On positive scalar curvature bordism

PAOLO PIAZZA, THOMAS SCHICK, AND VITO FELICE ZENOBI

Using standard results from higher (secondary) index theory, we prove that the positive scalar curvature bordism groups $\mathrm{Pos}_{4n}^{\mathrm{spin}}(G \times \mathbb{Z})$ are infinite for any $n \geq 1$ and G a group with non-trivial torsion. We construct representatives of each of these classes which are connected and with fundamental group $G \times \mathbb{Z}$. We get the same result for $\mathrm{Pos}_{4n+2}^{\mathrm{spin}}(G \times \mathbb{Z})$ if G is finite and contains an element which is not conjugate to its inverse. This generalizes the main result of Kazaras, Ruberman, Saveliev, "On positive scalar curvature cobordism and the conformal Laplacian on end-periodic manifolds" to arbitrary even dimensions and arbitrary groups with torsion.

1. Introduction

The classification of Riemannian metrics of positive scalar curvature (up to suitable natural equivalence relations) is an active object of study in geometry. One popular way to organize this uses the *Stolz positive scalar curvature exact sequence* (compare [11, Proposition 1.27]

$$(1.1) \quad \cdots \to \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma) \to \Omega_n^{\mathrm{spin}}(B\Gamma) \to R_n^{\mathrm{spin}}(B\Gamma) \to \mathrm{Pos}_{n-1}^{\mathrm{spin}}(B\Gamma) \to \cdots$$

Here, $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$ is one of our main objects of study, the group of closed n-dimensional spin manifolds (M, f, g) with a reference map $f \colon M \to B\Gamma$ and a Riemannian metric g of positive scalar curvature. The equivalence relation is bordism, where the bordisms have to carry the corresponding structure. We use throughout the usual convention that Riemannian metrics on manifolds with boundary (e.g. on a bordism) must have product structure near the boundary.

In this setup, Γ is just an arbitrary group. If the starting point is a connected smooth manifold M, typically one chooses $\Gamma = \pi_1(M)$. Moreover,

$B\Gamma$ is a classifying space for Γ and the map f contains essentially the same information as the homomorphism $f_*\colon \pi_1(M) \to \Gamma$.

The group $\Omega_n^{\mathrm{spin}}(B\Gamma)$ is the usual bordism group, whereas $R_n^{\mathrm{spin}}(B\Gamma)$ is Stolz' R-group, defined as the set of bordism classes (W, f, g) where W is a compact $(n+1)$-dimensional spin-manifold, possibly with boundary, with a reference map $f\colon W \to B\Gamma$, and with a positive scalar curvature metric *on the boundary* when the latter is non-empty.

Our main goal is to show that the positive scalar curvature bordism groups $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$ are rich (more precisely, map onto an infinite cyclic group) in new situations. Because the starting point often is a fixed connected manifold M with fundamental group Γ, we will show in addition that the infinitely many different non-trivial representatives can be chosen to be connected and with fundamental group Γ (mapped bijectively under the reference map to $B\Gamma$), even in dimension 4. Indeed, we are taking up the main result [9, Theorem 1] of Kazaras, Ruberman, and Saveliev, which says

Theorem 1.2. *Let $n = 4$ or 6 and $\{1\} \neq G$ be a fundamental group of a 3-dimensional spherical space form if $n = 4$, or more generally a finite group with at least one element not conjugate to its inverse if $n = 6$. Set $\Gamma := G \times \mathbb{Z}$.*

Then $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$ contains infinitely many elements, represented by maps $f\colon M \to B\Gamma$ where M is connected and f induces an isomorphism in π_1.

We improve that theorem by allowing more general groups and all even dimensions bigger than 2.

Theorem 1.3. *Let $n > 2$ be an even integer. If $n = 4k$ is divisible by 4, let G be an arbitrary finitely presented group which contains a non-trivial torsion element. If $n = 4k + 2$ let G be a finite group such that at least one of its elements is not conjugate to its inverse. Set $\Gamma := G \times \mathbb{Z}$.*

Then $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$ contains infinitely many elements x_j, represented by connected manifolds M_j with fundamental group Γ as in the Theorem of Kazaras, Ruberman, and Saveliev.

Even better, we have a non-trivial homomorphism $\mathrm{Ind}_\rho\colon R_{n+1}^{\mathrm{spin}}(B\Gamma) \to \mathbb{R}$ and the x_j lift to elements in $R_{n+1}^{\mathrm{spin}}(B\Gamma)$ whose image under Ind_ρ form an infinite cyclic subgroup of $\mathbb{R}$.

Theorem 1.2 of [9] is based on the beautiful, but rather complicated and technical index theory for manifolds with periodic ends. The associated

bordism invariance is very delicate and uses, for the time being, minimal hypersurface techniques which cause the dimension restriction in Theorem 1.2

The main point we want to make in this note is that this delicate theory is actually not necessary for the result at hand. Instead, it can be derived from well known (and not too complicated) results in higher index theory, and along the way one gets the more general result. Related techniques and results are developed in [18] which can be used to get even stronger results, compare the discussion below.

2. Secondary higher index

The distinction of bordism classes of metrics of positive scalar curvature in the literature typically relies on secondary index invariants of the Dirac operator, and this is precisely how we prove the main part of Theorem 1.3.

Recall that the classical (higher) index of the Dirac operator can be defined for closed manifolds, but also for manifolds with boundary provided the boundary operator is invertible [11, Section 2.2]. In particular, we have a commutative diagram

$$(2.1) \quad \begin{array}{ccc} \Omega_n^{\mathrm{spin}}(B\Gamma) & \longrightarrow & R_n^{\mathrm{spin}}(B\Gamma) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Ind}} \\ K_n(B\Gamma) & \xrightarrow{\ \mu\ } & K_n(C^*\Gamma) \end{array}$$

Here, μ is the Baum-Connes assembly map which in this setup is *not* an isomorphism if Γ is not torsion-free.

This diagram is the easy and elementary part of the diagram "mapping the Stolz exact sequence to analysis" developed in [11]. Recall that $\Gamma = G \times \mathbb{Z}$. The diagram (2.1) and the corresponding one for G are closely related via a kind of Künneth theorem. Indeed, we have a transformation, given by product with S^1 or its fundamental K-theory class (and using that $B\Gamma = BG \times S^1$ with $S^1 = B\mathbb{Z}$)

$$(2.2) \quad \begin{array}{ccc} \Omega_n^{\mathrm{spin}}(BG) & \longrightarrow & R_n^{\mathrm{spin}}(BG) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Ind}} \\ K_n(BG) & \xrightarrow{\ \mu\ } & K_n(C^*G) \end{array} \xrightarrow{\times [S^1]} \begin{array}{ccc} \Omega_{n+1}^{\mathrm{spin}}(B\Gamma) & \longrightarrow & R_{n+1}^{\mathrm{spin}}(B\Gamma) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Ind}} \\ K_{n+1}(B\Gamma) & \xrightarrow{\ \mu\ } & K_{n+1}(C^*\Gamma) \end{array}$$

Lemma 2.3. *After removing the term $R_*^{\mathrm{spin}}(\cdot)$, the map $\times[S^1]$ of (2.2) is an embedding as direct summand of the left diagram into the right. In other*

words, the maps $\times[S^1]$ are injective and the map μ is compatible with a suitable direct sum decomposition of the right hand side into the left hand side and a complement.

Proof. This is certainly well known. For the convenience of the reader we give here a detailed proof.

For a multiplicative generalized homology theory E_* and for natural transformations of such, in particular for $\Omega_*^{\mathrm{spin}} \to K_*$, this is a consequence of the natural Künneth theorem for the special product $X \times S^1$ or equivalently of the suspension isomorphism. The desired splitting of $E_*(X) \xrightarrow{\times[S^1]} E_{*+1}(X \times S^1)$ is obtained as the composition

$$E_{*+1}(X \times S^1) \xrightarrow{pr} \tilde{E}_{*+1}(X \times S^1/X \times *)$$
$$\xrightarrow{=} \tilde{E}_{*+1}(\Sigma(X_+)) \xrightarrow{\sigma^{-1}}_{\cong} \tilde{E}_*(X_+) = E_*(X).$$

Here pr is the canonical projection, X_+ is the disjoint union of X and an additional basepoint, $\Sigma(X_+)$ its reduced suspension, and σ is the suspension isomorphism. The complementary summand is the (injective) image $i_*(E_{*+1}(X)) \subset E_{*+1}(X \times S^1)$ for $i\colon X \hookrightarrow X \times S^1; x \mapsto (x, *)$ (or the exterior product with the unit 1), so that we get in particular the direct sum decomposition

$$K_{*+1}(X \times S^1) = K_{*+1}(X) \otimes 1 \oplus K_*(X) \otimes [S^1].$$

For the C^*-algebra K-theory, the Künneth theorem [15, Theorem 4.1] specializes in our situation to the isomorphism (given by external tensor product) $K_*(C^*G) \otimes K_*(C^*\mathbb{Z}) \to K_*(C^*G \otimes C^*\mathbb{Z}) = K_*(C^*\Gamma)$. Using $K_*(C^*\mathbb{Z}) = 1 \cdot \mathbb{Z} \oplus [S^1] \cdot \mathbb{Z}$ with unit $1 \in K_0$ and $[S^1] \in K_1(C^*\mathbb{Z})$, the image of $[S^1]$ under the Baum-Connes isomorphism $\mu_{S^1}\colon K_*(S^1) \xrightarrow{\cong} K_*(C^*\mathbb{Z})$, we obtain the splitting

$$K_{*+1}(C^*\Gamma) = K_{*+1}(C^*G) \otimes 1 \oplus K_*(C^*\Gamma) \otimes [S^1].$$

The Baum-Connes map μ is compatible with external products, proving the rest of the claims. $\square$

The strategy is now the following.

(1) We recall a suitable homomorphism $\mathrm{Ind}_\rho\colon K_n(C^*G) \to \mathbb{R}$, a delocalized index. "Delocalized" means in particular that $\mathrm{Ind}_\rho \circ \mu\colon K_n(BG) \to \mathbb{R}$ is zero.

(2) We will construct an appropriate element $[W, f, g] \in R_n^{\mathrm{spin}}(BG)$ with $\mathrm{Ind}_\rho(\mathrm{Ind}[W, f, g]) \neq 0$. It generates an infinite cyclic subgroup of $R_n^{\mathrm{spin}}(BG)$ detected in $K_n(C^*G)$ and then in $\mathbb{R}$ as image of the maps Ind and $\mathrm{Ind}_\rho \circ \mathrm{Ind}$, respectively.

(3) By the commutativity of (2.1) and using item (1), no non-trivial element of this infinite cyclic subgroup is in the image of $\Omega_n^{\mathrm{spin}}(BG) \to R_n^{\mathrm{spin}}(BG)$, and therefore this infinite cyclic subgroup is mapped injectively to $\mathrm{Pos}_{n-1}^{\mathrm{spin}}(BG)$.

Finally, we take the Cartesian product with S^1.

Corollary 2.4. *The element $W \times S^1$ generates an infinite cyclic subgroup of $R_{n+1}^{\mathrm{spin}}(B\Gamma)$ detected in $K_{n+1}(C^*\Gamma)$ as image of the map* Ind.
None of its non-zero elements is in the image of $\Omega_{n+1}^{\mathrm{spin}}(B\Gamma)$, and so this infinite cyclic group generated by $W \times S^1$ injects into $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$.

Proof. This follows from the commutativity of (2.1) and the fact that the Künneth map (2.2) gives an inclusion as direct summand of the whole map $K_n(BG) \xrightarrow{\mu} K_n(C^*G)$ into $K_{n+1}(B\Gamma) \xrightarrow{\mu} K_{n+1}(C^*\Gamma)$. $\qquad\square$

It remains to construct W and Ind_ρ with the appropriate properties. In all cases, W will be a null bordism of a disjoint union of lens spaces or lens space bundles, we describe this explicitly later.

Case $n \equiv 0 \pmod 4$. Essentially, this case has been treated in [10] and we refer to this paper for more details. We choose $\mathrm{Ind}_\rho \colon K_0(C^*G) \to \mathbb{R}$ to be equal to the difference of the homomorphisms induced by the standard trace and by the trivial representation. Recall that the first one gives the L^2-index and it is induced by the canonical trace $\mathrm{tr}_G \colon C^*G \to \mathbb{C}$, while the second one gives the ordinary index and is induced by the trivial homomorphism $G \to \{1\}$.

It is a direct consequence of Atiyah's L^2-index theorem [1, (1.1)] that Ind_ρ defined in this way vanishes on the image of μ.

Small caveat: we have to use a C^*-completion C^*G of $\mathbb{C}[G]$ such that the trivial homomorphism extends to this completion. For this, one could use the maximal C^*-completion or a smaller, more geometric one (based on the direct sum of the regular representation and the trivial representation). The relevant theory is well known, compare e.g. [14].

We have to compute Ind_ρ for a compact manifold W of dimension n with boundary L of dimension $n - 1$, where the boundary is equipped with a metric of positive scalar curvature, and where W is equipped with a reference

map to BG. Again, this is a classical result, an easy special case of the (L^2)-Atiyah-Patodi-Singer index theorem [13]: this index is the Cheeger-Gromov L^2-ρ-invariant of the positive scalar curvature metric of the boundary L, where we use the G-covering pulled back via the map to BG.

Finally, for a cyclic group C_p of order p which we choose to be prime, the L^2-ρ-invariant is just the Atiyah-Patodi-Singer ρ-invariant associated to $\alpha := \frac{1}{p}\rho_{reg} - \rho_1$, the linear combination of the trivial and the regular representation in the complexified representation ring.

For the group G with non-trivial torsion element, we choose an embedding $\iota\colon C_p \to G$ with induced map $B\iota\colon BC_p \to BG$. The L^2-ρ-invariant is compatible with "induction", meaning that for a map $L \to BC_p \xrightarrow{B\iota} BG$ the L^2-ρ-invariants with respect to the covering pulled back from BC_p and from BG coincide. In this situation, therefore, $\rho_{(2)}(L) = \rho_\alpha(L)$, see [10, Lemma 2.22].

Finally, we come to the explicit construction of W. We choose for the boundary L the disjoint union of an appropriate number N_L of copies of the lens space $L(p; 1, \ldots, 1)$, the quotient of $S^{n-1} \subset \mathbb{C}^{n/2}$ by the action of C_p where the generator acts by multiplication with a fixed p-th root of unity, equipped with the quotient metric g_L and with map $u\colon L(p; 1, \ldots, 1) \to BC_p$ inducing an isomorphism on the fundamental group. This space has a spin structure (unique if $p > 2$). The computation of the η-invariant of the Dirac operator is classical, we get:

Lemma 2.5.

$$\rho_{(2)}(L(p, 1, \ldots, 1)) = \rho_\alpha(L) = (-1)^{\frac{n}{4}+1}\frac{1}{p}\sum_{j=1}^{p-1}\frac{1}{|\zeta^j - 1|^{n/2}} \neq 0.$$

Here, ζ is a primitive p-th root of unity.

Proof. This follows from the equivariant Lefschetz fixed point formula of Donnelly [4, Proposition 4.1] (applied to S^{n-1} bounding an n-dimensional hemisphere), as observed in [2, Lemma 2.3], compare also [5, Lemma 2.1] or [6]. Note that the formulas hold in $\mathbb{R}$ and not only in $\mathbb{R}/\mathbb{Z}$ because we have positive scalar curvature and hence no harmonic spinors. We use that for our virtual representation ρ we have $\rho(g) = -1$ if $g \neq e$ and $\rho(e) = 0$. $\square$

The group $\Omega_{n-1}^{\mathrm{spin}}(BC_p)$ is finite because $n-1$ is odd, as one deduces readily from the Atiyah-Hirzebruch spectral sequence. Therefore, we find N_L and a spin null bordism $U\colon W \to BC_p$ of $L := N_L \cdot L(p; 1, \ldots, 1) =$

$\bigsqcup_{i=1}^{N_L} L(p; 1, \ldots, 1)$. Together with the metric g on the boundary this represents the desired element (W, U, g) of $R_n^{\mathrm{spin}}(BC_p)$.

Remark 2.6. In particular, [2, Theorem 0.1] —the non-triviality of $R_n^{\mathrm{spin}}(BC_p)$— extends to $n = 4$ without any difficulty. Indeed, Botvinnik and Gilkey could have stated the result in this form and their proof would have worked. However, their main focus was on [2, Theorem 0.2] —that the space of metrics of positive scalar curvature on a given manifold has infinitely many component. And this latter result we still can only prove for dimensions ≥ 5.

Case $n \equiv 2 \pmod 4$. Here, we have nothing new to offer, the construction of Ind_ρ and L are given in [2]. One uses an appropriate virtual representation ρ of dimension 0 and the associated twisted index (which then vanishes on the image of μ) and a suitable explicitly constructed L. For the readers convenience we recall the construction. Start with the sphere bundle S of the bundle $\eta \oplus \eta \to \mathbb{C}P^1$, where η is the tautological complex line bundle over $\mathbb{C}P^1$. Use the diagonal $U(1)$-action to divide by the action of the cyclic subgroup C_p to obtain a manifold X. Then set $L_0 := X \times K^r$ where K is the Kummer surface, a 4-dimensional simply connected spin manifold with non-vanishing $\hat{A}$-genus. In [2, Lemma 2.3] a spin structure, positive scalar curvature metric and classifying map to BC_p are constructed for L_0. We can now continue exactly as in the case $n \equiv 0 \pmod 4$, using L_0 instead of $L(p, 1, \ldots, 1)$ to obtain (W, U, g) in the new case. In [2], the associated η-invariant homomorphism $\eta_\rho \colon \mathrm{Pos}_{n-1}^{\mathrm{spin}}(BG) \to \mathbb{R}$ is used and computed. It is not stated, but implicit in [2] (and a direct consequence of the Atiyah-Patodi-Singer index theorem) that the composition $R_n^{\mathrm{spin}}(BG) \to \mathrm{Pos}_{n-1}^{\mathrm{spin}}(BG) \xrightarrow{\eta_\rho} \mathbb{R}$ equals $R_n^{\mathrm{spin}}(BG) \xrightarrow{\mathrm{Ind}} K_n(C^*G) \xrightarrow{\mathrm{Ind}_\rho} \mathbb{R}$, so that the program described above goes through.

Remark 2.7. The map Ind_ρ out of $K_n(C^*G)$ is really just a tiny special case of the idea to map the diagram (1.1) first to analysis/K-theory as in (2.1) but then further to non-commutative homology, to then pair with non-commutative cohomology, as carried out in [12] in general, and also with a different approach in [3, 17]. Very similar constructions based on non-commutative cohomology and higher index theory, for groups constructed even more generally than $\Gamma = G \times \mathbb{Z}$, have been used in [7] to prove non-triviality of $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$.

Remark 2.8. If one assumes something more about G, it is possible to prove even finer results. Let G be a group with torsion. Under the assumption

that for G holds the Strong Novikov Conjecture, namely that the assembly map for G is rationally injective, [18, Corollary 3.13] gives a good lower bound for the rank of $\mathrm{Pos}_n^{\mathrm{spin}}(B(G \times \mathbb{Z}))$ for any $n \geq 3$ and not only for its cardinality.

On the other hand, let us consider the weaker assumption that G is finitely embeddable into a Hilbert space (see [16, Definition 1.3]). If G contains elements of pairwise different prime order $\{p_1, \ldots, p_l\}$, then the method of Weinberger and Yu allows to construct a free abelian subgroup of rank l in $R_{4k}^{\mathrm{spin}}(BG)$ for $k \geq 1$. This free abelian subgroup is mapped injectively under Ind to $K_0(C^*G)/\mathrm{im}(\mu)$. Therefore, in this situation our methods apply to get, just taking Cartesian product with S^1, a free abelian subgroup of rank l in $R_{4k+1}^{\mathrm{spin}}(B(G \times \mathbb{Z}))$ mapping injectively to $\mathrm{Pos}_{4k}^{\mathrm{spin}}(B(G \times \mathbb{Z}))$. It is worth to mention that the relevant statements (and of course also the proof) in [16] are not completely correct. However, in [18, Section 3.2] the authors address the gaps in the original proof of [16, Corollary 4.2] for manifolds of dimension $4k + 3$, which we are using here.

3. Positive scalar curvature bordism

To complete the proof of Theorem 1.3, one has to show that we can actually find representatives which are connected and such that the map to $B\Gamma$ induces an isomorphism $\pi_1(M) \to \Gamma$. However, this is a general and well-known fact from the Gromov-Lawson surgery construction of positive scalar curvature metrics and is already carried out in [9, Section 5] and [8, Section 9.3]. As we have nothing new to offer here, we don't repeat this proof.

Remark 3.1. In dimension $n \geq 5$, one can do even better: given an arbitrary connected closed spin manifold M with reference map $f \colon M \to B\Gamma$ which is an isomorphism on π_1 and a manifold $f \colon M' \to B\Gamma$ with positive scalar curvature metric g' bordant to $M \to B\Gamma$, one can find a metric g on M of positive scalar curvature such that $[M', f', g'] = [M, f, g] \in \mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$. In other words: one can often choose the underlying manifold M representing a class in $\mathrm{Pos}_n^{\mathrm{spin}}(B\Gamma)$.

If $n = 4$, however, this method breaks down. Nonetheless, some trace of this remains true, as shown in [8, Section 9]: in the above situation, one can represent $[M', f', g']$ by $(\tilde{M}, \tilde{f}, g)$ where $\tilde{M}$ is the connected sum of M with an (unspecified) number of copies of $S^2 \times S^2$.

References

[1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Soc. Math. France, Paris, 1976, pp. 43–72. Astérisque, No. 32-33.

[2] Boris Botvinnik and Peter B. Gilkey, *The eta invariant and metrics of positive scalar curvature*, Math. Ann. **302** (1995), no. 3, 507–517.

[3] Xiaoman Chen, Jinmin Wang, Zhizhang Xie, and Guoliang Yu, *Delocalized eta invariants, cyclic cohomology and higher rho invariants*, 2019. arXiv:1901.02378.

[4] Harold Donnelly, *Eta invariants for G-spaces*, Indiana Univ. Math. J. **27** (1978), no. 6, 889–918.

[5] P. B. Gilkey, *The eta invariant and the K-theory of odd-dimensional spherical space forms*, Invent. Math. **76** (1984), no. 3, 421–453.

[6] Andrew J. Hanson and Hartmann Römer, *Gravitational instanton contribution to Spin 3/2 axial anomaly*, Phys. Lett. **80B** (1978), 58–60.

[7] Eric Leichtnam and Paolo Piazza, *On higher eta-invariants and metrics of positive scalar curvature*, K-Theory **24** (2001), no. 4, 341–359.

[8] Tomasz Mrowka, Daniel Ruberman, and Nikolai Saveliev, *An index theorem for end-periodic operators*, Compos. Math. **152** (2016), no. 2, 399–444.

[9] Demetre Kazaras, Daniel Ruberman, and Nikolai Saveliev, *On positive scalar curvature cobordisms and the conformal Laplacian on end-periodic manifolds*, Commun. Anal. Geom. **30** (2022), no. 4, 869–890.

[10] Paolo Piazza and Thomas Schick, *Groups with torsion, bordism and rho invariants*, Pacific J. Math. **232** (2007), no. 2, 355–378.

[11] ———, *Rho-classes, index theory and Stolz' positive scalar curvature sequence*, J. Topol. **7** (2014), no. 4, 965–1004.

[12] Paolo Piazza, Thomas Schick, and Vito Zenobi, *Mapping Analytic Surgery to Homology, Higher Rho Numbers and Metrics of Positive Scalar Curvature*, 2019. arXiv:1905.11861v5, to appear in Memoirs of the Amer. Math. Soc.

[13] Mohan Ramachandran, *von Neumann index theorems for manifolds with boundary*, J. Differential Geom. **38** (1993), no. 2, 315–349.

[14] Thomas Schick and Mehran Seyedhosseini, *On an Index Theorem of Chang, Weinberger and Yu*, Münster J. Math. **2** (2021), 123–154.

[15] Claude Schochet, *Topological methods for C^*-algebras. II. Geometric resolutions and the Künneth formula*, Pacific J. Math. **98** (1982), no. 2, 443–458.

[16] Shmuel Weinberger and Guoliang Yu, *Finite part of operator K-theory for groups finitely embeddable into Hilbert space and the degree of non-rigidity of manifolds*, Geom. Topol. **19** (2015), no. 5, 2767–2799.

[17] Zhizhang Xie and Guoliang Yu, *Delocalized eta invariants, algebraicity, and K-theory of group C^*-algebras*, Int. Math. Res. Not. **15** (2021), 11731–11766.

[18] Zhizhang Xie, Guoliang Yu, and Rudolf Zeidler, *On the range of the relative higher index and the higher rho-invariant for positive scalar curvature*, Adv. Math. (2021). Paper No. 107897, 24pp.

Universita' di Roma "La Sapienza"
Dipartimento di Matematica "Guido Castelnuovo"
Piazzale Aldo Moro 2, 00185 Roma, Italy
E-mail address: `piazza@mat.uniroma1.it`

Mathematisches Institut, Universtät Göttingen
Bunsenstr 3, 37037 Göttingen, Germany
E-mail address: `thomas.schick@math.uni-goettingen.de`

Universita' di Roma "La Sapienza"
Dipartimento di Matematica "Guido Castelnuovo"
Piazzale Aldo Moro 2, 00185 Roma, Italy
E-mail address: `vitofelice.zenobi@uniroma1.it`