal complement of $\mathfrak{k}$. This fact, in its turn, shows that $\bigoplus_{j \in J_{\mathfrak{k}}} \mathfrak{m}_j$ is a subset of $\mathfrak{k} \ominus \mathfrak{h}$. Consequently, formula (2.9) holds.

Assume $Q(\mathfrak{m}'_k, \mathfrak{m}_l) \neq \{0\}$. By Schur's lemma, the map π'_{kl} is then an isomorphism. Since l lies in $J_{\mathfrak{k}}$, there exists $q \leq p$ such that π_{lq} is an isomorphism as well. Evidently, $k \neq q$. Consider the map $\pi_{lq} \pi'_{kl} : \mathfrak{m}'_k \to \mathfrak{m}_q$. It is an $\mathrm{Ad}(H)$-invariant isomorphism. However, the existence of such an isomorphism contradicts requirement 1 of Hypothesis 2.3. □

Corollary 2.13. *The Lie algebra $\mathfrak{g}$ has at most 2^s distinct Lie subalgebras containing $\mathfrak{h}$.*

Define $J_{\mathfrak{h}} = \emptyset$. Observe that $J_{\mathfrak{g}} = \{1, \ldots, s\}$. It will be convenient for us to set

$$J_{\mathfrak{j}} = J_{\mathfrak{g}} \setminus J_{\mathfrak{k}}, \qquad J_{\mathfrak{j}'} = J_{\mathfrak{g}} \setminus J_{\mathfrak{k}'}, \qquad J_{\mathfrak{l}} = J_{\mathfrak{k}} \setminus J_{\mathfrak{k}'}. \tag{2.11}$$

Evidently,

$$\mathfrak{j} = \bigoplus_{j \in J_{\mathfrak{j}}} \mathfrak{m}_j, \qquad \mathfrak{j}' = \bigoplus_{j \in J_{\mathfrak{j}'}} \mathfrak{m}_j, \qquad \mathfrak{l} = \bigoplus_{j \in J_{\mathfrak{l}}} \mathfrak{m}_j, \qquad \mathfrak{n} = \bigoplus_{j \in J_{\mathfrak{k}'}} \mathfrak{m}_j, \tag{2.12}$$

which implies

$$\omega(\mathfrak{n}) = \min_{j \in J_{\mathfrak{k}'}} d_j. \tag{2.13}$$

Given $T \in \mathcal{M}$, it is always possible to choose the decomposition (2.8) so that

$$T = \sum_{i=1}^{s} z_i \pi^*_{\mathfrak{m}_i} Q, \qquad z_i > 0;$$

see [28, page 180]. If this formula holds, then

$$\lambda_-(T|_{\mathfrak{n}}) = \min_{i \in J_{\mathfrak{k}'}} z_i, \qquad \operatorname{tr}_Q T|_{\mathfrak{l}} = \sum_{i \in J_{\mathfrak{l}}} d_i z_i. \tag{2.14}$$

Recall that B denotes the Killing form of $\mathfrak{g}$. For every $i = 1, \ldots, s$, because $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is irreducible, there exists $b_i \geq 0$ such that

$$B|_{\mathfrak{m}_i} = -b_i Q|_{\mathfrak{m}_i}. \tag{2.15}$$

It is clear that

$$\operatorname{tr}_Q B|_{\mathfrak{n}} = -\sum_{j \in J_{\mathfrak{k}'}} d_j b_j, \qquad \operatorname{tr}_Q B|_{\mathfrak{l}} = -\sum_{j \in J_{\mathfrak{l}}} d_j b_j. \tag{2.16}$$

Given $i, j, k \in \{1, \ldots, s\}$, define

$$[ijk] = \langle \mathfrak{m}_i \mathfrak{m}_j \mathfrak{m}_k \rangle.$$

Note that $[ijk]$ is symmetric in all three indices. The numbers $([ijk])_{i,j,k=1}^{s}$ are often called the *structure constants* of the homogeneous space M. If

$$\mathfrak{u} = \bigoplus_{i \in \mathcal{J}_{\mathfrak{u}}} \mathfrak{m}_i, \qquad \mathfrak{v} = \bigoplus_{i \in \mathcal{J}_{\mathfrak{v}}} \mathfrak{m}_i, \qquad \mathfrak{w} = \bigoplus_{i \in \mathcal{J}_{\mathfrak{w}}} \mathfrak{m}_i, \tag{2.17}$$

where $\mathcal{J}_{\mathfrak{u}}$, $\mathcal{J}_{\mathfrak{v}}$ and $\mathcal{J}_{\mathfrak{w}}$ are subsets of $\{1, \ldots, s\}$, then

$$\langle \mathfrak{u}\mathfrak{v}\mathfrak{w} \rangle = \sum_{i \in \mathcal{J}_{\mathfrak{u}}} \sum_{j \in \mathcal{J}_{\mathfrak{v}}} \sum_{k \in \mathcal{J}_{\mathfrak{w}}} [ijk]. \tag{2.18}$$

(We interpret the sum over the empty set as 0.)

Lemma 2.14. *If $i \in J_{\mathfrak{l}}$ and $j, k \in J_{\mathfrak{k}'}$, then $[ijk] = 0$.*

Proof. The inclusion $j, k \in J_{\mathfrak{k}'}$ implies that $\mathfrak{m}_j$ and $\mathfrak{m}_k$ are subspaces of the Lie algebra $\mathfrak{k}'$. Therefore, the map

$$\mathfrak{m}_j \times \mathfrak{m}_k \ni (X, Y) \mapsto [X, Y]$$

takes values in $\mathfrak{k}'$. Since $i \in J_{\mathfrak{l}}$, the Q-orthogonal projection of $\mathfrak{k}'$ onto $\mathfrak{m}_i$ equals $\{0\}$. This means the tensor $\Delta(\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k)$ given by (2.3) is the zero tensor. Thus, the assertion of the lemma holds. □

Fix a Q-orthonormal basis $(w_j)_{j=1}^{\dim \mathfrak{h}}$ of the Lie algebra $\mathfrak{h}$. Given $i = 1, \ldots, s$, consider the Casimir operator $C_{\mathfrak{m}_i, Q|_{\mathfrak{h}}} : \mathfrak{m}_i \to \mathfrak{m}_i$ defined by the formula

$$C_{\mathfrak{m}_i, Q|_{\mathfrak{h}}}(X) = -\left(\sum_{j=1}^{\dim \mathfrak{h}} \operatorname{ad} w_j \circ \operatorname{ad} w_j \right)(X), \qquad X \in \mathfrak{m}_i.$$

The irreducibility of $\operatorname{Ad}(H)|_{\mathfrak{m}_i}$ implies the existence of $\zeta_i \geq 0$ such that

$$C_{\mathfrak{m}_i, Q|_{\mathfrak{h}}}(X) = \zeta_i X, \qquad X \in \mathfrak{m}_i. \tag{2.19}$$

Note that $\zeta_i = 0$ if and only if $\operatorname{Ad}(H)|_{\mathfrak{m}_i}$ is trivial. According to [28, Lemma (1.5)], the arrays $(b_i)_{i=1}^s$, $([ijk])_{i,j,k=1}^s$ and $(\zeta_i)_{i=1}^s$ are related to each other by the equality

$$d_i b_i = 2 d_i \zeta_i + \sum_{j,k=1}^{s} [ijk]. \tag{2.20}$$

The following result shows that the numbers $\eta(\mathfrak{k}, \mathfrak{k}')$ introduced in Subsection 2.2 are well-defined and non-negative.

Lemma 2.15. *One has*

$$
\begin{aligned}
&-2\operatorname{tr}_Q B|_{\mathfrak{l}} - \langle \mathfrak{lll}\rangle - 2\langle \mathfrak{ljj}\rangle \\
&\quad = \sum_{j\in J_{\mathfrak{l}}} \Big(4d_j\zeta_j + \sum_{k,l\in J_{\mathfrak{l}}} [jkl] + 4\sum_{k\in J_{\mathfrak{k}'}} \sum_{l\in J_{\mathfrak{l}}} [jkl]\Big) > 0, \\
&-2\operatorname{tr}_Q B|_{\mathfrak{n}} - 2\langle \mathfrak{nj}'\mathfrak{j}'\rangle - \langle \mathfrak{nnn}\rangle \\
&\quad = \sum_{j\in J_{\mathfrak{k}'}} \Big(4d_j\zeta_j + \sum_{k,l\in J_{\mathfrak{k}'}} [jkl]\Big) \geq 0.
\end{aligned}
$$

Proof. Equalities (2.16), (2.18) and (2.20), together with Lemma 2.14, yield

$$
\begin{aligned}
-2\operatorname{tr}_Q B|_{\mathfrak{l}} - \langle \mathfrak{lll}\rangle - 2\langle \mathfrak{ljj}\rangle &= 2\sum_{j\in J_{\mathfrak{l}}} d_j b_j - \sum_{j,k,l\in J_{\mathfrak{l}}} [jkl] - 2\sum_{j\in J_{\mathfrak{l}}} \sum_{k,l\in J_{\mathfrak{j}}} [jkl] \\
&= \sum_{j\in J_{\mathfrak{l}}} \Big(4d_j\zeta_j + \sum_{k,l\in J_{\mathfrak{l}}} [jkl] + 4\sum_{k\in J_{\mathfrak{k}'}} \sum_{l\in J_{\mathfrak{l}}} [jkl]\Big). \qquad (2.21)
\end{aligned}
$$

The expression in the last line must be non-negative because the numbers d_j, ζ_j and $[jkl]$ are non-negative by definition. If it is 0, then $\zeta_j = 0$ for every $j \in J_{\mathfrak{l}}$. Consequently, the representation $\operatorname{Ad}(H)|_{\mathfrak{m}_j}$ is trivial for every such j. Since $\operatorname{Ad}(H)|_{\mathfrak{m}_j}$ is also irreducible, this means $d_j = 1$. Moreover, in view of Lemma 2.14, if the expression in the last line of (2.21) is 0, then

$$
[jkl] = 0, \qquad j \in J_{\mathfrak{l}},\ k,l \in J_{\mathfrak{k}}.
$$

This implies

$$
[\mathfrak{m}_j, \mathfrak{k}'] = \{0\}, \qquad j \in J_{\mathfrak{l}}.
$$

However, the commutation $[\mathfrak{m}_j, \mathfrak{k}']$ must be non-trivial by requirement 2 of Hypothesis 2.3. Thus, the expression in the last line of (2.21) cannot be 0, and the first formula in the statement of the lemma holds.

Next, we use (2.16), (2.18), (2.20) and Lemma 2.14 again to compute

$$
\begin{aligned}
&-2\operatorname{tr}_Q B|_{\mathfrak{n}} - 2\langle \mathfrak{nj}'\mathfrak{j}'\rangle - \langle \mathfrak{nnn}\rangle \\
&\quad = 2\sum_{j\in J_{\mathfrak{k}'}} d_j b_j - 2\sum_{j\in J_{\mathfrak{k}'}} \sum_{k,l\in J_{\mathfrak{j}'}} [jkl] - \sum_{j,k,l\in J_{\mathfrak{k}'}} [jkl] \\
&\quad = \sum_{j\in J_{\mathfrak{k}'}} \Big(4d_j\zeta_j + \sum_{k,l\in J_{\mathfrak{k}'}} [jkl]\Big) \geq 0.
\end{aligned}
$$

□

Remark 2.16. If $\mathfrak{g}$ had a Lie subalgebra $\mathfrak{s}$ containing $\mathfrak{h}$ as a proper subset and satisfying the first requirement of Hypothesis 2.3 but not the second, then the formulation of Theorem 2.9 would become meaningless. Indeed, in this case, it would be possible to find an $\mathrm{Ad}(H)$-invariant 1-dimensional subspace $\mathfrak{r}$ of $\mathfrak{g} \ominus \mathfrak{s}$ such that $[\mathfrak{r}, \mathfrak{s}] = \{0\}$. By Remark 2.6,

$$\mathfrak{g} \supset \mathfrak{r} \oplus \mathfrak{s} \supset \mathfrak{s} \supset \mathfrak{h}$$

would be a simple chain. However, employing (2.21), we would be able to demonstrate that $\eta(\mathfrak{r} \oplus \mathfrak{s}, \mathfrak{s})$ is not well-defined.

Example 2.17. Let us compute the quantities in Lemma 2.15 explicitly assuming $M = G_2/U(2)$ (here, $U(2)$ corresponds to the long root of G_2), $\mathfrak{k} = \mathfrak{g}$ and $\mathfrak{k}' = \mathfrak{su}_3$. We discuss the space $G_2/U(2)$ in Section 4. In particular, we show that it satisfies Hypothesis 2.3 and observe that $s = 3$. If the decomposition (2.8) is as in [1] and $Q = -B$, then

$$\mathfrak{k}' = \mathfrak{m}_3 \oplus \mathfrak{h}, \qquad J_{\mathfrak{l}} = \{1, 2\}, \qquad J_{\mathfrak{k}'} = \{3\}.$$

Moreover,

$$\begin{aligned} d_1 = d_3 = 4, \qquad d_2 = 2, \qquad & [112] = [121] = [211] = 2/3, \\ & [123] = [231] = [312] = [321] \\ & \quad = [213] = [132] = 1/2, \end{aligned}$$

and the rest of the structure constants are 0. Formula (2.20) implies

$$\zeta_1 = 5/24, \qquad \zeta_2 = 1/12, \qquad \zeta_3 = 3/8.$$

In this setting, the quantities appearing in Lemma 2.15 are

$$\begin{aligned} -2\,\mathrm{tr}_Q\, B|_{\mathfrak{l}} - \langle \mathfrak{lll} \rangle - 2\langle \mathfrak{ljj} \rangle &= \left(4d_1\zeta_1 + 4d_2\zeta_2 + \sum_{j,k,l=1}^{2} [jkl] + 4\sum_{j,l=1}^{2} [3jl] \right) = 10, \\ -2\,\mathrm{tr}_Q\, B|_{\mathfrak{n}} - 2\langle \mathfrak{n}\mathfrak{j}'\mathfrak{j}' \rangle - \langle \mathfrak{nnn} \rangle &= 4d_3\zeta_3 = 6. \end{aligned} \tag{2.22}$$

The following result provides insight into the nature of the numbers $\eta(\mathfrak{k}, \mathfrak{k}')$. It will help us establish a corollary of Theorem 2.9 in Subsection 2.8.

Proposition 2.18. *Assume (2.4) is a simple chain. The number $\eta(\mathfrak{k}, \mathfrak{k}')$ is 0 if and only if the Lie algebra $\mathfrak{k}'$ is isomorphic to the direct sum of $\mathbb{R}$ and $\mathfrak{h}$.*

Proof. Assume $\eta(\mathfrak{k},\mathfrak{k}')=0$. This means the numerator in (2.6) must be 0. Therefore, in view of Lemma 2.15,

$$\sum_{j\in J_{\mathfrak{k}'}}\left(4d_j\zeta_j+\sum_{k,l\in J_{\mathfrak{k}'}}[jkl]\right)=0.$$

Since the numbers d_j, ζ_j and $[jkl]$ are all non-negative, $\zeta_j=0$ for all $j\in J_{\mathfrak{k}'}$. As a consequence, the representation $\mathrm{Ad}(H)|_{\mathfrak{m}_j}$ is trivial for such j. We will use this fact to prove that $\mathfrak{k}'$ is isomorphic to the direct sum of $\mathbb{R}$ and $\mathfrak{h}$.

Fix $i\in J_{\mathfrak{k}'}$. The irreducibility of $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ implies that the dimension d_i equals 1. Consequently,

$$\mathfrak{k}''=\mathfrak{m}_i\oplus\mathfrak{h}$$

is a Lie subalgebra of $\mathfrak{k}'$. Our next step is to show that $\mathfrak{k}''$ is, in fact, equal to $\mathfrak{k}'$.

Choose $k\in J_{\mathfrak{k}'}$. The dimension of $\mathfrak{m}_k$ is 1. Because the representations $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_k}$ are both trivial, they are equivalent. Clearly, $\mathfrak{m}_i$ coincides with $\mathfrak{k}''\ominus\mathfrak{h}$. If $k\neq i$, then $\mathfrak{m}_k$ must lie in $\mathfrak{g}\ominus\mathfrak{k}''$. However, this means $\mathfrak{k}''$ does not meet requirement 1 of Hypothesis 2.3. Thus, i is the only element in $J_{\mathfrak{k}'}$. We conclude that $\mathfrak{k}''$ equals $\mathfrak{k}'$. It is clear that $\mathfrak{k}''$ is isomorphic to the direct sum of $\mathbb{R}$ and $\mathfrak{h}$. This proves the "only if" portion of the lemma. Next, we turn to the converse statement.

Assume $\mathfrak{k}'$ is isomorphic to the direct sum of $\mathbb{R}$ and $\mathfrak{h}$. Let us show that $\eta(\mathfrak{k},\mathfrak{k}')=0$. According to (2.6) and Lemma 2.15,

$$\eta(\mathfrak{k},\mathfrak{k}')=\frac{\sum_{j\in J_{\mathfrak{k}'}}\left(4d_j\zeta_j+\sum_{k,l\in J_{\mathfrak{k}'}}[jkl]\right)}{\omega(\mathfrak{n})\sum_{j\in J_{\mathfrak{l}}}\left(4d_j\zeta_j+\sum_{k,l\in J_{\mathfrak{l}}}[jkl]+4\sum_{k\in J_{\mathfrak{k}'}}\sum_{l\in J_{\mathfrak{l}}}[jkl]\right)}. \tag{2.23}$$

The proof will be complete if we demonstrate that the numerator is 0.

Lemma 2.12 and the existence of an isomorphism between $\mathfrak{k}'$ and the direct sum of $\mathbb{R}$ and $\mathfrak{h}$ imply that

$$\mathfrak{k}'=\mathfrak{m}_i\oplus\mathfrak{h}$$

for some $i=1,\ldots,s$. Moreover, the dimension of $\mathfrak{m}_i$ is 1. Consequently, $J_{\mathfrak{k}'}$ is the set $\{i\}$, and

$$\sum_{j,k,l\in J_{\mathfrak{k}'}}[jkl]=[iii]=0.$$

This formula implies that the numerator on the right-hand side of (2.23) equals

$$4 \sum_{j \in J_{\mathfrak{k}'}} d_j \zeta_j = 4 d_i \zeta_i.$$

The proof will be complete if we demonstrate that $\zeta_i = 0$. It suffices to show that the representation $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is trivial.

Choose a nonzero $X \in \mathfrak{m}_i$ and some $Y \in \mathfrak{h}$. Since $\mathfrak{m}_i$ is $\mathrm{Ad}(H)$-invariant and 1-dimensional, the commutator $[X, Y]$ equals τX for some $\tau \in \mathbb{R}$. The fact that Q is induced by a bi-invariant metric on G implies

$$\tau = \frac{Q([X,Y],X)}{Q(X,X)} = -\frac{Q([X,X],Y)}{Q(X,X)} = 0.$$

Thus, $[X, Y]$ vanishes for $X \in \mathfrak{m}_i$ and $Y \in \mathfrak{h}$, which means $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is trivial. $\square$

Example 2.19. Let us compute the number $\eta(\mathfrak{k}, \mathfrak{k})$ in the setting of Example 2.17. Now, $M = G_2/U(2)$, $\mathfrak{k} = \mathfrak{g}$ and $\mathfrak{k}' = \mathfrak{su}_3$. Observing that $\omega(\mathfrak{n}) = d_2 = 2$ and using (2.22), we find $\eta(\mathfrak{k}, \mathfrak{k}') = 3/10$.

2.4. The scalar curvature and related functionals

The proof of Theorem 2.9 relies on the analysis of two functionals related to the scalar curvature of metrics in $\mathcal{M}$. Let us introduce the first of these functionals. Suppose g is an $\mathrm{Ad}(H)$-invariant scalar product on an $\mathrm{Ad}(H)$-invariant subspace $\mathfrak{u} \subset \mathfrak{m}$. Define

$$S(g) = -\frac{1}{2}\operatorname{tr}_g B|_{\mathfrak{u}} - \frac{1}{4}|\Delta(\mathfrak{u}, \mathfrak{u}, \mathfrak{u})|_g^2. \tag{2.24}$$

In this formula, $\Delta(\mathfrak{u}, \mathfrak{u}, \mathfrak{u})$ is given by (2.3), and $|\cdot|_g$ is the norm on $\mathfrak{u} \otimes \mathfrak{u}^* \otimes \mathfrak{u}^*$ induced by g. If $\mathfrak{u} = \mathfrak{m}$, then we identify g with a Riemannian metric in $\mathcal{M}$. The quantity on the right-hand side of (2.24) is then equal to the scalar curvature of this metric; see, e.g., [8, Corollary 7.39]. Thus, the notation (2.24) is consistent with the notation introduced in the beginning of Subsection 2.1. The following result provides a handy formula for $S(g)$; cf. [28, §1], [22, Section 1] and [25, Section 3].

Lemma 2.20. *Let $\mathfrak{u}$ satisfy the first equality in (2.17) for some $\mathcal{J}_{\mathfrak{u}} \subset \{1, \ldots, s\}$. Suppose the scalar product g and the decomposition (2.8) are* *such that*

$$g = \sum_{i \in \mathcal{J}_\mathfrak{u}} x_i \pi_{\mathfrak{m}_i}^* Q, \qquad x_i > 0.$$

Then

$$\operatorname{tr}_g B|_\mathfrak{u} = -\sum_{i \in \mathcal{J}_\mathfrak{u}} \frac{d_i b_i}{x_i}, \qquad |\Delta(\mathfrak{u},\mathfrak{u},\mathfrak{u})|_g^2 = \sum_{i,j,k \in \mathcal{J}_\mathfrak{u}} [ijk] \frac{x_k}{x_i x_j},$$

$$S(g) = \frac{1}{2} \sum_{i \in \mathcal{J}_\mathfrak{u}} \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k \in \mathcal{J}_\mathfrak{u}} [ijk] \frac{x_k}{x_i x_j}. \tag{2.25}$$

Proof. Let $(e_i)_{i=1}^n$ be a Q-orthonormal basis of $\mathfrak{m}$ adapted to the decomposition (2.8). For every $i = 1, \ldots, n$, define $\tilde{e}_i = \frac{1}{\sqrt{x_{\iota(i)}}} e_i$, where $\iota(i)$ is the number between 1 and s such that e_i lies in $\mathfrak{m}_{\iota(i)}$. Then $(\tilde{e}_i)_{i=1}^n$ is a g-orthonormal basis of $\mathfrak{m}$. We compute

$$\begin{aligned}
\operatorname{tr}_g B|_\mathfrak{u} &= \sum_{i \in \Gamma(\mathfrak{u})} B(\tilde{e}_i, \tilde{e}_i) = \sum_{i \in \Gamma(\mathfrak{u})} \frac{1}{x_{\iota(i)}} B(e_i, e_i) = -\sum_{i \in \mathcal{J}_\mathfrak{u}} \frac{d_i b_i}{x_i}, \\
|\Delta(\mathfrak{u},\mathfrak{u},\mathfrak{u})|_g^2 &= \sum_{i,j \in \Gamma(\mathfrak{u})} g(\Delta(\mathfrak{u},\mathfrak{u},\mathfrak{u})(\tilde{e}_i, \tilde{e}_j), \Delta(\mathfrak{u},\mathfrak{u},\mathfrak{u})(\tilde{e}_i, \tilde{e}_j)) \\
&= \sum_{i,j \in \Gamma(\mathfrak{u})} \sum_{k \in \mathcal{J}_\mathfrak{u}} \frac{x_k}{x_{\iota(i)} x_{\iota(j)}} Q(\Delta(\mathfrak{m}_k,\mathfrak{u},\mathfrak{u})(e_i, e_j), \Delta(\mathfrak{m}_k,\mathfrak{u},\mathfrak{u})(e_i, e_j)) \\
&= \sum_{i,j,k \in \mathcal{J}_\mathfrak{u}} [ijk] \frac{x_k}{x_i x_j}.
\end{aligned}$$

In the first three lines,

$$\Gamma(\mathfrak{u}) = \{i \in [1,n] \cap \mathbb{N} \,|\, e_i \in \mathfrak{u}\} = \{i \in [1,n] \cap \mathbb{N} \,|\, \iota(i) \in \mathcal{J}_\mathfrak{u}\}.$$

The last formula in (2.25) follows from the definition of S. $\square$

Let us introduce one more functional related to the scalar curvature of metrics in $\mathcal{M}$. As in Subsection 2.3, we consider distinct Lie subalgebras $\mathfrak{k}$ and $\mathfrak{k}'$ of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k}' \subset \mathfrak{k}$. The spaces $\mathfrak{j}$, $\mathfrak{j}'$, $\mathfrak{l}$ and $\mathfrak{n}$ are given by (2.5). The sets $J_\mathfrak{k}$, $J_{\mathfrak{k}'}$, $J_\mathfrak{j}$, $J_{\mathfrak{j}'}$ and $J_\mathfrak{l}$ appearing below are introduced in Lemma 2.12 and after Corollary 2.13.

Denote by $\mathcal{M}(\mathfrak{k})$ the space of $\operatorname{Ad}(H)$-invariant scalar products on $\mathfrak{k} \ominus \mathfrak{h}$. There is a natural identification between $\mathcal{M}(\mathfrak{g})$ and $\mathcal{M}$. In what follows, we assume $\mathcal{M}(\mathfrak{k})$ is equipped with the topology inherited from the second tensor power of $(\mathfrak{k}\ominus\mathfrak{h})^*$. If g lies in $\mathcal{M}(\mathfrak{k})$, set

$$\hat{S}(g) = S(g) - \frac{1}{2}|\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})|^2_{QgQ}.$$

The notation $|\cdot|_{QgQ}$ stands for the norm on $\mathfrak{j}\otimes(\mathfrak{k}\ominus\mathfrak{h})^*\otimes\mathfrak{j}^*$ induced by $Q|_\mathfrak{j}$ and $g|_{\mathfrak{k}\ominus\mathfrak{h}}$. One can easily verify that $\hat{S}$ is a continuous map from $\mathcal{M}(\mathfrak{k})$ to $\mathbb{R}$. If g lies in $\mathcal{M}(\mathfrak{g})$, then $\hat{S}(g)$ equals $S(g)$.

Lemma 2.21. *Suppose the scalar product $g\in\mathcal{M}(\mathfrak{k})$ and the decomposition (2.8) are such that*

$$g = \sum_{i\in J_\mathfrak{k}} x_i\pi^*_{\mathfrak{m}_i}Q, \qquad x_i>0. \tag{2.26}$$

Then

$$\hat{S}(g) = \frac{1}{2}\sum_{i\in J_\mathfrak{k}}\frac{d_ib_i}{x_i} - \frac{1}{2}\sum_{i\in J_\mathfrak{k}}\sum_{j,k\in J_\mathfrak{j}}\frac{[ijk]}{x_i} - \frac{1}{4}\sum_{i,j,k\in J_\mathfrak{k}}[ijk]\frac{x_k}{x_ix_j}. \tag{2.27}$$

Proof. As in the proof of Lemma 2.20, we choose a Q-orthonormal basis $(e_i)_{i=1}^n$ of $\mathfrak{m}$ adapted to the decomposition (2.8). For every $i=1,\dots,n$, the vector $\tilde{e}_i$ is defined as $\frac{1}{\sqrt{x_{\iota(i)}}}e_i$, where $\iota(i)$ is such that $e_i\in\mathfrak{m}_{\iota(i)}$. To establish (2.27), it suffices to take note of (2.25) and observe that

$$\begin{aligned}
|\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})|^2_{QgQ} &= \sum_{i\in\Gamma(\mathfrak{k})}\sum_{j\in\Gamma(\mathfrak{j})} Q(\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})(\tilde{e}_i,e_j),\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})(\tilde{e}_i,e_j))\\
&= \sum_{i\in\Gamma(\mathfrak{k})}\sum_{j\in\Gamma(\mathfrak{j})}\frac{1}{x_{\iota(i)}}Q(\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})(e_i,e_j),\Delta(\mathfrak{j},\mathfrak{k}\ominus\mathfrak{h},\mathfrak{j})(e_i,e_j))\\
&= \sum_{i\in J_\mathfrak{k}}\sum_{j,k\in J_\mathfrak{j}}\frac{[ijk]}{x_i}.
\end{aligned}$$

In the first two lines,

$$\Gamma(\mathfrak{k}) = \{i\in[1,n]\cap\mathbb{N}\,|\,e_i\in\mathfrak{k}\ominus\mathfrak{h}\} = \{i\in[1,n]\cap\mathbb{N}\,|\,\iota(i)\in J_\mathfrak{k}\}.$$
$$\Gamma(\mathfrak{j}) = \{i\in[1,n]\cap\mathbb{N}\,|\,e_i\in\mathfrak{j}\} = \{i\in[1,n]\cap\mathbb{N}\,|\,\iota(i)\in J_\mathfrak{j}\}.$$

□

The following estimate for S was essentially proven in [24]. Recall that the notation $\lambda_-(R')$ and $\lambda_+(R')$, where R' is a bilinear form on a nonzero subspace of $\mathfrak{m}$, was introduced by (2.2).

Lemma 2.22. *Suppose* $\mathfrak{h}$ *is a maximal Lie subalgebra of* $\mathfrak{k}$*. Given* $g \in \mathcal{M}(\mathfrak{k})$ *and* $\tau_1, \tau_2 > 0$*, assume that*

$$\lambda_-(g) \leq \tau_1, \qquad \lambda_+(g) \geq \tau_2.$$

Then

$$S(g) \leq A - D\lambda_+(g)^b,$$

where $A > 0$*,* $D > 0$ *and* $b > 0$ *are constants depending only on* G*,* H*,* $\mathfrak{k}$*,* Q*,* τ_1 *and* τ_2*.*

Proof. Without loss of generality, let the decomposition (2.8) satisfy formula (2.26); cf. [28, page 180]. The quantity $S(g)$ is then given by Lemma 2.20. It is easy to see that

$$\lambda_-(g) = \min_{j \in J_\mathfrak{k}} x_j, \qquad \lambda_+(g) = \max_{j \in J_\mathfrak{k}} x_j. \tag{2.28}$$

The estimate

$$S(g) \leq \frac{\tilde{A}}{\min_{j \in J_\mathfrak{k}} x_j} - \frac{D}{(\min_{j \in J_\mathfrak{k}} x_j)^a} - D\big(\max_{j \in J_\mathfrak{k}} x_j\big)^b \tag{2.29}$$

holds with the constants $\tilde{A} > 0$, $D > 0$, $a > 1$ and $b > 0$ depending only on G, H, $\mathfrak{k}$, Q, τ_1 and τ_2. Indeed, to obtain (2.29), it suffices to repeat the proof of [24, Lemma 2.4] with only elementary modifications to the argument. The function

$$y \mapsto \frac{\tilde{A}}{y} - \frac{D}{y^a}$$

is bounded above on $(0, \infty)$. In light of (2.28) and (2.29), this fact implies

$$S(g) \leq A - D\big(\max_{j \in J_\mathfrak{k}} x_j\big)^b = A - D\lambda_+(g)^b$$

for some $A > 0$ depending only on G, H, $\mathfrak{k}$, Q, τ_1 and τ_2. □

We will require the following identity and estimate for S and $\hat{S}$.

Lemma 2.23. *Suppose the scalar product $g \in \mathcal{M}(\mathfrak{k})$ and the decomposition (2.8) are such that (2.26) holds. Then*

$$\hat{S}(g) = S(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) - \frac{1}{2} \sum_{i \in J_{\mathfrak{k}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} - \frac{1}{4} \sum_{i,j \in J_{\mathfrak{l}}} \sum_{k \in J_{\mathfrak{k}'}} [ijk]\Big(\frac{x_k}{x_i x_j} + 2\frac{x_i}{x_j x_k}\Big), \tag{2.30}$$

$$\hat{S}(g) \leq \hat{S}(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}). \tag{2.31}$$

Proof. By direct computation, Lemmas 2.20 and 2.21 imply

$$\begin{aligned}\hat{S}(g) = S(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) &- \frac{1}{2} \sum_{i \in J_{\mathfrak{k}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} \\ &- \frac{1}{4} \sum_{i,j \in J_{\mathfrak{l}}} \sum_{k \in J_{\mathfrak{k}'}} [ijk]\Big(\frac{x_k}{x_i x_j} + 2\frac{x_i}{x_j x_k}\Big) \\ &- \frac{1}{4} \sum_{i \in J_{\mathfrak{l}}} \sum_{j,k \in J_{\mathfrak{k}'}} [ijk]\Big(2\frac{x_k}{x_i x_j} + \frac{x_i}{x_j x_k}\Big).\end{aligned}$$

The last of the five terms on the right-hand side vanishes. Indeed, Lemma 2.14 shows that the coefficients $[ijk]$ in this term are all 0. Thus, the identity in the first line of (2.30) must hold. To prove the estimate, observe that

$$\sum_{i,j \in J_{\mathfrak{l}}} [ijk]\frac{x_i}{x_j x_k} = \frac{1}{2} \sum_{i,j \in J_{\mathfrak{l}}} \frac{[ijk]}{x_k}\Big(\frac{x_i}{x_j} + \frac{x_j}{x_i}\Big) \geq \sum_{i,j \in J_{\mathfrak{l}}} \frac{[ijk]}{x_k}, \qquad k \in J_{\mathfrak{k}'}.$$

Consequently,

$$\begin{aligned}\hat{S}(g) &= S(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) - \frac{1}{2} \sum_{i \in J_{\mathfrak{k}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} - \frac{1}{4} \sum_{i,j \in J_{\mathfrak{l}}} \sum_{k \in J_{\mathfrak{k}'}} [ijk]\frac{x_k}{x_i x_j} \\ &\quad - \frac{1}{2} \sum_{i,j \in J_{\mathfrak{l}}} \sum_{k \in J_{\mathfrak{k}'}} [ijk]\frac{x_i}{x_j x_k} \\ &\leq S(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) - \frac{1}{2} \sum_{i \in J_{\mathfrak{k}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} - \frac{1}{2} \sum_{i,j \in J_{\mathfrak{l}}} \sum_{k \in J_{\mathfrak{k}'}} \frac{[ijk]}{x_k} \\ &= S(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) - \frac{1}{2} \sum_{i \in J_{\mathfrak{k}'}} \sum_{j,k \in J_{\mathfrak{j}'}} \frac{[ijk]}{x_i} - \frac{1}{2} \sum_{i \in J_{\mathfrak{l}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} \\ &= \hat{S}(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}) - \frac{1}{2} \sum_{i \in J_{\mathfrak{l}}} \sum_{j,k \in J_{\mathfrak{j}}} \frac{[ijk]}{x_i} \leq \hat{S}(g|_{\mathfrak{n}}) + S(g|_{\mathfrak{l}}).\end{aligned}$$

□

Fix $T \in \mathcal{M}$. Given a scalar product $g \in \mathcal{M}(\mathfrak{k})$ and a subspace $\mathfrak{u}$ of $\mathfrak{k} \ominus \mathfrak{h}$, the notation $g|_{\mathfrak{u}}$ stands for the restriction of g to $\mathfrak{u}$. If R is a bilinear form on $\mathfrak{m}$, let $\operatorname{tr}_g R|_{\mathfrak{u}}$ be the trace of $R|_{\mathfrak{u}}$ with respect to $g|_{\mathfrak{u}}$. Define

$$\mathcal{M}_T(\mathfrak{k}) = \{g \in \mathcal{M}(\mathfrak{k}) \mid \operatorname{tr}_g T|_{\mathfrak{k}\ominus\mathfrak{h}} = 1\}.$$

In what follows, we assume $\mathcal{M}_T(\mathfrak{k})$ carries the topology inherited from $\mathcal{M}(\mathfrak{k})$. There is a natural identification between $\mathcal{M}_T(\mathfrak{g})$ and $\mathcal{M}_T$. We will need the following bounds on $\lambda_-(g)$, $S(g)$ and $\hat{S}(g)$.

Lemma 2.24. *If g lies in $\mathcal{M}_T(\mathfrak{k})$ and $\mathfrak{u}$ is a nonzero subspace of $\mathfrak{k} \ominus \mathfrak{h}$, then*

$$\lambda_-(g) \geq \omega(\mathfrak{k}\ominus\mathfrak{h})\lambda_-(T|_{\mathfrak{k}\ominus\mathfrak{h}}), \qquad \hat{S}(g|_{\mathfrak{u}}) \leq S(g|_{\mathfrak{u}}) \leq -\frac{1}{2}\operatorname{tr}_g B|_{\mathfrak{u}} \leq -\frac{\lambda_-(B)}{2\lambda_-(T)}.$$

Proof. We may assume without loss of generality that the decomposition (2.8) satisfies (2.26); cf. [28, page 180]. Let q be a number in $J_{\mathfrak{k}}$ such that

$$\lambda_-(g) = \min\{x_i \mid i \in J_{\mathfrak{k}}\} = x_q.$$

Fix a Q-orthonormal basis $(e_j)_{j=1}^{d_q}$ of $\mathfrak{m}_q$. The inclusions $g \in \mathcal{M}_T(\mathfrak{k})$ and $T \in \mathcal{M}$ imply

$$\begin{aligned} 1 = \operatorname{tr}_g T|_{\mathfrak{k}\ominus\mathfrak{h}} &\geq \operatorname{tr}_g T|_{\mathfrak{m}_q} \\ &= \sum_{j=1}^{d_q} \frac{T(e_j,e_j)}{g(e_j,e_j)} \geq \frac{d_q\lambda_-(T|_{\mathfrak{m}_q})}{\lambda_-(g)} \geq \frac{\omega(\mathfrak{k}\ominus\mathfrak{h})\lambda_-(T|_{\mathfrak{k}\ominus\mathfrak{h}})}{\lambda_-(g)}. \end{aligned}$$

Thus, the first estimate must hold.

It is obvious that $\hat{S}(g|_{\mathfrak{u}}) \leq S(g|_{\mathfrak{u}})$. By formula (2.24),

$$S(g|_{\mathfrak{u}}) \leq -\frac{1}{2}\operatorname{tr}_g B|_{\mathfrak{u}} \leq -\frac{1}{2}\lambda_-(B|_{\mathfrak{u}})\operatorname{tr}_g Q|_{\mathfrak{u}}.$$

The inclusion $T \in \mathcal{M}$ implies

$$1 \geq \operatorname{tr}_g T|_{\mathfrak{u}} \geq \lambda_-(T|_{\mathfrak{u}})\operatorname{tr}_g Q|_{\mathfrak{u}}.$$

Therefore,

$$-\frac{1}{2}\lambda_-(B|_{\mathfrak{u}})\operatorname{tr}_g Q|_{\mathfrak{u}} \leq -\frac{\lambda_-(B|_{\mathfrak{u}})}{2\lambda_-(T|_{\mathfrak{u}})} \leq -\frac{\lambda_-(B)}{2\lambda_-(T)}.$$

□

We will also need the following simple consequence of (2.20).

Lemma 2.25. *The quantity*

$$\sup\left\{\hat{S}(h)\,\middle|\,h\in\mathcal{M}_T(\mathfrak{k})\right\}$$

is non-negative.

Proof. Denote $\psi=\operatorname{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}}$. Because

$$\operatorname{tr}_{\psi Q} T|_{\mathfrak{k}\ominus\mathfrak{h}}=\frac{1}{\psi}\operatorname{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}}=1,$$

the tensor $\psi Q|_{\mathfrak{k}\ominus\mathfrak{h}}$ lies in $\mathcal{M}_T(\mathfrak{k})$. Using Lemma 2.21 and formula (2.20), we obtain

$$\begin{aligned}\sup\left\{\hat{S}(h)\,\middle|\,h\in\mathcal{M}_T(\mathfrak{k})\right\}&\geq\hat{S}(\psi Q|_{\mathfrak{k}\ominus\mathfrak{h}})\\&=\frac{1}{2\psi}\sum_{i\in J_{\mathfrak{k}}}\Bigg(d_ib_i-\sum_{j,k\in J_{\mathfrak{j}}}[ijk]-\frac{1}{2}\sum_{j,k\in J_{\mathfrak{k}}}[ijk]\Bigg)\\&=\frac{1}{2\psi}\sum_{i\in J_{\mathfrak{k}}}\Bigg(2d_i\zeta_i+\frac{1}{2}\sum_{j,k\in J_{\mathfrak{k}}}[ijk]\Bigg)\geq 0.\end{aligned}$$

□

Let us conclude this subsection with one more auxiliary result about scalar products from $\mathcal{M}_T(\mathfrak{k})$.

Lemma 2.26. *Given $\tau>0$, the set*

$$\mathcal{C}(\mathfrak{k},\tau)=\{g\in\mathcal{M}_T(\mathfrak{k})\,|\,\lambda_+(g)\leq\tau\}$$

is compact in $\mathcal{M}_T(\mathfrak{k})$.

Proof. Lemma 2.24 yields the inclusion

$$\mathcal{C}(\mathfrak{k},\tau)\subset\mathcal{D}(\mathfrak{k},\tau)=\{g\in\mathcal{M}(\mathfrak{k})\,|\,\omega(\mathfrak{k}\ominus\mathfrak{h})\lambda_-(T|_{\mathfrak{k}\ominus\mathfrak{h}})\leq\lambda_-(g)\leq\lambda_+(g)\leq\tau\}.$$

Exploiting the fact that the set of $k\times k$ matrices with eigenvalues in some bounded closed interval is compact in $\mathbb{R}^{k^2}$ for $k\geq 1$, one can easily verify that $\mathcal{D}(\mathfrak{k},\tau)$ is compact in $\mathcal{M}(\mathfrak{k})$. It is clear that $\mathcal{C}(\mathfrak{k},\tau)$ is closed in $\mathcal{M}(\mathfrak{k})$. Therefore, $\mathcal{C}(\mathfrak{k},\tau)$ must be compact in $\mathcal{M}(\mathfrak{k})$. The assertion of the lemma now follows from the fact that the topology of $\mathcal{M}_T(\mathfrak{k})$ is inherited from $\mathcal{M}(\mathfrak{k})$. □

2.5. The key estimate

Throughout Subsections 2.5–2.6, we suppose $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as a proper subset. Recall that, by assumption, $\mathfrak{k}$ must meet the requirements of Hypothesis 2.3. Let $\mathfrak{k}_1, \dots, \mathfrak{k}_r$ be all the maximal Lie subalgebras of $\mathfrak{k}$ containing $\mathfrak{h}$ as a proper subset. In Subsection 2.5, we suppose that at least one such subalgebra exists. The fact that there are only finitely many follows from Corollary 2.13. It is clear that

$$\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{k}_i \supset \mathfrak{h} \tag{2.32}$$

is a simple chain for every $i = 1, \dots, r$.

Our first main objective in this subsection is to estimate the values of the functional $\hat{S}$ on $\mathcal{M}_T(\mathfrak{k})$ in terms of its values on $\mathcal{M}_T(\mathfrak{k}_1), \dots, \mathcal{M}_T(\mathfrak{k}_r)$. We achieve this objective in Lemma 2.31. Afterwards, we use the obtained result to show that $\hat{S}$ has a global maximum on $\mathcal{M}_T(\mathfrak{k})$ if it has global maxima on $\mathcal{M}_T(\mathfrak{k}_1), \dots, \mathcal{M}_T(\mathfrak{k}_r)$ and the conditions of Theorem 2.9 are satisfied. This is the content of Lemma 2.33. It will be convenient for us to denote

$$\mathfrak{l}_i = \mathfrak{k} \ominus \mathfrak{k}_i.$$

Let $\Theta(\mathfrak{k})$ be the class of $\mathrm{Ad}(H)$-invariant proper subspaces $\mathfrak{u} \subset \mathfrak{k} \ominus \mathfrak{h}$ such that

$$\mathfrak{u} \cap \mathfrak{l}_i \neq \{0\}$$

for each $i = 1, \dots, r$. Observe that $\mathfrak{u} \oplus \mathfrak{h}$ cannot be a Lie subalgebra of $\mathfrak{k}$ if $\mathfrak{u} \in \Theta(\mathfrak{k})$.

The following result will help us estimate $\hat{S}$. Roughly speaking, it is a consequence of the compactness of the set of decompositions of the form (2.8).

Lemma 2.27. *The number*

$$\theta = \begin{cases} \inf\{\langle \mathfrak{u}\mathfrak{u}\mathfrak{q} \rangle \,|\, \mathfrak{u} \in \Theta(\mathfrak{k}) \text{ and } \mathfrak{q} = \mathfrak{k} \ominus (\mathfrak{u} \oplus \mathfrak{h})\} & \text{if } \Theta(\mathfrak{k}) \neq \emptyset \\ 1 & \text{if } \Theta(\mathfrak{k}) = \emptyset \end{cases}$$

is greater than 0.

Proof. Assume the contrary. Then there exists a sequence $(\mathfrak{u}_j)_{j=1}^{\infty} \subset \Theta(\mathfrak{k})$ such that

$$\lim_{j\to\infty} \langle \mathfrak{u}_j \mathfrak{u}_j \mathfrak{q}_j \rangle = 0, \qquad \mathfrak{q}_j = \mathfrak{k} \ominus (\mathfrak{u}_j \oplus \mathfrak{h}). \tag{2.33}$$

The inclusion $(\mathfrak{u}_j)_{j=1}^{\infty} \subset \Theta(\mathfrak{k})$ implies

$$\mathfrak{u}_j \cap \mathfrak{l}_i \neq \{0\}, \qquad j \in \mathbb{N},\ i = 1, \ldots, r. \tag{2.34}$$

Replacing $(\mathfrak{u}_j)_{j=1}^{\infty}$ with a subsequence if necessary, we may assume that the dimension of $\mathfrak{u}_j$ is independent of j. We denote this dimension by m.

For every $j \in \mathbb{N}$, choose a Q-orthonormal basis $\mathcal{E}_j = (e_k^j)_{k=1}^m$ of the space $\mathfrak{u}_j$. The sequence $(\mathcal{E}_j)_{j=1}^{\infty}$ has a subsequence converging in $(\mathfrak{k} \ominus \mathfrak{h})^m$ to some

$$\mathcal{E}_\infty = (e_k^\infty)_{k=1}^m \in (\mathfrak{k} \ominus \mathfrak{h})^m.$$

Let $\mathfrak{u}_\infty$ be the linear span of $\mathcal{E}_\infty$. One can verify that $\mathfrak{u}_\infty$ is $\mathrm{Ad}(H)$-invariant. Formula (2.33) implies

$$\langle \mathfrak{u}_\infty \mathfrak{u}_\infty \mathfrak{q}_\infty \rangle = 0, \qquad \mathfrak{q}_\infty = \mathfrak{k} \ominus (\mathfrak{u}_\infty \oplus \mathfrak{h}).$$

Consequently, $\mathfrak{u}_\infty \oplus \mathfrak{h}$ must be a Lie subalgebra of $\mathfrak{k}$. Because $(\mathfrak{u}_j)_{j=1}^{\infty} \subset \Theta(\mathfrak{k})$,

$$\dim \mathfrak{k} \ominus \mathfrak{h} > \dim \mathfrak{u}_j = m = \dim \mathfrak{u}_\infty, \qquad j \in \mathbb{N}.$$

Therefore, $\mathfrak{u}_\infty \oplus \mathfrak{h}$ is a *proper* Lie subalgebra of $\mathfrak{k}$. We conclude that $\mathfrak{u}_\infty \oplus \mathfrak{h}$ is contained in $\mathfrak{k}_i$ for some $i = 1, \ldots, r$. Our next step is to show that this is impossible. The contradiction will complete the proof.

For every $j \in \mathbb{N}$, formula (2.34) yields the existence of a vector

$$X_j \in \mathfrak{u}_j \cap \mathfrak{l}_1$$

with $Q(X_j, X_j) = 1$. The sequence $(X_j)_{j=1}^{\infty}$ has a subsequence converging to some X_∞ in $\mathfrak{k}$. It is clear that

$$X_\infty \in \mathfrak{u}_\infty \cap \mathfrak{l}_1$$

and $Q(X_\infty, X_\infty) = 1$. Thus, $\mathfrak{u}_\infty$ is not contained in $\mathfrak{k}_1$. Similar arguments show that $\mathfrak{u}_\infty$ is not in $\mathfrak{k}_i$ for $i = 2, \ldots, r$. $\square$

Example 2.28. Suppose $M = G_2/U(2)$ is as in Examples 2.17 and 2.19. Let $\mathfrak{k} = \mathfrak{g}$ and $Q = -B$. Choose the decomposition (2.8) as in [1]. We may assume $\mathfrak{l}_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_3$ and $\mathfrak{l}_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_2$; see Section 4. It is easy to understand that $\Theta(\mathfrak{k}) = \{\mathfrak{m}_1, \mathfrak{m}_1 \oplus \mathfrak{m}_2, \mathfrak{m}_1 \oplus \mathfrak{m}_3, \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3\}$. Consequently,

$$\begin{aligned}\theta &= \min\{[112] + [113], [113] + 2[123] + [223], [112] + 2[123] + [233]\} \\ &= \min\{[112], 2[123]\} = 2/3.\end{aligned}$$

Our next result involves the sets $J_{\mathfrak{k}}$ and $\mathcal{C}(\mathfrak{k}, \tau)$ given by Lemmas 2.12 and 2.26. We also need the function $\alpha : (0,\infty) \to (0,\infty)$ defined by the formula

$$\alpha(\epsilon) = \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^{2^s-1}\epsilon, \qquad \epsilon > 0, \tag{2.35}$$

where s is the number of summands in (2.8).

Lemma 2.29. *Let the scalar product $g \in \mathcal{M}_T(\mathfrak{k})$ and the decomposition (2.8) satisfy (2.26). Suppose $\mathcal{J}$ is a subset of $J_{\mathfrak{k}}$ such that the space*

$$\mathfrak{m}_{\mathcal{J}} = \bigoplus_{u \in \mathcal{J}} \mathfrak{m}_u$$

lies in $\Theta(\mathfrak{k})$. Given $\epsilon > 0$, assume $\lambda_+(g|_{\mathfrak{m}_{\mathcal{J}}}) < \epsilon$ and $\hat{S}(g) > 0$. Then g lies in $\mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$.

Proof. The inclusion $\mathfrak{m}_{\mathcal{J}} \in \Theta(\mathfrak{k})$, Lemma 2.27 and formula (2.18) imply

$$\sum_{u,v \in \mathcal{J}} \sum_{w \in J_{\mathfrak{k}} \setminus \mathcal{J}} [uvw] \geq \theta > 0.$$

Consequently, there exists $i \in J_{\mathfrak{k}} \setminus \mathcal{J}$ such that

$$\sum_{u,v \in \mathcal{J}} [uvi] \geq \frac{\theta}{|J_{\mathfrak{k}} \setminus \mathcal{J}|} > \frac{\theta}{s}.$$

According to Lemmas 2.20 and 2.24,

$$\begin{aligned}\hat{S}(g) \leq S(g) &\leq -\frac{1}{2}\operatorname{tr}_g B|_{\mathfrak{k}} - \frac{1}{4}\sum_{u,v,q \in J_{\mathfrak{k}}} [uvq]\frac{x_q}{x_u x_v} \\ &\leq -\frac{\lambda_-(B)}{2\lambda_-(T)} - \frac{1}{4}\sum_{u,v \in \mathcal{J}} [uvi]\frac{x_i}{x_u x_v}.\end{aligned}$$

Since

$$\max_{u\in\mathcal{J}} x_u = \lambda_+(g|_{\mathfrak{m}_{\mathcal{J}}}) < \epsilon \tag{2.36}$$

and $\hat{S}(g) > 0$, the formula

$$\begin{aligned} x_i &\le -4\frac{(\max_{u\in\mathcal{J}} x_u)^2}{\sum_{u,v\in\mathcal{J}}[uvi]}\left(\hat{S}(g) + \frac{\lambda_-(B)}{2\lambda_-(T)}\right) \\ &< -\frac{2}{\sum_{u,v\in\mathcal{J}}[uvi]}\frac{\lambda_-(B)}{\lambda_-(T)}\epsilon^2 < -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon^2 \\ &\le \max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\epsilon \end{aligned} \tag{2.37}$$

holds. Suppose $\mathfrak{m}_i \oplus \mathfrak{m}_{\mathcal{J}} \oplus \mathfrak{h}$ coincides with $\mathfrak{k}$. In this case,

$$\begin{aligned} \lambda_+(g) &= \max\left\{x_i, \max_{u\in\mathcal{J}} x_u\right\} \\ &< \max\left\{\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\epsilon, \epsilon\right\} \\ &= \max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\epsilon \le \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^{2^s-1}\epsilon = \alpha(\epsilon). \end{aligned}$$

Thus, g is in $\mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$, and the assertion of the lemma holds.

Suppose $\mathfrak{m}_i \oplus \mathfrak{m}_{\mathcal{J}} \oplus \mathfrak{h}$ and $\mathfrak{k}$ are distinct. The inclusion $\mathfrak{m}_{\mathcal{J}} \in \Theta(\mathfrak{k})$ implies $\mathfrak{m}_i \oplus \mathfrak{m}_{\mathcal{J}} \in \Theta(\mathfrak{k})$. Employing Lemma 2.27 and formula (2.18), we conclude that

$$\sum_{u,v\in\mathcal{J}\cup\{i\}} \sum_{w\in J_{\mathfrak{k}}\setminus(\mathcal{J}\cup\{i\})} [uvw] \ge \theta > 0.$$

This means there exists $j \in J_{\mathfrak{k}} \setminus (\mathcal{J}\cup\{i\})$ such that

$$\sum_{u,v\in\mathcal{J}\cup\{i\}} [uvj] \ge \frac{\theta}{|J_{\mathfrak{k}} \setminus (\mathcal{J}\cup\{i\})|} > \frac{\theta}{s}.$$

Lemmas 2.20 and 2.24 imply

$$\hat{S}(g) \le -\frac{\lambda_-(B)}{2\lambda_-(T)} - \frac{1}{4}\sum_{u,v\in\mathcal{J}\cup\{i\}} [uvj]\frac{x_j}{x_u x_v}.$$

In light of (2.36), (2.37) and the assumption $\hat{S}(g) > 0$, we conclude that

$$x_j < -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\epsilon\right)^2 \leq \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^3\epsilon.$$

Let $\mathfrak{m}_i \oplus \mathfrak{m}_j \oplus \mathfrak{m}_{\mathcal{J}} \oplus \mathfrak{h}$ equal $\mathfrak{k}$. Then s is no less than $|\mathcal{J}| + 2 > 2$, and

$$\begin{aligned}
\lambda_+(g) &= \max\left\{x_i, x_j, \max_{u\in\mathcal{J}} x_u\right\} \\
&< \max\left\{\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\epsilon, \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^3\epsilon, \epsilon\right\} \\
&= \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^3\epsilon \\
&\leq \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^{2^s-1}\epsilon = \alpha(\epsilon).
\end{aligned}$$

Thus, the assertion of the lemma holds.

Suppose $\mathfrak{m}_i \oplus \mathfrak{m}_j \oplus \mathfrak{m}_{\mathcal{J}} \oplus \mathfrak{h}$ and $\mathfrak{k}$ are distinct. The inclusion $\mathfrak{m}_{\mathcal{J}} \in \Theta(\mathfrak{k})$ shows that $\mathfrak{m}_i \oplus \mathfrak{m}_j \oplus \mathfrak{m}_{\mathcal{J}} \in \Theta(\mathfrak{k})$. Continuing to argue as above, we demonstrate that

$$\begin{aligned}
\lambda_+(g) &< \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^{2^{|J_{\mathfrak{k}}\setminus\mathcal{J}|}-1}\epsilon \\
&\leq \left(\max\left\{1, -\frac{2s\lambda_-(B)}{\theta\lambda_-(T)}\epsilon\right\}\right)^{2^s-1}\epsilon = \alpha(\epsilon).
\end{aligned}$$

This completes the proof. □

Denote

$$\mathfrak{n}_i = \mathfrak{k}_i \ominus \mathfrak{h}, \qquad i = 1, \ldots, r.$$

Lemma 2.12 implies the existence of sets $J_{\mathfrak{k}_1}, \ldots, J_{\mathfrak{k}_r}$ such that

$$\mathfrak{n}_i = \bigoplus_{j\in J_{\mathfrak{k}_i}} \mathfrak{m}_j, \qquad i = 1, \ldots, r.$$

It will be convenient for us to define

$$J_{\mathfrak{l}_i} = J_{\mathfrak{k}} \setminus J_{\mathfrak{k}_i}, \qquad i = 1, \ldots, r.$$

Our next result shows that, roughly speaking, a scalar product $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$ satisfying $\hat{S}(g) > 0$ must be “large” outside of $\mathfrak{k}_i$ for some $i =$

$1, \ldots, r$. This result is an important ingredient in the proof of our key estimate for $\hat{S}$.

Lemma 2.30. *Given $\epsilon > 0$, consider $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$ such that $\hat{S}(g) > 0$. Assume the decomposition (2.8) satisfies (2.26). Then the set*

$$\mathcal{I}(g, \epsilon) = \{j \in J_{\mathfrak{k}} \,|\, x_j < \epsilon\}$$

is contained in $J_{\mathfrak{k}_i}$ for some $i = 1, \ldots, r$.

Proof. Denote

$$\mathfrak{m}_{\mathcal{I}(g,\epsilon)} = \bigoplus_{j \in \mathcal{I}(g,\epsilon)} \mathfrak{m}_j.$$

It is clear that

$$\lambda_+(g|_{\mathfrak{m}_{\mathcal{I}(g,\epsilon)}}) = \max_{j \in \mathcal{I}(g,\epsilon)} x_j < \epsilon.$$

By assumption, $\hat{S}(g)$ is positive. The inclusion $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$ and Lemma 2.29 imply that $\mathfrak{m}_{\mathcal{I}(g,\epsilon)}$ does not lie in $\Theta(\mathfrak{k})$. Therefore, either $\mathfrak{m}_{\mathcal{I}(g,\epsilon)}$ coincides with $\mathfrak{k} \ominus \mathfrak{h}$ or there exists $i = 1, \ldots, r$ such that

$$\mathfrak{m}_{\mathcal{I}(g,\epsilon)} \cap \mathfrak{l}_i = \{0\}. \tag{2.38}$$

In the former case, $\mathcal{I}(g, \epsilon)$ must equal $J_{\mathfrak{k}}$, and

$$\lambda_+(g) = \max_{j \in J_{\mathfrak{k}}} x_j = \max_{j \in \mathcal{I}(g,\epsilon)} x_j < \epsilon.$$

On the other hand, the inclusion $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \alpha(\epsilon))$ yields

$$\lambda_+(g) > \alpha(\epsilon) \geq \epsilon.$$

Thus, $\mathfrak{m}_{\mathcal{I}(g,\epsilon)}$ cannot coincide with $\mathfrak{k} \ominus \mathfrak{h}$. We conclude that there exists $i = 1, \ldots, r$ satisfying (2.38). For any such i, the intersection $\mathcal{I}(g, \epsilon) \cap J_{\mathfrak{l}_i}$ is empty, which means $\mathcal{I}(g, \epsilon) \subset J_{\mathfrak{k}_i}$. □

Define functions $\beta : (0, \infty) \to (0, \infty)$ and $\kappa : (0, \infty) \to (0, \infty)$ by setting

$$\beta(\epsilon) = -\frac{n\lambda_-(B) - 1}{2\epsilon}, \qquad \kappa(\epsilon) = \alpha(\beta(\epsilon)), \qquad \epsilon > 0,$$

where n is the dimension of M and $\alpha(\cdot)$ is given by (2.35). We are now ready to state our key estimate on $\hat{S}$.

Lemma 2.31. *Given $\epsilon > 0$, the formula*

$$\hat{S}(g) \leq \epsilon + \max_{i=1,\dots,r} \sup \big\{ \hat{S}(h) \,\big|\, h \in \mathcal{M}_T(\mathfrak{k}_i) \big\} \tag{2.39}$$

holds for every $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \kappa(\epsilon))$.

Remark 2.32. Lemma 2.24 implies that the set

$$\big\{ \hat{S}(h) \,\big|\, h \in \mathcal{M}_T(\mathfrak{k}_i) \big\}$$

is bounded above for every $i = 1, \dots, r$. Therefore, the quantity on the right-hand of (2.39) is always finite.

Proof of Lemma 2.31. Choose $g \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \kappa(\epsilon))$. We will show that (2.39) holds for g. Without loss of generality, suppose the decomposition (2.8) satisfies (2.26); cf. [28, page 180]. If $\hat{S}(g) \leq 0$, then (2.39) follows from Lemma 2.25. Thus, we may assume $\hat{S}(g) > 0$. Throughout the remainder of the proof, we fix i with $\mathcal{I}(g, \beta(\epsilon)) \subset J_{\mathfrak{k}_i}$. Such an i exists by Lemma 2.30. It is clear that $J_{\mathfrak{l}_i}$ is contained in $J_{\mathfrak{k}} \setminus \mathcal{I}(g, \beta(\epsilon))$.

According to Lemmas 2.23 and 2.20,

$$\begin{aligned}
\hat{S}(g) &\leq \hat{S}(g|_{\mathfrak{n}_i}) + S(g|_{\mathfrak{l}_i}) \\
&\leq \hat{S}(g|_{\mathfrak{n}_i}) + \frac{1}{2} \sum_{j \in J_{\mathfrak{l}_i}} \frac{d_j b_j}{x_j} \leq \hat{S}(g|_{\mathfrak{n}_i}) - \frac{\lambda_-(B)}{2} \sum_{j \in J_{\mathfrak{l}_i}} \frac{d_j}{x_j} \\
&\leq \hat{S}(g|_{\mathfrak{n}_i}) - \frac{n\lambda_-(B)}{2 \min_{j \in J_{\mathfrak{l}_i}} x_j}.
\end{aligned}$$

Recalling the definition of $\mathcal{I}(g, \beta(\epsilon))$, we find

$$\min_{j \in J_{\mathfrak{l}_i}} x_j \geq \min_{j \in J_{\mathfrak{k}} \setminus \mathcal{I}(g, \beta(\epsilon))} x_j \geq \beta(\epsilon).$$

Therefore,

$$\hat{S}(g) \leq \hat{S}(g|_{\mathfrak{n}_i}) - \frac{n\lambda_-(B)}{2\beta(\epsilon)} < \hat{S}(g|_{\mathfrak{n}_i}) + \epsilon.$$

Let us show that

$$\hat{S}(g|_{\mathfrak{n}_i}) \leq \sup \big\{ \hat{S}(h) \,\big|\, h \in \mathcal{M}_T(\mathfrak{k}_i) \big\}.$$

Inequality (2.39) will follow immediately. If $\psi_i = \operatorname{tr}_g T|_{\mathfrak{n}_i}$, then

$$\operatorname{tr}_{\psi_i g} T|_{\mathfrak{n}_i} = \frac{1}{\psi_i} \operatorname{tr}_g T|_{\mathfrak{n}_i} = 1,$$

which means the scalar product $\psi_i g|_{\mathfrak{n}_i}$ lies in $\mathcal{M}_T(\mathfrak{k}_i)$. Keeping in mind that $g \in \mathcal{M}_T(\mathfrak{k})$, we estimate

$$\psi_i = \operatorname{tr}_g T|_{\mathfrak{n}_i} < \operatorname{tr}_g T|_{\mathfrak{k}\ominus\mathfrak{h}} = 1.$$

As a consequence,

$$\begin{aligned} \hat{S}(g|_{\mathfrak{n}_i}) = \psi_i \hat{S}(\psi_i g|_{\mathfrak{n}_i}) &\leq \psi_i \sup \{\hat{S}(h) \,|\, h \in \mathcal{M}_T(\mathfrak{k}_i)\} \\ &< \sup \{\hat{S}(h) \,|\, h \in \mathcal{M}_T(\mathfrak{k}_i)\}. \end{aligned}$$

□

Our goal in Subsection 2.6 will be to show that $\hat{S}$ has a global maximum on $\mathcal{M}_T(\mathfrak{k})$ under the assumptions of Theorem 2.9. We will do so using induction in the dimension of $\mathfrak{k}$. The following lemma will help us prove the inductive step. As above, we define $\mathfrak{j}$ and $J_{\mathfrak{j}}$ by the first formulas in (2.5) and (2.11). It will be convenient for us to set

$$\mathfrak{j}_i = \mathfrak{g} \ominus \mathfrak{k}_i, \qquad J_{\mathfrak{j}_i} = J_{\mathfrak{g}} \setminus J_{\mathfrak{k}_i}, \qquad i = 1, \ldots, r.$$

Lemma 2.33. *Assume that the following statements are satisfied for each $i = 1, \ldots, r$:*

1) *The restriction of $\hat{S}$ to $\mathcal{M}_T(\mathfrak{k}_i)$ has a global maximum.*
2) *The inequality*

$$\frac{\lambda_-(T|_{\mathfrak{n}_i})}{\operatorname{tr}_Q T|_{\mathfrak{l}_i}} > \eta(\mathfrak{k}, \mathfrak{k}_i)$$

holds.

Then the restriction of $\hat{S}$ to $\mathcal{M}_T(\mathfrak{k})$ has a global maximum.

Proof. Fix an index i such that

$$\sup \{\hat{S}(h) \,|\, h \in \mathcal{M}_T(\mathfrak{k}_i)\} = \max_{j=1,\ldots,r} \sup \{\hat{S}(h) \,|\, h \in \mathcal{M}_T(\mathfrak{k}_j)\}.$$

By hypothesis, there exists $g_0 \in \mathcal{M}_T(\mathfrak{k}_i)$ satisfying

$$\hat{S}(g_0) = \sup \{\hat{S}(h) \,|\, h \in \mathcal{M}_T(\mathfrak{k}_i)\}.$$

Without loss of generality, suppose the decomposition (2.8) is such that

$$g_0 = \sum_{j\in J_{\mathfrak{k}_i}} y_j \pi_{\mathfrak{m}_j}^* Q, \qquad y_j > 0.$$

Given $t > \operatorname{tr}_Q T|_{\mathfrak{l}_i}$, define $g(t) \in \mathcal{M}_T(\mathfrak{k})$ by the formulas

$$g(t) = \sum_{j\in J_{\mathfrak{k}_i}} \phi(t) y_j \pi_{\mathfrak{m}_j}^* Q + \sum_{j\in J_{\mathfrak{l}_i}} t\pi_{\mathfrak{m}_j}^* Q, \qquad \phi(t) = \frac{t}{t - \operatorname{tr}_Q T|_{\mathfrak{l}_i}}.$$

We will show that $\hat{S}(g(t)) > \hat{S}(g_0)$ for some t. Together with Lemma 2.31, this will imply the existence of a global maximum of $\hat{S}$ on $\mathcal{M}_T(\mathfrak{k})$.

Using (2.25), (2.27) and the first line in (2.30), we compute

$$\begin{aligned}
\lim_{t\to\infty} \hat{S}(g(t)) &= \lim_{t\to\infty} \big(S(g(t)|_{\mathfrak{n}_i}) + S(g(t)|_{\mathfrak{l}_i})\big) \\
&\quad - \frac{1}{2}\lim_{t\to\infty}\Bigg(\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}}} \frac{[jkl]}{\phi(t)y_j} + \sum_{j\in J_{\mathfrak{l}_i}}\sum_{k,l\in J_{\mathfrak{j}}} \frac{[jkl]}{t}\Bigg) \\
&\quad - \frac{1}{4}\lim_{t\to\infty}\sum_{j,k\in J_{\mathfrak{l}_i}}\sum_{l\in J_{\mathfrak{k}_i}} [jkl]\left(\frac{\phi(t)y_l}{t^2} + \frac{2}{\phi(t)y_l}\right) \\
&= \lim_{t\to\infty}\Bigg(\frac{S(g_0)}{\phi(t)} + \frac{1}{2}\sum_{j\in J_{\mathfrak{l}_i}} \frac{d_j b_j}{t} - \frac{1}{4}\sum_{j,k,l\in J_{\mathfrak{l}_i}} \frac{[jkl]}{t}\Bigg) \\
&\quad - \frac{1}{2}\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}}} \frac{[jkl]}{y_j} - \frac{1}{2}\sum_{j,k\in J_{\mathfrak{l}_i}}\sum_{l\in J_{\mathfrak{k}_i}} \frac{[jkl]}{y_l} \\
&= S(g_0) - \frac{1}{2}\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}_i}} \frac{[jkl]}{y_j} = \hat{S}(g_0).
\end{aligned}$$

To prove that $\hat{S}(g(t)) > \hat{S}(g_0)$ for some t, it suffices to demonstrate that $\frac{d}{dt}\hat{S}(g(t)) < 0$ when t is large. Observe that

$$\begin{aligned}
&\frac{d}{dt}\phi(t) = -\frac{\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{(t - \operatorname{tr}_Q T|_{\mathfrak{l}_i})^2}, \qquad \frac{d}{dt}\frac{1}{\phi(t)} = \frac{\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{t^2}, \\
&\frac{d}{dt}\frac{\phi(t)}{t^2} = -\frac{2t - \operatorname{tr}_Q T|_{\mathfrak{l}_i}}{(t^2 - t\operatorname{tr}_Q T|_{\mathfrak{l}_i})^2}.
\end{aligned}$$

Computing as above and utilising (2.16), (2.18) and Lemma 2.14, we obtain

$$\begin{aligned}
\frac{d}{dt}\hat{S}(g(t)) &= \frac{d}{dt}\Big(\frac{S(g_0)}{\phi(t)} + \frac{1}{2}\sum_{j\in J_{\mathfrak{l}_i}}\frac{d_jb_j}{t} - \frac{1}{4}\sum_{j,k,l\in J_{\mathfrak{l}_i}}\frac{[jkl]}{t}\Big) \\
&\quad - \frac{1}{2}\frac{d}{dt}\Big(\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}}}\frac{[jkl]}{\phi(t)y_j} + \sum_{j\in J_{\mathfrak{l}_i}}\sum_{k,l\in J_{\mathfrak{j}}}\frac{[jkl]}{t}\Big) \\
&\quad - \frac{1}{4}\frac{d}{dt}\sum_{j,k\in J_{\mathfrak{l}_i}}\sum_{l\in J_{\mathfrak{k}_i}}[jkl]\Big(\frac{\phi(t)y_l}{t^2} + \frac{2}{\phi(t)y_l}\Big) \\
&= \frac{S(g_0)\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{t^2} + \frac{\operatorname{tr}_Q B|_{\mathfrak{l}_i}}{2t^2} + \frac{\langle \mathfrak{l}_i\mathfrak{l}_i\mathfrak{l}_i\rangle}{4t^2} - \frac{\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{2t^2}\Big(\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}}}\frac{[jkl]}{y_j}\Big) \\
&\quad + \frac{1}{2t^2}\langle \mathfrak{l}_i\mathfrak{j}\mathfrak{j}\rangle - \frac{1}{4}\sum_{j,k\in J_{\mathfrak{l}_i}}\sum_{l\in J_{\mathfrak{k}_i}}[jkl]\Big(-\frac{2t-\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{(t^2 - t\operatorname{tr}_Q T|_{\mathfrak{l}_i})^2}y_l + \frac{2\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{t^2 y_l}\Big) \\
&= \frac{\hat{S}(g_0)\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{t^2} + \frac{\operatorname{tr}_Q B|_{\mathfrak{l}_i}}{2t^2} + \frac{\langle \mathfrak{l}_i\mathfrak{l}_i\mathfrak{l}_i\rangle}{4t^2} + \frac{1}{2t^2}\langle \mathfrak{l}_i\mathfrak{j}\mathfrak{j}\rangle \\
&\quad + \frac{2t-\operatorname{tr}_Q T|_{\mathfrak{l}_i}}{4(t^2 - t\operatorname{tr}_Q T|_{\mathfrak{l}_i})^2}\sum_{j,k\in J_{\mathfrak{l}_i}}\sum_{l\in J_{\mathfrak{k}_i}}[jkl]y_l.
\end{aligned}$$

It is obvious that $\frac{d}{dt}\hat{S}(g(t)) < 0$ if and only if $t^2\frac{d}{dt}\hat{S}(g(t)) < 0$. Thus, to prove that $\frac{d}{dt}\hat{S}(g(t)) < 0$ for large t, it suffices to show that

$$\lim_{t\to\infty} t^2\frac{d}{dt}\hat{S}(g(t)) < 0. \tag{2.40}$$

Using the above expression for $\frac{d}{dt}\hat{S}(g(t))$, we calculate

$$4\lim_{t\to\infty} t^2\frac{d}{dt}\hat{S}(g(t)) = 4\hat{S}(g_0)\operatorname{tr}_Q T|_{\mathfrak{l}_i} + 2\operatorname{tr}_Q B|_{\mathfrak{l}_i} + \langle \mathfrak{l}_i\mathfrak{l}_i\mathfrak{l}_i\rangle + 2\langle \mathfrak{l}_i\mathfrak{j}\mathfrak{j}\rangle.$$

Lemmas 2.21 and 2.24, along with (2.16), (2.18) and Lemma 2.14, imply

$$\begin{aligned}
4\hat{S}(g_0) &= 2\sum_{j\in J_{\mathfrak{k}_i}}\frac{d_jb_j}{y_j} - 2\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}_i}}\frac{[jkl]}{y_j} - \sum_{j,k,l\in J_{\mathfrak{k}_i}}[jkl]\frac{y_l}{y_jy_k} \\
&= 2\sum_{j\in J_{\mathfrak{k}_i}}\frac{d_jb_j}{y_j} - 2\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}_i}}\frac{[jkl]}{y_j} - \frac{1}{2}\sum_{j,k,l\in J_{\mathfrak{k}_i}}\frac{[jkl]}{y_j}\Big(\frac{y_l}{y_k} + \frac{y_k}{y_l}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in J_{\mathfrak{k}_i}} \frac{1}{y_j}\Big(2d_jb_j - 2\sum_{k,l\in J_{\mathfrak{j}_i}}[jkl] - \sum_{k,l\in J_{\mathfrak{k}_i}}[jkl]\Big) \\
&\leq \frac{1}{\lambda_-(g_0)}\Big(2\sum_{j\in J_{\mathfrak{k}_i}} d_jb_j - 2\sum_{j\in J_{\mathfrak{k}_i}}\sum_{k,l\in J_{\mathfrak{j}_i}}[jkl] - \sum_{j,k,l\in J_{\mathfrak{k}_i}}[jkl]\Big) \\
&\leq \frac{-2\operatorname{tr}_Q B|_{\mathfrak{n}_i} - 2\langle \mathfrak{n}_i\mathfrak{j}_i\mathfrak{j}_i\rangle - \langle \mathfrak{n}_i\mathfrak{n}_i\mathfrak{n}_i\rangle}{\omega(\mathfrak{n}_i)\lambda_-(T|_{\mathfrak{n}_i})}.
\end{aligned}
$$

(The penultimate estimate exploits the formula

$$2d_jb_j - 2\sum_{k,l\in J_{\mathfrak{j}_i}}[jkl] - \sum_{k,l\in J_{\mathfrak{k}_i}}[jkl] = 4d_j\zeta_j + \sum_{k,l\in J_{\mathfrak{k}_i}}[jkl] \geq 0, \qquad j\in J_{\mathfrak{k}_i},$$

a consequence of (2.20).) Therefore, to prove (2.40), it suffices to show that

$$\frac{-2\operatorname{tr}_Q B|_{\mathfrak{n}_i} - 2\langle \mathfrak{n}_i\mathfrak{j}_i\mathfrak{j}_i\rangle - \langle \mathfrak{n}_i\mathfrak{n}_i\mathfrak{n}_i\rangle}{\omega(\mathfrak{n}_i)\lambda_-(T|_{\mathfrak{n}_i})}\operatorname{tr}_Q T|_{\mathfrak{l}_i} + 2\operatorname{tr}_Q B|_{\mathfrak{l}_i} + \langle \mathfrak{l}_i\mathfrak{l}_i\mathfrak{l}_i\rangle + 2\langle \mathfrak{l}_i\mathfrak{j}\mathfrak{j}\rangle < 0.$$

After some elementary transformations, this becomes

$$\frac{\lambda_-(T|_{\mathfrak{n}_i})}{\operatorname{tr}_Q T|_{\mathfrak{l}_i}} > \frac{2\operatorname{tr}_Q B|_{\mathfrak{n}_i} + 2\langle \mathfrak{n}_i\mathfrak{j}_i\mathfrak{j}_i\rangle + \langle \mathfrak{n}_i\mathfrak{n}_i\mathfrak{n}_i\rangle}{\omega(\mathfrak{n}_i)(2\operatorname{tr}_Q B|_{\mathfrak{l}_i} + \langle \mathfrak{l}_i\mathfrak{l}_i\mathfrak{l}_i\rangle + 2\langle \mathfrak{l}_i\mathfrak{j}\mathfrak{j}\rangle)} = \eta(\mathfrak{k},\mathfrak{k}_i),$$

which is satisfied by hypothesis. Thus, (2.40) holds, and $\frac{d}{dt}\hat{S}(g(t)) < 0$ for large t. It is easy to establish the existence of $t_0 > \operatorname{tr}_Q T|_{\mathfrak{l}_i}$ such that

$$\hat{S}(g(t_0)) > \lim_{t\to\infty}\hat{S}(g(t)) = \hat{S}(g_0).$$

Applying Lemma 2.31 with

$$\epsilon = \frac{\hat{S}(g(t_0)) - \hat{S}(g_0)}{2} > 0,$$

we conclude that $\hat{S}(g) < \hat{S}(g(t_0))$ for all $g\in\mathcal{M}_T(\mathfrak{k})\setminus\mathcal{C}(\mathfrak{k},\kappa(\epsilon))$. To complete the proof, we need to demonstrate that $\hat{S}$ has a global maximum on $\mathcal{C}(\mathfrak{k},\kappa(\epsilon))$. However, this is an immediate consequence of the compactness of $\mathcal{C}(\mathfrak{k},\kappa(\epsilon))$. $\square$

Remark 2.34. The proof of the lemma shows that $\hat{S}(g(t))$ converges to $\hat{S}(g_0)$ as t goes to infinity. Therefore, the inclusion

$$\hat{S}(\{g(t)\,|\,t \geq 2\,\mathrm{tr}_Q T|_{\mathfrak{l}_i}\}) \subset \big[\hat{S}(g_0) - \sigma, \hat{S}(g_0) + \sigma\big]$$

holds for some $\sigma > 0$. We conclude that the preimage of the interval $\big[\hat{S}(g_0) - \sigma, \hat{S}(g_0) + \sigma\big]$ under $\hat{S}$ has a non-compact intersection with $\mathcal{M}_T(\mathfrak{k})$. This means the restriction of $\hat{S}$ to $\mathcal{M}_T(\mathfrak{k})$ cannot be proper.

2.6. The existence of global maxima

As in Subsection 2.5, suppose $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as a proper subset. Recall that $\mathfrak{k}$ must meet the requirements of Hypothesis 2.3. Our next goal is to prove by induction that $\hat{S}$ has a global maximum on $\mathcal{M}_T(\mathfrak{k})$ under the assumptions of Theorem 2.9. The following result will enable us to take the basis step and help with the inductive step.

Lemma 2.35. *If $\mathfrak{h}$ is a maximal Lie subalgebra of $\mathfrak{k}$, then there exists $g \in \mathcal{M}_T(\mathfrak{k})$ such that $\hat{S}(g) \geq \hat{S}(h)$ for all $h \in \mathcal{M}_T(\mathfrak{k})$.*

Proof. The formulas

$$\frac{1}{\lambda_+(h)}\,\mathrm{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}} \leq \mathrm{tr}_h T|_{\mathfrak{k}\ominus\mathfrak{h}} \leq \frac{1}{\lambda_-(h)}\,\mathrm{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}}$$

and $\mathrm{tr}_h T|_{\mathfrak{k}\ominus\mathfrak{h}} = 1$ hold whenever h lies in $\mathcal{M}_T(\mathfrak{k})$. As a consequence,

$$\lambda_-(h) \leq \mathrm{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}} \leq \lambda_+(h), \qquad h \in \mathcal{M}_T(\mathfrak{k}).$$

Applying Lemma 2.22 with $\tau_1 = \tau_2 = \mathrm{tr}_Q T|_{\mathfrak{k}\ominus\mathfrak{h}}$, we find

$$\hat{S}(h) \leq S(h) \leq A - D\lambda_+(h)^b, \qquad h \in \mathcal{M}_T(\mathfrak{k}), \tag{2.41}$$

where the constants $A > 0$, $D > 0$ and $b > 0$ depend only on G, H, $\mathfrak{k}$, Q and T. Fix $h_0 \in \mathcal{M}_T(\mathfrak{k})$ and suppose

$$\tau = \left|\frac{A - \hat{S}(h_0)}{D}\right|^{\frac{1}{b}} + 1 > 0.$$

According to Lemma 2.26, the set $\mathcal{C}(\mathfrak{k}, \tau)$ is compact in $\mathcal{M}_T(\mathfrak{k})$. Consequently, there exists $g \in \mathcal{C}(\mathfrak{k}, \tau)$ such that $\hat{S}(g) \geq \hat{S}(h)$ for all $h \in \mathcal{C}(\mathfrak{k}, \tau)$.

Formula (2.41) implies that $\hat{S}(h_0) > \hat{S}(h)$ if h lies in $\mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \tau)$. This means h_0 is in $\mathcal{C}(\mathfrak{k}, \tau)$ and

$$\hat{S}(g) \geq \hat{S}(h_0) > \hat{S}(h)$$

for all $h \in \mathcal{M}_T(\mathfrak{k}) \setminus \mathcal{C}(\mathfrak{k}, \tau)$. Thus, the global maximum of $\hat{S}$ on $\mathcal{M}_T(\mathfrak{k})$ exists and is attained at g. $\square$

The following result concludes our analysis of $\hat{S}$.

Lemma 2.36. *Assume that*

$$\frac{\lambda_-(T|_{\mathfrak{k}''\ominus\mathfrak{h}})}{\operatorname{tr}_Q T|_{\mathfrak{k}'\ominus\mathfrak{k}''}} > \eta(\mathfrak{k}', \mathfrak{k}'')$$

for every simple chain

$$\mathfrak{g} \supset \mathfrak{k}' \supset \mathfrak{k}'' \supset \mathfrak{h}$$

such that $\mathfrak{k}' \subset \mathfrak{k}$. Then the restriction of $\hat{S}$ to $\mathcal{M}_T(\mathfrak{k})$ has a global maximum.

Proof. We proceed by induction. If $\dim \mathfrak{k} \ominus \mathfrak{h}$ equals 1, then $\mathfrak{h}$ must be a maximal Lie subalgebra of $\mathfrak{k}$. In this case, the existence of $g \in \mathcal{M}_T(\mathfrak{k})$ such that

$$\hat{S}(g) \geq \hat{S}(h), \qquad h \in \mathcal{M}_T(\mathfrak{k}), \tag{2.42}$$

follows from Lemma 2.35. This is the basis of induction.

Fix $m = 1, \ldots, n-1$, where n is the dimension of M. Suppose $\hat{S}$ has a global maximum on $\mathcal{M}_T(\mathfrak{s})$ for every Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ satisfying the formulas

$$\mathfrak{h} \subset \mathfrak{s}, \qquad 1 \leq \dim \mathfrak{s} \ominus \mathfrak{h} \leq m.$$

This is the induction hypothesis.

Let $\dim \mathfrak{k} \ominus \mathfrak{h}$ equal $m+1$. If $\mathfrak{h}$ is a maximal Lie subalgebra of $\mathfrak{k}$, then the existence of $g \in \mathcal{M}_T(\mathfrak{k})$ satisfying (2.42) follows from Lemma 2.35. Suppose $\mathfrak{h}$ is not. As in Subsection 2.5, denote by $\mathfrak{k}_1, \ldots, \mathfrak{k}_r$ the maximal Lie subalgebras of $\mathfrak{k}$ containing $\mathfrak{h}$ as a proper subset. It is clear that

$$1 \leq \dim \mathfrak{k}_i \ominus \mathfrak{h} \leq m, \qquad i = 1, \dots, r.$$

By the induction hypothesis, the restriction of $\hat{S}$ to $\mathfrak{k}_i$ has a global maximum for each i. The existence of $g \in \mathcal{M}_T(\mathfrak{k})$ satisfying (2.42) follows from this fact and Lemma 2.33. □

Remark 2.37. Some of the results in Subsections 2.3–2.6 hold under milder assumptions than those imposed above. In particular, Lemmas 2.12, 2.14, 2.20–2.31 and 2.35 do not use requirement 2 of Hypothesis 2.3.

2.7. The completion of the proof of Theorem 2.9

Setting $\mathfrak{k} = \mathfrak{g}$ in Lemma 2.36, we conclude that the restriction of $\hat{S}$ to $\mathcal{M}_T$ has a global maximum. By definition, the maps $\hat{S}$ and S coincide on $\mathcal{M}_T$. Ergo, there exists $g \in \mathcal{M}_T$ such that $S(g) \geq S(h)$ for all $h \in \mathcal{M}_T$. Lemma 2.1 tells us that the Ricci curvature of g equals cT for some $c \in \mathbb{R}$. To complete the proof of Theorem 2.9, we need to show that $c > 0$.

By Bochner's theorem (see [8, Theorem 1.84]), the space M cannot support a G-invariant Riemannian metric with negative-definite Ricci curvature. It follows that $c \geq 0$. Let us show that M cannot support a Ricci-flat G-invariant metric. This will immediately imply that $c > 0$.

We argue by contradiction. Assume there exists a Ricci-flat G-invariant metric on M. Employing Bochner's theorem again, we conclude that the isometry group of M with respect to this metric must be abelian. It follows that

$$\gamma\gamma'\mu = \gamma'\gamma\mu, \qquad \gamma, \gamma' \in G,\ \mu \in M.$$

Replacing γ' with $\chi \in H$ and choosing $\mu = \gamma^{-1}H$, we obtain

$$\gamma\chi\gamma^{-1}H = H, \qquad \gamma \in G,\ \chi \in H.$$

This formula implies

$$[\mathfrak{m}, \mathfrak{h}] \subset [\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}.$$

At the same time, $[\mathfrak{m}, \mathfrak{h}]$ is contained in $\mathfrak{m}$ because $\mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. Thus, $[\mathfrak{m}, \mathfrak{h}]$ is equal to $\{0\}$.

Let us turn our attention to the decomposition (2.8). Given $i = 1, \ldots, s$, the representation $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ is trivial. Its irreducibility implies that $d_i = 1$. In light of (2.1), this means $s \geq 3$. The space $\mathfrak{m}_1 \oplus \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as a proper subset. Clearly,

$$\mathfrak{m}_1 \subset \mathfrak{m}_1 \oplus \mathfrak{h}, \qquad \mathfrak{m}_2 \subset \mathfrak{g} \ominus (\mathfrak{m}_1 \oplus \mathfrak{h}).$$

Because the representations $\mathrm{Ad}(H)|_{\mathfrak{m}_1}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_2}$ are both trivial, they must be equivalent. However, this contradicts requirement 1 of Hypothesis 2.3.

2.8. A corollary

The following observation provides an alternative to condition (2.7).

Corollary 2.38. *Suppose Hypothesis 2.3 is satisfied for M. Given $T \in \mathcal{M}$, if*

$$\frac{\lambda_-(T|_{\mathfrak{n}})}{\lambda_+(T|_{\mathfrak{l}})} > \eta(\mathfrak{k}, \mathfrak{k}') \dim \mathfrak{l}$$

for every simple chain of the form (2.4), then there exist $g \in \mathcal{M}_T$ such that $S(g) \geq S(h)$ for all $h \in \mathcal{M}_T$. The Ricci curvature of g equals cT with $c > 0$.

Proof. The corollary follows from Theorem 2.9 and the obvious estimate $\mathrm{tr}_Q\, T|_{\mathfrak{l}} \leq \lambda_+(T|_{\mathfrak{l}}) \dim \mathfrak{l}$. □

3. The case of two inequivalent irreducible summands

Theorem 2.9 provides a sufficient condition for the existence of a metric $g \in \mathcal{M}$ whose Ricci curvature equals cT with $c > 0$. We will show that this condition is necessary when the isotropy representation of M splits into two inequivalent irreducible summands. Our argument will rely on [24, Proposition 3.1].

Suppose $s = 2$ in every decomposition of the form (2.8), i.e.,

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2.$$

Let $\mathrm{Ad}(H)|_{\mathfrak{m}_1}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_2}$ be inequivalent. According to Theorem 2.9, finding a metric whose Ricci curvature equals cT for some $c > 0$ is always possible if $\mathfrak{h}$ is maximal in $\mathfrak{g}$. Thus, we may assume that there exists a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that

$$\mathfrak{g} \supset \mathfrak{s} \supset \mathfrak{h}, \qquad \mathfrak{h} \neq \mathfrak{s}, \qquad \mathfrak{s} \neq \mathfrak{g}.$$

It is clear that $\mathfrak{s} \ominus \mathfrak{h}$ is a proper $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{m}$. The only such subspaces are $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Therefore, $\mathfrak{s}$ must equal $\mathfrak{m}_1 \oplus \mathfrak{h}$ or $\mathfrak{m}_2 \oplus \mathfrak{h}$. Suppose

$$\mathfrak{s} = \mathfrak{m}_1 \oplus \mathfrak{h}. \tag{3.1}$$

If $\mathfrak{m}_2 \oplus \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then $[112] = [221] = 0$. In this case, all the metrics in $\mathcal{M}$ have the same Ricci curvature, and the problem of solving equation (1.2) becomes trivial; see, e.g., [25, Subsection 4.2]. Thus, we may assume $\mathfrak{m}_2 \oplus \mathfrak{h}$ is not closed under the Lie bracket. This implies $[221] > 0$.

Let us verify Hypothesis 2.3. It is clear that $\mathfrak{s}$ given by (3.1) is the unique proper Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{s}$ and $\mathfrak{h} \neq \mathfrak{s}$. The only nonzero $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{s} \ominus \mathfrak{h}$ is $\mathfrak{m}_1$, and the only such subspace of $\mathfrak{g} \ominus \mathfrak{s}$ is $\mathfrak{m}_2$. Since $\mathrm{Ad}(H)|_{\mathfrak{m}_1}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_2}$ are inequivalent, $\mathfrak{s}$ meets requirement 1 of Hypothesis 2.3. If

$$[\mathfrak{m}_2, \mathfrak{s}] = \{0\},$$

then $[112] = [221] = 0$, which contradicts the formula $[221] > 0$. Thus, $\mathfrak{s}$ meets requirement 2 of Hypothesis 2.3.

It is easy to see that

$$\mathfrak{g} \supset \mathfrak{g} \supset \mathfrak{s} \supset \mathfrak{h}$$

is the only simple chain associated with M. Setting $\mathfrak{k} = \mathfrak{g}$ and $\mathfrak{k}' = \mathfrak{s}$ in (2.4), we obtain

$$\mathfrak{j} = \{0\}, \qquad \mathfrak{j}' = \mathfrak{l} = \mathfrak{m}_2, \qquad \mathfrak{n} = \mathfrak{m}_1.$$

Given $T \in \mathcal{M}$, the equality

$$T = z_1 \pi_{\mathfrak{m}_1}^* Q + z_2 \pi_{\mathfrak{m}_2}^* Q$$

holds for some $z_1, z_2 > 0$. It is obvious that

$$\lambda_-(T|_{\mathfrak{n}}) = z_1, \qquad \operatorname{tr}_Q T|_{\mathfrak{l}} = d_2 z_2.$$

A straightforward computation involving (2.16) and (2.20) yields

$$\begin{aligned}\eta(\mathfrak{g},\mathfrak{s}) &= \frac{2\,\mathrm{tr}_Q\, B|_{\mathfrak{m}_1} + 2\langle \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_2\rangle + \langle \mathfrak{m}_1\mathfrak{m}_1\mathfrak{m}_1\rangle}{\omega(\mathfrak{m}_1)(2\,\mathrm{tr}_Q\, B|_{\mathfrak{m}_2} + \langle \mathfrak{m}_2\mathfrak{m}_2\mathfrak{m}_2\rangle)} \\ &= \frac{2d_1b_1 - 2[122] - [111]}{d_1(2d_2b_2 - [222])} = \frac{4d_1\zeta_1 + [111]}{d_1(4d_2\zeta_2 + [222] + 4[122])}.\end{aligned}$$

Theorem 2.9 asserts that it is possible to find a metric $g \in \mathcal{M}$ whose Ricci curvature equals cT for some $c > 0$ if

$$z_1/z_2 > d_2\eta(\mathfrak{g},\mathfrak{s}) = \frac{d_2(4d_1\zeta_1 + [111])}{d_1(4d_2\zeta_2 + [222] + 4[122])}. \tag{3.2}$$

According to [24, Proposition 3.1], this condition is, in fact, sufficient *and* necessary for the existence of g.

Example 3.1. Suppose that $M = SU(4)/SU(2)$ as in [28, Example 5] and that $Q = -B$. The assumptions of Section 3 hold, and

$$\begin{aligned} &d_1 = 7, \qquad d_2 = 5, \qquad b_1 = b_2 = 1, \\ &[111] = 21/20, \qquad [122] = 7/4, \qquad [222] = 0; \end{aligned}$$

see [28]. By the above reasoning, a Riemannian metric with Ricci curvature equal to cT for some $c > 0$ exists if and only if $z_1/z_2 > 27/40$.

4. Generalised flag manifolds

In this section, we discuss the case where M is a generalised flag manifold. Our first objective is to verify Hypothesis 2.3. After that, we will consider a class of examples to illustrate the use of Theorem 2.9. For the definition and some properties of a generalised flag manifold, see, e.g., [2, Chapter 7]. We will also rely on the classification results obtained in [1, 20] and collected in [1].

Proposition 4.1. *Suppose M is a generalised flag manifold. Then M satisfies Hypothesis 2.3.*

Proof. Choose a decomposition of the form (2.8). Since M is a generalised flag manifold, the representations $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_j}$ are inequivalent whenever $i \neq j$; see, e.g., [2, Chapter 7, Section 5]. The summands $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$

are determined uniquely up to order. Consequently, every nonzero $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{m}$ is the direct sum of $\mathfrak{m}_i$ with the index i running through some non-empty subset of $\{1, \dots, s\}$.

Let us verify Hypothesis 2.3. Consider a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ containing $\mathfrak{h}$ as a proper subset. It is obvious that $\mathfrak{s} \ominus \mathfrak{h}$ is $\mathrm{Ad}(H)$-invariant. Therefore, for some $J_{\mathfrak{s}} \subset \{1, \dots, s\}$,

$$\mathfrak{s} \ominus \mathfrak{h} = \bigoplus_{i \in J_{\mathfrak{s}}} \mathfrak{m}_i, \qquad \mathfrak{g} \ominus \mathfrak{s} = \bigoplus_{i \in \{1, \dots, s\} \setminus J_{\mathfrak{s}}} \mathfrak{m}_i.$$

As we noted above, $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_j}$ are inequivalent for $i \neq j$. It follows that $\mathfrak{s}$ meets requirement 1 of Hypothesis 2.3.

As explained in [2, Chapter 7, Section 5], for every $i = 1, \dots, s$, the complexification of $\mathfrak{m}_i$ is the sum of two complex vector spaces of the same dimension. Consequently, d_i is even. We conclude that $\mathfrak{m}$ does not have any $\mathrm{Ad}(H)$-invariant 1-dimensional subspaces. This means $\mathfrak{s}$ meets requirement 2 of Hypothesis 2.3. $\square$

Let M be a generalised flag manifold. Suppose $s = 2$ in every decomposition of the form (2.8). It is easy to understand that the assumptions of Section 3 are satisfied; see [1]. The structure constants [111], [112] and [222] vanish. Thus, condition (3.2) becomes

$$z_1/z_2 > \frac{d_2\zeta_1}{d_2\zeta_2 + [122]}.$$

Assume that $s = 3$ in every decomposition of the form (2.8) and that M is of type I in the terminology of [1]. Our next goal is to write down explicit formulas for the numbers $\eta(\mathfrak{k}, \mathfrak{k}')$ associated with simple chains of the form (2.4). This will lead up to the application of Theorem 2.9. Analogous reasoning works if M is of type II in the terminology of [1] or if the isotropy representation of M splits into four or five irreducible summands. We provide further details in Remark 4.2 below.

Consider a decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$$

of the form (2.8). It will be convenient for us to assume that this decomposition is the same as in [1, Subsection 2.4]. The definition of a generalised flag manifold requires the group G to be semisimple. This enables us to set

$Q=-B$. According to [1, Formulas (11), (13) and (15)],

$$
\begin{aligned}
&[112]=[121]=[211]=\frac{d_1d_2+2d_1d_3-d_2d_3}{d_1+4d_2+9d_3}, \\
(4.1)\qquad &[123]=[231]=[312]=[321]=[213]=[132]=\frac{(d_1+d_2)d_3}{d_1+4d_2+9d_3},
\end{aligned}
$$

and the rest of the structure constants are 0. The dimensions d_1, d_2, d_3 for concrete spaces are listed in [1, Table 4].

Remark 4.2. The reader will find the structure constants of generalised flag manifolds with two irreducible isotropy summands in [3, 5], three summands in [1, 20], four summands in [4] and five summands in [6].

As we mentioned in the proof of Proposition 4.1, the representations $\mathrm{Ad}(H)|_{\mathfrak{m}_i}$ and $\mathrm{Ad}(H)|_{\mathfrak{m}_j}$ are inequivalent for $i\neq j$. Consequently, every nonzero $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}\ominus\mathfrak{h}$ is the direct sum of some of the spaces $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$. This fact and formulas (4.1) imply that the proper Lie subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ as a proper subset are

$$\mathfrak{s}_1=\mathfrak{m}_2\oplus\mathfrak{h},\qquad \mathfrak{s}_2=\mathfrak{m}_3\oplus\mathfrak{h}.$$

It follows that the simple chains associated with M are

$$\mathfrak{g}\supset\mathfrak{g}\supset\mathfrak{s}_1\supset\mathfrak{h},\qquad \mathfrak{g}\supset\mathfrak{g}\supset\mathfrak{s}_2\supset\mathfrak{h}.$$

Given $T\in\mathcal{M}$, the equality

$$T=-z_1\pi^*_{\mathfrak{m}_1}B-z_2\pi^*_{\mathfrak{m}_2}B-z_3\pi^*_{\mathfrak{m}_3}B$$

holds for some $z_1,z_2,z_3>0$. Setting $\mathfrak{k}=\mathfrak{g}$ and $\mathfrak{k}'=\mathfrak{s}_i$ in (2.4), we obtain

$$
\begin{aligned}
&\mathfrak{j}=\{0\},\qquad \mathfrak{j}'=\mathfrak{l}=\mathfrak{m}_1\oplus\mathfrak{m}_{4-i},\qquad \mathfrak{n}=\mathfrak{m}_{1+i}, \\
&\qquad\lambda_-(T|_{\mathfrak{n}})=z_{1+i},\qquad \mathrm{tr}_{-B}\,T|_{\mathfrak{l}}=d_1z_1+d_{4-i}z_{4-i},\qquad i=1,2.
\end{aligned}
$$

A computation involving (2.16), (2.18) and (4.1) yields

$$
\begin{aligned}
\eta(\mathfrak{g},\mathfrak{s}_1) &= \frac{2\operatorname{tr}_{-B} B|_{\mathfrak{m}_2}+2(\langle\mathfrak{m}_2\mathfrak{m}_1\mathfrak{m}_1\rangle+\langle\mathfrak{m}_2\mathfrak{m}_3\mathfrak{m}_3\rangle+2\langle\mathfrak{m}_2\mathfrak{m}_1\mathfrak{m}_3\rangle)+\langle\mathfrak{m}_2\mathfrak{m}_2\mathfrak{m}_2\rangle}{\omega(\mathfrak{m}_2)(2\operatorname{tr}_{-B} B|_{\mathfrak{m}_1\oplus\mathfrak{m}_3}+\langle\mathfrak{m}_1\mathfrak{m}_1\mathfrak{m}_1\rangle+\langle\mathfrak{m}_3\mathfrak{m}_3\mathfrak{m}_3\rangle+3\langle\mathfrak{m}_1\mathfrak{m}_1\mathfrak{m}_3\rangle+3\langle\mathfrak{m}_1\mathfrak{m}_3\mathfrak{m}_3\rangle)} \\
&= \frac{-d_2+[112]+2[123]}{d_2(-d_1-d_3)} = \frac{-4d_2^2-8d_2d_3+4d_1d_3}{-d_2(d_1+d_3)(d_1+4d_2+9d_3)}, \\
\eta(\mathfrak{g},\mathfrak{s}_2) &= \frac{2\operatorname{tr}_{-B} B|_{\mathfrak{m}_3}+2(\langle\mathfrak{m}_3\mathfrak{m}_1\mathfrak{m}_1\rangle+\langle\mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_2\rangle+2\langle\mathfrak{m}_3\mathfrak{m}_1\mathfrak{m}_2\rangle)+\langle\mathfrak{m}_3\mathfrak{m}_3\mathfrak{m}_3\rangle}{\omega(\mathfrak{m}_3)(2\operatorname{tr}_{-B} B|_{\mathfrak{m}_1\oplus\mathfrak{m}_2}+\langle\mathfrak{m}_1\mathfrak{m}_1\mathfrak{m}_1\rangle+\langle\mathfrak{m}_2\mathfrak{m}_2\mathfrak{m}_2\rangle+3\langle\mathfrak{m}_1\mathfrak{m}_1\mathfrak{m}_2\rangle+3\langle\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_2\rangle)} \\
&= \frac{-2d_3+4[123]}{d_3(-2d_1-2d_2+3[112])} = \frac{-2d_1+4d_2+18d_3}{2d_1^2+8d_2^2+7d_1d_2+12d_1d_3+21d_2d_3}.
\end{aligned}
$$

Theorem 2.9 tells us that a Riemannian metric with Ricci curvature equal to cT for some $c > 0$ exists if

$$
\begin{aligned}
\frac{z_2}{d_1z_1+d_3z_3} &> \frac{-4d_2^2-8d_2d_3+4d_1d_3}{-d_2(d_1+d_3)(d_1+4d_2+9d_3)}, \\
\frac{z_3}{d_1z_1+d_2z_2} &> \frac{-2d_1+4d_2+18d_3}{2d_1^2+8d_2^2+7d_1d_2+12d_1d_3+21d_2d_3}.
\end{aligned}
$$

Example 4.3. Suppose M is the generalised flag manifold $G_2/U(2)$ in which $U(2)$ corresponds to the long root of G_2. According to [1, Table 4], in this case, $d_1 = d_3 = 4$ and $d_2 = 2$. Theorem 2.9 implies that a Riemannian metric with Ricci curvature equal to cT for some $c > 0$ exists if

$$
\frac{z_2}{z_1+z_3} > \frac{1}{12}, \qquad \frac{z_3}{2z_1+z_2} > \frac{3}{10}.
$$

5. Prospects

There are many compact homogeneous spaces that do not satisfy Hypothesis 2.3. The isotropy representations of such spaces necessarily have equivalent non-trivial irreducible subrepresentations. If Hypothesis 2.3 is not satisfied, the assertion of Lemma 2.12 may fail to hold for some subalgebras $\mathfrak{k}$ and decompositions of the form (2.8). As a consequence, crucial equalities and inequalities for the structure constants break down, and one runs into trouble trying to prove the key estimate in Section 2.5. Thus, in order to relax or remove Hypothesis 2.3, one would have to find a new way to arrive at Lemma 2.31. The authors intend to study this in a forthcoming paper. Several other questions remain open concerning the prescribed Ricci curvature problem on compact homogeneous spaces. These include, for instance, investigating the uniqueness of solutions up to scaling, finding necessary conditions for the existence of solutions in the case of three or more irreducible isotropy summands, and discovering critical points of $S|_{\mathcal{M}_T}$ that are not global maxima.

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