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Existence and multiplicity of solutions for a class of indefinite variational problems

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In this paper we study the existence and multiplicity of solutions for the following class of strongly indefinite problems

$$(P)_k \qquad \begin{cases} -\Delta u + V(x)u = A(x/k)f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 1$, $k \in \mathbb{N}$ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $V, A : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some technical conditions. Assuming that V is a $\mathbb{Z}^N$-periodic function, $0 \notin \sigma(-\Delta + V)$ the spectrum of $-\Delta + V$, we show how the "shape" of the graph of function A affects the number of nontrivial solutions.

1. Introduction

This paper concerns with the existence and multiplicity of solutions for the following class of problems

$$(P)_k \qquad \begin{cases} -\Delta u + V(x)u = A(x/k)f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 1$, $k \in \mathbb{N}$ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $V, A : \mathbb{R} \to \mathbb{R}$ are continuous functions verifying some technical conditions.

In the whole paper, function V is $\mathbb{Z}^N$-periodic with

$$(V_1) \qquad 0 \notin \sigma(-\Delta + V), \quad \text{the spectrum of } -\Delta + V,$$

which means that the problem is strongly indefinite.

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Related to function A, we assume the following conditions:

$(H1)$ Function $A : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{|x|\to\infty} A(x) = A_\infty,$$

with $0 < A_\infty < A(x)$ for any $x \in \mathbb{R}^N$.

$(H2)$ There exist l points $z_1, z_2, ..., z_l$ in $\mathbb{Z}^N$ with $z_1 = 0$ such that

$$1 = A(z_i) = \max_{x\in\mathbb{R}^N} A(x), \quad \text{for} \ \ 1 \le i \le l.$$

The present paper was motivated by some recent results that studied the existence of ground state solutions for related problems to $(P)_k$, more precisely, for the strongly indefinite problems of type

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

In [8], Kryszewski and Szulkin studied the existence of ground state solution for (1.1) by supposing condition (V_1). Involving the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, they assumed that the function f is continuous, $\mathbb{Z}^N$-periodic in x with

$$(h_1) \qquad |f(x,t)| \le c(|t|^{q-1} + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N$$

and

$$(h_2) \ \ 0 < \alpha F(x,t) \le t f(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^*, \quad F(x,t) = \int_0^t f(x,s)\,ds$$

for some $c > 0$, $\alpha > 2$ and $2 < q < p < 2^*$ where $2^* = \dfrac{2N}{N-2}$ if $N \ge 3$ and $2^* = +\infty$ if $N = 1, 2$. The above hypotheses guarantee that the energy functional associated to (1.1) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \ \forall u \in H^1(\mathbb{R}^N),$$

is well defined and belongs to $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. By assumption (V_1), there is an equivalent inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. In order to prove the existence of solutions for (P_1), Kryszewski and Szulkin introduced a new and interesting generalized linking theorem. In [11], Li and Szulkin improved this generalized linking theorem and proved the existence of solution for a class of strongly indefinite problem with f being asymptotically linear at infinity.

The linking theorems mentioned above were applied in many papers, we would like to mention Chabrowski and Szulkin [4], do Ó and Ruf [6], Furtado and Marchi [7], Tang [20, 21] and the references therein.

Pankov and Pflüger [13] also considered the existence of solution for problem (P_1) with the same conditions introduced in [8], however the approach was based on an approximation technique of periodic function together with the linking theorem due to Rabinowitz [14]. Later, Pankov [12] studied the existence of solution for problems of the type

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

by supposing conditions (V_1), $(h_1) - (h_2)$ and employing the same approach explored in [13]. In [12] and [13], the existence of ground state solutions was established by assuming that f is C^1 and there exists $\theta \in (0,1)$ such that

$$(h_3) \qquad 0 < t^{-1}f(x,t) \le \theta f_t'(x,t), \quad \forall t \neq 0 \quad \text{and} \quad x \in \mathbb{R}^N.$$

In [12], Pankov obtained a ground state solution by minimizing the energy functional J on the set

$$\mathcal{O} = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ J'(u)u = 0 \text{ and } J'(u)v = 0, \forall\ v \in E^- \right\}.$$

The reader is invited to see that if J is strongly definite, that is, when $E^- = \{0\}$, the set $\mathcal{O}$ is exactly the Nehari manifold associated to J. Hereafter, we say that $u_0 \in H^1(\mathbb{R}^N)$ is a *ground state solution* if

$$J'(u_0) = 0, \quad u_0 \in \mathcal{O} \quad \text{and} \quad J(u_0) = \inf_{w \in \mathcal{O}} J(w).$$

Szulkin and Weth [15] established the existence of ground state solution for problem (1.1) by completing the study made in [12]. In fact they also minimize the energy functional on $\mathcal{O}$, however they have used more weaker conditions on f, for example f is continuous, $\mathbb{Z}^N$-periodic in x and satisfies

$$(h_4) \qquad |f(x,t)| \leq C(1+|t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N$$

for some $C > 0$ and $p \in (2, 2^*)$.

$$(h_5) \qquad f(x,t) = o(t) \text{ uniformly in } x \text{ as } |t| \to 0.$$

$$(h_6) \qquad F(x,t)/|t|^2 \to +\infty \text{ uniformly in } x \text{ as } |t| \to +\infty,$$

and

$$(h_7) \qquad t \mapsto f(x,t)/|t| \text{ is strictly increasing on } \mathbb{R} \setminus \{0\}.$$

The same result was also established by Yang in [25] by applying a monotone trick. Finally, for the perturbed periodic Schrödinger equation, Alves and Germano in [1–3] studied the existence and concentration of solution for strongly indefinite problem like

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where V is as above and A is a continuous function satisfying some technical conditions.

For the definite case, Cao and Noussair [5] considered the existence and multiplicity of solutions for the following class of problem

$$\begin{cases} -\Delta u + u = A(\epsilon x)|u|^{p-2}u \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.4}$$

This equation is strongly definite because $V = 1$. Using Ekeland's variational principle and concentration compactness principle of Lions [10], Cao and Noussair proved that if A has l equal maximum points, then problem (1.4) has at least l positive solutions and l nodal solutions if ϵ is small enough. Later Wu in [24] proved the existence of at least ℓ positive solutions for the perturbed problem

$$\begin{cases} -\Delta u + u = h(\varepsilon x)|u|^{r-2}u + \lambda g(\varepsilon x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where λ is a positive small parameter, $q \in [1,2)$ and $g : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous satisfying

$$g(x) \to 0 \quad \text{and} \quad |x| \to +\infty.$$

In [24], an important fact is that the energy functional associated with (1.5) satisfies the mountain pass geometry and has a well defined Nehari manifold.

Inspired by the results in [5], the aim of the present paper is to prove a similar result for $(P)_k$, one can see how the "shape" of A affects the number of nontrivial solutions. However, we would like point out that one of the main difficulties is the loss of the mountain pass geometry, because we are working with a strongly indefinite problem. Then, if I_k denotes the energy functional associated with $(P)_k$, we need to carry out a careful study involving the behavior of number c_k given by

$$c_k = \inf_{u \in \mathcal{M}_k} I_k(u) \tag{1.6}$$

where

$$\mathcal{M}_k = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ I_k'(u)u = 0 \text{ and } I_k'(u)v = 0, \forall\ v \in E^- \right\}. \tag{1.7}$$

The understanding of the behavior of c_k is a key point in our approach to show the existence of multiple solutions for k large enough.

Hereafter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that verifies the following assumptions:

(f_1) $\dfrac{f(t)}{t} \to 0$ as $t \to 0$.

(f_2) $\displaystyle\limsup_{|t| \to +\infty} \frac{|f(t)|}{|t|^q} < +\infty$ for some $q \in (1, 2^* - 1)$.

(f_3) $t \mapsto f(t)/t$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$.

(f_4) (Ambrosetti-Rabinowitz) There exists $\theta > 2$ such that

$$0 < \theta F(t) \leq f(t)t, \ \forall\ t \neq 0$$

where $F(t) := \displaystyle\int_0^t f(s)ds$.

Our main result is the following theorem.

Theorem 1.1. *There is $k^* > 0$ such that $(P)_k$ has at least l nontrivial solution for $k > k^*$.*

In the proof Theorem 1.1 we will use some arguments applied in [2], because the energy functional here has the same geometry explored in that paper. We would like to point out that our paper completes the study made in that paper in the sense that here we are assuming different assumptions on function A, for example, in [2] the function A satisfies the condition

$$(A_1) \qquad 0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x),$$

which is not assumed in the present paper.

The plan of the paper is as follows: In Section 2 we show a compactness result for the autonomous problem, which is a crucial result in our approach, see Theorem 2.3. In Section 3 we show the existence of multiple solutions for $(P)_k$.

Notation. In this paper, we will use the following notations:

- $o_n(1)$ denotes a sequence that converges to zero.
- If g is a mensurable function, the integral $\int_{\mathbb{R}^N} g(x)\,dx$ will be denoted by $\int g(x)\,dx$.
- $B_R(z)$ denotes the open ball with center z and radius R in $\mathbb{R}^N$.
- The usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ will be denoted by $\|\ \ \|_{H^1(\mathbb{R}^N)}$ and $|\ \ |_p$ respectively.
- For each $u \in H^1(\mathbb{R}^N)$, the equality $u = u^+ + u^-$ yields $u^+ \in E^+$ and $u^- \in E^-$.

2. A compactness result for the autonomous problem.

In this section our main goal is to prove a compactness result for the autonomous equation that will be used later on. In order to do that, we need to recall some results that were proved in [2]. Let us consider the following autonomous problem

$$(AP)_\lambda \qquad \begin{cases} -\Delta u + V(x)u = \lambda f(u) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $\lambda > 0$ and V, f verify the conditions (V_1) and $(f_1) - (f_4)$ respectively. Associated to equation $(AP)_\lambda$ we define the energy functional

$J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)|u|^2)\,dx - \lambda \int F(u)\,dx,$$

equivalently

$$J_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \lambda \int F(u)\,dx.$$

In what follows, let us denote by d_λ the real number defined by

$$d_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u); \tag{2.8}$$

where

$$\mathcal{N}_\lambda = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ J'_\lambda(u)u = 0 \text{ and } J'_\lambda(u)v = 0, \forall\, v \in E^- \right\}. \tag{2.9}$$

Moreover, for each $u \in H^1(\mathbb{R}^N)$, the sets $E(u)$ and $\hat{E}(u)$ designate

$$E(u) = E^- \oplus \mathbb{R}u \quad \text{and} \quad \hat{E}(u) = E^- \oplus [0, +\infty)u. \tag{2.10}$$

The reader is invited to observe that $E(u)$ and $\hat{E}(u)$ are independent of λ, more precisely they depend on only of the operator $-\Delta + V$. This remark is very important because these sets will be used in the next sections as well as the lemma below.

Lemma 2.1. *For all $u = u^+ + u^- \in H^1(\mathbb{R}^N)$ and $y \in \mathbb{Z}^N$, if $u_y(x) := u(x+y)$ then $u_y \in H^1(\mathbb{R}^N)$ with $u_y^+(x) = u^+(x+y)$ and $u_y^-(x) = u^-(x+y)$.*

In [15], Szulkin and Weth have proved that for each $\lambda > 0$, the problem $(AP)_\lambda$ possesses a ground state solution $u_\lambda \in H^1(\mathbb{R}^N)$, that is,

$$u_\lambda \in \mathcal{N}_\lambda, \quad J_\lambda(u_\lambda) = d_\lambda \quad \text{and} \quad J'_\lambda(u) = 0.$$

Still in [15], the authors also proved that

$$0 < d_\lambda = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} J_\lambda(u). \tag{2.11}$$

Moreover, an interesting and important fact is that for each $u \in H^1(\mathbb{R}^N) \setminus E^-$, $\mathcal{N}_\lambda \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $J_\lambda|_{\hat{E}(u)}$, that is, there are $t^* \geq 0$ and $v^* \in E^-$ such that

$$J_\lambda(t^*u + v^*) = \max_{w \in \widehat{E}(u)} J_\lambda(w). \tag{2.12}$$

By (2.12), we can define the function

$$m_\lambda : E^+ \setminus \{0\} \to \mathcal{N}_\lambda \quad \text{where} \quad m_\lambda(u) = t^*u + v^* \in \hat{E}(u) \cap \mathcal{N}_\lambda. \tag{2.13}$$

Proposition 2.2. *The function $\lambda \mapsto d_\lambda$ is decreasing and continuous on $(0, +\infty)$.*

The next result is an important compactness result, which is well known for the strongly definite case, for example if $\inf_{x \in \mathbb{R}^N} V(x) > 0$. Here we prove that it also holds for the strongly indefinite case and its proof follows as in [2, Proposition 4.5]. For the completeness of the paper, we outline the proof here for readers' convenience.

Theorem 2.3. *(Compactness result) Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_{d_\lambda}$ sequence for J_λ with $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then, one of the following two cases holds:*

(i) *$u_n \to u$ in $H^1(\mathbb{R}^N)$,*
or

(ii) *There exists $\{y_n\} \subset \mathbb{Z}^N$ with $|y_n| \to \infty$ such that the sequence $\tilde{u}_n(x) = u_n(. + y_n)$ is strongly convergent to a function $H^1(\mathbb{R}^N)$ for some $v \in H^1(\mathbb{R}^N) \setminus \{0\}$.*

Proof. We begin the proof by showing that (i) holds if $u \neq 0$. In fact, since $\{u_n\}$ is a $(PS)_{d_\lambda}$ sequence, we know that $J'_\lambda(u) = 0$. Then, if $u \neq 0$, we know that $u \in \mathcal{N}_{d_\lambda}$, and so,

$$\begin{aligned}
d_\lambda \leq J_\lambda(u) = J_\lambda(u) - \frac{1}{2}J'_\lambda(u)u &= \int \left(\frac{1}{2}f(u)u - F(u)\right) dx \\
&\leq \liminf_{n \to +\infty} \int \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) dx \\
&\leq \limsup_{n \to +\infty} \int \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) dx \\
&= \limsup_{n \to +\infty} \left(J_\lambda(u_n) - \frac{1}{2}J'_\lambda(u_n)u_n\right) = d_\lambda.
\end{aligned}$$

Therefore

$$\lim_{n\to+\infty}\int\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx = \int\left(\frac{1}{2}f(u)u - F(u)\right)dx.$$

Since

$$\frac{1}{2}f(u_n)u_n - F(u_n) \geq 0, \quad \forall n \in \mathbb{N},$$

and supposing that

$$u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N,$$

we deduce that

$$\frac{1}{2}f(u_n)u_n - F(u_n) \to \frac{1}{2}f(u)u - F(u) \text{ in } L^1(\mathbb{R}^N).$$

Thus, up to subsequence, there exists $H \in L^1(\mathbb{R}^N)$ such that

$$\frac{1}{2}f(u_n)u_n - F(u_n) \leq H \quad \text{a.e. in } \mathbb{R}^N$$

for all $n \in \mathbb{N}$. Then, by (f_4),

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) f(u_n)u_n \leq H, \quad \forall n \in \mathbb{N}.$$

Consequently, there exists $c > 0$ such that

$$f(u_n)u_n \leq c\,H, \quad \forall n \in \mathbb{N}.$$

In what follows, we set

$$Q_n := f(u_n)u_n^+ - f(u)u^+.$$

Our goal is to prove that

$$\int |Q_n| dx \to 0.$$

First of all, as f is of subcritical growth,

$$\int_{B_R(0)} |Q_n| dx \to 0, \quad \forall R > 0. \tag{2.14}$$

On the other hand, for each $\tau > 0$, we can fix R large enough a such way that

$$\int_{B_R(0)^c} |f(u)u^+| dx < \tau.$$

Claim 2.4. Increasing R if necessary, we also have

$$\int_{B_R(0)^c} |f(u_n)u_n^+|dx < 2\Theta\tau, \quad \forall n \in \mathbb{N}$$

where

$$\Theta := \sup_{n\in\mathbb{N}} \left\{ \left(\int |u_n^+|^{q+1} dx \right)^{\frac{1}{q+1}}, \int |u_n u_n^+| dx \right\}.$$

In fact, by [2, Lemma 4.4], for each $\tau > 0$ the exists $c_\tau > 0$ such that

$$|g_\tau(t)| \le \tau|t| \quad \text{and} \quad |j_\tau(t)|^r \le c_\tau t f(t), \quad \forall t \in \mathbb{R},$$

where $r = \dfrac{q+1}{q}$ with q given in (f_2). Here,

$$g_\tau(t) := \chi_\delta(t) f(t) \quad \text{and} \quad j_\tau(t) := \tilde{\chi}_\delta(t) f(t),$$

where χ_δ is the characteristic function on $(-\delta, \delta)$ and $\tilde{\chi}_\delta(t) = 1 - \chi_\delta(t)$ and $\delta > 0$ is fixed of a such way that

$$\frac{|f(t)|}{|t|} < \tau, \quad \forall t \in (-\delta, \delta).$$

Using the above functions, we have that

$$\begin{aligned}
&\int_{B_R(0)^c} |f(u_n)u_n^+|dx = \int_{B_R(0)^c} |g_\tau(u_n)||u_n^+|dx + \int_{B_R(0)^c} |j_\tau(u_n)||u_n^+|dx \\
&\le \tau \int_{B_R(0)^c} |u_n||u_n^+|dx + \left(\int_{B_R(0)^c} |j_\tau(u_n)|^r dx \right)^{1/r} \left(\int_{B_R(0)^c} |u_n^+|^{q+1} dx \right)^{1/(q+1)} \\
&\le \tau\Theta + \left(\int_{B_R(0)^c} c_\tau f(u_n) u_n dx \right)^{1/r} \Theta \le \tau\Theta + c_\tau \left(\int_{B_R(0)^c} cH \ dx \right)^{1/r} \Theta.
\end{aligned}$$

Now, increasing R if necessary, a such way that

$$c_\tau \left(\int_{B_R(0)^c} cH \ dx \right)^{1/r} < \tau,$$

we get

$$\int_{B_R(0)^c} |f(u_n)u_n^+|dx \le 2\tau\Theta,$$

proving Claim 2.4. From (2.14) and Claim 2.4,

$$\int |Q_n|\, dx \to 0.$$

Therefore

$$f(u_n)u_n^+ \to f(u)u^+ \text{ in } L^1(\mathbb{R}^N).$$

Analogously,

$$f(u_n)u_n^- \to f(u)u^- \text{ in } L^1(\mathbb{R}^N).$$

Since $J'_\lambda(u_n)u_n^+ = o_n(1)$, it follows that

$$||u_n^+||^2 = \int f(u_n)u_n^+ dx \to \int f(u)u^+ dx = ||u^+||^2,$$

showing that $u_n^+ \to u^+$ in $H^1(\mathbb{R}^N)$, because $u_n^+ \rightharpoonup u^+$ in $H^1(\mathbb{R}^N)$. Likewise $u_n^- \to u^-$ in $H^1(\mathbb{R}^N)$. Thereby $u_n = u_n^+ + u_n^- \to u^+ + u^- = u$ in $H^1(\mathbb{R}^N)$, proving that (i) holds.

Now, we are going to prove that if $u = 0$, then $ii)$ holds. Assuming that $u = 0$ and arguing as in the proof of Proposition 2.2, there are $r, \eta > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\int_{B_r(y_n)} |u_n^+|^2\, dx \geq \eta > 0.$$

Since $u_n^+ \rightharpoonup 0$, the last inequality implies that $\{y_n\}$ is an unbounded sequence. Setting $\tilde{u}_n(x) = u_n(x + y_n)$, it follows that $\{\tilde{u}_n\}$ is also a $(PS)_{d_\lambda}$ sequence for J_λ with $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^N)$ and $\tilde{u}^+ \neq 0$. Repeating the same arguments explored in the proof of item $i)$, we have that $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^N)$, finishing the proof. □

3. Existence of multiple solutions for $(P)_k$.

Hereafter, for each $k \in \mathbb{N}$, we denote by $I_k : H^1(\mathbb{R}^N) \to \mathbb{R}$ the energy functional associated to $(P)_k$ defined as

$$I_k(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)|u|^2)\, dx - \int A(x/k)F(u)\, dx,$$

or equivalently

$$I_k(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int A(x/k)F(u)\, dx.$$

The same idea explored in [15, Lemma 2.4] shows that

$$0 < c_k = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} I_k(u). \tag{3.15}$$

By [2, Lemma 2.2], we can argue as [15, Lemma 2.6] to prove that for each $u \in H^1(\mathbb{R}^N) \setminus E^-$, $\mathcal{M}_k \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $I_k|_{\hat{E}(u)}$, that is, there are $\tilde{t} \geq 0$ and $\tilde{v} \in E^-$ such that

$$I_k(\tilde{t}u + \tilde{v}) = \max_{w \in \widehat{E}(u)} I_k(w). \tag{3.16}$$

By (3.16), we can define the function

$$m_k : E^+ \setminus \{0\} \to \mathcal{M}_k \quad \text{where} \quad m_k(u) = \tilde{t}u + \tilde{v} \in \hat{E}(u) \cap \mathcal{M}_k. \tag{3.17}$$

Hereafter,

$$\mathcal{M}_k = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ I_k'(u)u = 0 \text{ and } I_k'(u)v = 0, \forall\, v \in E^- \right\}. \tag{3.18}$$

Our first lemma shows an important relation between c_k and $d_{A(0)}$, however we will omit its proof because it can be done as [2, Lemma 3.1].

Lemma 3.1. *The minimax values c_k satisfies the limit below*

$$\lim_{k \to +\infty} c_k = d_{A(0)}.$$

This together with Proposition 2.2 imply the corollary below.

Corollary 3.2. *There exists $k^* > 0$ such that $c_k < d_{A_\infty}$ for all $k \geq k^*$, where $A_\infty = \lim_{|x| \to +\infty} A(x)$.*

Our next result shows that I_k is coercive on $\mathcal{M}_k$, since its proof follows the same steps found in [2, Proposition 3.4] it will also be omitted.

Proposition 3.3. *I_k is coercive on $\mathcal{M}_k$.*

Hereafter, we consider the functional $\hat{\Psi}_k : E^+ \setminus \{0\} \to \mathbb{R}$ defined by $\hat{\Psi}_k(u) := I_k(m_k(u))$. We know that $\hat{\Psi}_k$ is continuous by previous lemma. In the sequel, we denote by $\Psi_k : S^+ \to \mathbb{R}$ the restriction of $\hat{\Psi}_k$ to $S^+ = B_1(0) \cap E^+$.

The next two results establish some important properties involving the functionals Ψ_k and $\hat{\Psi}_k$ and their proofs follow as in [15].

Lemma 3.4. *$\hat{\Psi}_k \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, and*

$$\hat{\Psi}'_k(y)z = \frac{||m_k(y)^+||}{||y||} I'_k(m_k(y))z, \ \forall y, z \in E^+, \ y \neq 0. \tag{3.19}$$

Corollary 3.5. *The following properties hold:*

(a) *$\Psi_k \in C^1(S^+)$, and*

$$\Psi'_k(w)z = ||m(y)^+|| I'_k(m(w))z, \ \text{for } z \in T_y S^+.$$

Hence, $w \in S^+$ is a critical point of Ψ_k on S^+ if, and only if, $u = m(w)$ is a critical point of I_k in $H^1(\mathbb{R}^N)$.

(b) *$\{w_n\}$ is a $(PS)_c$ sequence for Ψ_k if and only if $\{m_k(w_n)\}$ is a $(PS)_c$ sequence for I_k.*

(c) *$(PS)_c$ condition holds for Ψ_k if, and only if, $(PS)_c$ condition holds for I_k.*

Then we can establish the following compactness criteria for Ψ_k.

Lemma 3.6. *The functional Ψ_k satisfies the $(PS)_c$ condition for $c \leq d_1 + \gamma$, where $\gamma = \frac{1}{2}(d_{A_\infty} - d_1)$.*

Proof. Let $\{\omega_n\} \subset S^+$ be a $(PS)_c$ sequence for Ψ_k. From Corollary 3.5, $u_n = m_k(\omega_n)$ is also a $(PS)_c$ sequence for functional I_k. Then by Proposition 3.3, $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, and passing to a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^N) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, it is possible to prove that

$$I_k(u_n) - I_k(v_n) - I_k(u) = o_n(1) \tag{3.20}$$

and

$$\|I'_k(u_n) - I'_k(v_n) - I'_k(u)\| = o_n(1), \tag{3.21}$$

where $v_n = u_n - u$.

Recalling that $I_k'(u) = 0$ and $I_k(u) \geq 0$, it follows from (3.20)–(3.21) that

$$\|I_k'(v_n)\| = o_n(1) \quad and \quad I_k(v_n) \to c^* = c - I_k(u). \tag{3.22}$$

This implies that $\{v_n\}$ is a $(PS)_{c^*}$ sequence for I_k with $c^* \leq d_1 + \gamma$.
Claim 1. For each $R > 0$ fixed, we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx = 0. \tag{3.23}$$

Assuming for instance the claim, we can apply Lions [9] to deduce that $v_n \to 0$ in $L^q(\mathbb{R}^N)$, or equivalently, $u_n \to u$ in $L^q(\mathbb{R}^N)$. From this, $f(u_n)u_n^{\pm} \to f(u)u^{\pm}$ in $L^1(\mathbb{R}^N)$, which implies that $u_n^{\pm} \to u_n^{\pm}$ in $H^1(\mathbb{R}^N)$, proving the lemma.

Now, we are going to prove Claim 1. If the claim does not hold, there are $\xi > 0$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\limsup_{n\to+\infty} \int_{B_R(y_n)} |v_n|^2 dx \geq \xi > 0.$$

The last limit ensures that $\{y_n\}$ is an unbounded sequence.

Setting $\tilde{v}_n = v_n(. + y_n)$, we have that $\{\tilde{v}_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$. Hence, there exist $\tilde{v} \in L^{p'}(\mathbb{R}^N) \setminus \{0\}$ and subsequence of $\{\tilde{v}_n\}$, still denoted by itself, such that

$$\tilde{v}_n \rightharpoonup \tilde{v} \quad \text{in } H^1(\mathbb{R}^N),$$
$$\tilde{v}_n(x) \to \tilde{v}(x) \quad \text{a.e. in } \mathbb{R}^N$$

and

$$\int_{B_R(0)} |\tilde{v}|^2 \, dx \geq \xi > 0.$$

Since $I_k'(v_n)\psi(. - y_n) = o_n(1)$, $\forall \psi \in H^1(\mathbb{R}^N)$, the above limits ensure that $J_{A_\infty}'(\tilde{v})\psi = 0$, then $\tilde{v}$ is a nontrivial critical point of solution of J_{A_∞}. As a consequence,

$$
\begin{aligned}
d_{A_\infty} &\leq J_{A_\infty}(\tilde{v}) \\
&= J_{A_\infty}(\tilde{v}) - \frac{1}{2}J'_{A_\infty}(\tilde{v})\tilde{v} \\
&= \lim_{n\to\infty}\int A((x+y_n)/k)(\frac{1}{2}f(\tilde{v}_n)\tilde{v}_n - F(\tilde{v}_n))dx \\
&\leq \liminf_{n\to\infty}[I_k(v_n) - \frac{1}{2}I'_k(v_n)v_n] \\
&= c_* \leq d_1 + \gamma,
\end{aligned}
$$

which contradicts the fact that $\gamma < d_{A_\infty} - d_1$. Hence Ψ_k satisfies $(PS)_c$ condition for $c \leq d_1 + \gamma$. $\square$

In what follows, let us fix $\rho_0, r_0 > 0$ such that $\overline{B_{\rho_0}(a_i)} \cap \overline{B_{\rho_0}(a_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, ..., l\}$, $\bigcup_{i=1}^{l} B_{\rho_0}(a_i) \subset B_{r_0}(0)$ and $K_{\frac{\rho_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho_0}{2}}(a_i)}$. Moreover, we also set function $Q_k : H^1(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ by

$$
Q_k(u) = \frac{\int_{\mathbb{R}^N} \chi(x/k)|u|^2\,dx}{\int_{\mathbb{R}^N} |u|^2\,dx},
$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$
\chi(x) = \begin{cases} x, & \text{if } \ |x| \leq r_0, \\ r_0\dfrac{x}{|x|}, & \text{if } \ |x| > r_0. \end{cases}
$$

The following lemma is very useful to obtain $(PS)_c$ sequences for Ψ_k.

Lemma 3.7. *There exist $\alpha_0 > 0$ and $k_1 > 0$ such that if $u \in S^+$ and $\Psi_k(u) \leq d_1 + \alpha_0$, then $Q_k(u) \in K_{\frac{\rho_0}{2}}$, $\quad \forall k \geq k_1$.*

Proof. If the lemma is not true, then there exist $\alpha_n \to 0$, $k_n \to +\infty$ and $\omega_n \in S^+$ such that

$$
\Psi_{k_n}(\omega_n) \leq c_\infty + \alpha_n
$$

and

$$
Q_{k_n}(\omega_n) \notin K_{\frac{\rho_0}{2}}.
$$

Hereafter, we consider the functional $\hat{\Theta}_\lambda : E^+ \setminus \{0\} \to \mathbb{R}$ defined by $\hat{\Theta}_\lambda(u) := J_\lambda(m_\lambda(u))$ and denote by $\Theta_\lambda : S^+ \to \mathbb{R}$ the restriction of $\hat{\Theta}_\lambda$ to $S^+ = B_1(0) \cap E^+$, where m_λ was given in (2.13)

From definition of Θ_1 and Ψ_k,

$$d_1 \le \Theta_1(\omega_n) \le \Psi_{k_n}(\omega_n) \le d_1 + \alpha_n, \quad \forall n \in \mathbb{N}.$$

Then,

$$\{\omega_n\} \subset S^+ \quad \text{and} \quad \Theta_1(\omega_n) \to d_1.$$

From Ekeland's Variational principle, we can assume that $\Theta_1'(\omega_n) \to 0$. Hence, $u_n = m_1(\omega_n)$ verifies

$$\{u_n\} \subset \mathcal{N}_1, \quad J_1(u_n) \to d_1 \quad \text{and} \quad J_1'(u_n) \to 0.$$

By virtue of Theorem 2.3, we need to consider the following two cases:

(i) $u_n \to u \neq 0$ in $H^1(\mathbb{R}^N)$,
or

(ii) There exists $\{y_n\} \subset \mathbb{Z}^N$ with $|y_n| \to \infty$ such that the sequence $v_n = u_n(. + y_n)$ is strongly convergent to a function $H^1(\mathbb{R}^N)$ for some $v \in H^1(\mathbb{R}^N) \setminus \{0\}$.

It is easy to see that if (i) holds, then $\omega_n \to \omega \neq 0$ in $H^1(\mathbb{R}^N)$. However, if (ii) holds, we must have $\omega_n(. + y_n) \to \hat{\omega}$ in $H^1(\mathbb{R}^N)$.

Analysis of (i): Applying Lebesgue's dominated convergence theorem

$$Q_{k_n}(\omega_n) = \frac{\int_{\mathbb{R}^N} \chi(x/k_n)|\omega_n|^2 dx}{\int_{\mathbb{R}^N} |\omega_n|^2 dx} \to \frac{\int_{\mathbb{R}^N} \chi(0)|\omega|^2 dx}{\int_{\mathbb{R}^N} |\omega|^2 dx} = 0 \in K_{\frac{\rho_0}{2}}.$$

From this, $Q_{k_n}(\omega_n) = Q_{k_n}(u_n) \in K_{\frac{\rho_0}{2}}$ for n large enough, which is a contradiction.

Analysis of (ii): Setting $u_n = m_{k_n}(\omega_n)$, $v_n(x) = u_n(x + y_n)$ and $\hat{\omega}_n(x) = \omega_n(x + y_n)$, we have that $I'_{k_n}(v_n)v_n = 0$, $I'_{k_n}(v_n)\phi = 0$ for all $\phi \in E^-$.

Next, we distinguish two cases:

(I) $|y_n/k_n| \to +\infty$
or

(II) $y_n/k_n \to y$ for some $y \in \mathbb{R}^N$, for some subsequence.

If (I) holds, since there are $s > 0$ and $z \in E^-$ such that $s\hat{\omega}^+ + z \in \mathcal{M}_\infty$, it follows from $\hat{\omega}_n^+ \to \hat{\omega}^+$ in $H^1(\mathbb{R}^N)$ that

$$\begin{aligned} d_{A_\infty} &\leq J_{A_\infty}(s\hat{\omega}^+ + z) \\ &= \lim_{n\to+\infty}\left(\frac{1}{2}s^2\|\hat{\omega}_n^+\|^2 - \frac{1}{2}\|z\|^2 - \int A((x+y_n)/k_n)F(s\hat{\omega}_n^+ + z)dx\right) \\ &= \lim_{n\to\infty} I_{k_n}(s\omega_n^+ + z) \leq \lim_{n\to\infty} \Psi_{k_n}(\omega_n) = d_1 \end{aligned}$$

which contradicts Lemma 3.1.

Now, if (II) holds, the previous argument yields

$$d_{A(y)} \leq d_1, \tag{3.24}$$

where $d_{A(y)}$ is the mountain pass level of the functional $J_{A(y)} : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_{A(y)}(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - A(y)\int F(u)\,dx.$$

One can see that

$$d_{A(y)} = \inf_{u\in\mathcal{M}_{A(y)}} J_{A(y)}(u),$$

where

$$\mathcal{M}_{A(y)} = \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : J'_{A(y)}(u)u = 0\}.$$

If $A(y) < 1$, it is possible to prove that $d_{A(y)} > d_1$, which contradicts (3.24). Then $A(y) = 1$ and $y = z_i$ for some $i = 1, ..., l$. Hence

$$Q_{k_n}(\omega_n) = \frac{\int_{\mathbb{R}^N}\chi(x/k_n)|\omega_n|^2dx}{\int_{\mathbb{R}^N}|\omega_n|^2dx} = \frac{\int_{\mathbb{R}^N}\chi((x+y_n)/k_n)|\hat{\omega}_n|^2dx}{\int_{\mathbb{R}^N}|\hat{\omega}_n|^2dx} \quad \to a_i \in K_{\frac{\rho_0}{2}}$$

from where it follows that $Q_{k_n}(\omega_n) \in K_{\frac{\rho_0}{2}}$ for n large, which is absurd, because we are assuming that $Q_{k_n}(\omega_n) \notin K_{\frac{\rho_0}{2}}$. This finishes the proof. □

Next, we specify the following symbols.

$$\begin{aligned} \Omega_k^i &= \{u \in S^+ : |Q_k(u) - z_i| < \rho_0\}, \\ \partial\Omega_k^i &= \{u \in S^+ : |Q_k(u) - z_i| = \rho_0\}, \\ \alpha_k^i &= \inf_{u\in\Omega_k^i} \Psi_k(u), \\ \tilde{\alpha}_k^i &= \inf_{u\in\partial\Omega_k^i} \Psi_k(u). \end{aligned}$$

Lemma 3.8. *There exists $k_2 \in \mathbb{N}$ such that*

$$\alpha_k^i < d_1 + \gamma \quad and \quad \alpha_k^i < \tilde{\alpha}_k^i,$$

for all $k \geq k_2$, where $\gamma = \frac{1}{2}(d_{A_\infty} - d_1) > 0$.

Proof. Let $u \in H^1(\mathbb{R}^N)$ be a ground state critical point for Ψ_1, i.e.,

$$w \in S^+, \quad \Psi_1(w) = d_1 \quad \text{and} \quad \Psi_1'(w) = 0 \quad (\text{See } Corollary \text{ 3.5}).$$

For $1 \leq i \leq l$ and $k \in \mathbb{N}$, we define the function $\tilde{w}_k^i : \mathbb{R}^N \to \mathbb{R}$ by $\tilde{w}_k^i(x) = w(x - kz_i)$. Clearly $\tilde{w}^i \in S^+$. Setting $u_k^i = m_k(\tilde{w}_k^i)$, we have
Claim 2. For all $1 \leq i \leq l$, we have

$$\limsup_{k \to +\infty} \Psi_k(\tilde{w}_k^i) = \limsup_{k \to +\infty} I_k(u_k^i) \leq d_1.$$

By definition, $u_k^i = t_k(w_k^i)^+ + v_k$ with $t_k > 0$ and $v_k \in E^-$, changing variable, we have

$$I_k(u_k^i) = \frac{t_k^2}{2}\|w^+\|^2 - \frac{1}{2}\|\hat{v}_k\|^2 - \int A((x + kz_i)/k)F(t_k w + \hat{v}_k)\, dx.$$

Supposing that $t_k \to t_0$ and $\hat{v}_k \rightharpoonup \hat{v}$ in $H^1(\mathbb{R}^N)$, we get

$$\limsup_{k \to +\infty} I_k(u_k^i) \leq \frac{t_0^2}{2}\|w^+\|^2 - \frac{1}{2}\|\hat{v}\|^2 - \int F(t_0 w + \hat{v})\, dx \leq \Psi_1(w) = d_1$$

for $i \in \{1, ..., l\}$ and Claim 2 is proved.

Once $Q_k(\tilde{w}_k^i) \to z_i$ as $k \to +\infty$, it means that $\tilde{u}_k^i \in \Omega_k^i$ for k sufficiently large. On the other hand, from Claim 2,

$$\limsup_{k \to +\infty} \Psi_k(\tilde{u}_k^i) < d_1 + \frac{\alpha_0}{4}.$$

Hence, there is $k_* \in \mathbb{N}$ such that

$$\alpha_k^i < d_1 + \frac{\alpha_0}{4}, \quad \forall k \geq k_*. \tag{3.25}$$

Then, decreasing α_0 if necessary,

$$\alpha_k^i < d_1 + \gamma, \quad \forall k \geq k_*,$$

which is the first inequality. To obtain the second one, note that if $u \in \partial\Omega_k^i$, then

$$u \in S^+ \quad \text{and} \quad |Q_k(u) - z_i| = \rho_0 > \frac{\rho_0}{2},$$

that is, $Q_k(u) \notin K_{\frac{\rho_0}{2}}$. Thus, from Lemma 3.7

$$\Psi_k(u) > d_1 + \alpha_0 \; for \; all \; u \in \partial\Omega_k^i \; and \; k \geq k_1,$$

and so

$$\tilde{\alpha}_k^i = \inf_{u \in \partial\Omega_k^i} \Psi_k(u) \geq d_1 + \alpha_0, \quad \forall k \geq k_1. \tag{3.26}$$

Consequently, from (3.25)–(3.26),

$$\alpha_k^i < \tilde{\alpha}_k^i, \quad \forall k \geq k_1,$$

and the results are derived by fixing $k_2 = \max\{k_1, k_*\}$. □

Proof of Theorem 1.1. From Lemma 3.4, there exists $k_2 \in \mathbb{N}$ such that

$$\alpha_k^i < \tilde{\alpha}_k^i \;\; for \;\; \forall k \geq k_2.$$

Arguing as in [5, Proof of Therem 2.1], the above inequality permits to use Ekeland's variational principle to get a $(PS)_{\alpha_k^i}$ sequence $\{u_n^i\} \subset \Omega_k^i$ for Ψ_k. Noting that $\alpha_k^i < d_1 + \rho$, from Lemma 3.2 there exists u^i such that $u_n^i \to u^i$ in $H^1(\mathbb{R}^N)$. So

$$u^i \in \Omega_k^i, \quad \Psi_k(u^i) = \alpha_k^i \;\; and \;\; \Psi_k'(u^i) = 0.$$

Since

$$Q_k(u^i) \in \overline{B_{\rho_0}(z_i)}, \; Q_k(u^j) \in \overline{B_{\rho_0}(z_j)},$$
$$\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} = \emptyset \quad \text{for} \quad i \neq j.$$

We deduce that $u^i \neq u^j$ for $i \neq j$ for $1 \leq i, j \leq l$. Hence Ψ_k possess at least l nontrivial critical points for all $k \geq k_*$ on S^+, with $k_* \geq k_2$. From Corollary 3.5, I_k possess at least l nontrivial critical points for all $k \geq k_2$ in $H^1(\mathbb{R}^N)$, finishing the proof. □

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