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A generating function of a complex Lagrangian cone in $\mathbf{H}^n$

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We formulate the space of multivalued branched minimal immersions of compact Riemann surfaces of genus $\gamma \geq 2$ into $\mathbf{R}^n$, and show that it is a complex analytic set. If an irreducible component of the complex analytic set admits a non-degenerate critical point, then we construct a complex Lagrangian cone in $\mathbf{H}^{n\gamma}$ derived from the complex period map, and obtain its applications as follows: The irreducible component can be divided among some open connected components of non-degenerate critical points, and each connected component admits a special pseudo Kähler structure with the signature (p, q). We induce a sharp inequality between q and the Morse index of a minimal surface which are two invariants of the connected component. Furtheremore, we obtain an algorithm to compute the Morse index and the signature.

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1. Introduction

Let M be a compact Riemann surface of genus γ. Throughout this paper, an n-tuple ${}^t(\psi_1, ..., \psi_n)$ of holomorphic 1-forms on M is called a Weierstrass data if $\sum_{k=1}^n \psi_k^2 = 0$. If $\gamma = 0$, then there exists no Weierstrass data. For a Weierstrass data ${}^t(\psi_1, ..., \psi_n)$,

$$S(p) = \mathrm{Re} \int_{p_0}^{p} {}^t(\psi_1, ..., \psi_n)$$

is a multivalued branched minimal immersion of M into $\mathbf{R}^n$. If $\gamma = 1$, then such a map is totally geodesic. Hence, we may assume $\gamma \geq 2$. For a canonical homology basis $\{A_1, ..., A_\gamma, B_1, ..., B_\gamma\}$ of M, the real period matrix L of S is defined by $\left(\int_{A_1} dS, ..., \int_{A_\gamma} dS, \int_{B_1} dS, ..., \int_{B_\gamma} dS\right)$, that is,

$$\mathrm{Re}\Big(\int_{A_1} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, ..., \int_{A_\gamma} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, \int_{B_1} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}, ..., \int_{B_\gamma} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}\Big).$$

A multivalued branched minimal immersion of M into $\mathbf{R}^n$ is called to be full if the image is not in any affine subspace in $\mathbf{R}^n$, which is equivalent to $\mathrm{rank}\, L = n$. If the column vectors of L span a lattice $\langle L \rangle$ of $\mathbf{R}^n$, then we get the full branched minimal immersion of M into the n-dimensional flat torus $\mathbf{R}^n / \langle L \rangle$.

We can consider the Jacobi operator of a multivalued branched minimal immersion of M into $\mathbf{R}^n$ (for example, see [11] and [23]). Let $index_a$ be the number of the negative eigenvalues and $nullity_a$ the number of the zero-eigenvalue of the Jacobi operator (counted with multiplicity). $index_a$ is called the Morse index of a minimal surface and an eigenvector for the zero-eigenvalue is said to be a Jacobi field. If the map is not totally geodesic, then there exist n independent Jacobi fields caused by parallel tanslations in $\mathbf{R}^n$ which are called trivial Jacobi fields. Thus $nullity_a \geq n$ holds and $nullity_a = n$ if and only if the minimal surface has only trivial Jacobi fields.

There exists no compact orientable immersed minimal surface of genus 2 in a 3-dimensional flat torus. However, there exist many compact orientable immersed minimal surfaces of genus $\gamma \geq 3$. For example, Schwarz' P-surface, Schwarz' D-surface, Schoen's Gyroid and Schwarz' CLP-surface are widely known as embedded minimal surfaces of genus 3. It is important to indicate some results on $index_a$ of minimal surfaces in 3-dimensional flat tori.

Ross [31] proved that Schwarz' P-surface is CMC-stable and hence Schwarz' D-surface and Schoen's Gyroid are also CMC-stable, which leads Schwarz' P-surface, Schwarz' D-surface and Schoen's Gyroid to $index_a = 1$ and $nullity_a = 3$. Montiel and Ros [23] proved that Schwarz' CLP-surface has $index_a = 3$ and $nullity_a = 3$.

Ritoré and Ros [28] proved the compactness of the space of index one embedded minimal surfaces in flat tori. Ritoré [27] proved that index one immersed minimal surfaces of genus γ in flat tori satisfy $\gamma \leq 4$. Ros [29], [30] obtained $\gamma \leq 3$, in general, $index_a \geq \frac{2\gamma-3}{3}$.

Große-Brauckmann and Wohlgemuth [17] constructed a deformation of CMC-stable non-minimal surfaces from Schoen's Gyroid in the same ambient flat torus. Morgan and Ros [24] proved that there are nearby L^1-local minimizers of the Cahn-Hilliard energy for Schwarz' P-surface, Schwarz' D-surface and Schoen's Gyroid which are CMC-stable.

Inevitably it is of great significance to classify index one immersed minimal surfaces, index one embedded minimal surfaces and CMC-stable embedded minimal surfaces of genus 3 in 3-dimensional flat tori. However, little is known about an embedded minimal surface of genus 3 with $index_a = 1$ in a 3-dimensional flat torus for the last few decades since Ross' result [31].

We review a geometric meaning of $nullity_a$. Let $\mathcal{M}_\gamma$ be the space of full multivalued branched minimal immersions of non-hyperelliptic Riemann surfaces of genus γ into $\mathbf{R}^n$ and $\mathcal{N}_\gamma$ the space of full multivalued branched minimal immersions of hyperelliptic Riemann surfaces of genus γ into $\mathbf{R}^n$, which are the spaces of equivalence classes of triples of a Riemann surface, a Weierstrass data and a canonical homology basis [26], [4]. By real period matrices, we define the real period maps π from $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ to the space $L_{n,2\gamma}$ of $n \times 2\gamma$ real matrices. Here, we refer to Meeks' conjecture 50 in [20].

Conjecture 50 (Meeks). *The differential of the natural map from the moduli space $\mathcal{M} = \{M, (\omega_1, \omega_2, \omega_3) \,|\, M$ is a compact hyperelliptic Riemann surface of genus 3 with three independent holomorphic 1-forms satisfying $\sum_{i=1}^{3} \omega_i^2 = 0\}$ to the space of real periods $\subset \mathbf{R}^{18}$ of the forms ω_i evaluated on a basis of $H_1(M, \mathbf{Z})$ has rank 18 almost everywhere.*

We note that $\mathcal{M} = \mathcal{N}_3$ for $n = 3$ and the natural map is the real period map.

Pirola [26], Arezzo and Pirola [4] studied $\mathcal{M}_\gamma$, $\mathcal{N}_\gamma$ and π for full multivalued minimal immersions and proved an important formula

$$nullity_a = \mathrm{dimKer}\pi_* + n$$

for a “smooth point”. In particular, if $nullity_a = n$, then $\dim_R \mathcal{M}_\gamma = 2n\gamma$, $\dim_R \mathcal{N}_\gamma = 2n\gamma$ and $\mathrm{Ker}\pi_* = \{0\}$. Thus, a connected component of $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ admitting $nullity_a = n$ is locally a graph on an open set in $L_{n,2\gamma}$. In particular, since $nullity_a = 3$ for Schwarz’ P-surface and Schwarz’ CLP-surface ([31], [23]), Meeks’ conjecture 50 is true.

In this paper, we focus attention on such a connected component. We define the complex period maps from $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ to the space $K_{n,2\gamma}$ of complex $n \times 2\gamma$ matrices by $(\int_{A_1} dS^{1,0} \cdots \int_{A_\gamma} dS^{1,0}, \int_{B_1} dS^{1,0} \cdots \int_{B_\gamma} dS^{1,0})$, where $dS^{1,0}$ is the $(1,0)$-component of dS, that is,

$$\frac{1}{2}\Big(\int_{A_1} \begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}, \ldots, \int_{A_\gamma} \begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}, \int_{B_1} \begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}, \ldots, \int_{B_\gamma} \begin{pmatrix}\psi_1\\ \vdots\\ \psi_n\end{pmatrix}\Big).$$

Each complex period map induces a special pseudo Kähler structure on a connected component (see Theorem 6.2), which gives the relation between the special pseudo Kähler structure and $index_a$ as follows: Let (p,q) be the signature of the pseudo Kähler metric associated with the special pseudo Kähler structure on a connected component, where $p + q = n\gamma$. We prove

$$index_a \leq 6\gamma - 6 - 2q$$

for the connected component of $\mathcal{M}_\gamma$. For a connected component of $\mathcal{N}_\gamma$, we get

$$\alpha \leq index_a \leq 4\gamma - 2 - 2q + \alpha \ (0 \leq \alpha \leq \gamma - 2) \leq 5\gamma - 4 - 2q.$$

$\alpha = \gamma - 2$ holds except a complex analytic set in the connected component. Micallef [22] proved that a hyperelliptic stable minimal surface is a holomorphic curve, that is, a non-holomorphic hyperelliptic minimal surface has $index_a \geq 1$.

In the case where $\gamma = 3$, $n = 3$, we obtain $1 \leq index_a \leq 11 - 2q$. In particular, $index_a = 1$ if $q = 5$. Hence, some examples mentioned below imply that the inequality is sharp.

Shoda and the author [12] apply the algorithm (see Theorem 5.5 and its application in Subsection 6.2) to compute the signature and $index_a$ of some one-parameter families of embedded minimal surfaces of genus 3 in 3-dimensional flat tori as follows:

For tP family, which includes Schwarz' P-surface [34], [36], $index_a = 1, (p,q) = (4,5)$, $index_a = 2, (p,q) = (5,4)$ for some parameter values. Since tD family, which includes Schwarz' D-surface, consists of conjugate minimal surfaces of minimal surfaces in tP family, we obtain the same result.

For H family [34], [36], $index_a = 1, (p,q) = (4,5)$, $index_a = 2, (p,q) = (5,4)$ and $index_a = 3, (p,q) = (6,3)$ for some parameter values.

For tCLP family, which includes Schwarz' CLP-surface [34], [36], $index_a = 3$, $(p,q) = (6,3)$ for all parameter values.

For rPD family (Karcher's TT-surface) [19], [34], [36], $index_a = 1, (p,q) = (4,5)$ and $index_a = 2, (p,q) = (5,4)$ for some parameter values. Schwarz' P-surface and Schwarz' D-surface are contained in rPD family.

The set of parameter values corresponding to $index_a = 1$ minimal surfaces in tP, tD, H and rPD family are bounded closed intervals which are contained in the compact set in the Ritoré and Ros compactness theorem [28].

Finally, we prove that the space of full multivalued holomorphic maps of non-hyperelliptic Riemann surfaces of genus $\gamma \geq 3$ into $\mathbf{R}^6$ with suitable orthogonal complex structures has a special pseudo Kähler structure of the signature $(3\gamma+3, 3\gamma-3)$ and the space of full multivalued holomorphic maps of hyperelliptic Riemann surfaces of genus $\gamma \geq 2$ into $\mathbf{R}^4$ with suitable orthogonal complex structures has a special pseudo Kähler structure of the signature $(2\gamma+1, 2\gamma-1)$. Furthermore, our inequality is sharp for non-holomorphic stable minimal surfaces in flat tori, which are constructed by Arezzo and Micallef [3].

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2. A quaternion structure of $L_{n,2\gamma} \times L_{n,2\gamma}$

We investigate a quaternion structure of $L_{n,2\gamma} \times L_{n,2\gamma}$, a complex symplectic form ω_1 on $L_{n,2\gamma} \times L_{n,2\gamma}$ and a complex Lagrangian subspace in $L_{n,2\gamma} \times L_{n,2\gamma}$.

2.1. A quaternion structure of $L_{n,2\gamma} \times L_{n,2\gamma}$

We consider $L_{n,2\gamma}$ as the linear space of $n \times 2\gamma$ real matrices. We denote by (L_1, L_2) and (L_1, L_2, L_3, L_4), where $L_1, L_2, L_3, L_4 \in L_{n,\gamma}$, an element of $L_{n,2\gamma}$ and an element of $L_{n,2\gamma} \times L_{n,2\gamma}$, respectively. Then the canonical inner product $\langle\ ,\ \rangle$ on $L_{n,2\gamma} \times L_{n,2\gamma}$ is defined by

$$\langle (L_1, L_2, L_3, L_4), (L'_1, L'_2, L'_3, L'_4) \rangle = \mathrm{tr} \sum_{i=1}^{4} {}^t L_i L'_i.$$

We define two complex structures I, J on $L_{n,2\gamma} \times L_{n,2\gamma}$ as

$$\begin{aligned} I(L_1, L_2, L_3, L_4) = (L_4, -L_3, L_2, -L_1),\ \ J(L_1, L_2, L_3, L_4) \\ = (-L_3, -L_4, L_1, L_2). \end{aligned}$$

Lemma 2.1. *I and J preserve $\langle\ ,\ \rangle$.*

We define the symplectic form ω compatible with J by

$$\omega((L_1, L_2, L_3, L_4), (L'_1, L'_2, L'_3, L'_4)) = \langle J(L_1, L_2, L_3, L_4), (L'_1, L'_2, L'_3, L'_4) \rangle.$$

Lemma 2.2. *$IJ = -JI$ holds and hence $1, I, J$ and $K(= IJ)$ give a quaternion structure on $L_{n,2\gamma} \times L_{n,2\gamma}$.*

We consider $K_{n,\gamma}$ as the complex linear space of $n \times \gamma$ complex matrices. The real linear isomorphism $\varphi : K_{n,\gamma} \times K_{n,\gamma} \to L_{n,2\gamma} \times L_{n,2\gamma}$ is defined by $\varphi(K_1, K_2) = \mathrm{Re}(-iK_2, iK_1, K_1, K_2)$ for $(K_1, K_2) \in K_{n,\gamma} \times K_{n,\gamma}$. $\varphi^{-1}(L_1, L_2, L_3, L_4) = (L_3 - iL_2, L_4 + iL_1)$ holds.

Lemma 2.3. *The complex structures I_1, J_1 on $K_{n,\gamma} \times K_{n,\gamma}$ induced by φ and I, J are given by $I_1(K_1, K_2) = (iK_1, iK_2)$ and $J_1(K_1, K_2) = (i\overline{K_2}, -i\overline{K_1})$.*

φ allows us to identify $L_{n,2\gamma} \times L_{n,2\gamma}$ with $K_{n,\gamma} \times K_{n,\gamma}$ as two complex linear spaces for I, I_1. On the other hand, $K_{n,\gamma} \times K_{n,\gamma}$ admits the canonical Hermitian form

$$\langle (K_1, K_2), (K'_1, K'_2) \rangle_1 = \mathrm{tr}({}^t K_1 \overline{K'_1} + {}^t K_2 \overline{K'_2})$$

and the canonical complex symplectic form

$$\omega_1((K_1, K_2), (K'_1, K'_2)) = \mathrm{tr}({}^t K_2 K'_1 - {}^t K_1 K'_2)$$

for $(K_1, K_2), (K'_1, K'_2) \in K_{n,\gamma} \times K_{n,\gamma}$. The inner product and the symplectic form induced by φ satisfy

Lemma 2.4.

$$\langle \varphi(K_1, K_2), \varphi(K_1', K_2') \rangle = \mathrm{Re}\langle (K_1, K_2), (K_1', K_2') \rangle_1 \quad \textit{and}$$
$$\omega(\varphi(K_1, K_2), \varphi(K_1', K_2')) = \mathrm{Im}\, \omega_1((K_1, K_2), (K_1', K_2')).$$

Proof. Let

$$W_1 = \mathrm{Re}(-iK_2, iK_1, K_1, K_2) \quad \text{and}$$
$$W_2 = \mathrm{Re}(-iK_2', iK_1', K_1', K_2') \in L_{n,2\gamma} \times L_{n,2\gamma}.$$

Then $\omega(W_1, W_2) = \langle JW_1, W_2 \rangle = \mathrm{trIm}({}^tK_2K_1' - {}^tK_1K_2')$. □

Furthermore, we obtain

Lemma 2.5.

$$\langle J_1(K_1, K_2), J_1(K_1', K_2') \rangle_1 = \overline{\langle (K_1, K_2), (K_1', K_2') \rangle_1} \quad \textit{and}$$
$$\langle J_1(K_1, K_2), (K_1', K_2') \rangle_1 = i\, \overline{\omega_1((K_1, K_2), (K_1', K_2'))}.$$

Conversely, the complex symplectic form $(\varphi^{-1})^*\omega_1$ induced on $L_{n,2\gamma} \times L_{n,2\gamma}$ is given by

$$(\varphi^{-1})^*\omega_1(W_1, W_2) = -\langle KW_1, W_2 \rangle + i\langle JW_1, W_2 \rangle$$

for $W_1, W_2 \in L_{n,2\gamma} \times L_{n,2\gamma}$.

Let $GL(n, \mathbf{C})$ be the general linear group consisting of $n \times n$ regular matrices. Then $g \in GL(n, \mathbf{C})$ acts on $K_{n,\gamma} \times K_{n,\gamma}$ by $g(K_1, K_2) = (gK_1, gK_2)$. Let E_n be the $n \times n$ identity matrix. Let $O(n, \mathbf{C}) = \{A \in GL(n, \mathbf{C}) \mid {}^tAA = E_n\}$ be the complex orthogonal group and $SO(n, \mathbf{C})$ the subgroup $\{A \in O(n, \mathbf{C}) \mid \det A = 1\}$.

Lemma 2.6. *$O(n, \mathbf{C})$ is a complex symplectic transformation group with respect to ω_1 of $K_{n,\gamma} \times K_{n,\gamma}$.*

Let $J_0 = \begin{pmatrix} 0 & -E_\gamma \\ E_\gamma & 0 \end{pmatrix}$. Let $Sp(\gamma, \mathbf{C})$ be the $\mathbf{C}$-symplectic group $\{B \in GL(2\gamma, \mathbf{C}) \mid BJ_0\,{}^tB = J_0\}$ and $Sp(\gamma, \mathbf{R})$ the $\mathbf{R}$-symplectic group consisting of real matrices in $Sp(\gamma, \mathbf{C})$.

Lemma 2.7. *$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_p(\gamma, \mathbf{C})$ if and only if $a\,{}^t d - b\,{}^t c = E_\gamma$, $a^t b$ and $c^t d$ are symmetric, which is equivalent to $B^{-1} = \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}$.*

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{C})$ acts on $K_{n,\gamma} \times K_{n,\gamma}$ as follows:

$$(K_1, K_2)B = (K_1 a + K_2 c, K_1 b + K_2 d).$$

Lemma 2.8. *$Sp(\gamma, \mathbf{C})$ is a complex symplectic transformation group with respect to ω_1.*

Proof. By Lemma 2.7, we get

$$\begin{aligned} &\omega_1((K_1 a + K_2 c, K_1 b + K_2 d), (K_1' a + K_2' c, K_1' b + K_2' d)) \\ &\quad = \mathrm{tr}\{{}^t(K_1 b + K_2 d)(K_1' a + K_2' c) - {}^t(K_1 a + K_2 c)(K_1' b + K_2' d)\} \\ &\quad = \omega_1((K_1, K_2), (K_1', K_2')). \end{aligned}$$

□

Consequently, Lemmas 2.6 and 2.8 imply

Proposition 2.1. *$O(n, \mathbf{C})$ and $Sp(\gamma, \mathbf{C})$ are complex linear, complex symplectic transformation groups of $K_{n,\gamma} \times K_{n,\gamma}$ with respect to ω_1.*

Let $(K_1, K_2) \in K_{n,\gamma} \times K_{n,\gamma}$ and (v_k', v_k'') denote the k-th row vector of (K_1, K_2). Then we make an isomorphism $\Theta : K_{n,\gamma} \times K_{n,\gamma} \to K_{1,n\gamma} \times K_{1,n\gamma}$ by $\Theta((K_1, K_2)) = (v_1', ..., v_n', v_1'', ..., v_n'')$. We may consider an action of $O(n, \mathbf{C})$ and $Sp(\gamma, \mathbf{C})$ on $K_{1,n\gamma} \times K_{1,n\gamma}$ such that the following holds.

Proposition 2.2. *Θ is an $O(n, \mathbf{C})$, $Sp(\gamma, \mathbf{C})$-equivariant, complex symplectic isomorphism from $K_{n,\gamma} \times K_{n,\gamma}$ to $K_{1,n\gamma} \times K_{1,n\gamma}$.*

Proposition 2.2 enable us to apply some results on $K_{1,\gamma'} \times K_{1,\gamma'}$ (or $K_{1,\gamma} \times K_{1,\gamma}$) to $K_{n,\gamma} \times K_{n,\gamma}$.

Let $T^* L_{n,2\gamma} \xrightarrow{\pi_1} L_{n,2\gamma}$ be the cotangent bundle over $L_{n,2\gamma}$. Let ℓ_{jk} be the (j,k) entry of $L \in L_{n,2\gamma}$. Then we obtain the canonical coordinate system $\{\ell_{jk}\}$ in $L_{n,2\gamma}$. Any point of $T^* L_{n,2\gamma}$ is given by $\sum_{j=1}^n \sum_{k=1}^{2\gamma} p_{jk} d\ell_{jk}$ and, hence, $\{p_{jk}, \ell_{jk}\}$ is the canonical coordinate system in $T^* L_{n,2\gamma}$.

$\sum_{j=1}^{n}\sum_{k=1}^{2\gamma} p_{jk}d\ell_{jk}$ on $T^*L_{n,2\gamma}$ is called the Liouville-form and the canonical symplectic form on $T^*L_{n,2\gamma}$ is given by $\sum_{j=1}^{n}\sum_{k=1}^{2\gamma} dp_{jk}\wedge d\ell_{jk}$. Since $\{p_{jk}\}$ may be the canonical coordinate system in $L_{n,2\gamma}$, we can identify $T^*L_{n,2\gamma}$ with $L_{n,2\gamma}\times L_{n,2\gamma}$ whose canonical coordinate system is $\{p_{jk},\ell_{jk}\}$. The latter $L_{n,2\gamma}$ is the base space of $T^*L_{n,2\gamma}$. The tangent space at any point of $T^*L_{n,2\gamma}$ is identified with $L_{n,2\gamma}\times L_{n,2\gamma}$. Hence (L_1,L_2,L_3,L_4), where $L_1,\dots,L_4\in L_{n,\gamma}$, may be a tangent vector. Then the canonical symplectic form $\sum_{j=1}^{n}\sum_{k=1}^{2\gamma} dp_{jk}\wedge d\ell_{jk}$ is ω as follows: Since $dp_{jk}(L_1,L_2,L_3,L_4)$ is the (j,k) entry of (L_1,L_2) and $d\ell_{jk}(L_1,L_2,L_3,L_4)$ is the (j,k) entry of (L_3,L_4),

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{2\gamma} & dp_{jk}\wedge d\ell_{jk}((L_1,L_2,L_3,L_4),(L_1',L_2',L_3',L_4'))\\ &= \mathrm{tr}\{(L_1,L_2)^t(L_3',L_4')-(L_1',L_2')^t(L_3,L_4)\}\\ &= \omega((L_1,L_2,L_3,L_4),(L_1',L_2',L_3',L_4')).\end{aligned}$$

Let Z_{1jk} be the (j,k) entry of $Z_1\in K_{n,\gamma}$. Then we obtain the canonical complex coordinate system $\{Z_{1jk}\}$ in $K_{n,\gamma}$. A point of the complex cotangent bundle $T^*K_{n,\gamma}$ over $K_{n,\gamma}$ is given by $\sum_{j=1}^{n}\sum_{k=1}^{\gamma} Z_{2jk}dZ_{1jk}$ and, hence, $\{Z_{1jk},Z_{2jk}\}$ is the canonical complex coordinate system in $T^*K_{n,\gamma}$. The canonical complex symplectic form on $T^*K_{n,\gamma}$ is given by $\sum_{j=1}^{n}\sum_{k=1}^{\gamma} dZ_{2jk}\wedge dZ_{1jk}$. Since $\{Z_{2jk}\}$ may be the canonical complex coordinate system in $K_{n,\gamma}$, we identify $T^*K_{n,\gamma}$ with $K_{n,\gamma}\times K_{n,\gamma}$. The tangent space at any point in $T^*K_{n,\gamma}$ is identified with $K_{n,\gamma}\times K_{n,\gamma}$. Hence (K_1,K_2), where $K_1,K_2\in K_{n,\gamma}$, may be a tangent vector at the point. Since $dZ_{1jk}(K_1,K_2)=K_{1jk}$ and $dZ_{2jk}(K_1,K_2)=K_{2jk}$, we get

$$\sum_{j=1}^{n}\sum_{k=1}^{\gamma} dZ_{2jk}\wedge dZ_{1jk}((K_1,K_2),(K_1',K_2'))=\omega_1((K_1,K_2),(K_1',K_2')).$$

Thus, the canonical complex symplectic form on the tangent space of $T^*K_{n,\gamma}$ is ω_1 defined on the complex linear space $K_{n,\gamma}\times K_{n,\gamma}$.

Let $dZ_1(K_1,K_2)=K_1$ and $dZ_2(K_1,K_2)=K_2$. Then dZ_1 and dZ_2 are $K_{n,\gamma}$-valued holomorphic 1-forms on $K_{n,\gamma}\times K_{n,\gamma}$ such that $\omega_1=\mathrm{tr}\,{}^t dZ_2\wedge dZ_1$.

2.2. The Grassmann manifold of complex Lagrangian subspaces

Let T be a complex linear subspace of complex dimension $n\gamma$ in $\mathbf{H}^{n\gamma}=\mathbf{C}^{n\gamma}\times\mathbf{C}^{n\gamma}=K_{n,\gamma}\times K_{n,\gamma}$ with the Hermitian form $\langle\ ,\ \rangle_1$. When ω_1 vanishes on T, T is called a complex Lagrangian subspace. From Lemmas 2.2 and 2.5, we obtain

Lemma 2.9. *Let S be a real $2n\gamma$-dimensional subspace S in $L_{n,2\gamma} \times L_{n,2\gamma}$. Then the following conditions are equivalent:*

(1) *S is a complex Lagrangian subspace,*

(2) *$\langle KX, Y\rangle = 0$ and $\langle JX, Y\rangle = 0$ for any X, $Y \in S$, and*

(3) *S is I-invariant and real Lagrangian, that is, ω induced on S vanishes.*

The condition (2) above is called the bilagrangian condition by Hitchin [18].

Let $U(2\gamma)$ be the unitary subgroup of $GL(2\gamma, \mathbf{C})$ and $Sp(\gamma)$ the subgroup $Sp(\gamma, \mathbf{C}) \cap U(2\gamma)$. Let Lag^C be the space of complex Lagrangian subspaces in $K_{1,\gamma} \times K_{1,\gamma}$. Then, from Lemma 2.5, we obtain

Lemma 2.10. *Let $T \in Lag^C$ and $\{u_1, ..., u_\gamma\}$ a unitary basis of T. Then $\{u_i, -iJ_1u_i\}$ is a unitary and symplectic basis, that is, $\omega_1(u_i, -iJ_1u_j) = \delta_{ij}$, of $K_{1,\gamma} \times K_{1,\gamma}$ and hence ${}^t({}^tu_1, ..., {}^tu_\gamma, {}^t(iJ_1u_1), ..., {}^t(iJ_1u_\gamma)) \in Sp(\gamma)$.*

Thus $Sp(\gamma)$ transitively acts on Lag^C and $U(\gamma)$ is the isotropy group of the complex Lagrangian subspace spanned by $(e_1, 0), ..., (e_\gamma, 0)$, where $\{e_1, ..., e_\gamma\}$ is the canonical basis of $\mathbf{C}^\gamma$.

Lemma 2.11. *Lag^C is a Hermitian symmetric space $U(\gamma)\backslash Sp(\gamma)$ of compact type and rank γ.*

Let $V \xrightarrow{p_1} Lag^C$ be the tautological vector bundle over Lag^C given by $\{(T, v) \in Lag^C \times \mathbf{H}^\gamma \mid T \in Lag^C,\ v \in T\}$. Let S^2_C be the space of $\gamma \times \gamma$ complex symmetric matrices and $RS^2_C = \{\tau \in S^2_C \mid \operatorname{Im}\tau \text{ is regular}\}$. The Siegel upper half space H_γ is defined by $\{\tau \in S^2_C \mid \operatorname{Im}\tau > 0\}$. Let $\eta : K_{1,\gamma} \times K_{1,\gamma} \to K_{1,\gamma} \times \{0\}$ be the projection.

Lemma 2.12. *Assume that the restriction of η to $T \in Lag^C$ is surjective. Then there exists $\tau \in S^2_C$ such that $T = \{(K, K\tau) \in K_{1,\gamma} \times K_{1,\gamma} \mid K \in K_{1,\gamma}\}$.*

Proof. A basis $\{\alpha_1, ..., \alpha_\gamma\}$ of T is given by $\alpha_i = (e_i, (a_{i1}, ..., a_{i\gamma}))$, $(a_{i1}, ..., a_{i\gamma}) \in \mathbf{C}^\gamma$ by the assumption. $T \in Lag^C$ if and only if $\operatorname{tr} {}^t dZ_2 \wedge dZ_1(\alpha_i, \alpha_j) = a_{ij} - a_{ji} = 0$ and hence $(a_{ij}) \in S^2_C$. $\square$

Let Lag_o^C be the set of the subspaces which satisfy the assumption as in Lemma 2.12. Then Lag_o^C is identified with S_C^2.

Lemma 2.13. *Lag_o^C is a Zariski open set of Lag^C.*

Proof. Let $T \in Lag^C$ and $f_1, ..., f_\gamma$ a local holomorphic frame fields on a neighborhood U of T. Then there exist holomorphic functions a_{kj} on U such that $\eta(f_j) = \sum_{k=1}^{\gamma} a_{kj} e_k$. $T' \in U \cap Lag_o^C$ if and only if $\det(a_{jk}) \neq 0$ at T'. □

Lemma 2.14. *$p_1^{-1}(Lag_o^C) = \{(\tau, (K, K\tau)) | \tau \in S_C^2,\ K \in K_{1,\gamma}\}$. Hence, $S_C^2 \times K_{1,\gamma}$ is identified with $p_1^{-1}(Lag_o^C)$.*

We consider the bundle $V^n = V \bigoplus \cdots \bigoplus V \xrightarrow{p_1} Lag^C$. We define a holomorphic map Φ of V^n into $\mathbf{H}^{n\gamma}$ as $\Phi((T, v_1), ..., (T, v_n)) = {}^t({}^tv_1, ..., {}^tv_n)$. $\Phi(V^n)$ is a set of complex Lagrangian subspaces $T \bigoplus \cdots \bigoplus T \subset \mathbf{H}^\gamma \bigoplus \cdots \bigoplus \mathbf{H}^\gamma = \mathbf{H}^{n\gamma}$. $S_C^2 \times K_{n,\gamma}$ may be considered as $p_1^{-1}(Lag_o^C)$.

Lemma 2.15. *Φ on $S_C^2 \times K_{n,\gamma}$ satisfies $\Phi(\tau, Z) = (Z, Z\tau)$, where $(\tau, Z) \in S_C^2 \times K_{n,\gamma}$.*

$d\tau$ is an S_C^2-valued holomorphic 1-form and dZ is a $K_{n,\gamma}$-valued holomorphic 1-form on $S_C^2 \times K_{n,\gamma}$.

Lemma 2.16. *$\Phi^*\omega_1 = \mathrm{tr}(d\tau \wedge {}^tZdZ)$.*

Proof. Since τ is a symmetric matrix, we get

$$\begin{aligned}\Phi^*\omega_1 &= \mathrm{tr}({}^td(Z\tau) \wedge dZ) = \mathrm{tr}({}^t(dZ\tau + Zd\tau) \wedge dZ) \\ &= \mathrm{tr}(\tau {}^tdZ \wedge dZ + d\tau \wedge^t ZdZ) = \mathrm{tr}(d\tau \wedge {}^tZdZ).\end{aligned}$$

□

In general, Φ_* is not surjective.

Lemma 2.17. *If $n = 1$, then Φ_* is surjective for a generic point.*

Proof. We calculate Φ_* at (τ, e_1). Let $K' \in K_{1,\gamma}$ and $\tau' = (\tau'_{ij}) \in S_C^2$. Then we obtain $\Phi_*(0, K') = (K', K'\tau)$ and $\Phi_*(\tau', 0) = (0, (\tau'_{11}, ..., \tau'_{1\gamma}))$. Therefore, Φ_* is surjective at (τ, e_1). □

$GL(n, \mathbf{C})$ acts on V^n by $g(\tau, K) = (\tau, gK)$, where $g \in GL(n, \mathbf{C})$.

Lemma 2.18. *Φ is $GL(n,\mathbf{C})$-equivariant.*

Let $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma,\mathbf{C})$. Then we have

Lemma 2.19. *If $a+\tau c$ is regular, then $B(\tau)=(a+\tau c)^{-1}(b+\tau d)\in S^2_C$.*

Thus $Sp(\gamma,\mathbf{C})$ is a fractional linear transformation group of S^2_C, moreover, we can define the action of B on $S^2_C \times K_{n,\gamma}$ and hence V^n as

$$B(\tau,K)=((a+\tau c)^{-1}(b+\tau d),K(a+\tau c)).$$

Lemma 2.20. *$\Phi(B(\tau,K))=\Phi(\tau,K)B$.*

Remark 2.1. If $n=1$, then Lemmas 2.17, 2.18 and 2.20 imply that Φ_* is surjective at each point except the zero-section of V, that is, each point of $\mathbf{C}^{2\gamma}$ except 0 is a regular value of Φ.

Lemma 2.21. *$A\in O(n,\mathbf{C})$ and $B\in Sp(\gamma,\mathbf{C})$ preserve $\Phi^*\omega_1$.*

For $\tau\in RS^2_C$, we define the $2\gamma\times 2\gamma$ symmetric matrix $P(\tau)\in Sp(\gamma,\mathbf{R})$ by

$$P(\tau)=\begin{pmatrix} (\mathrm{Im}\tau)+(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}(\mathrm{Re}\tau) & -(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1} \\ -(\mathrm{Im}\tau)^{-1}(\mathrm{Re}\tau) & (\mathrm{Im}\tau)^{-1} \end{pmatrix}.$$

Lemmas 2.9 and 2.12 give two points of view on complex Lagrangian subspaces.

Theorem 2.1. *Let P be a $2\gamma\times 2\gamma$ real matrix. Then the subspace T defined as $\{(LP,L)\in L_{n,2\gamma}\times L_{n,2\gamma}\,|\,L\in L_{n,2\gamma}\}$ is a Lagrangian subspace in $L_{n,2\gamma}\times L_{n,2\gamma}$ if and only if P is symmetric. We set $P=\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$, A and C are $\gamma\times\gamma$ real symmetric matrices and B is a $\gamma\times\gamma$ real matrix. Then T is complex Lagrangian if and only if $P\in Sp(\gamma,\mathbf{R})$. Let J_P be the almost complex structure on T with respect to I. Then*

$$J_P((L_1,L_2)P,(L_1,L_2))=((L_1,L_2)J_0,(L_1,L_2)PJ_0).$$

If C is regular, then there exists $\tau\in RS^2_C$ such that $P=P(\tau)$ and the complex Lagrangian subspace corresponds to τ. In particular, if $C>0$, *then* $\tau \in H_\gamma$. *In this situation, the real linear map of* $\{(LP(\tau), L) \in L_{n,2\gamma} \times L_{n,2\gamma} \,|\, L \in L_{n,2\gamma}\}$ *onto* $\{(K, K\tau) \in K_{n,\gamma} \times K_{n,\gamma} \,|\, K \in K_{n,\gamma}\}$ *is given by*

$$((L_1, L_2)P(\tau), (L_1, L_2)) \mapsto (K, K\tau),$$

where $K = L_1 + i[L_1 \mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1}$.

Proof. The first statement is similar to the argument in Lemma 2.12. Let (Z_1, Z_2) be the element corresponding to $((L_1, L_2)P, (L_1, L_2))$ by φ^{-1}. We have $Z_1 = L_1 - i(L_1 B + L_2 C)$, $Z_2 = L_2 + i(L_1 A + L_2 {}^t B)$. Since $\varphi^{-1}(T)$ is a real subspace where the symplectic form associated with J_1 vanishes, $\varphi^{-1}(T)$ is complex Lagrangian if and only if it is a complex subspace in $K_{n,\gamma} \times K_{n,\gamma}$, which is equivalent to $(iZ_1, iZ_2) \in \varphi^{-1}(T)$. Therefore there exists (L_1', L_2') such that $(iZ_1, iZ_2) = \varphi^{-1}((L_1', L_2')P, (L_1', L_2'))$. We get $L_1' = L_1 B + L_2 C$, $L_1 = -L_1' B - L_2' C$, $L_2' = -L_1 A - L_2 {}^t B$ and $L_2 = L_1' A + L_2' {}^t B$. Immediately, $L_1(E_\gamma + B^2 - AC) + L_2(CB - {}^t BC) = 0$, $L_1(BA - A {}^t B) + L_2(CA - {}^t B^2 - E_\gamma) = 0$ hold for all $(L_1, L_2) \in L_{n,2\gamma}$. Hence we get $E_\gamma + B^2 - AC = 0$, $CB - {}^t BC = 0$ and $BA - A {}^t B = 0$, which are equivalent to $P \in Sp(\gamma, \mathbf{R})$. Since $L_1' = L_1 B + L_2 C$ and $L_2' = -(L_1 A + L_2 {}^t B)$, we get

$$\begin{aligned}(L_1', L_2') = (L_1, L_2)\begin{pmatrix} B & -A \\ C & -{}^t B\end{pmatrix} &= (L_1, L_2)\begin{pmatrix} A & B \\ {}^t B & C\end{pmatrix}\begin{pmatrix} 0 & -E_\gamma \\ E_\gamma & 0\end{pmatrix} \\ &= (L_1, L_2) P J_0.\end{aligned}$$

Consequently, $J_P((L_1, L_2)P, (L_1, L_2)) = ((L_1, L_2)J_0, (L_1, L_2)PJ_0)$ holds.

If C is regular, then we set $\mathrm{Im}\tau = C^{-1}$. Since CB is symmetric, we set $\mathrm{Re}\tau = -\mathrm{Im}\tau \, CB \, \mathrm{Im}\tau$. Then $\tau = -BC^{-1} + iC^{-1}$ and $P = P(\tau)$. Furthermore, $Z_1 = L_1 + i[L_1 \mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1}$ and $Z_2 = Z_1 \tau$. □

If $\gamma = 1$, then A, B, C are real numbers satisfying $1 + B^2 = AC$ and hence $C \neq 0$. Thus, the complex Lagrangian subspace $\in RS^2_C$. However, we note

Remark 2.2. $\left\{\left((L_1, L_2)\begin{pmatrix} 0 & J_0 \\ -J_0 & 0\end{pmatrix}, (L_1, L_2)\right) \,\middle|\, (L_1, L_2) \in L_{n,4\gamma}\right\}$ and

$$\left\{\left((L_1, L_2)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 \\ 0 & 0 & 1 & 0 \\ 0 & -J_0 & 0 & 0\end{pmatrix}, (L_1, L_2)\right) \,\middle|\, (L_1, L_2) \in L_{n,4\gamma+2}\right\}$$

are complex Lagrangian subspaces $\notin RS^2_C$ by Theorem 2.1.

Corollary 2.1. *The space of complex Lagrangian subspaces consisting of* $\{((L_1, L_2)P, (L_1, L_2)) \mid (L_1, L_2) \in L_{n,2\gamma}\}$ *in* $L_{n,2\gamma} \times L_{n,2\gamma}$ *corresponds to the space of symmetric matrices in* $Sp(\gamma, \mathbf{R})$.

Corollary 2.2. *The space of complex Lagrangian subspaces consisting of* $\{((L_1, L_2)P, (L_1, L_2)) \mid (L_1, L_2) \in L_{n,2\gamma},\ P > 0\}$ *in* $L_{n,2\gamma} \times L_{n,2\gamma}$ *corresponds to* $H_\gamma = Sp(\gamma, \mathbf{R})/U(\gamma)$.

Proof. If $\tau \in H_\gamma$, then $P(\tau) > 0$. Conversely, if $P > 0$, then $C > 0$. Thus, it follows from Theorem 2.1 that there exists $\tau \in H_\gamma$ such that $P = P(\tau)$. $\square$

3. Energy function

The energy function E on $RS_C^2 \times L_{n,2\gamma}$ is defined by $E(\tau, L) = \frac{1}{2}\mathrm{tr}(P(\tau)\,{}^tLL)$, where $\tau \in RS_C^2$ and $L \in L_{n,2\gamma}$ in [8]. Let M be a complex submanifold in RS_C^2 and E_M the restriction of E to $M \times L_{n,2\gamma}$. We study critical points of the function E_M on M for each fixed $L \in L_{n,2\gamma}$.

3.1. A diffeomorphism of $RS_C^2 \times L_{n,2\gamma}$ onto $RS_C^2 \times K_{n,\gamma}$

The diffeomorphism $\Psi : RS_C^2 \times L_{n,2\gamma} \to RS_C^2 \times K_{n,\gamma}$ is defined by

$$\Psi(\tau, (L_1, L_2)) = (\tau, \frac{1}{2}(L_1 + i[L_1(\mathrm{Re}\tau) - L_2](\mathrm{Im}\tau)^{-1})),$$

where $L = (L_1, L_2) \in L_{n,2\gamma}$.

Remark 3.1. By Theorem 2.1, we may define Ψ by $(\tau, (L_1, L_2)) \mapsto (\tau, (L_1 + i[L_1(\mathrm{Re}\tau) - L_2](\mathrm{Im}\tau)^{-1}))$. However, we adjust Ψ to the energy function (see Lemma 3.4). Furthermore, for a multivalued branched minimal immersion S of M into $\mathbf{R}^n$, dS corresponds to $dS^{1,0}$ by Ψ (see Subsection 6.1).

Lemma 3.1. $\Psi^{-1}(\tau, K) = (\tau, (\mathrm{Re}(2K), \mathrm{Re}(2K\tau)))$.

Remark 3.2. $GL(n, \mathbf{C})$ acts on $RS_C^2 \times L_{n,2\gamma}$ by, for $g \in GL(n, \mathbf{C})$,

$$(\tau, (\mathrm{Re}(2K), \mathrm{Re}(2K\tau))) \mapsto (\tau, (\mathrm{Re}(2gK), \mathrm{Re}(2gK\tau))).$$

Then Ψ is $GL(n, \mathbf{C})$-equivariant.

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{R})$ acts on $L_{n,2\gamma}$ as follows:

$$(L_1, L_2)g = (L_1 a + L_2 c, L_1 b + L_2 d).$$

Lemma 3.2. *Ψ is $Sp(\gamma, \mathbf{R})$-equivariant.*

Proof. Let $\Psi(\tau, (L_1, L_2)) = (\tau, K)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{R})$. Set

$$\begin{aligned}&A = \mathrm{Re}((a+\tau c)^{-1}(b+\tau d)), \quad B = \mathrm{Im}((a+\tau c)^{-1}(b+\tau d))\\ &\text{and} \quad (\tau g, \widetilde{K}) = \Psi(\tau g, (L_1, L_2)g).\end{aligned}$$

We shall prove $\widetilde{K} = K(a + \tau c)$.

We first calculate $\widetilde{K}$ as follows:

$$\begin{aligned}\widetilde{K} &= \frac{1}{2}(L_1 a + L_2 c + i[(L_1 a + L_2 c)A - (L_1 b + L_2 d)]B^{-1})\\ &= \frac{1}{2}(L_1(a + iaAB^{-1} - ibB^{-1}) + L_2(c + icAB^{-1} - idB^{-1})).\end{aligned}$$

Since $A + iB = (a+\tau c)^{-1}(b+\tau d)$, we have $(a+\tau c)(A+iB) = b + \tau d$. Hence, we obtain $aAB^{-1} - bB^{-1} + \tau(cAB^{-1} - dB^{-1}) = -ia - i\tau c$, that is, $a + \tau c = iaAB^{-1} - ibB^{-1} + \tau(icAB^{-1} - idB^{-1})$, and therefore, we get

$$\begin{aligned}\mathrm{Re}(a+\tau c) &= -(\mathrm{Im}\tau)(cAB^{-1} - dB^{-1}) \text{ and}\\ \mathrm{Im}(a+\tau c) &= aAB^{-1} - bB^{-1} + (\mathrm{Re}\tau)(cAB^{-1} - dB^{-1}).\end{aligned}$$

As $cAB^{-1} - dB^{-1} = -(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c)$, we obtain

$$aAB^{-1} - bB^{-1} = \mathrm{Im}(a+\tau c) + (\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c).$$

Consequently, we get

$$\begin{aligned}\widetilde{K} = \frac{1}{2}(&L_1(a + i\mathrm{Im}(a+\tau c) + i(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c))\\ &+ L_2(c - i(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c))).\end{aligned}$$

$K(a + \tau c)$ is given by

$$\frac{1}{2}(L_1((a+\tau c) + i(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}(a+\tau c)) + L_2(-i(\mathrm{Im}\tau)^{-1})(a+\tau c)).$$

Since

$$\begin{aligned}
&(a+\tau c)+i(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}(a+\tau c)\\
&\qquad = a+i\mathrm{Im}(a+\tau c)+i(\mathrm{Re}\tau)(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c)\\
&\text{and } \; -i(\mathrm{Im}\tau)^{-1}(a+\tau c)=c-i(\mathrm{Im}\tau)^{-1}\mathrm{Re}(a+\tau c),
\end{aligned}$$

we obtain $\tilde{K}=\frac{1}{2}(L_1+i[L_1\mathrm{Re}\tau-L_2](\mathrm{Im}\tau)^{-1})(a+\tau c)$. □

3.2. An energy function E on RS^2_C

Theorem 2.1 implies

Lemma 3.3.

$$(L_1,L_2)P(\tau)=2\mathrm{Re}(-iK\tau,iK),$$

where $K=\frac{1}{2}(L_1+i[L_1\mathrm{Re}\tau-L_2](\mathrm{Im}\tau)^{-1})$.

A Hermitian form η_2 defined in [7] is $-i\omega_1((K_1,K_2),\overline{(K_1',K_2')})$. It follows from Proposition 2.1 that η_2 is $Sp(\gamma,\mathbf{R})$-invariant.

Lemma 3.4. *The square norm of the vector* $\Phi(\Psi(\tau,L))$ *with respect to* η_2 *is the energy function, which is* $Sp(\gamma,\mathbf{R})$*-invariant.*

Proof. We know $-i\omega_1((K_1,K_2),\overline{(K_1,K_2)})=2\mathrm{Imtr}(K_1^*K_2)$ and hence $\Phi^*(2\mathrm{Imtr}(K_1^*K_2))=2\mathrm{Imtr}(K^*K\tau)$. Since $K=\frac{1}{2}(L_1+i[L_1(\mathrm{Re}\tau)-L_2](\mathrm{Im}\tau)^{-1})$, we get $(\Phi\circ\Psi)^*(2\mathrm{Imtr}(K_1^*K_2))=E(\tau,L)$. □

Lemma 3.5. $2\mathrm{Imtr}(K^*K\tau)$ *is* $U(n)$*-invariant.*

Let M be a k-dimensional complex submanifold in RS^2_C and τ the holomorphic immersion of M into RS^2_C. Then, we consider the function E_M on $M\times L_{n,2\gamma}$ as the restriction of E and define the set $C(E_M)$ of critical points as

$$\{(q,L)\in M\times L_{n,2\gamma}\,|\,\frac{\partial E_M}{\partial z^1}(q,L)=\cdots=\frac{\partial E_M}{\partial z^k}(q,L)=0\},$$

where $(z^1,...,z^k)$ is a local complex coordinate system in M. Since $M\times K_{n,\gamma}$ is a complex submanifold in $RS^2_C\times K_{n,\gamma}$, Ψ induces a complex structure on $M\times L_{n,2\gamma}$. Theorem 2.1 implies

Proposition 3.1. *$M \times L_{n,2\gamma}$ is a holomorphic trivial vector bundle over M with respect to the complex structure induced by Ψ. Let J_τ be the complex structure of the fibre at $q \in M, \tau = \tau(q)$. Then J_τ is given by $J_\tau(L_1, L_2) = (L_1, L_2)P(\tau)J_0$.*

In [8], we calculated the gradient vector field of E for a fixed $L \in L_{n,2\gamma}$ on RS^2_C with respect to the Hermitian form $\langle A, B\rangle = \mathrm{tr}A\overline{B}$, $A, B \in S^2_C$, which is

$$\mathrm{grad}E(\tau, L) = 2i \times \overline{{}^tKK}, \text{where}\, K = \frac{1}{2}(L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1}).$$

Lemma 3.6. *$dE_M = 2\mathrm{Imtr}(d\tau^tKK)$, where $K = \frac{1}{2}(L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})$.*

Proof. We get $dE(X) = \langle\tau_*(X), \mathrm{grad}E\rangle = \mathrm{trRe}(\tau_*(X)\overline{(2i \times \overline{{}^tKK})}) = 2\mathrm{Imtr}(\tau_*(X)\,{}^tKK)$, where $X \in T_qM$. $\square$

By the complex structure on $M \times L_{n,2\gamma}$ induced by Ψ, we obtain

Theorem 3.1. *$C(E_M)$ is a complex analytic set in $M \times L_{n,2\gamma}$.*

Proof. It follows from Lemma 3.6 that $(q, (L_1, L_2)) \in C(E_M)$ if and only if $\mathrm{tr}(\frac{\partial\tau}{\partial z^\ell}(q)\,{}^tKK) = 0$, $\ell = 1, ..., k$, where $(z^1, ..., z^k)$ is a local complex coordinate system in M. $\Psi(C(E_M))$ is a complex analytic set in $M \times K_{n,\gamma}$ since $\mathrm{tr}(\frac{\partial\tau}{\partial z^\ell}\,{}^tKK)$ is a holomorphic function on $M \times K_{n,\gamma}$. $\square$

We define a holomorphic 1-form Ξ on $S^2_C \times K_{n,\gamma}$ as $\mathrm{tr}(d\tau\,{}^tKK)$, which is $O(n, \mathbf{C})$-invariant. Then $dE = 2\mathrm{Im}\Xi$ on $RS^2_C \times K_{n,\gamma}$ holds. We can extend the 1-form to a holomorphic 1-form on V^n, which is also denoted by Ξ.

Lemma 3.7. *Ξ is an $Sp(\gamma, \mathbf{C})$-invariant holomorphic 1-form on V^n such that $d\Xi = -2\Phi^*\omega_1$.*

Proof. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{C})$. Then

$$\begin{aligned} B^*\Xi &= \mathrm{tr}\Big([-(a+\tau c)^{-1}d\tau c(a+\tau c)^{-1}(b+\tau d) \\ &\quad + (a+\tau c)^{-1}d\tau d]({}^ta + {}^tc\tau)\,{}^tKK(a+\tau c)\Big) \\ &= \mathrm{tr}\Big(d\tau[-c(a+\tau c)^{-1}(b+\tau d) + d]({}^ta + {}^tc\tau)\,{}^tKK\Big) = \Xi, \end{aligned}$$

because of $[-c(a+\tau c)^{-1}(b+\tau d) + d]({}^ta + {}^tc\tau) = E_\gamma$. Lemma 2.16 completes $d\Xi = -2\Phi^*\omega_1$. $\square$

We consider the induced 1-form Ξ_M of Ξ on $M \times K_{n,\gamma}$.

Lemma 3.8. *$\Psi(C(E_M))$ is the set of zero points of Ξ_M on $M \times K_{n,\gamma}$. In particular, $\Phi^*\omega_1 = 0$ on $\Psi(C(E_M))$. $\Psi(C(E_M))$ and $C(E_M)$ are $O(n, \mathbf{C}) \times \mathbf{C}^*$-invariant and E_M is constant on the $U(1)$-orbit in $\Psi(C(E_M))$ and $C(E_M)$. If M is invariant by a subgroup G of $Sp(\gamma, \mathbf{C})$, then so is $\Psi(C(E_M))$ and $C(E_M)$.*

Proof. The set of critical points is the set of $\Xi_M = 0$. By Lemma 3.7, we get $\Phi^*\omega_1 = 0$. The set of $\Xi_M = 0$ is $O(n, \mathbf{C}) \times \mathbf{C}^*$-invariant, where $\mathbf{C}^*$ acts on $M \times K_{n,\gamma}$ as a subgroup $\{\alpha E_n \,|\, \alpha \in \mathbf{C}^*\}$ of $GL(n, \mathbf{C})$. Although $U(n)$ preserves E_M by Lemma 3.5, $U(n)$ may not preserve $\Psi(C(E_M))$. However, the subgroup $U(1) = \mathbf{C}^* \cap U(n)$ preserves $\Psi(C(E_M))$ and E_M. Ξ_M is G-invariant by Lemma 3.7 and so is $\Xi_M = 0$. $\square$

We put $\widetilde{\mathbf{K}_n} = \{K \in K_{n,\gamma} \,|\, {}^tKK = 0\}$.

Proposition 3.2. $\Psi(C(E_{RS^2_C})) = \{(\tau, K) \,|\, \tau \in RS^2_C,\ K \in \widetilde{\mathbf{K}_n}\}$.

Proof. By the proof of Theorem 3.1, $\Psi^{-1}(\tau, K)$ is a critical point if and only if $\mathrm{tr}(\hat{\tau}\, {}^tKK) = 0$ for any $\hat{\tau} \in S^2_C$. Thus ${}^tKK = 0$. $\square$

4. A complex isotropic submanifold in $T^*L_{n,2\gamma}$

Let X be a real submanifold in $T^*L_{n,2\gamma}$. Then X is called a complex isotropic submanifold if $IT_pX = T_pX$ and $JT_pX \perp T_pX$ for all $p \in X$. Since $IJ = K$ implies $KT_pX \perp T_pX$, this condition is equivalent to that X is a complex submanifold where $(\varphi^{-1})^*\omega_1 = 0$ by Lemma 2.5. When $\dim_R X = 2n\gamma$, T_pX is a complex Lagrangian subspace by Lemma 2.9, thus X is called a complex Lagrangian submanifold.

Since $C(E_M)$ is an analytic set, we can consider an irreducible component N of $C(E_M)$. Since $SO(n, \mathbf{C}) \times \mathbf{C}^*$ is a connected subgroup of $O(n, \mathbf{C}) \times \mathbf{C}^*$ preserving $\Psi(C(E_M))$, $SO(n, \mathbf{C}) \times \mathbf{C}^*$ preserves $\Psi(N)$ and hence N. By Lemmas 2.18 and 3.8, we obtain

Theorem 4.1. *$\Phi \circ \Psi(N)$ is an $SO(n, \mathbf{C}) \times \mathbf{C}^*$-invariant complex isotropic cone possibly with singularities in $\Phi(V^n|_M)$.*

Corollary 4.1. *$\Phi(\{(\tau, K) \,|\, K \in \widetilde{\mathbf{K}_n}, \tau \in S^2_C\})(\supset \Phi(C(E_{RS^2_C})))$ is a $\mathbf{C}^*$, $O(n, \mathbf{C})$, $Sp(\gamma, \mathbf{C})$-invariant complex isotropic cone, furthermore, it gives a compact horizontal complex submanifold* [14] *possibly with singularities in $CP^{2n\gamma-1}$.*

Proof. Let $T' \notin Lag_o^C$. Then we have an element $g \in Sp(\gamma, \mathbf{C})$ and $T \in Lag_o^C$ such that $T' = g(T)$ by Lemma 2.11. There exists $\tau \in S_C^2$ which corresponds to T. $g(\{(\tau, K) \,|\, K \in \widetilde{\mathbf{K}_n}\})$ is some subset of the fibre of T' in V^n. This subset is independent of a choice for g and T because if τ and $(a+\tau c)^{-1}(b+\tau d)$ are elements of Lag_o^C for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{C})$, then $K \in \widetilde{\mathbf{K}_n}$ is equivalent to $K(a+\tau c) \in \widetilde{\mathbf{K}_n}$. Hence we can extend $\{(\tau, K) \,|\, K \in \widetilde{\mathbf{K}_n}, \tau \in S_C^2\}$ to a $\mathbf{C}^*$-invariant closed set in V^n on Lag^C. It is the set satisfying $\Xi = 0$ in V^n whose image is compact in $CP^{2n\gamma-1}$ since Lag^C is compact. □

Let $\mathbf{K}_{2m} = \{K \in \widetilde{\mathbf{K}_{2m}} \,|\, \mathrm{rank} K = m\}$. Let $O(2m)$ be the real orthogonal group. Arezzo and Micallef [3] determined $\mathbf{K}_{2m}$ for $\gamma \geq m$.

Lemma 4.1. *For $\gamma \geq m$, $\mathbf{K}_{2m} = (O(2m) \times K'_{m,\gamma})/U(m)$, where $K'_{m,\gamma}$ is the subset of $K_{m,\gamma}$ consisting of matrices of rank m. $\mathbf{K}_{2m}$ is open dense in $\widetilde{\mathbf{K}_{2m}}$ and hence $\dim_C \widetilde{\mathbf{K}_{2m}} = \frac{1}{2}m(m-1) + m\gamma$.*

Proof. Let $K \in \widetilde{\mathbf{K}_{2m}}$ and $k_1, ..., k_\gamma \in \mathbf{C}^{2m}$ the column vectors of K. We denote by $\langle\, , \rangle$ the complex bilinear extension on $\mathbf{C}^{2m}$ of the canonical inner product on $\mathbf{R}^{2m}$. Then ${}^tKK = 0$ is equivalent to $\langle k_i, k_j \rangle = 0$ for all i, j. For this reason $\langle\, , \rangle$ vanishes on the subspace spanned by $k_1, ..., k_\gamma$. Let T_K be the maximal subspace containing the subspace such that $\langle\, , \rangle$ vanishes. Then T_K is called to be totally isotropic and $\dim_C T_K = m$ holds. We can choose a unitary basis of T_K such that $\{\frac{1}{\sqrt{2}}(f_1 + if_{m+1}), ..., \frac{1}{\sqrt{2}}(f_m + if_{2m})\}$, where $\{f_1, ..., f_{2m}\}$ is an orthonormal basis of $\mathbf{R}^{2m}$. Thus there exists $a \in K_{m,\gamma}$ $(\gamma \geq m)$ such that $K = \frac{1}{\sqrt{2}}((f_1 + if_{m+1}) \cdots (f_m + if_{2m}))\, a$. We put $g = (f_1, ..., f_{2m})$. Then $g \in O(2m)$ and $K = \frac{1}{\sqrt{2}} g \begin{pmatrix} a \\ ia \end{pmatrix}$, which implies $\mathrm{rank}\, K \leq m$ and $\mathrm{rank}\, a \leq m$. For the map $G' : O(2m) \times K_{m,\gamma} \longrightarrow \widetilde{\mathbf{K}_{2m}}$ as $G'(g, a) = g \begin{pmatrix} a \\ ia \end{pmatrix}$, the surjectivity of G' holds. Furthermore, the restriction G' to $O(2m) \times K'_{m,\gamma}$ is also a surjective map to $\mathbf{K}_{2m}$. If $G'(g, a) = G'(h, b)$ for a, $b \in K'_{m,\gamma}$, then $h^{-1}g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(2m)$ satisfies $Aa + iBa = b$ and $Ca + iDa = ib$. Therefore, we get $(A + iB)a = (D - iC)a$. Since $\mathrm{rank}\, a = m$, we obtain $A + iB = D - iC$ and hence $h^{-1}g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in SO(2m)$ is a unitary matrix u. By using the action of $U(m)$ on $O(2m) \times K'_{m,\gamma}$ defined by $u(g, K) = (gu^{-1}, uK)$, we obtain $\mathbf{K}_{2m} = (O(2m) \times K'_{m,\gamma})/U(m)$ which is open dense in $\widetilde{\mathbf{K}_{2m}}$. □

Let $\widetilde{E}=\{(\tau,K)\,|\,\tau\in S^2_C, K\in\widetilde{\mathbf{K}_{2m}}\}$.

Theorem 4.2. *For $\gamma\geq m\geq 2$, $\Phi(\widetilde{E})$ consists of two $\mathbf{C}^*$, $SO(2m,\mathbf{C})$, $Sp(\gamma,\mathbf{C})$-invariant, non-totally geodesic, complex Lagrangian cones possibly with singularities, which give compact non-totally geodesic, horizontal complex submanifolds possibly with singularities in $CP^{4m\gamma-1}$. For $\gamma\geq m=1$, it is contained in the two complex Lagrangian planes.*

Proof. Corollary 4.1 implies that $\Phi(\widetilde{E})$ is a complex isotropic cone. We define the map $G: S^2_C\times SO(2m)\times K_{m,\gamma}\to K_{2m,\gamma}\times K_{2m,\gamma}$ as

$$G(\tau,g,a)=(g\begin{pmatrix}a\\ ia\end{pmatrix},g\begin{pmatrix}a\\ ia\end{pmatrix}\tau)$$

and hG, where $h\in O(2m)$ such that $\det h=-1$. Since h induces a complex symplectic isometry of $K_{2m,\gamma}\times K_{2m,\gamma}$ by Lemma 2.6, it is enough to prove our assertion on G. Let p be $(iE_\gamma,E_{2m},\begin{pmatrix}E_m & 0\end{pmatrix})\in S^2_C\times SO(2m)\times K_{m,\gamma}$.

We first construct a basis of $T_{G,p}:=G_*(T_p(S^2_C\times SO(2m)\times K_{m,\gamma}))$. $T_{G,p}$ is spanned by the following vectors:

$$\text{type 1}\quad (\begin{pmatrix}X+iY & 0\\ -{}^tY+iZ & 0\end{pmatrix},\begin{pmatrix}X+iY & 0\\ -{}^tY+iZ & 0\end{pmatrix}i),$$
$$\text{type 2}\quad (\begin{pmatrix}\alpha & \beta\\ i\alpha & i\beta\end{pmatrix},\begin{pmatrix}\alpha & \beta\\ i\alpha & i\beta\end{pmatrix}i),\;\; \text{type 3}\quad (0,\begin{pmatrix}A & B\\ iA & iB\end{pmatrix}),$$

where X,Z are real skew $m\times m$ matrices, Y is a real $m\times m$ matrix, α is an $m\times m$ matrix, β is an $m\times(\gamma-m)$ matrix, A is a symmetric $m\times m$ matrix and B is an $m\times(\gamma-m)$ matrix. A vector of type 2 is divided into a sum of vectors of types 4 and 5 as follows:

$$\text{type 4}\quad (\begin{pmatrix}\alpha & 0\\ i\alpha & 0\end{pmatrix},\begin{pmatrix}\alpha & 0\\ i\alpha & 0\end{pmatrix}i),\quad \text{type 5}\quad (\begin{pmatrix}0 & \beta\\ 0 & i\beta\end{pmatrix},\begin{pmatrix}0 & \beta\\ 0 & i\beta\end{pmatrix}i).$$

As a result, $T_{G,p}$ is spanned by vectors of types $1,3,4$ and 5. Note that A,B of type 3 and β of type 5 are free except the condition that A is symmetric. The common vector of types 1 and 4 satisfies $X+iY=\alpha$ and $-{}^tY+iZ=i\alpha$, and thus $X+iY=\alpha$ is skew Hermitian because $X=Z$ and Y is symmetric. We give type 6 as

$$\text{type 6}\quad (\begin{pmatrix}\alpha & 0\\ i\alpha & 0\end{pmatrix},\begin{pmatrix}\alpha & 0\\ i\alpha & 0\end{pmatrix}i),$$

where α is Hermitian. Hence $T_{G,p}$ is spanned by vectors of types $1, 3, 5, 6$ and the real dimension is $2m^2 - m + m(m+1) + 2m(\gamma - m) + 2m(\gamma - m) + m^2 = 4m\gamma$. Namely, the complex dimension of $T_{G,p}$ is $2m\gamma$. Hence, $T_{G,p}$ is a complex Lagrangian subspace.

Since the maximal rank of G_* is $2m\gamma$ by the complex isotropicness of G, G_* has the maximal rank $2m\gamma$ for generic points of $\widetilde{E}$. Each neighborhood of the generic points admits a decomposition of $U \times V$ such that

(1) $U \subset \mathbf{C}^{2m\gamma}$ and $V \subset \mathbf{C}^{\frac{\gamma(\gamma+1)}{2}+\frac{m(m-1)}{2}-m\gamma}$ are open sets,

(2) $G(U \times \{v\})$ for each $v \in V$ is a complex Lagrangian submanifold in $K_{2m,\gamma} \times K_{2m,\gamma}$,

(3) $G(\{u\} \times V)$ for each $u \in U$ is a point.

Thus, the image of $\widetilde{E}$ is a complex Lagrangian cone possibly with singularities.

We next prove that some second differential of G at p is not containd in $T_{G,p}$ if $m \geq 2$ as follows:

$$\frac{\partial^2 G}{\partial \tau \partial a} = (0, \begin{pmatrix} \alpha & \beta \\ i\alpha & i\beta \end{pmatrix} \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}) = (0, \begin{pmatrix} \alpha A + \beta\, {}^tB & \alpha B + \beta C \\ i\alpha A + i\beta\, {}^tB & i\alpha B + i\beta C \end{pmatrix})$$

holds, where A and C are symmetric. Let α be a matrix such that the $(1, m)$ entry is 1 and the others are 0, $\beta = 0$. Let A be a matrix such that $(2, m), (m, 2)$ entries are 1 and the others are 0 and $B = C = 0$. Then $\alpha A + \beta\, {}^tB$ is an $m \times m$ matrix with the $(1, 2)$ entry $= 1$ and the $(2, 1)$ entry $= 0$, that is, is not symmetric. Thus, the second differential is not contained in $T_{G,p}$. If $\Phi(\widetilde{E})$ is a totally geodesic, complex Lagrangian subspace through $G(p)$, then the complex Lagrangian subspace is $T_{G,p}$ through $G(p)$. Furthermore, the second differential is tangent to $T_{G,p}$. This is a contradiction.

Assume $m = 1$. Let $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2, \mathbf{R})$. Then $g \begin{pmatrix} a \\ ia \end{pmatrix} = \begin{pmatrix} e^{-i\theta} a \\ ie^{-i\theta} a \end{pmatrix}$. Therefore $G(\tau, g, a)$ is $(\begin{pmatrix} e^{-i\theta} a \\ ie^{-i\theta} a \end{pmatrix}, \begin{pmatrix} e^{-i\theta} a\tau \\ ie^{-i\theta} a\tau \end{pmatrix})$, which is contained in the complex Lagrangian subspace $\{\begin{pmatrix} Z_1 & Z_2 \\ iZ_1 & iZ_2 \end{pmatrix}\}$. □

One of two complex Lagrangian cones obtained in Theorem 4.2 is denoted by $M_{\gamma,m}$ for $\gamma \geq m \geq 2$. The other complex Lagrangian cone is the image of $M_{\gamma,m}$ by the complex symplectic isometry h.

5. A special pseudo Kähler structure

Let $index_{E_M}$ and $nullity_{E_M}$ be the index and the nullity of the Hessian of E_M at a point of $C(E_M)$, respectively. Let N be an irreducible component of $C(E_M)$. If N has a non-degenerate critical point, then a connected component of non-degenerate critical points of N admits a special pseudo Kähler structure as seen below. We give a relation between the special pseudo Kähler structure and $index_{E_M}$, $nullity_{E_M}$.

5.1. A review on the set of critical points

Let $\mathbf{R}^k$ be the k-dimensional Euclidean space, $(q^1, ..., q^k)$ the canonical coordinate system in $\mathbf{R}^k$ and U a neighborhood of the origin $0 \in \mathbf{R}^k$. Let $(\lambda^1, ..., \lambda^n)$ be the canonical coordinate system in $\mathbf{R}^n$ and V a neighborhood of the origin $0 \in \mathbf{R}^n$. Let F be a real valued function on $U \times V$ such that $F(0,0) = 0$. Assume that $0 \in U$ is a critical point of $F|_{U\times\{0\}}$. When we consider that F is an unfolding of $F|_{U\times\{0\}}$ such that $q^1, ..., q^k$ are innervariables and $\lambda^1, ..., \lambda^n$ are parameters, the set $C(F)$ of critical points is defined as

$$C(F) = \{(q,\lambda) \in U \times V \,|\, \frac{\partial F}{\partial q^1} = \cdots = \frac{\partial F}{\partial q^k} = 0\}.$$

The origin $0 \in U$ is called a non-degenerate critical point if the Hessian of $F|_{U\times\{0\}}$ at $0 \in U$ is non-degenerate. We also call $(0,0) \in C(F)$ to be non-degenerate. Then, some neighborhood of $(0,0) \in C(F)$ is an n-dimensional submanifold in $U \times V$ as follows: The Jacobian matrix at $(0,0)$ of the map $K : U \times V \to \mathbf{R}^k$ defined by $(\frac{\partial F}{\partial q^1}, ..., \frac{\partial F}{\partial q^k})$ is non-degenerate, that is, K_* is surjective at $(0,0)$ and, by the implicit function theorem, a neighborhood of $(0,0) \in C(F)$ is a graph over a neighborhood of $0 \in V$. In general, if K_* is surjective at $(0,0)$, then, again, a neighborhood of $(0,0)$ in $C(F)$ is a submanifold of dimension n in $U \times V$. Such a function F is called a Morse family.

Let $\pi : U \times V \longrightarrow V$ be the projection. Then the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ is a subspace of $T_0U \times \{0\}$. When a neighborhood of $(0,0)$ in $C(F)$ is a submanifold, we can consider $\pi : C(F) \subset U \times V \longrightarrow V$ the restriction of π. The subspace $\{X \in T_{(0,0)}C(F) \,|\, \pi_*(X) = 0\}$ is called the null space of $C(F)$ at $(0,0)$, which is contained in the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ since $K_*(X) = 0$ for $X \in T_{(0,0)}C(F)$ such that $\pi_*(X) = 0$

If F is a Morse family, then we obtain that the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ is the null space of $C(F)$ at $(0,0)$. Furthermore, we can construct the Lagrangian embedding of this submanifold into the cotangent bundle $T^*\mathbf{R}^n$ by

$$(q,\lambda)\mapsto(\frac{\partial F}{\partial\lambda^1}(q,\lambda),...,\frac{\partial F}{\partial\lambda^n}(q,\lambda),\lambda^1,...,\lambda^n)$$

as follows:

Let $(p^1,...,p^n,\lambda^1,..,\lambda^n)$ be the canonical coordinate system in $T^*\mathbf{R}^n$. Then the Liouville-form is $\sum_{j=1}^n p^j d\lambda^j$ and the symplectic form of $T^*\mathbf{R}^n$ is given by $\sum_{j=1}^n dp^j\wedge d\lambda^j = d(\sum_{j=1}^n p^j d\lambda^j)$. The Liouville-form induced on $C(F)$ is $\sum_{j=1}^n \frac{\partial F}{\partial\lambda^j}d\lambda^j = d(F|_{C(F)})$, that is, exact and hence the symplectic form induced on $C(F)$ vanishes. If $(0,0)$ be a non-degenerate critical point, the above map is a Lagrangian embedding since the image is a graph over a neighborhood of $(0,0)$.

So we consider the case that $(0,0)$ is not a non-degenerate critical point. Let m be the nullity of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ and assume that $\frac{\partial}{\partial q^\alpha}$, $\alpha=1,...,m$ are a basis of the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$. Since K_* is surjective,

$$\operatorname{rank}\begin{pmatrix} A' & B' & C' \\ D' & E' & F' \end{pmatrix} = k$$

holds, where, for $\alpha,\beta=1,...,m$, $\alpha',\beta'=m+1,...,k$, $\delta=1,...,n$,

$$A'=\left(\frac{\partial^2 F}{\partial q^\alpha\partial q^\beta}\right),\quad B'=\left(\frac{\partial^2 F}{\partial q^\alpha\partial q^{\beta'}}\right),\quad C'=\left(\frac{\partial^2 F}{\partial q^\alpha\partial\lambda^\delta}\right),$$
$$D'=\left(\frac{\partial^2 F}{\partial q^{\alpha'}\partial q^\beta}\right),\quad E'=\left(\frac{\partial^2 F}{\partial q^{\alpha'}\partial q^{\beta'}}\right),\quad F'=\left(\frac{\partial^2 F}{\partial q^{\alpha'}\partial\lambda^\delta}\right).$$

By the assumption, we get $A'=B'=D'=0$ and $\operatorname{rank}E'=k-m$. Hence, $\operatorname{rank}C'=m$.

We set $C'=(C'_1,C'_2)$, where C'_1 is an $m\times(n-m)$ matrix and C'_2 is an $m\times m$ matrix. Without loss of generality, we may assume that C'_2 is regular. Then the submanifold in a neighborhood of $(0,0)$ may be given by

$$q^{\alpha'}=q^{\alpha'}(q^\alpha,\lambda^{\delta'}),\quad \lambda^{\delta''}=\lambda^{\delta''}(q^\alpha,\lambda^{\delta'}),$$

where $\delta'=1,...,n-m$, $\delta''=n-m+1,...,n$. Since

$$\frac{\partial F}{\partial q^\alpha}(q^\beta,q^{\beta'}(q^\beta,\lambda^{\delta'}),\lambda^{\delta'},\lambda^{\delta''}(q^\beta,\lambda^{\delta'}))=0,$$

its differential at $(0,0)$ implies $C_2'\left(\frac{\partial\lambda^{\delta''}}{\partial q^{\beta}}\right)=0$. Since C_2' is regular, we get $\frac{\partial\lambda^{\delta''}}{\partial q^{\beta}}=0$ for each β. Since

$$\frac{\partial F}{\partial q^{\alpha'}}(q^{\beta},q^{\beta'}(q^{\beta},\lambda^{\delta'}),\lambda^{\delta'},\lambda^{\delta''}(q^{\beta},\lambda^{\delta'}))=0,$$

its differential at $(0,0)$ implies $E'\left(\frac{\partial q^{\beta'}}{\partial q^{\beta}}\right)=0$. Similarly, since E' is regular, we get $\frac{\partial q^{\beta'}}{\partial q^{\beta}}=0$ for each β. These show that the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ is the null space of $T_{(0,0)}C(F)$.

Furthermore,

$$(q,\lambda)\mapsto(\frac{\partial F}{\partial\lambda^1}(q,\lambda),...,\frac{\partial F}{\partial\lambda^n}(q,\lambda),\lambda^1,...,\lambda^n)$$

is an embedding of a neighborhood of $(0,0)$ of $C(F)$, which completes the proof.

We call such an F a generating function with respect to this Lagrangian submanifold in $T^*(\mathbf{R}^n)$.

Conversely, the result above is a sufficient condition that F is a Morse family. In fact, if some neighborhood of $(0,0)$ in $C(F)$ is an n-dimensional submanifold, the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$ is the null space of $T_{(0,0)}C(F)$ and the map defined by

$$f(q,\lambda)=(\frac{\partial F}{\partial\lambda^1}(q,\lambda),...,\frac{\partial F}{\partial\lambda^n}(q,\lambda),\lambda^1,...,\lambda^n)$$

an embedding, then F is a Morse family at $(0,0)$ as follows:

Let m be the nullity of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$. Without loss of generality, we assume that $\{\frac{\partial}{\partial q^{\alpha}}\}$ is a basis of the null space of the Hessian of $F|_{U\times\{0\}}$ at $(0,0)$. Let $(u^1,...,u^m,u^{m+1},...,u^n)$ be a local coordinate system at $(0,0)$ in the n-dimensional submanifold such that $\frac{\partial}{\partial u^{\alpha}}=\frac{\partial}{\partial q^{\alpha}}$ by the assumption. Then, we get, at $(0,0)$,

$$f_*(\frac{\partial}{\partial u^{\alpha}})=((\sum_{\beta}\frac{\partial^2 F}{\partial\lambda^{\delta}\partial q^{\beta}}\frac{\partial q^{\beta}}{\partial u^{\alpha}}),0)=(\left(\frac{\partial^2 F}{\partial\lambda^{\delta}\partial q^{\beta}}\right)\left(\frac{\partial q^{\beta}}{\partial u^{\alpha}}\right),0).$$

By the assumption, $\operatorname{rank}\left(\frac{\partial^2 F}{\partial\lambda^{\delta}\partial q^{\beta}}\right)=m$ and $\operatorname{rank}\left(\frac{\partial^2 F}{\partial q^{\alpha'}\partial q^{\beta'}}\right)=(k-m)$. Therefore, $\operatorname{rank}K_*=k$.

Proposition 5.1. *F is a Morse family at $(0,0)$ if and only if*

(1) *some neighborhood of* $(0,0)$ *in* $C(F)$ *is an* n*-dimensional submanifold,*

(2) *the null space of the Hessian of* $F|_{U\times\{0\}}$ *at* $(0,0)$ *is the null space of* $T_{(0,0)}C(F)$*,*

(3) $f(q,\lambda) = (\frac{\partial F}{\partial \lambda^1}(q,\lambda), ..., \frac{\partial F}{\partial \lambda^n}(q,\lambda), \lambda^1, ..., \lambda^n)$ *is an embedding.*

5.2. A complex Lagrangian cone

Now we consider a complex submanifold $M \subset RS^2_C$ in place of U and $L_{n,2\gamma}$ in place of V in Subsection 5.1. Let $\pi : C(E_M) \subset RS^2_C \times L_{n,2\gamma} \to L_{n,2\gamma}$ be the projection and its restriction.

Let N be an irreducible component of $C(E_M)$. If N has a non-degenerate critical point $(\tau, (L_1, L_2))$, then some neighborhood of $(\tau, (L_1, L_2))$ is a graph over an open set of $(L_1, L_2) \in L_{n,2\gamma}$. Therefore we see $\dim_C N = n\gamma$.

N may have a singular locus S_1, which is of real codim ≥ 2. Let S_2 be the set of degenerate critical points in N. Since a neighborhood of a point $\notin S_2$ in N is a submanifold, $S_1 \subset S_2$ holds. Since a neighborhood of a point $\in S_2 \setminus S_1$ in N is a submanifold and the zero set of the Jacobian of π of the submanifold is a real analytic set, $S_2 \setminus S_1$ may be a real hypersurface in N possibly with singularities. If E_M is a Morse family at $(\tau, (L_1, L_2)) \in S_2 \setminus S_1$, then, a neighborhood of $(\tau, (L_1, L_2))$ is a submanifold and there exists a Lagrangian embedding of the neighborhood as in Proposition 5.1. However, we note

Lemma 5.1. *If* N *has a non-degenerate critical point, then a Lagrangian immersion* ψ *of* N *into* $T^*L_{n,2\gamma}$ *possibly with singularities is given by*

$$\psi : (\tau, (L_1, L_2)) \in N \mapsto ((L_1, L_2)P(\tau), (L_1, L_2)) \in T^*L_{n,2\gamma}.$$

In particular, $\psi = 2\varphi \circ \Phi \circ \Psi$*. The obtained Lagrangian submanifold possibly with singularities is a complex Lagrangian cone possibly with singularities. Some neighborhood of a non-degenerate critial point is the graph :* $L \mapsto (LP(\tau(L)), L)$ *on an open set of* $L_{n,2\gamma}$*, where* $LP(\tau(L)) = \Big(\frac{\partial E_M}{\partial \ell_{ij}}\Big)(\tau(L), L)$ *for the canonical coordinate system* $\{\ell_{ij}\}$ *in* $L_{n,2\gamma}$*.*

Proof. By $\psi = 2\varphi \circ \Phi \circ \Psi$, the obtained Lagrangian submanifold possibly with singularities is a cone. Since $(\tau(L), L)$ is a critical point of E_M,

$$\frac{\partial E_M}{\partial \ell_{ij}} = \mathrm{tr}(P(\tau(L))\,{}^t e_{ij} L) = \sum_{k=1}^{2\gamma} P(\tau(L))_{kj} L_{ik} = (LP(\tau(L))_{ij},$$

where $e_{ij} \in L_{n,2\gamma}$ such that the (i,j) entry is 1 and the others are 0. $\square$

Let $T_{n,2\gamma} = \{L \in L_{n,2\gamma} \,|\, \text{column vectors of } L \text{ span a lattice}\}$. Then $T_{n,2\gamma}$ is dense in $L_{n,2\gamma}$ since $T_{n,2\gamma}$ contains L with rank $= n$ whose entries are rational numbers.

Proposition 5.2. *Let N be an irreducible component of $C(E_M)$ with a non-degenerate critical point. Then we have*

(1) $\dim_C N = n\gamma$,

(2) *ψ gives a complex Lagrangian cone possibly with singularities,*

(3) *$\pi^{-1}(T_{n,2\gamma})$ is dense in N.*

5.3. Non-degenerate condition

We review non-degenerate condition (see Cortés [7]). The canonical coordinate system in $K_{1,\gamma} \times K_{1,\gamma}$ is denoted by $\{z^1, ..., z^\gamma, w^1, ..., w^\gamma\}$. A complex symplectic form ω_2 is given by $\sum_{j=1}^{\gamma} dz^j \wedge dw^j = -\omega_1$. A Hermitian form with the signature (γ, γ) is defined as $\eta_2(x, y) = i\omega_2(x, \overline{y})$, which is $Sp(\gamma, \mathbf{R})$-invariant. A complex Lagrangian subspace is called to be non-degenerate if the Hermitian form induced from η_2 is non-degenerate. Then, its signature (p, q) is called the signature of the complex Lagrangian subspace. $Sp(\gamma, \mathbf{R})$ acts on the space of non-degenerate complex Lagrangian subspaces and preserves signatures.

Let Lag_1^C denote the set of non-degenerate complex Lagrangian subspaces $\subset K_{1,\gamma} \times K_{1,\gamma}$. The complement of Lag_1^C in Lag^C is a real analytic hypersurface possibly with singularities, because, for a local holomorphic frame fields $f_1, ..., f_\gamma$ of V on a neighborhood $U \subset Lag^C$, $p \in U \cap Lag_1^C$ if and only if $\det(\eta_2(f_j, f_k)) \neq 0$ at p.

Lemma 5.2. *$\tau \in S_C^2$ is a non-degenerate complex Lagrangian subspace if and only if $\tau \in RS_C^2$. Then the signature (p, q) of τ is that of* $\mathrm{Im}\tau$. *In particular, a complex Lagrangian subspace with $q = 0$ corresponds to $\tau \in H_\gamma$. RS_C^2 is identified with $Lag_o^C \cap Lag_1^C$.*

Proof. Let $(e_j, e_j\tau)$, $1 \le j \le \gamma$ be a basis of the complex Lagrangian subspace. Then $\eta_2((e_j, e_j\tau), (e_k, e_k\tau)) = i\mathrm{tr}^t dZ_1 \wedge dZ_2((e_j, e_j\tau), (e_k, e_k\overline{\tau})) = 2\mathrm{Im}\tau_{jk}$. □

Let W be a complex Lagrangian subspace of $K_{1,\gamma} \times K_{1,\gamma}$. Then, in general, η_2 induced on W may be degenerate. Let W_{η_2} be the null space and $nullity_{\eta_2}$ its complex dimension. Cortés gave the following (Proposition 1.1 [7]).

Proposition 5.3. *Let W be a complex Lagrangian subspace. Then $W \cap \overline{W} = W_{\eta_2}$. In particular, W is non-degenerate if and only if $W \cap \overline{W} = \{0\}$.*

The previous half of Proposition 5.3 is Proposition 1.4 in [7]. $a \in W \cap \overline{W}$ if and only if $\mathrm{Re}\, a \pm i\mathrm{Im}\, a \in W$, that is, $\mathrm{Re}\, a$, $\mathrm{Im}\, a \in W$. Consequently, $W \cap \overline{W} = \{0\}$ is equivalent to that W does not contain non-zero real vector. Combining these with Corollaries 2.1 and 2.2, we obtain

Proposition 5.4. *$W \subset K_{1,\gamma} \times K_{1,\gamma}$ is non-degenerate if and only if the projection $\pi_1 : (K_1, K_2) \in W \mapsto 2(\mathrm{Re}K_1, \mathrm{Re}K_2) \in L_{1,2\gamma}$ is surjective. The space of non-degenerate complex Lagrangian subspaces is identified with the space of $2\gamma \times 2\gamma$ symmetric matrices in $Sp(\gamma, \mathbf{R})$. H_γ is the space of complex Lagrangian subspaces such that η_2 is positive definite.*

A complex Lagrangian submanifold X with the non-degenerate Hermitian form η_2 in $\mathbf{C}^{2\gamma}$ is called a Lagrangian pseudo Kähler submanifold, which admits a special pseudo Kähler structure [7]. Freed [15] (see, for example, [18]) formalized a special pseudo Kähler structure as follows:

A special pseudo Kähler manifold is a complex manifold with the complex structure J and

(a) a pseudo Kähler metric g with a pseudo Kähler form ω,

(b) a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and

$$(\nabla_X J)(Y) - (\nabla_Y J)(X) = 0.$$

Propositions 2.2 and 5.4 imply

Corollary 5.1. *Let Y be a complex Lagrangian submanifold in $K_{n,\gamma} \times K_{n,\gamma}$ and $\pi_1 : Y \to L_{n,2\gamma}$. Then Y is a Lagrangian pseudo Kähler submanifold if and only if π_{1*} at each point of Y is bijective.*

Let N be an irreducible component with a non-degenerate critical point of $C(E_M)$. Then a connected component of non-degenerate critical points of N may be locally a graph on a neighborhood in $L_{n,2\gamma}$. From Lemmas 2.18, 2.20, 3.8 and 5.1, Theorem 4.1, Corollary 5.1 and that $SO(n), U(1)$ and $Sp(\gamma, \mathbf{R})$ preserve η_2, we obtain

Theorem 5.1. *The connected component gives the Lagrangian pseudo Kähler cone with $SO(n) \times U(1)$ as holomorphic isometries. If M is invariant by a subgroup G of $Sp(\gamma, \mathbf{R})$, then an element of G acts on $C(E_M)$. An element of G may change an irreducible component N_1 to another irreducible component N_2. Then, the element of G changes a connected component of N_1 to a connected component of N_2 as a holomorphic isometry preserving the special pseudo Kähler structures. If G is connected, then G is a group of holomorphic isometries on connected components with special pseudo Kähler structures.*

Remark 5.1. $SO(n, \mathbf{C})$ preserves an irreducible component, however, may not preserve a connected component.

It is worth noting that the energy function is defined on a complex submanifold M in Lag_1^C.

Proposition 5.5. *When we identify Lag_1^C with the space of $2\gamma \times 2\gamma$ symmetric matrices in $Sp(\gamma, \mathbf{R})$, we can define the energy function as $E(P, L) = \frac{1}{2}\mathrm{tr}(P\,{}^tLL)$ for $P \in Lag_1^C$ and $L \in L_{n,2\gamma}$.*

5.4. A complex Lagrangian graph in $T^*L_{n,2\gamma}$

We study the signature of a complex Lagrangian graph.

Corollary 5.2. *If a non-degenerate complex Lagrangian subspace in $L_{n,2\gamma} \times L_{n,2\gamma}$ is given by a symmetric $2\gamma \times 2\gamma$ real matrix $P = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix} \in Sp(\gamma, \mathbf{R})$ and $((L_1, L_2)P, (L_1, L_2))$ is a vector in the complex Lagrangian subspace, then,*

$$\eta_2(((L_1, L_2)P, (L_1, L_2)), ((L_1, L_2)P, (L_1, L_2))) = \frac{1}{2}\mathrm{tr}\Big((L_1, L_2)P \begin{pmatrix} {}^tL_1 \\ {}^tL_2 \end{pmatrix}\Big)$$

and the signature of η_2 is the half of $n \times$ (the signature of P). The complex structure induced by I is given by PJ_0 (see Theorem 2.1).

Proof. Let $K_1 = \frac{1}{2}(L_1 - i(L_1B + L_2C))$ and $K_2 = \frac{1}{2}(L_2 + i(L_1A + L_2\,{}^tB))$. Then we obtain

$$\begin{aligned}\eta_2((K_1, K_2), (K_1, K_2)) =& i\,\mathrm{tr}\,{}^t dZ_1 \wedge dZ_2((K_1, K_2), \overline{(K_1, K_2)}) \\ =& \frac{1}{2}\mathrm{tr}\Big((L_1, L_2)P\begin{pmatrix} {}^tL_1 \\ {}^tL_2 \end{pmatrix}\Big).\end{aligned}$$

□

Let $(L_1, ..., L_{2\gamma})$ and $(L'_1, ..., L'_{2\gamma})$ be tangent vectors at each point of $L_{1,2\gamma}$. Then

$$\omega_0((L_1, ..., L_{2\gamma}), (L'_1, ..., L'_{2\gamma})) = \frac{1}{2}(L_1, ..., L_{2\gamma})\,{}^tJ_0\,{}^t(L'_1, ..., L'_{2\gamma})$$

is a 2-form on $L_{1,2\gamma}$, which is $Sp(\gamma, \mathbf{R})$-invariant for the action:

$$(L_1, L_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (L_1a + L_2c, L_1b + L_2d) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(\gamma, \mathbf{R}),$$

where $L_1 = (L_1, ..., L_\gamma)$ and $L_2 = (L_{\gamma+1}, ..., L_{2\gamma})$.

Let $(\ell_1, ..., \ell_{2\gamma})$ be the canonical coordinate system in $L_{1,2\gamma}$ and ϕ a function on an open set $U \subset L_{1,2\gamma}$. Then $(\frac{\partial\phi}{\partial\ell_1}, ..., \frac{\partial\phi}{\partial\ell_{2\gamma}}, \ell_1, ..., \ell_{2\gamma})$ is a Lagrangian graph L_ϕ on U into $T^*L_{1,2\gamma}$ and $(\ell_1, ..., \ell_{2\gamma})$ is a local coordinate system in L_ϕ. $(L_1, ..., L_{2\gamma}) \in L_{1,2\gamma}$ may be a tangent vector of L_ϕ. The following gives the special pseudo Kähler structure on L_ϕ.

Proposition 5.6. *L_ϕ in $T^*L_{1,2\gamma}$ is a Lagrangian pseudo Kähler submanifold if and only if the Hessian of ϕ is an element of $Sp(\gamma, \mathbf{R})$. Then we have*

(1) *The complex structure J_ϕ is given by $(\frac{\partial^2\phi}{\partial\ell_j\partial\ell_k})J_0$,*

(2) *The pseudo Kähler metric g_ϕ is given by*

$$\frac{1}{2}(L_1, ..., L_{2\gamma})(\frac{\partial^2\phi}{\partial\ell_j\partial\ell_k})^t(L'_1, ..., L'_{2\gamma}),$$

(3) *The pseudo Kähler form ω_ϕ is ω_0,*

(4) *The canonical connection on $L_{1,2\gamma}(= \mathbf{R}^{2\gamma})$ satisfies* (b) *in the definition of the special pseudo Kähler structure.*

Proof. (1) is obtained by Theorem 2.1. Corollary 5.2 implies (2). It is enough to calculate the pseudo Kähler form to prove (3). By Corollary 5.2 and that the Hessian of ϕ is an element of $Sp(\gamma, \mathbf{R})$, we get

$$\begin{aligned}\frac{1}{2}(L_1, ..., L_{2\gamma})(\frac{\partial^2\phi}{\partial\ell_j\partial\ell_k})\,{}^tJ_0(\frac{\partial^2\phi}{\partial\ell_j\partial\ell_k})\,{}^t(L_1', ..., L_{2\gamma}')\\ = \frac{1}{2}(L_1, ..., L_{2\gamma})\,{}^tJ_0\,{}^t(L_1', ..., L_{2\gamma}').\end{aligned}$$

We choose the canonical connection on $L_{1,2\gamma}$ as a flat torsin-free connection. Then the pseudo Kähler form is parallel and

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial\ell_j}}J_\phi)(\frac{\partial}{\partial\ell_k}) - (\nabla_{\frac{\partial}{\partial\ell_k}}J_\phi)(\frac{\partial}{\partial\ell_j})\\ = \sum_{p,q}(\frac{\partial^3\phi}{\partial\ell_j\partial\ell_k\partial\ell_p} - \frac{\partial^3\phi}{\partial\ell_k\partial\ell_j\partial\ell_p})J_{0pq}\frac{\partial}{\partial\ell_q} = 0,\end{aligned}$$

where J_{0pq} is the (p,q) entry of J_0. □

Let N be an irreducible component of $C(E_M)$ with a non-degenerate critical point. Then N gives a complex Lagrangian cone posssibly with singularities. Let N_1 be a connected component of non-degenerate critical points of N. Then N_1 admits a special pseudo Kähler structure of the signature (p,q), where $p+q = n\gamma$. Hence, (p,q) and $index_{E_M}$ at a non-degenerate critical point are invariants of N_1. We investigate a relation between the two invariants. By Lemma 5.1, N_1 is locally a graph $\{(\tau(L), L) \mid L \in U\}$ on an open set $U \subset L_{n,2\gamma}$. The complex Lagrangian cone is given by $\{(LP(\tau(L)), L) \mid L \in U\}$.

Let ℓ_{jk} be the (j,k) entry of $L \in L_{n,2\gamma}$. We denote by $\{\ell_{jk}\}$ the canonical coordinate system in $L_{n,2\gamma}$. Review $e_{jk} \in L_{n,2\gamma}$ such that the (j,k) entry is 1 and the others are 0. Let $\{\tau^1, ..., \tau^\alpha\}$, where $\alpha = 2\dim_C M$, be a local real coordinate system in M.

We consider the case $\phi = E_M$. By Propositions 2.2 and 5.6, $\frac{1}{2}\mathrm{tr}(P(\tau(L))\,{}^tLL)$ has the pseudo Kähler potential $a(L)$. By using $\tau(L) \in M$, we calculate

$$\begin{aligned}\frac{\partial a}{\partial\ell_{jk}} &= \mathrm{tr}(P(\tau(L))\,{}^te_{jk}L),\\ \frac{\partial^2 a}{\partial\ell_{\ell m}\partial\ell_{jk}} &= \sum_q \mathrm{tr}(\frac{\partial P(\tau(L))}{\partial\tau^q}\,{}^te_{jk}L)\frac{\partial\tau^q}{\partial\ell_{\ell m}} + \mathrm{tr}(P(\tau(L))\,{}^te_{jk}e_{\ell m}).\end{aligned}$$

Differentiating $\mathrm{tr}(\frac{\partial P(\tau(L))}{\tau^q}\,{}^tLL)=0$ by ℓ_{jk}, we get

$$\sum_r \mathrm{tr}(\frac{\partial^2 P(\tau)}{\partial\tau^q\partial\tau^r}\,{}^tLL)\frac{\partial\tau^r}{\partial\ell_{jk}}+2\mathrm{tr}(\frac{\partial P(\tau)}{\partial\tau^q}\,{}^te_{jk}L)=0 \text{ and hence,}$$

$$\sum_q \mathrm{tr}(\frac{\partial P(\tau)}{\partial\tau^q}\,{}^te_{jk}L)\frac{\partial\tau^q}{\partial\ell_{\ell m}}=-\frac{1}{2}\sum_{q,r}\mathrm{tr}(\frac{\partial^2 P(\tau)}{\partial\tau^q\partial\tau^r}\,{}^tLL)\frac{\partial\tau^r}{\partial\ell_{jk}}\frac{\partial\tau^q}{\partial\ell_{\ell m}}.$$

Thus, we obtain

$$\frac{\partial^2 a}{\partial\ell_{\ell m}\partial\ell_{jk}}=-\frac{1}{2}\sum_{q,r}\mathrm{tr}(\frac{\partial^2 P(\tau)}{\partial\tau^q\partial\tau^r}\,{}^tLL)\frac{\partial\tau^r}{\partial\ell_{jk}}\frac{\partial\tau^q}{\partial\ell_{\ell m}}+\mathrm{tr}(P(\tau(L))\,{}^te_{jk}e_{\ell m}).$$

We define the non-degenerate pseudo inner product by $\mathrm{Re}\eta_2$ on $T^*L_{n,2\gamma}$, which corresponds to g_ϕ in Proposition 5.6. Then, we get an important formula by Corollary 5.2 and Proposition 5.6.

Theorem 5.2.

$$\begin{aligned}\sum_{q,r}\mathrm{Hess}E_M(\frac{\partial}{\partial\tau^q},\frac{\partial}{\partial\tau^r})\frac{\partial\tau^q}{\partial\ell_{jk}}\frac{\partial\tau^r}{\partial\ell_{\ell m}}\\ =\mathrm{tr}\Big(P(\tau(L))\,{}^te_{jk}e_{\ell m}\Big)-\mathrm{Hess}\,a(\frac{\partial}{\partial\ell_{jk}},\frac{\partial}{\partial\ell_{\ell m}}).\end{aligned}$$

(1) $\frac{1}{2}\mathrm{tr}\Big(P(\tau(L))\,{}^te_{jk}e_{\ell m}\Big)$ *is the Gram matrix of the basis* $\{\frac{\partial}{\partial\ell_{jk}}\}$ *with respect to* $\mathrm{Re}\eta_2$ *on the complex Lagrangian subspace corresponding to* $\tau(L)$.

(2) $\frac{1}{2}\mathrm{Hess}\,a(\frac{\partial}{\partial\ell_{jk}},\frac{\partial}{\partial\ell_{\ell m}})$ *is the Gram matrix of the basis* $\{\frac{\partial}{\partial\ell_{jk}}\}$ *with respect to* $\mathrm{Re}\eta_2$ *on the tangent space at* $(LP(\tau(L)),L)$ *of the complex Lagrangian graph.*

Note that the tangent space at $(LP(\tau(L)),L)$ of the complex Lagrangian graph in Theorem 5.2 may not be a complex Lagrangian subspace $(n\geq 2)$ which is investigated in Theorem 2.1.

5.5. $\boldsymbol{index_{E_M}}$ of a complex Lagrangian graph

Assume $M\subset H_\gamma$. Let N be an irreducible component of $C(E_M)$ with a non-degenerate critical point. Since $C(E_M)\subset M\times L_{n,2\gamma}$, we obtain $\tau\colon N\longrightarrow M$, which is a holomorphic map except the singular locus S_1 in N.

(i) Surjective condition at a non-degenerate critical point $\in N$ is that τ_* at the point is surjective.

(ii) Surjective condition on an irreducible component N is that generic non-degenerate critical points of N satisfy surjective condition.

If surjective condition is satisfied at a non-degenerate critical point of N, then, generic non-degenerate points of N satisfy surjective condition since τ is holomorphic, that is, N satisfies surjective condition.

Theorem 5.3. *Let N be an irreducible component satisfy surjective condition. Then*

$$2q + index_{E_M} \leq 2\dim_C M$$

for each connected component N_1, where (p, q) is the signature of the special pseudo Kähler metric. In particular, the special pseudo Kähler metric on $\mathrm{Ker}\tau_*$ *at a non-degenerate critical point $\in N_1$ satisfying surjective condition is positive definite.*

Proof. By the asumption, there exists a non-degenerate critical point (τ_0, L_0) satisfying surjective condition in N_1. Thus we obtain a neighborhood of (τ_0, L_0) in N_1 is given by $\{(\tau(L), L) \mid L \in U\}$, where $\tau(L_0) = \tau_0$ and U is an open set $\subset L_{n,2\gamma}$. It is enough to prove the inequality at $(\tau(L_0), L_0)$.

We consider the symmetric bilinear form on $T_{(\tau(L_0),L_0)}N$ of the left hand side of the formula in Theorem 5.2. We get the subspace $T_f = \mathrm{Ker}\tau_*$, which is containd in the kernel of the symmetric bilinear form. Let T_b be the subspace where the bilinear form is negative-definite. Therefore, the symmetric bilinear form on $T_f + T_b$ is non-positive. By Theorem 2.1, $P(\tau(L_0))$ is positive definite. Thus, Theorem 5.2 implies that the Hessian of a is positive on $T_f + T_b$. Therefore, $2p \geq \dim_R(T_f + T_b)$. Since surjective condition is satisfied, $\dim_R T_b = index_{E_M}$ and $\dim_R T_f = 2n\gamma - 2\dim_C M$. Consequently, $2p \geq index_{E_M} + 2n\gamma - 2\dim_C M$, together with $p + q = n\gamma$, completes the proof. $\square$

We can identify $L \in L_{n,2\gamma}$ with ${}^t({}^tL\,0) \in L_{n+1,2\gamma}$, that is, $L_{n,2\gamma} \subset L_{n+1,2\gamma}$. Then, $E_M(\tau, L) = E_M(\tau, {}^t({}^tL\,0))$ and hence $C(E_M)$ for $L_{n,2\gamma}$ is contained in $C(E_M)$ for $L_{n+1,2\gamma}$. If (τ, L) is a non-degenerate critical point, then $(\tau, {}^t({}^tL\,0))$ is also a non-degenerate critical point. Thus we see that the graph G_1 of non-degenerate critical points on a neighborhood of L in $L_{n,2\gamma}$ is $G_2 \cap \{(\tau, {}^t({}^tL\,0)) | \tau \in M, L \in L_{n,2\gamma}\}$, where G_2 is a graph of non-degenerate critical points on a neighborhood of ${}^t({}^tL\,0)$ in $L_{n+1,2\gamma}$.

It is a simple method to obtain a non-degenerate critical point for $L_{n+1,2\gamma}$ from a non-degenerate critical point for $L_{n,2\gamma}$, which is called swelling in [10]. The obtained irreducible component is called a degenerative component in [4]. The point may not satisfy surjective condition, however, generic points of the irreducible component obtained by swelling may satisfy surjective condition. In the case, since swelling preserves $index_{E_M}$, it is enough to calculate $index_{E_M}$ at a critical point satisfying surjective condition close to $(\tau, {}^t({}^tL, 0))$.

Theorem 5.3 implies

Corollary 5.3. *Let (p_1, q_1) be the signature of G_1 and (p_2, q_2) the signature of G_2. Then $q_1 \leq q_2$ holds. If G_2 satisfies surjective condition, $index_{E_M} \leq 2\dim_C M - 2q_2$ in place of q_1. In particular, $index_{E_M} \leq 2\dim_C M - 2q_1$. If $q_1 = \dim_C M$, then $q_1 = q_2$ and $index_{E_M} = 0$ on the connected component.*

We have the null space of the Hessian of E_M at a degenerate critical point and denote by $nullity_{E_M}$ its dimension. We recall that S_2 is the set of degenerate critical points of N. If N has different connected components of non-degenerate critical points in N, then S_2 is an hypersurface possibly with singularities. We consider a neighborhood of a point of the hypersurface except singularities. Propositions 2.2, 5.1 and 5.3 imply

Proposition 5.7. *If E_M is a Morse family at a point of $S_2 \setminus S_1$, then we obtain $nullity_{E_M} = nullity_{\eta_2}$.*

Theorem 5.4. *Let (τ, L) be a critical point of an irreducible component with a non-degenerate critical point of $C(E_M)$. Assume that E_M is a Morse family at (τ, L). Let $\{T_k\}$ be a basis of the tangent space of the corresponding complex Lagrangian submanifold above at $(K, K\tau)$ and W the Gram matrix $(\eta_2(T_i, T_j))$ of $\{T_k\}$ with respect to η_2. Similarly, we define the real basis $\{T_k, T'_k\}$ and the Gram matrix W_1 of the real basis $\{T_k, T'_k\}$ with respect to $\mathrm{Re}\eta_2$. Then nullity of W is $nullity_{E_M}$ and nullity of W_1 is $2\, nullity_{E_M}$. In particular, if W or W_1 is regular, then (τ, K) is a non-degenerate critical point and the signature is the signature of W or the half of that of W_1.*

Corollary 5.1 and Theorem 5.2 imply

Theorem 5.5 (Algorithm). *Let (τ, L) be a non-degenerate critical point of an irreducible component of $C(E_M)$ where surjective condition is satisfied at the critical point. Let $T_k = (A_k, B_k)$ be a basis of the tangent space at*

$(K, K\tau)$ of the corresponding complex Lagrangian cone in $\mathbf{C}^{2n\gamma}$. Set $T'_k = (iA_k, iB_k)$. Let

$$(C_k, D_k) = \mathrm{Re}(2A_k, 2B_k), \quad (C'_k, D'_k) = \mathrm{Re}(i2A_k, i2B_k).$$

Then $\{(C_k, D_k), (C'_k, D'_k)\}$ is a basis of $L_{n,2\gamma}$. Put

$$S_k = (E_k, E_k\tau), \quad S'_k = (E'_k, E'_k\tau),$$

where

$$\begin{aligned} E_k &= \frac{1}{2}(C_k + i[C_k\mathrm{Re}\tau - D_k](\mathrm{Im}\tau)^{-1}) \\ &= \mathrm{Re}(A_k) + i[\mathrm{Re}(A_k)\mathrm{Re}\tau - \mathrm{Re}(B_k)](\mathrm{Im}\tau)^{-1}, \\ E'_k &= \mathrm{Re}(iA_k) + i[\mathrm{Re}(iA_k)\mathrm{Re}\tau - \mathrm{Re}(iB_k)](\mathrm{Im}\tau)^{-1}. \end{aligned}$$

$\{S_k, S'_k\}$ is a real basis of a non-degenerate complex Lagrangian subspace defined by τ. Let W_2 denote the Gram matrix of the real basis $\{S_k, S'_k\}$ and W_1 the Gram matrix of the real basis $\{T_k, T'_k\}$ with respect to $\mathrm{Re}\eta_2$. *Then*

(1) $2n\gamma - 2\dim_C M$ *is the nullity of $W_2 - W_1$,*

(2) $\mathrm{Hess}E_M$ *is the difference between the tangent space of the non-degenerate complex Lagrangian cone and the complex Lagrangian subspace by τ,*

(3) *$index_{E_M}$ is the number of negative eigenvalues of $W_2 - W_1$.*

Finally we give a geometric property of the signature. Let N be an irreducible component of $C(E_M)$ with a non-degenerate critical point. Then a connected component N_1 of non-degenerate critical points of N admits a special pseudo Kähler structure of the signature (p, q). It is locally a graph $(LP(\tau(L)), L)$ on an open set $U \subset L_{n,2\gamma}$. We investigate q for $n = 1$ for the convenient because the proof is available for all n. N admits the orientation induced by the complex structure except S_1. We investigate whether the projection $\pi : N_1 \to L_{1,2\gamma}$ is orientation preserving or reversing for the orientation of $L_{1,2\gamma}$ by $\frac{1}{\gamma!}(-1)^{\frac{\gamma(\gamma+1)}{2}}\omega_0^\gamma$, where

$$\omega_0 = \frac{1}{2}(L_1, ..., L_{2\gamma})\, {}^tJ_0\, {}^t(L'_1, ..., L'_{2\gamma}).$$

Let $\widetilde{\omega_0}$ be the induced pseudo Kähler form of ω_0 on N_1, which is the pseudo Kähler form by Proposition 5.6. Then

$$\widetilde{\omega_0} = -ih(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}})dz^\alpha \wedge d\overline{z^\beta},$$

where $(z^1, ..., z^\gamma)$ is a local complex coordinate system in N_1, $(h(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}}))$ is the Hermitian matrix with the signature (p, q). For $z^k = x^k + iy^k$,

$$\frac{1}{\gamma!}(-1)^{\frac{\gamma(\gamma+1)}{2}}\widetilde{\omega_0}^\gamma = \det(h(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z^\beta}}))dx^1 \wedge \cdots \wedge dx^\gamma \wedge dy^1 \wedge \cdots \wedge dy^\gamma.$$

Proposition 5.8. *$\pi : N_1 \longrightarrow L_{1,2\gamma}$ is orientation preserving if $(-1)^q > 0$ and orientation reversing if $(-1)^q < 0$.*

6. Applications to minimal surfaces in flat tori

Sacks and Uhlenbeck [32], Schoen and Yau [33] constructed an incompressible minimal surface as a minimum point of the energy function on the Teichmüller space induced by harmonic maps. When the energy function is smooth, its critical point corresponds to a minimal surface. The index and the nullity of the Hessian of the energy function at the critical point are related to $index_a$ and $nullity_a$ of the minimal surface [8], respectively.

Furthermore, we studied a multivalued harmonic map from a compact Riemann surface M to $\mathbf{R}^n$ with a real period matrix L for a canonical homology basis $\{A_1, ..., A_\gamma, B_1, ..., B_\gamma\}$ of M and proved that the energy is $E(\tau, L) = \frac{1}{2}\mathrm{tr}(P(\tau)\,{}^tLL)$, where τ is its Riemann matrix $\in H_\gamma$. Let $RM_{non-hyper}$ be the space of Riemann matrices of non-hyperelliptic Riemann surfaces of genus $\gamma \geq 2$ and RM_{hyper} the space of Riemann matrices of hyperelliptic Riemann surfaces of genus $\gamma \geq 2$. Then, $\mathcal{M}_\gamma$ may be an open set in $C(E_{RM_{non-hyper}})$ and $\mathcal{N}_\gamma$ an open set in $C(E_{RM_{hyper}})$. Thus, our results obtained in Sections 2, 3, 4 and 5 are applicable in the study of compact orientable minimal surfaces in flat tori.

6.1. A minimal surface in an n-dimensional flat torus

Let M be a compact Riemann surface of genus γ and $\{A_i, B_i\}$ a canonical homology basis of M. Let $\{\psi_i\}$ be the basis of the space of holomorphic 1-forms on M such thst $\int_{A_i} \psi_j = \delta_{ij}$. The matrix $\tau = (\tau_{ij}) = (\int_{B_j} \psi_i)$ is called the Riemann matrix associated with M and $\{A_i, B_i\}$.

Ahlfors [1] proved that $RM_{non-hyper}$ is a $(3\gamma-3)$-dimensional complex submanifold, RM_{hyper} is a $(2\gamma-1)$-dimensional complex submanifold in H_γ and RM_{hyper} is the singularity of $RM_{non-hyper} \bigcup RM_{hyper}$. Thus we can consider $C(E_{RM_{non-hyper}})$ and $C(E_{RM_{hyper}})$, which are $Sp(\gamma, \mathbf{Z})$-invariant by Lemma 3.8 since $RM_{non-hyper}$ and RM_{hyper} are $Sp(\gamma, \mathbf{Z})$-invariant.

From a Rauch's result (see, for example, [8]), we note

Lemma 6.1. *Let τ be a Riemann matrix associated with M and $\{A_i, B_i\}$. Then $\overline{A} = \overline{(A_{ij})} \in S^2_C$ is a normal vector of $RM_{non-hyper}$, RM_{hyper} at τ in H_γ if and only if $\sum_{i,j=1}^{\gamma} A_{ij}\psi_i\psi_j = 0$, where each $A_{ij}\psi_i\psi_j$ means a holomorphic quadratic differential.*

We see that $(\tau, L) \in C(RM_{non-hyper})$ if and only if $\operatorname{grad} E(\tau, L) =$

$$2i\Big(\frac{1}{2}\overline{{}^t(L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})}\Big)\frac{1}{2}\Big(\overline{(L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})}\Big)$$

is a normal vector of $RM_{non-hyper}$ at τ. By Lemma 6.1, this is equivalent to

$$\begin{aligned}&{}^t((L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})^t(\psi_1, ..., \psi_\gamma))\times \\ &(L_1 + i[L_1\mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})^t(\psi_1, ..., \psi_\gamma) = 0.\end{aligned}$$

We consider the above equality as follows: We first determined a multivalued harmonic map S from M to $\mathbf{R}^n$ by integrating dS along a path from a fixed point whose real peroid matrix is (L_1, L_2) in [8]. In fact, one $\mathbf{R}^n$-valued harmonic 1-form dS as

$$(L_1, L_2)T_\tau^{-1}\,{}^t(\mathrm{Re}\psi_1, ..., \mathrm{Re}\psi_\gamma, \mathrm{Im}\psi_1, ..., \mathrm{Im}\psi_\gamma),$$

where $T_\tau = \begin{pmatrix} E_\gamma & \mathrm{Re}\tau \\ 0 & \mathrm{Im}\tau \end{pmatrix}$ and $T_\tau^{-1} = \begin{pmatrix} E_\gamma & -\mathrm{Re}\tau\mathrm{Im}\tau^{-1} \\ 0 & \mathrm{Im}\tau^{-1} \end{pmatrix}$ satisfies

$$(\int_{A_1} dS, ..., \int_{A_\gamma} dS, \int_{B_1} dS, ..., \int_{B_\gamma} dS) = (L_1, L_2).$$

However, by the ambiguity of a canonical homology basis for a non-hyperelliptic Riemann surface M, we obtain $(M, \{A_i, B_i\})$ and $(M, \{-A_i, -B_i\})$ for τ. The other is

$$-(L_1, L_2)T_\tau^{-1}\,{}^t(\mathrm{Re}\psi_1, ..., \mathrm{Re}\psi_\gamma, \mathrm{Im}\psi_1, ..., \mathrm{Im}\psi_\gamma)$$

for $(M, \{-A_i, -B_i\})$. Then

$$dS^{1,0} = \pm\frac{1}{2}(L_1 + i[L_1 \mathrm{Re}\tau - L_2](\mathrm{Im}\tau)^{-1})\, {}^t(\psi_1, ..., \psi_\gamma).$$

The above equality is equivalent to ${}^t dS^{1,0} dS^{1,0} = 0$, which implies that S is weakly conformal. The complex period map is given by $(\tau, K) \mapsto (K, K\tau)$. In this paper, we identify one multivalued branched minimal immersion of M into $\mathbf{R}^n$ with the other. Thus, $\mathcal{M}_\gamma = \{(\tau, L) \in C(E_{RM_{non-hyper}}) \,|\, \mathrm{rank}\, L = n\}$ for n.

By $C(E_{RM_{non-hyper}}) \xrightarrow{\pi} L_{n,2\gamma}$, we see that $C(E_{RM_{non-hyper}}) \cap \pi^{-1}(L)$ is the space of full multivalued branched minimal immersions of non-hyperelliptic Riemann surfaces of genus γ into $\mathbf{R}^n$ whose real periods are L. Let $G_L = \{g \in Sp(\gamma, \mathbf{Z}) \,|\, Lg = L\}$. Then G_L is the subgroup of $Sp(\gamma, \mathbf{Z})$, $C(E_{RM_{non-hyper}}) \cap \pi^{-1}(L)$ is G_L-invariant and a G_L-orbit gives the same full multivalued branched minimal immersions. If $L \in T_{n,2\gamma}$, then S is a full branched minimal immersion of M into the flat torus $\mathbf{R}^n/\langle L \rangle$.

We can conclude the similar result in the case of RM_{hyper} without the ambiguity.

Using φ in Lemma 2.3 and ψ in Proposition 5.2, we conclude the following.

Theorem 6.1. *The complex period map is given by*

$$\frac{1}{2}\varphi^{-1}\psi : C(E_{RM_{non-hyper}}) \bigcup C(E_{RM_{hyper}}) \longrightarrow \mathbf{H}^{n\gamma}.$$

As an application of Proposition 5.2 and Theorem 5.1, we prove

Theorem 6.2. *Let N be an irreducible component of $C(E_{RM_{non-hyper}})$ or $C(E_{RM_{hyper}})$ admitting a non-degenerate critical point. Then*

(1) $\dim_C N = n\gamma$,

(2) *ψ gives a complex Lagrangian cone in $T^*L_{n,2\gamma}$,*

(3) *A dense set of N gives full branched minimal immersions of compact Riemann surfaces of genus γ into n-dimensional flat tori.*

The connected component of non-degenerate critical points admits a special pseudo Kähler structure with holomorphic isometries S^1, $SO(n)$. An element of $Sp(\gamma, \mathbf{Z})$ gives a correspondence among irreducible components, furthermore, a holomorphic isometry among the special pseudo Kähler manifolds in the irreducible components.

Remark 6.1. The dense set of (3) for $RM_{non-hyper}$ gives non-branched minimal immersions [8].

We investigated $index_a$ and $nullity_a$ in [8].

Theorem 6.3. *Let S be a full minimal immersion of a compact Riemann surface M into an n-dimensional flat torus $\mathbf{R}^n/\langle L\rangle$ with a real period matrix L. If M is not hyperelliptic, then*

$$index_a = index_{E_{RM_{non-hyper}}}, \quad nullity_a = n + nullity_{E_{RM_{non-hyper}}}.$$

If M is hyperelliptic, then

$$\begin{aligned} index_a &= index_{E_{RM_{hyper}}} + \alpha, \\ nullity_a &= n + nullity_{E_{RM_{hyper}}} + 2\gamma - 4 - 2\alpha, \end{aligned}$$

where α is an integer satisfying $0 \leq \alpha \leq \gamma - 2$. If M has only trivial Jacobi fields, then $\alpha = \gamma - 2$ and hence $\gamma - 2 \leq index_a$ holds.

In [8], these results were proved in the case of a compact orientable minimal surface in an n-dimensional flat torus. We can formulate these results for multivalued branched minimal immersions of compact Riemann surfaces of genus γ into $\mathbf{R}^n$.

By Theorem 6.3, for an immersed non-hyperelliptic minimal surface, the immersion has only trivial Jacobi fields if and only if the corresponding critical point is non-degenerate.

However, we may not obtain the equivalence for an immersed hyperelliptic minimal surface. In fact, the Albanese map of a hyperelliptic Riemann surface has $2\gamma - 4$ non-trivial Jacobi fields ($\alpha = 0$) and $E_{RM_{hyper}}$ is non-degenerate.

In addition to the above example, there exist immersed hyperelliptic holomorphic curves of genus $\gamma \geq 2$ in complex flat tori of complex dimension 2 with $2\gamma - 4$ non-trivial Jacobi fields ($\alpha = 0$) [6], which implies $E_{RM_{hyper}}$ is non-degenerate. Hence these hyperelliptic holomorphic curves can not be deformed to hyperelliptic holomorphic curves in the same torus up to parallel translations, since a neighborhood of $\mathcal{N}_\gamma$ at the holomorphic curve is a graph on an open set in $L_{n,2\gamma}$.

On the other hand, we note the following by a result of Hitchin [18].

Proposition 6.1. *Let $\mathcal{C}$ be an immersed hyperelliptic holomorphic curve of genus $\gamma \geq 3$ in a 2-dimensional complex flat torus T. If $\mathcal{C}$ is a non-degenerate* *critical point of* $E_{RM_{hyper}}$*, then* $\mathcal{C}$ *can be deformed to non-hyperelliptic holomorphic curves in* T*.*

Proof. T has the canonical complex symplectic form. Then $\mathcal{C}$ is complex Lagrangian, since the complex symplectic form induced on $\mathcal{C}$ is a holomorphic 2-form and hence vanishes. Thus $\mathcal{C}$ admits the deformation space of real dimension $2\gamma - 4$ up to parallel translations in T [18]. Therefore $\mathcal{C}$ can be deformed to non-hyperelliptic holomorphic curves in T. $\square$

$index_{E_{RM_{non-hyper}}}$ and $index_{E_{RM_{hyper}}}$ are preserved by swelling. Assume that surjective condition is not satisfied. If we can choose the connected component satisfying surjective condition by swelling, then we can compute them. As an application of Theorems 5.3 and 6.3, we prove

Corollary 6.1. *For an irreducible component with a non-degenerate critical point, we have*

(1) *A connected component of non-degenerate critical points of an irreducible component* N *of* $\mathcal{M}_\gamma$ *admits a special pseudo Kähler structure of the signature* (p, q) *such that* $index_a \leq 6\gamma - 6 - 2q$*. Furthermore, if surjective condition is satisfied, then a tangent space of the complex Lagrangian cone corresponding to the connected component determines* $index_a$*.*

(2) *A connected component of non-degenerate critical points of an irreducible component* N *of* $\mathcal{N}_\gamma$ *admits a special pseudo Kähler structure of the signature* (p, q)*. Furthermore,* $index_a \leq 4\gamma - 2 - 2q + \alpha \leq 5\gamma - 4 - 2q$*. The subset of* $\mathcal{N}_\gamma$ *satisfying* $\alpha < \gamma - 2$ *is a complex analytic set.* $\gamma - 2 \leq index_a$ *holds on* N *except the complex analytic set. Moreover, if surjective condition is satisfied, then a tangent space of the complex Lagrangian cone corresponding to the connected component determines* $index_{E_{RM_{hyper}}}$ *and thus* $index_a = index_{E_{RM_{hyper}}} + \gamma - 2$ *except the complex analytic set.*

Proof. Irreducible components of $C(E_{RM_{non-hyper}})$ and $C(E_{RM_{hyper}})$ for $n = 2\gamma$ is unique [8], [2] (see Corollary 7.1 below). Since these two irreducible components admit surjective condition, we have two inequalities by applying swelling and Corollary 5.3.

We use B (in p.122 [8]). A symmetric bilinear form $2\mathrm{Imtr}(B(\mu, \nu)\, {}^tKK)$ is defined by Theorem 7.10 in [8]. Its real rank is 2α and hence the complex rank of $2\mathrm{tr}(B(\mu, \nu)\, {}^tKK)$ is α. As the determinant of $2\mathrm{tr}(B(\mu, \nu)\, {}^tKK)$ is holomorphic, the subset of $\mathcal{N}_\gamma$ satisfying $\alpha < \gamma - 2$ is a complex analytic set. $\square$

Remark 6.2. It is possible that the complex analytic set may be the empty set or N. If $2\mathrm{Imtr}(B(\mu,\nu)\,{}^tKK)$ at a critical point (τ, K) vanishes, the corresponding hyperelliptic minimal surface is holomorphic [8]. In particular, when $n = 3$ and $\gamma = 3$, we obtain $\alpha = 1$ and the complex analytic set is empty. For the case that the analytic set is N, see Theorem 7.3 below.

There exist irreducible components which satisfy or do not satisfy surjective condition [26], [4], [35]. We may obtain a better estimate by Corollary 5.3 for an irreducible connected component that does not satisfy surjective condition. Since $SO(n, \mathbf{C})$ does not preserve a connected component, it is interesting to study the change of $index_a$ and the signature with respect to the deformation of a minimal surface by $SO(n, \mathbf{C})$ (see, for example, [25] and [9]).

6.2. Minimal surfaces of genus 3 in 3-dimensional flat tori

6.2.1. How to use the algorithm.

Theorem 6.4. *$\mathcal{N}_3$ $(n = 3)$ satisfies the following.*

(1) *$\mathcal{N}_3$ consists of only one irreducible component,*

(2) *$\mathcal{N}_3$ admits a non-degenerate critical point, moreover, different connected components of non-degenerate critical points,*

(3) *$\mathcal{N}_3$ is an $SO(3, \mathbf{C}) \times \mathbf{C}^*$-bundle on RM_{hyper}, which satisfies surjective condition,*

(4) *$\mathcal{N}_3$ causes an embedded non-totally geodesic 9-dimensional complex Lagrangian cone in $\mathbf{C}^{18}$,*

(5) *$E_{RM_{hyper}}$ for any $L \in L_{3,6}$ with $\mathrm{rank}L = 3$ is a Morse family,*

(6) *Each connected component admits a special pseudo Kähler metric of signature (p, q) with holomorphic isometries S^1, $SO(3)$. $1 \leq index_a \leq 11 - 2q$ holds,*

(7) *There exists a dense set in RM_{hyper} whose point gives a minimal immersion into a flat torus.*

Proof. Each hyperelliptic Riemann surface of genus 3 admits a basis of holomorphic 1-forms ψ_i such that $\psi_1^2 + \psi_2^2 + \psi_3^2 = 0$ which has no other relation among the quadratic differentials $\psi_i\psi_j$. In fact, we consider different eight points $\{a_1, ..., a_8\} \subset \mathbf{C}$ and construct the plane curve

$$M : w^2 = (z - a_1) \cdots (z - a_8),$$

which is a hyperelliptic Riemann surface of genus 3. z is a meromorphic function on M and $\frac{dz}{w}$ is a holomorphic 1-form on M. Then

$$f(z) = \int^z {}^t\omega : M \longrightarrow J(M)$$

is the Albanese map, where $\omega = \frac{1}{2}((1 - z^2), (1 + z^2)i, 2z)\frac{dz}{w}$ and $J(M)$ is the Jacobi variety of M. ω gives a full multivalued minimal immersion of M in $\mathbf{R}^3$. Furthermore, $\alpha g\,{}^t\omega, \alpha \in \mathbf{C}^*, g \in SO(3, \mathbf{C})$ are different Weierstrass data, and thus $\mathcal{N}_3$ is an $SO(3, \mathbf{C}) \times \mathbf{C}^*$-bundle on $RM_{hyper} \subset RM_{hyper} \times L_{3,6}$, where surjective condition is satisfied. In particular, $\dim_C \mathcal{N}_3 = 9$.

As a local expression of the complex period map of $\mathcal{N}_3$ into $T^*L_{3,6} = K_{3,3} \times K_{3,3}$, we get

$$\alpha g\Big(\int_{A_1} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}, ..., \int_{B_3} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}\Big),$$

where $\alpha \in \mathbf{C}^*$, $g \in SO(3, \mathbf{C})$, $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ is a canonical homology basis, a_6, a_7, a_8 are fixed. We obtain the complex Lagrangian (branched) immersion $F_{\mathcal{N}_3}$ of $SO(3, \mathbf{C}) \times \mathbf{C}^* \times$ {the space of different five points (with fixed three points) in $\mathbf{C}$} into $K_{3,3} \times K_{3,3}$. When we choose another canonical homology basis, we obtain another local expression of $F_{\mathcal{N}_3}$.

We shall prove that $F_{\mathcal{N}_3}$ is an immersion. We can give $F_{\mathcal{N}_3}$ by $(\tau, K) \mapsto (K, K\tau) \in K_{3,3} \times K_{3,3}$, where $K = \alpha g K_1, \tau = K_1^{-1}K_2$,

$$K_1 = \Big(\int_{A_1} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}, \int_{A_2} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}, \int_{A_3} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}\Big),$$

$$K_2 = \Big(\int_{B_1} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}, \int_{B_2} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}, \int_{B_3} \begin{pmatrix} 1 - z^2 \\ i(1 + z^2) \\ 2z \end{pmatrix} \frac{dz}{w}\Big).$$

Therefore, each K is regular. Let $(\tau_0, K_0) \in \mathcal{N}_3$ and $(\tau(t), K(t))$ be a regular curve in $\mathcal{N}_3$ such that $\tau(0) = \tau_0, K(0) = K_0$. Then $F_{\mathcal{N}_3}(\tau(t), K(t))' = (K(t)', K(t)'\tau(t) + K(t)\tau(t)')$. If $F_{\mathcal{N}_3}(\tau(t), K(t))' = 0$ at $t = 0$, then $K(0)' =$

$0, \tau(0)' = 0$ because $K(0)$ is regular. Thus $F_{\mathcal{N}_3}$ is an immersion and, furthermore, an embedding.

Since $\alpha = 1$ in Remark 6.2, $nullity_a = 3 + nullity E_{RM_{hyper}}$ holds by $\gamma = 3$ and Theorem 6.3. Any null vector of the Hessian of $E_{RM_{hyper}}$ corresponds to a non-trivial Jacobi field [8]. We get an infinitesimal minimal deformation of multivalued immersion in $\mathbf{R}^3$ from the non-trivial Jacobi field by using the argument in [26] and the corresponding infinitesimal deformation in hyperelliptic Riemann surfaces of genus 3 [8]. Thus the null space of the Hessian of $E_{RM_{hyper}}$ is the null space of $\mathcal{N}_3$. Proposition 5.1 implies that $E_{RM_{hyper}}$ is a Morse family.

We prove that the obtained complex Lagrangian cone is not totally geodesic as follows: Assume that the complex Lagrangian cone is totally geodeic, that is, a non-degenerate complex Lagrangian subspace. Then, for the projection π_1 of the non-degenerate complex Lagrangian subspace into $L_{3,6}$, ${\pi_1}_*$ is isomorphism. The real period map of $\mathcal{N}_3$ is the composite of the complex period map of $\mathcal{N}_3$ into the non-degenerate complex Lagrangian subspace and π_1. Thus, a point admitting non-zero null spaces in $\mathcal{N}_3$ is a branched point of the complex period map. Therefore, the set of points admitting non-zero null spaces in $\mathcal{N}_3$ is a complex analytic set, which implies that the set of non-degenerate critical points in $\mathcal{N}_3$ is connected. This contradicts that Schwarz' P-surface has $index_{E_{RM_{hyper}}} = 0$ [31] and Schwarz' CLP-surface has $index_{E_{RM_{hyper}}} = 2$ [23].

The image of $\pi^{-1}(T_{3,6})$ by $\mathcal{N}_3 \longrightarrow RM_{hyper}$ is also a dense set. $\square$

We see $index_a = 1 + index_{E_{hyper}}$ for $\gamma = 3$ by Corollary 6.1 and Remark 6.2. We can state how to calculate $nullity_a$ and $index_a$ as follows:

Choose $a_1, ..., a_5$ and a canonical homology basis $\{A_1, A_2, A_3, B_1, B_2, B_3\}$.

$$T_i = \frac{\partial}{\partial a_i}\Big(\int_{A_k} {}^t\omega, \int_{B_k} {}^t\omega\Big) \text{ for } 1 \le i \le 5, \quad T_6 = \Big(\int_{A_k} {}^t\omega, \int_{B_k} {}^t\omega\Big),$$

$$T_7 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} T_6, \quad T_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} T_6, \quad T_9 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} T_6.$$

τ is $C_1^{-1}C_2$ for $T_6 = (C_1, C_2)$. If T_i for $1 \le i \le 9$ are linearly independent, then $\{\, T_i \,\}$ is a basis. W and $W_2 - W_1$ in Theorem 5.4 and Theorem 5.5 give

(1) $nullity_a$ is $3+$ the number of the zero-eigenvalue of W,

(2) $nullity_a$ is $3 + \frac{1}{2}$ the number of the zero-eigenvalue of W_1,

(3) If $|W| \neq 0$, then the critical point corresponding to the minimal surface is non-degenerate, its signature is the signature of W and $index_a$ is $1+$ the number of negative eigenvalues of $W_2 - W_1$.

Shoda and the author [12] gave W and $W_2 - W_1$ by periods of Abelian differentials of the first kind and the second kind.

6.2.2. The deformation space of embedded minimal surfaces of genus 3 in 3-dimensional flat tori. Let f be the same notation as in the proof of Theorem 6.4. Meeks [21] proved that if $\{a_1, ..., a_8\}$ satisfies $a_1a_2a_3a_4 > 0$ and

$$a_5 = -(1/\overline{a_1}), a_6 = -(1/\overline{a_2}), a_7 = -(1/\overline{a_3}), a_8 = -(1/\overline{a_4}),$$

then Ref and Imf give embedded hyperelliptic minimal surfaces of genus 3 in 3-dimensional flat tori and hence there exist two real 5-dimensional spaces of embedded hyperelliptic minimal surfaces of genus 3 in 3-dimensional flat tori. The spaces are called Meeks' family [36]. We remark that Schoen's Gyroid, Schwarz' H-surface and the Lidinoid do not belong to Meeks' family.

Since Meeks' family is not a subset of $\mathcal{N}_3$ as it stands, we extend Meeks' family to two real 9-dimensional families by the homothety and $SO(3)$-action. Furthermore we consider the deformation space of a fixed minimal surface ($\in$ Meeks' family) with a fixed canonical homology basis and construct two 9-dimensional deformation spaces of embedded hyperelliptic minimal surfaces of genus 3 in 3-dimensional flat tori. Thus we can consider Meeks' family as two 9-dimensional submanifolds in $\mathcal{N}_3$. Each deformation space may contain a minimal surface with different canonical homology bases. It is interesting to determine the subgroup of $Sp(3, \mathbf{Z})$ preserving the deformation space.

Furthermore, one Meeks' family is an image of the other by an element of $Sp(3, \mathbf{Z})$ as follows: The conjugate surface of Schwarz' CLP-surface is itself. Such an example also exists in rPD family (Karcher's TT-surfaces) $\subset$ Meeks' family. Thus, one Meeks' family intersect the other up to $Sp(3, \mathbf{Z})$ at the two minimal surfaces which admit no non-trivial Jacobi fields. Its deformations of such a minimal surface is a graph on a domain in a real 9-dimensional subspace $\subset L_{3,6}$ (see Proposition 6.2 below). Two Meeks' families contain the common neighborhood. By the real analyticity, one Meeks' family is identified with the other. Thus we obtain the uniqueness of Meeks' family up to $Sp(3, \mathbf{Z})$.

Finally, we focus attention on the deformation space of embedded minimal surfaces of genus 3 in 3-dimensional flat tori.

We first consider the deformation space of a 3-dimensional flat torus as follows: Column vectors of $L \in L_{3,6}$ span a lattice of $\mathbf{R}^3$ if and only if there exist a 3×3 real regular matrix X, a 3×6 real matrix g with integer entries and a 6×3 real matrix h with integer entries such that $L = Xg$ and $X = Lh$. Consequently we get $gh = E_3$. Therefore, we consider the deformation $L_s \in L_{3,6}$ of L such that $L_0 = L$ and column vectors of L_s span a lattice. Then there exists the 3×3 real regular matrix X_s such that $L_s = X_s g$ and $X_s = L_s h$ holds [8]. We define the deformation space $\{Xg\}$ of a 3-dimensional flat torus by $\{Xg \,|\, X \in GL(3, \mathbf{R}), \det X > 0\}$ or $\{Xg \,|\, X \in GL(3, \mathbf{R}), \det X < 0\}$ for g and h satisfying $gh = E_3$, which is an open subset of a linear subspace $\{Yg\} \subset L_{3,6}$, where Y is any 3×3 real matrix. Thus $\dim_R\{Xg\} = 9$. Note that $\{Xg\}$ contains its homothetic flat tori and $SO(3)$-orbit of a flat torus. We next define the deformation space of a hyperelliptic minimal surface of genus 3 in a 3-dimensional flat torus as a connected component of $\mathcal{N}_3 \cap \pi^{-1}(\{Xg\})$ containing the minimal surface. We may consider that some connected component of $\mathcal{N}_3 \cap \pi^{-1}(\{Xgq\})$, $q \in Sp(3, \mathbf{Z})$ gives the same deformation space.

By using Proposition 2.2, we can consider the same symplectic form on $L_{n,2\gamma} \times L_{n,2\gamma}$ as ω_0 in Proposition 5.6, which is also denoted by ω_0, as follows: For tangent vectors $(L_1, ..., L_{2\gamma})$ and $(L'_1, ..., L'_{2\gamma})$ at each point of $L_{n,2\gamma}$. Then

$$\omega_0((L_1, ..., L_{2\gamma}), (L'_1, ..., L'_{2\gamma})) = \frac{1}{2}\mathrm{tr}\Big((L_1, ..., L_{2\gamma})\, {}^tJ_0\, {}^t(L'_1, ..., L'_{2\gamma})\Big)$$

is an $Sp(\gamma, \mathbf{R})$-invariant symplectic form.

We obtain a criterion whether $\{Xg \,|\, X \in GL(3, \mathbf{R}), \det X > 0\}$ and $\{Xg \,|\, X \in GL(3, \mathbf{R}), \det X < 0\}$ is Lagrangian with respect to ω_0 for $n = 3$ and $\gamma = 3$.

Lemma 6.2. *Let $\{Xg\}$ be the deformation space of a flat torus, where $g = (A, B)$ and A, B are 3×3 matrices. The following statements are equivalent:*

(1) *$\{Xg\}$ is Lagrangian with respect to ω_0 in $L_{3,6}$,*

(2) *$A\,{}^tB$ is symmetric, and*

(3) *Let (L_1, L_2) be an element of $\{Xg\}$. Then $L_1\,{}^tL_2$ is symmetric.*

The set of $(L_1, L_2) \in L_{3,6}$ such that $L_1\,{}^tL_2$ is not symmetric is open dense in $L_{3,6}$.

The real period matrix of Schwarz' P-surface for a canonical homology basis is given by $L_0 = (IE_3)g_0$ such that

$$g_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \ h_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ g_0 h_0 = E_3$$

and I is the elliptic integral in [12]. Thus the deformation space of the ambient 3-dimensional flat torus containing Schwarz' P surface is $\{Xg_0\}$.

Thus the period matrices of minimal surfaces in Meeks' family are contained in $\{Xg_0\}$. Since g_0 satisfies the condition in Lemma 6.2, the part of Meeks' family contained in each connected component with the special pseudo Kähler structure of $\mathcal{N}_3$ is Lagrangian with respect to the pseudo Kähler form $\widetilde{\omega_0}$ on the connected component. The uniqueness of the Heegaard splitting of T^3 (Frohman and Hass [16], Boileau and Otal [5]) implies

Proposition 6.2. *The deformation space of an embedded minimal surface of genus* 3 *is a connected component of* $\pi^{-1}(\{Xg_0\}) \cap \mathcal{N}_3$ *containing the embedded minimal surface. Its singularity is in the sets of degenerate points. Each connected component of non-degenerate points in the deformation space is Lagrangian with respect to the pseudo Kähler form* $\widetilde{\omega_0}$ *on the connected component.*

Meeks' family is contained in a connected component of $\pi^{-1}(\{Xg_0\}) \cap \mathcal{N}_3$.

On the other hand, we remark that there exist deformation spaces of immersed minimal surfaces of genus 3 which are not Lagrangian by Lemma 6.2.

There exists a countable set (Property P in [21]) of the associate minimal surfaces of Schwarz' P-surface which are immersed in 3-dimensional flat tori, that is, there exists a countable set $\{\theta \in S^1\}$ such that $e^{i\theta}$ act on Schwarz' P surface as the associate minimal surface. Let $X_\theta g_\theta \in L_{3,6}$ be its real period matrix, where X_θ is a 3×3 real regular matrix, g_θ is a 3×6 real matrix with integer entries and has a 6×3 real matrix h_θ with integer entries such that $g_\theta h_\theta = E_3$. The condition in Lemma 6.2 for g_θ, h_θ is satisfied.

We define the deformation space of the associate minimal surface by the connected component of $\pi^{-1}(\{Xg_\theta\}) \cap \mathcal{N}_3$ containing the associate minimal surface. Since the associate minimal surface has only trivial Jacobi fields, the deformation space of the associate minimal surface is a 9-dimensional submanifold near the associate minimal surface. Furthermore, the connected component of non-degenerate points of the deformation space is also a 9-dimensional submanifold which is Lagrangian with respect to the pseudo Kähler form $\widetilde{\omega_0}$ in $\mathcal{N}_3$.

In particular, Schoen's Gyroid is an embedded asociate minimal surface of Schwarz' P-surface. By the embedding of Schoen's Gyroid and Proposition 6.2, the deformation space of Schoen's Gyroid is a connected component of $\pi^{-1}(\{Xg_0\}) \cap \mathcal{N}_3$ up to $Sp(3, \mathbf{Z})$. A neighborhood of Schoen's Gyroid in the deformation space is a 9-dimensional submanifold, which is a Lagrangian submanifold in $\mathcal{N}_3$.

On the other hand, a holomorphic isometry $e^{i\theta}$ of $\mathcal{N}_3$ in Theorem 6.4 also induces a Lagrangian submanifold containing Schoen's Gyroid as the image of Meeks' family by $e^{i\theta}$ action.

We show that the above two Lagrangian submanifolds are different as follows:

Lidinoid is an embedded associate minimal surface of Schwarz' H-surface. We can construct a one-parameter family of embedded minimal surfaces containing Schoen's Gyroid and Lidinoid, which is called rG family. rG family has an intersection point in Meeks' family up to $Sp(3, \mathbf{Z})$. Any minimal surface from Schoen's Gyroid, through Lidinoid, to the intersection point of the one-parameter family gives a non-degenerate critical point except the intersection point [13]. Thus there exists a 9-dimensional Lagrangian submanifold consisting of embedded minimal surfaces in 3-dimensional flat tori, which contains the one-parameter family from Schoen's Gyroid to the intersection point except itself. Since Lidinoid is not an associate minimal surface of a minimal surface in Meeks' family, we obtain two different Lagrangian submanifolds containing Schoen's Gyroid.

We are interested in the closure of the above 9-dimensional Lagrangian submanifold containing Schoen's Gyroid and Lidinoid in $\mathcal{N}_3$ and the intersection of it and Meeks' family since any point of the intersection is a degenerate critical point.

Proposition 6.3. *The intersection point of rG family and Meeks' family up to* $Sp(3, \mathbf{Z})$ *is a degenerate point.*

Proof. Assume that the intersection point is a non-degenerate point of Meeks' family. Since a neighborhood of the non-degenerate point in Meeks' family is the graph in $\mathcal{N}_3$ on an open set in the deformation of the ambient flat torus, some part near the non-degenerate point of rG family is contained in the graph. Hence any minimal surface of the part is contained in Meeks' family. By the real analyticity, Lidinoid is contained in Meeks' family. This is a contradiction. $\square$

7. The deformation space of a holomorphic curve in a complex flat torus

The deformation space of an immersed holomorphic curve of genus γ in a 2-dimensional complex flat torus is of complex dimension γ and admits a special Kähler structure by using a Hitchin's result [18].

Here, we consider the space of full multivalued holomorphic maps of compact Riemann surfaces of genus $\gamma \geq 2$ in $\mathbf{R}^{2m}$ with suitable orthogonal complex structures. If $\gamma < m$, then the holomorphic map is not full. Hence we may assume $m \leq \gamma$. We review the results obtained by Colombo and Pirola [6] and investigate special pseudo Kähler structures of the spaces.

7.1. The signature of $M_{\gamma,m}$

We will use notation as in the proof of Theorem 4.2. $T_{G(iE_\gamma, E_{2m}, a)}M_{\gamma,m}$, where $\operatorname{rank} a = m$, is spanned by (K, iK), where K is a tangent vector of $\mathbf{K}_{2m}$, and $(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a\tau \\ ia\tau \end{pmatrix})$, where $\tau \in S^2_C$. Let $a = (a_1, a_2)$, where a_1 is an $m \times m$ matrix and a_2 is an $m \times (\gamma - m)$ matrix. Suppose that a_1 is regular. Then we solve the equation: $(a_1, a_2)\begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix} = 0$ with respect to $\tau = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$. Then $A = a_1^{-1}a_2C\,{}^ta_2\,{}^ta_1^{-1}$ and $B = -a_1^{-1}a_2C$. Therefore $\dim_R\{(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a\tau \\ ia\tau \end{pmatrix}) \,|\, \tau = \begin{pmatrix} A & B \\ {}^tB & 0 \end{pmatrix} \in S^2_C\} = 2m\gamma - m^2 + m$. Lemma 4.1 implies $\dim_R \mathbf{K}_{2m} = 2m\gamma + m^2 - m$. It follows that the real dimension of tangent spaces of these points is $2m\gamma - m^2 + m + 2m\gamma + m^2 - m = 4m\gamma$.

In general, we consider $\operatorname{rank}(a_1, a_2) = m$. Since there is a $\gamma \times \gamma$ orthogonal matrix u such that $(a_1, a_2) = (a_1', a_2')u$ satisfies $\operatorname{rank} a_1' = m$. Let $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \in Sp(\gamma, \mathbf{R})$. Then

$$(\begin{pmatrix} a' \\ ia' \end{pmatrix}, i\begin{pmatrix} a' \\ ia' \end{pmatrix})\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = (\begin{pmatrix} a'u \\ ia'u \end{pmatrix}, i\begin{pmatrix} a'u \\ ia'u \end{pmatrix}) = (\begin{pmatrix} a \\ ia \end{pmatrix}, i\begin{pmatrix} a \\ ia \end{pmatrix}).$$

By Proposition 2.1, $\dim_R T_{G(iE_\gamma, E_{2m}, a)}M_{\gamma,m} = 4m\gamma$, where $\operatorname{rank} a = m$. Thus the tangent spaces are complex Lagrangian subspaces by Theorem 4.2.

We prove that the tangent spaces at these points satisfy non-degenerate condition. It is enough to give a proof for points satisfying $\operatorname{rank} a_1 = m$.

$$(\begin{pmatrix} X & Y \\ -{}^tY & Z \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}, i\begin{pmatrix} X & Y \\ -{}^tY & Z \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}),$$

where X, Z are real skew symmetric matrices,

$$(\begin{pmatrix} a_1' & a_2' \\ ia_1' & ia_2' \end{pmatrix}, \begin{pmatrix} ia_1' & ia_2' \\ -a_1' & -a_2' \end{pmatrix}), \quad (0, \begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}\begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & 0 \end{pmatrix})$$

span the tangent space. Proposition 5.4 implies that non-degenerate condition is equivalent to a claim that if the real part of the sum of these vectors is 0, then the sum is zero. We prove the claim as follows:

Assume that the real parts of the sum of these vectors is 0. Thus we obtain two equations:

$$\begin{aligned} \operatorname{Re}\{\begin{pmatrix} X & Y \\ -{}^tY & Z \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}\} + \operatorname{Re}\begin{pmatrix} a_1' & a_2' \\ ia_1' & ia_2' \end{pmatrix} &= 0 \text{ and} \\ \operatorname{Re} i\{\begin{pmatrix} X & Y \\ -{}^tY & Z \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}\} + \operatorname{Re}\begin{pmatrix} ia_1' & ia_2' \\ -a_1' & -a_2' \end{pmatrix} & \\ + \operatorname{Re}\begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix}\begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & 0 \end{pmatrix} &= 0. \end{aligned}$$

The former equation is given by

$$\begin{aligned} \operatorname{Re} a_1' &= -X\operatorname{Re} a_1 + Y\operatorname{Im} a_1, \quad \operatorname{Im} a_1' = -{}^tY\operatorname{Re} a_1 - Z\operatorname{Im} a_1 \text{ and} \\ \operatorname{Re} a_2' &= -X\operatorname{Re} a_2 + Y\operatorname{Im} a_2, \quad \operatorname{Im} a_2' = -{}^tY\operatorname{Re} a_2 - Z\operatorname{Im} a_2, \end{aligned}$$

which, together with the latter equation, implies

$$(X - Z - i({}^tY - Y))(a_1, a_2) = i\,\overline{(a_1, a_2)\begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & 0 \end{pmatrix}}.$$

It is equivalent to $\tilde{X}a_1 = i\,\overline{a_1\tau_{11}} + i\,\overline{a_2\,{}^t\tau_{12}}$, $\tilde{X}a_2 = i\,\overline{a_1\tau_{12}}$ where $\tilde{X} = (X - Z - i({}^tY - Y))$. $\tilde{X}$ is skew symmetric. Since ${}^t\overline{\tau_{12}} = i\,{}^ta_2\tilde{X}\,{}^t\overline{a_1}^{-1}$, we get $\tilde{X}a_1\,{}^t\overline{a_1} + \overline{a_2}\,{}^ta_2\tilde{X} = i\overline{a_1\tau_{11}}\,{}^t\overline{a_1}$. Thus $\tilde{X}a_1\,{}^t\overline{a_1} + \overline{a_2}\,{}^ta_2\tilde{X}$ is symmetric. It follows that $\tilde{X}(a_1\,{}^t\overline{a_1} + a_2\,{}^t\overline{a_2})$ is a symmetric matrix because $\tilde{X}$ is skew symmetric. We denote $(a_1\,{}^t\overline{a_1} + a_2\,{}^t\overline{a_2})$ by $\tilde{Y}$, which is a positive definite Hermitian matrix since $\operatorname{rank}(a_1, a_2) = m$. Note that $\tilde{S} = \tilde{X}\tilde{Y}$ is symmetric. We choose a unitary matrix U such that $U^*\tilde{Y}U$ is positive diagonal. We set $\tilde{X}' = {}^t U \tilde{X} U$ and $\tilde{Y}' = U^* \tilde{Y} U$. $\tilde{X}'\tilde{Y}' = {}^t U \tilde{S} U$ is symmetric. Since $\tilde{X}'$ is skew symmetric and $\tilde{Y}'$ is positive diagonal, $\tilde{X}' = 0$ and hence $\tilde{X} = 0$. Thus we obtain

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ ia_1 & ia_2 \end{pmatrix} + \begin{pmatrix} a_1' & a_2' \\ ia_1' & ia_2' \end{pmatrix} = 0 \text{ and } \begin{pmatrix} \tau_{11} & \tau_{12} \\ {}^t\tau_{12} & 0 \end{pmatrix} = 0,$$

which complete the proof of the claim.

We investigate the signature (p, q) of $M_{\gamma,m}$ at $(iE_\gamma, E_{2m}, (E_m\, 0))$. Its tangent space is spanned by

$$\text{type 1} \quad (\begin{pmatrix} X + iY & 0 \\ -{}^tY + iZ & 0 \end{pmatrix}, \begin{pmatrix} X + iY & 0 \\ -{}^tY + iZ & 0 \end{pmatrix} i), \quad \text{type 3} \quad (0, \begin{pmatrix} A & B \\ iA & iB \end{pmatrix}),$$

where X, Z are real skew, Y is real and A is symmetric.

$$\text{type 5} \quad (\begin{pmatrix} 0 & \beta \\ 0 & i\beta \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & i\beta \end{pmatrix} i), \quad \text{type 6} \quad (\begin{pmatrix} \alpha & 0 \\ i\alpha & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ i\alpha & 0 \end{pmatrix} i),$$

where α is Hermitian. The non-degenerate pseudo inner product $\mathrm{Re}\eta_2$ is null on the subspace spanned by type 3, whose real dimension is $m(m+1) + 2m(\gamma - m) = 2m\gamma - m^2 + m$. Consequently $2q \geq 2m\gamma - m^2 + m$. Three subspaces spanned by vectors of types 1, 5 and 6 are orthogonal each other, furthermore, the pseudo inner product is positive definite on three subspaces and hence $2p \geq m(m-1) + m^2 + 2m(\gamma - m) + m^2 = 2m\gamma + m^2 - m$. $p + q = 2m\gamma$ implies $2q = 2m\gamma - m^2 + m$ and $2p = 2m\gamma + m^2 - m$.

$\Phi : H_\gamma \times \mathbf{K}_{2m} \longrightarrow K_{2m,\gamma} \times K_{2m,\gamma}$ is an $Sp(\gamma, \mathbf{R})$, $SO(2m)$-equivariant map. Let $N_{\gamma,m}$ be $\Phi(H_\gamma \times \mathbf{K}_{2m})$ in $K_{2m,\gamma} \times K_{2m,\gamma}$ which is given by

$$G(\{(\tau, g, a) | \tau \in H_\gamma,\ g \in SO(2m),\ a \in K_{m,\gamma},\ \mathrm{rank}\, a = m\}).$$

The orbits of $G(iE_\gamma, E_{2m}, a)$ with $\mathrm{rank}\, a = m$ by $Sp(\gamma, \mathbf{R})$ and $SO(2m)$ is $N_{\gamma,m}$. Since $Sp(\gamma, \mathbf{R})$ and $SO(2m)$ are holomorphic isometries of $\mathrm{Re}\eta_2$, the signature was preserved.

Theorem 7.1. *For Φ, $\Phi_*(T_{(\tau,K)} H_\gamma \times \mathbf{K}_{2m})$ is a complex Lagrangian subspace with the signature $(m\gamma + \frac{m(m-1)}{2}, m\gamma - \frac{m(m-1)}{2})$.*

7.2. The signature of the space of multivalued holomorphic maps

The space of full multivalued holomorphic maps of non-hyperelliptic Riemann surfaces of genus γ in $\mathbf{R}^{2m}$ with suitable orthogonal complex structures is given by $E = \{(\tau, K) \,|\, \tau \in RM_{non-hyper}, K \in \mathbf{K}_{2m}\}$ [8], [3] up to h in the proof of Theorem 4.2. If $m \geq 3$, then there exists an immersed holomorphic curve with only trivial Jacobi fields [6] which is a non-degenerate critical point of E. Hence, non-degenerate critical points of E is an open dense set of E by the real analyticity of E.

By Theorem 6.2, the complex dimension of the irreducible component containing a non-degenerate critical point of E in $\mathcal{M}_\gamma$ for $n = 2m$ is $2m\gamma$. On the other hand, by Lemma 4.1 and $m \leq \gamma$, we obtain

$$\begin{aligned}\dim_C E - 2m\gamma &= 3\gamma - 3 + \frac{1}{2}m(m-1) + m\gamma - 2m\gamma \\ &= \frac{1}{2}(m-3)(m-(2\gamma-2)).\end{aligned}$$

Thus if $m \geq 4$, then E is not an irreducible component of $\mathcal{M}_\gamma$. Arezzo and Micallef [3] proved that the deformation of a non-hyperelliptic holomorphic curve with only trivial Jacobi fields can be deformed to a non-holomorphic, non-hyperelliptic stable minimal surface. If $m = 3$, then, Theorems 4.2 and 7.1 imply

Theorem 7.2. *E is not an irreducible component in $\mathcal{M}_\gamma$ for $\gamma \geq m \geq 4$. There exists an irreducible component of $\mathcal{M}_\gamma$ containing a non-degenerate critical point of E. The corresponding holomorphic curve with only trivial Jacobi fields can be deformed to a non-holomorphic, non-hyperelliptic stable minimal surface. For $\gamma \geq m = 3$, E has a special pseudo Kähler structure of signature $(3\gamma + 3, 3\gamma - 3)$ with holomorphic isometries S^1, $SO(6)$ and $Sp(\gamma, \mathbf{Z})$ possibly with a complex analytic set as singularities.*

Proof. We first prove that E for $\gamma \geq m = 3$ gives an irreducible component of $\mathcal{M}_\gamma$. In fact, an irreducible component containing a non-degenerate critical point of E contains an open set consisting of non-degenerate critical poins in E. Since $\eta \circ \Phi : (\tau, K) \to K$ is a holomorphic map such that ${}^t(\eta \circ \Phi)(\eta \circ \Phi)$ vanishes on the open set. Hence ${}^t(\eta \circ \Phi)(\eta \circ \Phi) = 0$ on the irreducible component. So, the irreducible component is contained in E.

We next calculate the signature. Since $\Phi : E(\subset H_\gamma \times \mathbf{K}_6) \to K_{6,\gamma} \times K_{6,\gamma}$ is holomorphic and E has a non-degenerate critical point, the singular locus of the map is an complex analytic set. For $(\tau, K) \in E$ except the complex analytic set, $T_{(\tau,K)}E$ is a non-degenerate complex Lagrangian subspace with the signature $(3\gamma+3, 3\gamma-3)$ by Theorem 7.1. □

For $m=2$, $\dim_C E = 4\gamma+\gamma-2 = \dim_C L_{4,2\gamma}+\gamma-2$. Note that $\gamma-2$ is the dimension of the deformation space (up to translations) for an immersed holomorphic curve of genus γ in a 2-dimensional complex flat torus. It is plausible that, for the projection π of E into $L_{4,2\gamma}$, π_* is surjective. Does E or $E\times\mathbf{C}^2$ containing parallel translations by vectors of $\mathbf{C}^2$ admit a special pseudo Kähler structure?

The space of full multivalued holomorphic maps of hyperelliptic Riemann surfaces of genus γ in $\mathbf{R}^{2m}$ with suitable orthogonal complex structures is $F=\{(\tau,K)\,|\,\tau\in RM_{hyper}, K\in\mathbf{K}_{2m}\}$ up to h in the proof of Theorem 4.2. If $\gamma\geq m\geq 2$, F has a non-degenerate critical point by [6]. Hence, non-degenerate critical points of F is an open dense set of F by real analyticity of F.

By Lemma 4.1,

$$\begin{aligned}\dim_C F - 2m\gamma &= 2\gamma-1+\frac{1}{2}m(m-1)+m\gamma-2m\gamma\\ &= \frac{1}{2}(m-2)(m-(2\gamma-1))\end{aligned}$$

and hence if $\gamma\geq m\geq 3$, then F is not an irreducible component of $\mathcal{N}_\gamma$. Micallef [22] proved that a non-holomorphic hyperelliptic minimal surface is unstable ($index_a\geq 1$). On the other hand, if $m=2$, then we get

Theorem 7.3. *F for $\gamma\geq m\geq 3$ can not give an irreducible component of $\mathcal{N}_\gamma$. There exists an irreducible component of $\mathcal{N}_\gamma$ containing a non-degenerate critical point of F. The corresponding hyperelliptic holomorphic curve can be deformed to a non-holomorphic hyperelliptic minimal surface, which is unstable. For $m=2$, F has a special pseudo Kähler structure of signature $(2\gamma+1, 2\gamma-1)$ with holomorphic isometries S^1, $SO(4)$ and $Sp(\gamma,\mathbf{Z})$ possibly with a complex analytic set as singularities.*

A hyperelliptic minimal surface is said to be hyperelliptically stable [8] if the Hessian of $E_{RM_{hyper}}$ is semi-positive at the critical point corresponding to the minimal surface. Corollary 5.3 and Theorems 7.2, 7.3 imply

Corollary 7.1. *There exist connected components of signature $(n\gamma-3\gamma+3, 3\gamma-3), \gamma\geq 4, 7\leq n\leq 2\gamma$ satisfying surjective condition for non-hyperelliptic minimal surfaces and connected components of signature $(n\gamma-$*

$2\gamma+1, 2\gamma-1), \gamma \geq 3, 5 \leq n \leq 2\gamma$ satisfying surjective condition for hyperelliptic minimal surfaces which contain non-holomorphic minimal surfaces. In particular, corresponding non-hyperelliptic minimal surfaces are stable and corresponding hyperelliptic minimal surfaces are hyperelliptically stable. For $n = 2\gamma$, there exists a unique irreducible component of non-hyperelliptic minimal surfaces and a unique irreducible component of hyperelliptic minimal surfaces. Then the above connected component is also unique in each irreducible component. The two connected components admit a holomorphic isometry group $Sp(\gamma, \mathbf{Z})$. Furthermore, the irreducible component of non-hyperelliptic minimal surfaces contains a connected component for unstable non-hyperelliptic minimal surfaces.

Proof. E in Theorem 7.2 is an irreducible component satisfying surjective condition of $\mathcal{M}_\gamma$ for $n = 6$ and $\gamma \geq 4$. Its signature is $(3\gamma+3, 3\gamma-3)$. Since E has the open dense set of non-degenerate critical points of E, we obtain an irreducible component satisfying surjective condition of $\mathcal{M}_\gamma$ for $n = 7$ and $\gamma \geq 4$ by swelling of a non-degenerate critical point. By Corollary 5.3, the signature of the connected component containing the swelled non-degenerate critical point is $(7\gamma - 3\gamma + 3, 3\gamma - 3)$. Repeatedly, we obtain the desired connected components for non-hyperelliptic minimal surfaces. Similarly, by using F in Theorem 7.3, we construct the connected component for hyperelliptic minimal surfaces.

For $n = 2\gamma$, $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ admit a unique irreducible component, which satisfies surjective condition. We review it [8]. Let $P(\tau)$ be an embedding of $RM_{non-hyper}$ and RM_{hyper} into the space S^2_R of $2\gamma \times 2\gamma$ real symmetric matrices and the energy function for L is the height function in the direction of ${}^tLL$. Then a point (τ, L), where $\operatorname{rank} L = 2\gamma$, is a critical point of $\frac{1}{2}\mathrm{tr}(P(\tau)\,{}^tLL)$ if and only if ${}^tLL$ is a normal vector of the submanifolds $P(RM_{non-hyper})$ and $P(RM_{hyper})$ in S^2_R. Thus $\mathcal{M}_\gamma/O(2\gamma)$, $\mathcal{N}_\gamma/O(2\gamma)$ are identified with the subset $(\tau, \widetilde{L})$ consisting of $\widetilde{L}$ which is a normal vector and a positive definite matrix, which is an open set of the normal bundle. In each normal space at $P(\tau)$, the subset $(\tau, \widetilde{L})$ consisting of $\widetilde{L}$ above is a non-empty convex set which contains the positive definite matrix corresponding to the Albanese map. If $(\tau, \widetilde{L_1})$ and $(\tau, \widetilde{L_2})$ are stable, then $(\tau, (1-s)\widetilde{L_1} + s\widetilde{L_2})$ for any $s \in [0, 1]$ is stable. Thus we get a uniqueness of such a connected component [8], [2]. We see that the action of $SO(n, \mathbf{C})$ deforms a stable minimal surface except a holomorphic curve to an unstable minimal surface [9]. $\square$

Remark 7.1. All obtained stable non-hyperelliptic minimal surfaces and all obtained hyperelliptically stable hyperelliptic minimal surfaces satisfy the equality in Theorem 5.3. We are very interested in its geometric meaning.

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