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A heat flow problem from Ericksen's model for nematic liquid crystals with variable degree of orientation, II

Chi-Cheung Poon

We study a heat flow problem for nematic liquid crystals with variable degree of orientation. Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary and $\mathcal{C}$ be the round cone in $\mathbb{R} \times \mathbb{R}^3$,

$$\mathcal{C} = \{(s, u) \in \mathbb{R} \times \mathbb{R}^3 : \quad s^2 = |u|^2\}.$$

Under certain conditions on the double-well potential function $W(s)$, we prove that there exist solutions $(s, u) : \Omega \times [0, \infty) \to \mathcal{C}$ which satisfy the system

$$\begin{aligned} s_t &= \Delta s - \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\, s - \frac{W'(s)}{s}\, s \\ u_t &= \Delta u + \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\, u - \frac{W'(s)}{s}\, u, \end{aligned}$$

with given initial-boundary data. Also, we prove that the solutions are Holder continuous.

1. Introduction

In a previous work [3], we studied a heat flow problem for nematic liquid crystals with variable degree of orientation. After some simplifications and choices of material constants, the problem is equivalent to consider the existence of solutions of harmonic heat flow into the round cone with a lower order term. Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary and

$$\mathcal{C} = \{(s,u) \in \mathbb{R} \times \mathbb{R}^3 : \quad s^2 = |u|^2\}.$$

We look for solutions $(s,u) : \Omega \times (0,\infty) \to \mathcal{C}$ which satisfy

$$\begin{aligned} s_t &= \Delta s - \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\, s - \frac{W'(s)}{s}\, s \\ u_t &= \Delta u + \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\, u - \frac{W'(s)}{s}\, u. \end{aligned} \tag{1.1}$$

The system (1.1) is the heat flow equation corresponding to the energy functional

$$\int_\Omega \left(|\nabla s|^2 + |\nabla u|^2 + W(s)\right) dx \tag{1.2}$$

for H^1 map $(s,u) : \Omega \to \mathcal{C}$. In the parabolic system (1.1) or the functional (1.2), the function $W(s)$ is usually assumed to be a double-well potential function. See [2]. In [3], when proving the existence of solutions, we assumed that $W(s)$ is of the form $W(s) = F(s^2)$ for some C^1 function F. Here, we will prove that solutions of (1.1) exist when the potential function $W(s)$ is really a double-well potential. Let $W(s)$ be a non-negative C^1 function defined for $s \in (-\frac{1}{2}, 1)$. We assume that there are $s_1 \in (-\frac{1}{2}, 0)$ and $s_2 \in (0,1)$ such that

$$W'(s) < 0 \quad \text{for} \quad s \in (-\frac{1}{2}, s_1), \qquad W'(s) > 0 \quad \text{for} \quad s \in (s_2, 1), \tag{1.3}$$

and

$$\lim_{s \to (-1/2)^+} W(s) = \infty, \quad \text{and} \quad \lim_{s \to 1^-} W(s) = \infty. \tag{1.4}$$

Also, we further assume that $W(s)$ and has a local minimum at $s = 0$ and

$$\sup \left\{ \left| \frac{W'(s)}{s} \right| : s \in (s_1, s_2) \right\} < \infty. \tag{1.5}$$

We will prove

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary. Let $W(s)$ be a C^1 function which satisfies (1.3), (1.4) and (1.5). Let (g,h)*

be a Lipschitz map from Ω *into the cone* $\mathcal{C}$ *and* $-\frac{1}{2} < g(x) < 1$ *for* $x \in \bar{\Omega}$. *Then, there is a continuous map* $(s,u) : \Omega \times [0,\infty) \to \mathcal{C}$ *such that at any point* (x_0,t_0) *where* $t_0 > 0$ *and* $s(x_0,t_0) \neq 0$, (s,u) *is a solution of (1.1) in a neighborhood of* (x_0,t_0). *Also,* (s,u) *satisfies the initial-boundary conditions,*

$$(s(x,0),u(x,0)) = (g(x),h(x)), \quad x \in \Omega, \tag{1.6}$$

$$(s(x,t),u(x,t)) = (g(x),h(x)), \quad x \in \partial\Omega, \quad t > 0, \tag{1.7}$$

in the sense of trace. Furthermore, there is a sequence t_j *such that* $t_j \to \infty$ *as* $j \to \infty$ *and* $(s(x,t_j),u(x,t_j))$ *converges a map* $(s_0(x),u_0(x))$ *uniformly on compact subsets in* Ω. *For each point* $x_0 \in \Omega$ *where* $s_0(x_0) \neq 0$, *in a neighborhood of* x_0, (s_0,u_0) *is a stationary solution of the system (1.1), and* (s_0,u_0) *satisfies the boundary condition(1.7) in the sense of trace.*

In [2], F.H. Lin proved that the minimizers of the functional are Holder continuous. Here we prove that the same result is true for solutions obtained in Theorem 1.1, if the potential function $W(s)$ is of C^2 and

$$\sup\left\{\left|\left(\frac{W'(s)}{s}\right)'\right| : s \in (s_1,s_2)\right\} < \infty. \tag{1.8}$$

Theorem 1.2. *Let* (s,u) *be the map obtained in Theorem 1.1. If we further assume that* $W(s)$ *is of* C^2 *and (1.8) holds, then* (s,u) *is Holder continuous in* $\Omega \times (0,\infty)$.

Note that (1.5) holds if $W'(s)$ is a Lipschitz function in (s_1,s_2). Also, (1.8) holds if $W''(s)$ is a Lipschitz function in (s_1,s_2). By our assumption that $W(s)$ has a local minimum at $s = 0$, we have $W'(0) = 0$. If $W'(s)$ is a Lipschitz function, then

$$|W'(s)| = |W'(s) - W'(0)| \leq C|s|.$$

This proves (1.5). If $W(s)$ is of C^2, by mean value theorem,

$$W'(s) = W'(s) - W'(0) = W''(\tilde{s})s$$

for some $\tilde{s}$ between s and 0. If $W''(s)$ is a Lipschitz function, then we have

$$|sW''(s) - W'(s)| = |W''(s) - W''(\tilde{s})||s| \leq C|s|^2.$$

This proves (1.8). Both (1.5) and (1.8) are not needed in [2]. To prove the existence of energy minimizer, Lin assumed that $W(s)$ is of C^1 and (1.4) holds. To prove that the energy minimizer is Holder continuous, Lin assumed that $W(s)$ is of C^2.

In [2], F.H. Lin also proved that the minimizers of the functional (1.2) are Holder continuous on the boundary. We wish to discuss the boundary regularity for the solutions obtained in Theorem 1.1 in an upcoming paper.

In this paper, we use the notation $B(x_0; R_0) = \{x \in \mathbb{R}^m : |x - x_0| < R_0\}$. When $x_0 = 0$, we simply write $B(R_0) = B(0; R_0)$.

2. Existence of solutions

We employ the penalization scheme in [3] and consider equations of the form

$$\partial_t s = \Delta s - 2K(s^2 - |u|^2)s - f(s)s, \tag{2.1}$$

$$\partial_t u = \Delta u + 2K(s^2 - |u|^2)u - f(s)u, \tag{2.2}$$

for some constant $K > 0$. We assume that the function $f(s)$ is a bounded function defined on $(-\infty, \infty)$: there is a constant $M > 0$ such that

$$|f(s)| \le M \quad \text{for} \quad s \in (-\infty, \infty). \tag{2.3}$$

Later, we will choose $f(s)$ to be a cut-off of the function $\frac{W'(s)}{s}$. See (2.19), (2.20) and (2.21). Let $(s_K, u_K) : \Omega \times [0, \infty) \to \mathbb{R} \times \mathbb{R}^3$ be solutions of (2.20) and (2.21) with initial-boundary condition (1.6) and (1.7). We will prove that the maps (s_K, u_K) are equicontinuous on each compact subsets in $\Omega \times (0, \infty)$, and there is a sequence that will converge to a map (s, u) with properties mentioned in the Theorem 1.1.

Let (s, u) be a solution of (2.1) and (2.2) in $B(R) \times (-R^2, 0]$. For $\lambda \in (0, 1)$, we define

$$\bar{s}(x, t) = s(\lambda x, \lambda^2 t), \quad \bar{u}(x, t) = u(\lambda x, \lambda^2 t). \tag{2.4}$$

Then $(\bar{s}, \bar{u})$ is defined on the set $B(R/\lambda) \times (-(R/\lambda)^2, 0]$ and is a solution of the equations

$$\partial_t \bar{s} = \Delta \bar{s} - 2\lambda^2 K(\bar{s}^2 - |\bar{u}|^2)\bar{s} - \lambda^2 f(\bar{s})\bar{s},$$

$$\partial_t \bar{u} = \Delta \bar{u} + 2\lambda^2 K(\bar{s}^2 - |\bar{u}|^2)\bar{u} - \lambda^2 f(\bar{s})\bar{u}.$$

Note that when $\lambda \in (0, 1)$ and $f(s)$ satisfies (2.3), then $\lambda^2 |f(\bar{s})| \le M$.

For any $\gamma > 0$, we define

$$\tilde{s}(x,t) = \frac{1}{\gamma}s(x,t), \quad \tilde{u}(x,t) = \frac{1}{\gamma}u(x,t). \tag{2.5}$$

Then $(\tilde{s}, \tilde{u})$ is a solution of

$$\partial_t \tilde{s} = \Delta\tilde{s} - 2\gamma^2 K(\tilde{s}^2 - |\tilde{u}|^2)\tilde{s} - f(\gamma\tilde{s})\tilde{s},$$

$$\partial_t \tilde{u} = \Delta\tilde{u} + 2\gamma^2 K(\tilde{s}^2 - |\tilde{u}|^2)\tilde{u} - f(\gamma\tilde{s})\tilde{u}.$$

Note that if $f(s)$ satisfies (2.3), then we also have $|f(\gamma\tilde{s})| \le M$.

Proposition 2.1. *Let (s_K, u_K) be a solution pair of (2.1) and (2.2) in $B(x_*; 4R_*) \times (t_* - (4R_*)^2, t_*]$. Suppose that*

$$\sup\{s_K^2(x,t) + |u_K|^2(x,t) : \\ (x,t) \in B(x_*; 4R_*) \times (t_* - (4R_*)^2, t_*]\} \le L_*. \tag{2.6}$$

Let $\omega(R)$ be a non-decreasing function defined for $R \in [0, 4R_*)$ and $\omega(0) = 0$ and $R^2 \le \omega(R)$ for $R \in [0, 4R_*)$. Suppose that for $R \in (0, R_*)$ and $(x_0, t_0) \in B(x_*, 2R_*) \times (t_* - (2R_*)^2, t_*]$,

$$R^{2-m}\int_{B(x_0;R)} \big(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\big)(x, t_0)\, dx \le \omega(R). \tag{2.7}$$

Then for $(x_1, t_1), (x_2, t_2) \in B(x_*, R_*) \times (t_* - R_*^2, t_*]$, we have

$$|s_K(x_1,t_1) - s_K(x_2,t_2)| + |u_K(x_1,t_1) - u_K(x_2,t_2)| \le C\sqrt{\omega(2\rho)}, \tag{2.8}$$

where $\rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$. The constant C depends only on m, R_*.

Proof. Let $(x_0, t_0) \in B(x_*; 2R_*)$. We claim that for any $R \in (0, R_*)$, we have

$$R^{2-m}\int_{t_0-R^2}^{t_0}\int_{B(x_0;R)} \big(|\partial_t s_K|^2 + |\partial_t u_K|^2\big)\, dx\, dt \le C\omega(2R), \tag{2.9}$$

where C is a constant depending only on m, M and L_*. Let $\eta(x)$ be a cutoff function such that $\eta(x) = 0$ when $x \in B(x_0; 2R)$ and $\eta(x) = 1$ when

$x \in B(x_0; R)$, $|\nabla \eta| \le C/R$. From (2.1) and (2.2),

$$\begin{aligned}
&\frac{d}{dt}\int_{B(x_0;2R)} \big(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 \\
&\qquad\qquad + K(s_K^2(x,t) - |u_K(x,t)|^2)^2\big)\eta^2(x)\ dx \\
&= 2\int_{B(x_0;2R)} \big(\nabla s_K \nabla \partial_t s_K + \nabla u_K \nabla \partial_t u_K \\
&\qquad\qquad + 2K(s_K^2 - |u_K|^2)(s_K \partial_t s_K - u_K \partial_t u_K)\big)\eta^2\ dx \\
&= -4\int_{B(x_0;2R)} \big(\nabla s_K \partial_t s_K + \nabla u_K \partial_t u_K\big)\eta \nabla \eta\ dx \\
&\quad - 2\int_{B(x_0;2R)} \big(\Delta s_K \partial_t s_K + \Delta u_K \partial_t u_K \\
&\qquad\qquad - 2K(s_K^2 - |u_K|^2)(s_K \partial_t s_K - u_K \partial_t u_K)\big)\eta^2\ dx \\
&= -4\int_{B(x_0;2R)} \big(\nabla s_K \partial_t s_K + \nabla u_K \partial_t u_K\big)\eta \nabla \eta\ dx \\
&\quad - 2\int_{B(x_0;2R)} \big(|\partial_t s_K|^2 + |\partial_t u_K|^2\big)\eta^2\ dx \\
&\quad - 2\int_{B(x_0;2R)} \big(f(s_K)(s_K \partial_t s_K + u_K \partial_t u_K)\big)\eta^2\ dx
\end{aligned}$$

Using (2.3) and (2.6), we have

$$\begin{aligned}
&\int_{B(x_0;2R)} \big(|\partial_t s_K|^2 + |\partial_t u_K|^2\big)\eta^2\ dx \\
&\quad \le -\frac{d}{dt}\int_{B(x_0;2R)} \big(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 \\
&\qquad\qquad + K(s_K^2(x,t) - |u_K(x,t)|^2)^2\big)\eta^2(x)\ dx \\
&\qquad + \frac{C}{R^2}\int_{B(x_0;2R)} \big(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2\big)\ dx + CR^m.
\end{aligned}$$

By (2.7),

$$\frac{1}{R^{m-2}}\int_{t_0-(2R)^2}^{t_0}\int_{B(x_0;2R)} \big(|\partial_t s_K|^2 + |\partial_t u_K|^2\big)\eta^2\ dx\ dt \le C\omega(2R) + CR^2,$$

where C is a constant depending on m, M and L_* only. This proves (2.9).

Now, we proceed to prove (2.8). Let (x_0, t_0) and $R \in (0, R_*)$ be fixed such that $P(x_0, t_0; 2R) \subset \Omega \times (0, \infty)$. Let (x_1, t_1) and (x_2, t_2) be points in

$P(x_0, t_0; R/4)$ and $t_2 \le t_1$. Let

$$\bar{x} = (x_1 + x_2)/2, \qquad \text{and} \qquad \rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}.$$

We note that $\rho < R$. For each $x \in B(\bar{x}; r)$, we observe that

$$\begin{aligned}|s_K(x_1, t_1) - s_K(x, t_1)| &= \left| \int_0^1 (x_1 - x) \cdot \nabla s_K(x_1 + \tau(x - x_1), t_1)\, d\tau \right| \\ &\le 4r \int_0^1 |\nabla s_K(x_1 + \tau(x - x_1))|\, d\tau.\end{aligned}$$

Let $\xi(x)$ be a non-negative smooth function such that $\xi(x) = 1$ when $x \in B(\bar{x}; \rho/2)$ and $\xi(x) = 0$ when x lies outside $B(\bar{x}; r)$, and $|\nabla \xi| \le C/\rho$. After interchanging the order of integration, we obtain

$$\begin{aligned}&\frac{1}{\rho^m} \int_{B(\bar{x};\rho)} |s_K(x_1, t_1) - s_K(x, t_1)| \xi(x)\, dx \\ &\quad \le \frac{1}{\rho^m} \int_{B(\bar{x};\rho)} |s_K(x_1, t_1) - s_K(x, t_1)|\, dx \\ &\quad \le \frac{4}{\rho^{m-1}} \int_{B(\bar{x};\rho)} \int_0^1 |\nabla s_K(x_1 + \tau(x - x_1), t_1)|\, d\tau\, dx.\end{aligned}$$

Let $y = x_1 + \tau(x - x_1)$ and $\bar{x}_\tau = x_1 + \tau(\bar{x} - x_1)$. We note that if $x \in B(\bar{x}; \rho)$, then $|y - \bar{x}_\tau| \le \tau\rho$, and $\bar{x}_\tau \in B(x_0; R)$ for all $0 < \tau < 1$. Thus, from (2.7), we have

$$\begin{aligned}&\frac{4}{\rho^{m-1}} \int_{B(\bar{x};\rho)} \int_0^1 |\nabla s_K(x_1 + \tau(x - x_1), t_1)|\, d\tau\, dx \\ &\quad \le C\rho^{1-m} \int_0^1 \int_{B(\bar{x}_\tau;\tau\rho)} |\nabla s_K(y, t_1)|\, dy\, d\tau \\ &\quad \le C\rho^{1-m} \int_0^1 (\tau\rho)^{m/2} \left(\int_{B(\bar{x}_\tau;\tau\rho)} |\nabla s_K(y, t_1)|^2\, dy \right)^{1/2} d\tau \\ &\quad \le C\rho^{1-m} \int_0^1 (\tau\rho)^{m-1} \sqrt{\omega(\tau\rho)}\, d\tau \\ &\quad \le C\sqrt{\omega(\rho)}.\end{aligned}$$

Let

$$\bar{s}_K(\bar{x}, t) = \frac{\int_{B(\bar{x};\rho)} s_K(x, t)\xi(x)\, dx}{\int_{B(\bar{x};\rho)} \xi(x)\, dx}.$$

The computations in the above implies that

$$|s_K(x_1,t_1) - \bar{s}_K(\bar{x},t_1)| \le C\sqrt{\omega(\rho)}.$$

Similarly, we also have

$$|s_K(x_2,t_2) - \bar{s}_K(\bar{x},t_2)| \le C\sqrt{\omega(\rho)}.$$

Since $|t_1 - t_2| \le \rho^2$, by (2.9),

$$\begin{aligned}|\bar{s}_K(\bar{x},t_1) - \bar{s}_K(\bar{x},t_2)| \le & C\rho^{-m}\int_{t_2}^{t_1}\int_{B(\bar{x};\rho)} |\partial_t s_K|\xi \; dx \; dt \\ \le & C\rho^{-m}\left(\int_{t_2}^{t_1}\int_{B(\bar{x};\rho)} |\partial_t s_K|^2 \; dx \; dt\right)^{1/2} \cdot \rho^{m/2+1} \\ \le & C\sqrt{\omega(2\rho)}\end{aligned}$$

This implies that

$$|s_K(x_1,t_1) - s_K(x_2,t_2)| \le C\sqrt{\omega(2\rho)}.$$

Similarly, we can prove that

$$|u_K(x_1,t_1) - u_K(x_2,t_2)| \le C\sqrt{\omega(2\rho)}.$$

This completes the proof. □

Theorem 2.2. *Let (s_K, u_K) be a solution pair of (2.1) and (2.2) in $B(x_*;4R_*) \times (t_* - (4R_*)^2, t_*]$. Suppose that (2.6) holds and for $t \in (t_* - (4R_*)^2, t_*]$,*

$$\begin{aligned}\int_{B(x_*;3R_*)} \big(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 \\ + K(s_K^2(x,t) - |u_K(x,t)|^2)^2\big) \; dx \le E_*.\end{aligned} \tag{2.10}$$

Then there is a constant C, depending only on m, M, R_*, L_* and E_* but independent of K, such that for any $(x_1,t_1),(x_2,t_2) \in B(x_*;R_*) \times (t_* - R_*^2, t_*]$,

$$|s_K(x_1,t_1) - s_K(x_2,t_2)| + |u_K(x_1,t_1) - u_K(x_2,t_2)| \le \left(\frac{C}{\ln(16R_*^2/\rho^2)}\right)^{1/2},$$

where $\rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$.

Proof. Let $\xi(x)$ be a cutoff function such that $\xi = 0$ outside $B(x_*; 3R_*)$, and $\xi = 1$ inside $B(x_*; 2R_*)$, and $|\nabla\xi| \le C/R_*$, and $|\nabla^2\xi| \le C/R_*^2$. Let

$$e_K(x,t) = |\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 + K(s_K^2(x,t) - |u_K(x,t)|^2)^2.$$

Let $(x_0, t_0) \in B(x_*; R_*) \times (t_* - R_*^2, t_*]$. For $t < t_0$, let

$$E_K(t; x_0, t_0) = |t - t_0| \int_\Omega e_K(x,t)\xi^2(x) G(x,t; x_0, t_0)\, dx, \tag{2.11}$$

$$I_K(t; x_0, t_0) = \int_\Omega (s_K^2(x,t) + |u_K(x,t)|^2)\xi^2(x) G(x,t; x_0, t_0)\, dx. \tag{2.12}$$

Here, $G(x,t; x_0, t_0)$ is the backward heat kernel on $\mathbb{R}^m$: for $t < t_0$,

$$G(x,t; x_0, t_0) = \frac{1}{|t - t_0|^{m/2}} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right).$$

After a translation, we assume that $(x_0, t_0) = (0, 0)$. Also, we write

$$I_K(t) = I_K(t; x_0, t_0) \quad E_K(t) = E_K(t; x_0, t_0), \quad G(x,t) = G(x,t; x_0, t_0).$$

Note that

$$\partial_t G(x,t) = -\Delta G(x,t), \quad \text{and} \quad \nabla G(x,t) = \frac{x}{2t} G.$$

We need to compute $I_K'(t)$ and $E_K'(t)$. The computations are basically the same as those in [3] section 2. From (2.1) and (2.2), one can compute that

$$\begin{aligned}\partial_t(s_K^2 + |u_K|^2) = \Delta(s_K^2 + |u_K|^2) - 2\left(|\nabla s_K|^2 + |\nabla u_K|^2\right) - 4K(s_K^2 - |u_K|^2)^2 \\ - 2f(s_K)(s_K^2 + |u_K|^2).\end{aligned}$$

Thus,

$$\begin{aligned}I_K'(t) &= \int_\Omega \partial_t(s_K^2 + |u_K|^2)\xi^2 G\, dx + \int_\Omega (s_K^2 + |u_K|^2)\xi^2 \partial_t G\, dx \\ &= \int_\Omega \Delta(s_K^2 + |u_K|^2)\xi^2 G\, dx - \int_\Omega (s_K^2 + |u_K|^2)\xi^2 \Delta G\, dx \\ &\quad - 2\int_\Omega (|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2)\xi^2 G\, dx \\ &\quad - 2\int_\Omega f(s_K)(s_K^2 + |u_K|^2)\xi^2 G\, dx\end{aligned}$$

Using integrating by parts, we have

$$
\begin{aligned}
(2.13)\qquad I_K'(t) = &-2\int_\Omega (|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2)\xi^2 G\,dx \\
&- 2\int_\Omega f(s_K)(s_K^2 + |u_K|^2)\xi^2 G\,dx \\
&+ 2\int_\Omega (s_K^2 + |u_K|^2)\xi\left(\frac{x}{2t}\nabla\xi\right) G\,dx \\
&- 4\int_\Omega (s_K\nabla s_K + u_K\nabla u_K)\xi\nabla\xi G\,dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{2}{|t|}E_K(t) \le &-I_K'(t) + 2M\int_\Omega (s_K^2 + |u_K|^2)\xi^2 G\,dx \\
&+ 2\int_\Omega (s_K^2 + |u_K|^2)\xi\left(\frac{x}{2t}\nabla\xi\right) G\,dx \\
&- 4\int_\Omega (s_K\nabla s_K + u_K\nabla u_K)\xi\nabla\xi G\,dx,
\end{aligned}
$$

where M is the constant in (2.3). Since $\nabla\xi(x) = 0$ when $x \in B(2R_*)$, using (2.6) and (2.10), we obtain

$$
\frac{1}{|t|}E_K(t) \le -I_K'(t) + 2MI_K(t) + C\exp\left(\frac{1}{6t}\right),
$$

where C is a constant depending only on the dimension m, M, R_*, L_* and E_*. In general, given (x_0, t_0), if $t \in (t_0 - R_*^2, t_0]$, we have

$$
\begin{aligned}
(2.14)\qquad \frac{1}{|t-t_0|}E_K(t; x_0, t_0) \le &-I_K'(t; x_0, t_0) + 2MI_K(t; x_0, t_0) \\
&+ C\exp\left(\frac{1}{6(t-t_0)}\right).
\end{aligned}
$$

This implies that

$$
I_K'(t; x_0, t_0) \le 2MI_K(t; x_0, t_0) + C\exp\left(\frac{1}{6(t-t_0)}\right).
$$

For any $t_1 \in (t_0 - R_*^2, t_0]$ and $t \in (t_1, t_0)$, we have

$$
(2.15)\qquad I_K(t; x_0, t_0) \le CI_K(t_1; x_0, t_0) + C\exp\left(\frac{1}{6(t-t_0)}\right),
$$

where C is a positive constant depending only on m, M, R_*, L_* and E_*, and is independent of K.

For the function $E(t)$, by a straightforward computation, we have

$$\begin{aligned}
E_K'(t) = &-\int_\Omega K(s_K^2-|u_K|^2)^2\xi^2 G\,dx\\
&-2|t|\int_\Omega \left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)^2+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)^2\right)\xi^2 G\,dx\\
&-4|t|\int_\Omega f(s_K)\left(s_K\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)+u_K\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\right)\xi^2 G\,dx\\
&-4|t|\int_\Omega \left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)\nabla s_K+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\nabla u_K\right)\nabla\xi\xi G\,dx\\
&+4|t|\int_\Omega \left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\xi\nabla\xi\frac{x}{2t}G\,dx.
\end{aligned}$$

See [3] section 2 or section 4 in this paper. It is not difficult to see that

$$\begin{aligned}
E_K'(t) \le &-\frac{1}{2}|t|\int_\Omega \left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)^2\xi^2(x)G\,dx\\
&-\frac{1}{2}|t|\int_\Omega \left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)^2\xi^2(x)G\,dx\\
&+C|t|I_K(t)+C\exp\left(\frac{1}{6t}\right),
\end{aligned} \tag{2.16}$$

and C is a positive constant depending on m, M, R_*, L_* and E_*. Thus, we obtain

$$\frac{d}{dt}E_K(t;x_0,t_0)\le C|t-t_0|I_K(t;x_0,t_0)+C\exp\left(\frac{1}{6(t-t_0)}\right). \tag{2.17}$$

Let $R\in(0,R_*)$. When $t\in(t_0-(4R_*)^2,t_0-R^2)$, using (2.6), we see that

$$E_K(t;x_0,t_0)\ge E_K(t_0-R^2;x_0,t_0)-C|t-t_0|.$$

This implies that

$$\int_{t_0-(4R_*)^2}^{t_0-R^2}\frac{E_K(t;x_0,t_0)}{t_0-t}\,dt\ge \ln\left(\frac{16R_*^2}{R^2}\right)E_K(t_0-R^2;x_0,t_0)-CR_*^2.$$

Using (2.14), we also have

$$\begin{aligned}&\int_{t_0-(4R_*)^2}^{t_0-R^2} \frac{E_K(t;x_0,t_0)}{t_0-t}\,dt\\ &\quad\le \frac{1}{2}(I_K(t_0-(4R_*)^2;x_0,t_0)-I_K(t_0-R^2;x_0,t_0))+C\\ &\quad\le C,\end{aligned}$$

and C is a positive constant depending on m, M, R_*, L_* and E_*. Thus,

$$E_K(t_0-R^2;x_0,t_0)\le \frac{2C}{\ln(16R_*^2/R^2)}.$$

We may replace t_0 by t_0+R^2 and obtain

$$E_K(t_0;x_0,t_0+R^2)\le \frac{2C}{\ln(16R_*^2/R^2)}.$$

This implies that

$$R^{2-m}\int_{B(x_0;R)} \left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\,dx\le \frac{C}{\ln(16R_*^2/R^2)}.$$

By Proposition 2.1, we have

$$|s_K(x_1,t_1)-s_K(x_2,t_2)|+|u_K(x_1,t_1)-u_K(x_2,t_2)|\le \left(\frac{C}{\ln(16R_*^2/\rho^2)}\right)^{1/2},$$

where $\rho=\sqrt{|x_1-x_2|^2+|t_1-t_2|}$. The positive constant C depends only on m, M, R_*, L_* and E_*, and is independent of K. □

Now, we begin to prove Theorem 1.1. Let $g(x)$ be the function in (1.6) and (1.7). Let g_1 and g_2 be constants such that $-\frac{1}{2}<g_1\le s_1$ and $s_2\le g_2<1$ and

$$-\frac{1}{2}<g_1\le g(x)\le g_2<1 \quad \text{for} \quad x\in\bar{\Omega}. \tag{2.18}$$

Let $s_*=\min\{s_1,g_1\}$, $s^*=\max\{s_2,g_2\}$ where s_1,s_2,g_1,g_2 are the constants in (1.3) and (2.18). Let

$$V(s)=\begin{cases} W'(s_*) & \text{when} \quad s\in(-\infty,s_*),\\ W'(s) & \text{when} \quad s\in[s_*,s^*],\\ W'(s^*) & \text{when} \quad s\in(s^*,\infty).\end{cases} \tag{2.19}$$

We consider solutions of equations

$$\partial_t s_K = \Delta s_K - 2K(s_K^2 - |u_K|^2)s_K - \frac{V(s_K)}{s_K}\, s_K, \tag{2.20}$$

$$\partial_t u_K = \Delta u_K + 2K(s_K^2 - |u_K|^2)u_K - \frac{V(s_K)}{s_K}\, u_K; \tag{2.21}$$

with initial-boundary conditions (1.6) and (1.7).

It is easy to check that solutions of (2.20) and (2.21) exist as long as the solutions stay bounded. Let (s_K, u_K) be a solution pair of (2.20) and (2.21) defined for $t \in (0, T]$. One can compute that

$$\begin{aligned}\partial_t(s_K^2 + |u_K|^2) = {} & \Delta(s_K^2 + |u_K|^2) - 2\left(|\nabla s_K|^2 + |\nabla u_K|^2\right) \\ & - 4K(s_K^2 - |u_K|^2)^2 - 2\frac{V(s_K)}{s_K}(s_K^2 + |u_K|^2).\end{aligned} \tag{2.22}$$

Let

$$M = \max\left\{\left|\frac{V(s)}{s}\right| :\ s \in (-\infty, \infty)\right\}.$$

From (2.22), we have

$$\partial_t[e^{2Mt}(s_K^2 + |u_K|^2)] \le \Delta[e^{2Mt}(s_K^2 + |u_K|^2)].$$

By the maximum principle, we obtain

$$\begin{aligned}&\sup\{(s_K^2 + |u_K|^2)(x,t) :\ (x,t) \in \Omega \times (0,T)\} \\ &\quad \le \max\{e^{2MT}(g^2 + |h|^2), g^2 + |h|^2\}.\end{aligned} \tag{2.23}$$

Therefore, for each $T > 0$, solutions of (2.20) and (2.21) exist for $t \in (0, T]$.

From (2.20) and (2.21), it is easy to see that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \left(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 + K(s_K^2(x,t) - |u_K(x,t)|^2)^2\right) dx \\ &= -2\int_\Omega \left(|\partial_t s_K|^2 + |\partial_t u_K|^2\right)\, dx \\ &\quad - 2\int_\Omega \frac{V(s_K)}{s_K}\left(s_K\partial_t s_K + u_K\partial_t u_K\right)\, dx \\ &\le -\int_\Omega \left(|\partial_t s_K|^2 + |\partial_t u_K|^2\right)\, dx + M^2\int_\Omega \left(s_K^2 + |u_K|^2\right)\, dx.\end{aligned} \tag{2.24}$$

Thus, for all $t \in (0,T]$, we have

$$\begin{aligned}(2.25)\quad &\int_\Omega \big(|\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 + K(s_K^2(x,t) - |u_K(x,t)|^2)^2\big)dx \\ &\qquad \le E_0 + CM^2T\max\{e^{2MT}(g_1^2+|h|^2), g^2+|h|^2\},\end{aligned}$$

where

$$E_0 = \int_\Omega \big(|\nabla g(x)|^2 + |\nabla h(x)|^2\big)\ dx.$$

Let T be a fixed positive number. By Theorem 2.2, the pairs $\{(s_K, u_K) : K \ge 1\}$ are equicontinuous on each compact set in $\Omega \times (0,T)$. We may choose a sequence K_i such that $K_i \to \infty$ as $i \to \infty$ and (s_{K_i}, u_{K_i}) converges uniformly to a pair (s,u) on each compact set in $\Omega \times (0,T]$. On each compact set in $\Omega \times (0,T]$, the pair (s,u) satisfies the estimate

$$(2.26)\quad |s(x_1,t_1) - s(x_2,t_2)| + |u(x_1,t_1) - u(x_2,t_2)| \le \left(\frac{C}{\ln(4R_*^2/\rho^2)}\right)^{1/2},$$

where $\rho = \sqrt{|x_1-x_2|^2 + |t_1-t_2|}$ and C is a positive constant depending on T, m, M, R_*, and the initial data. Moreover, we have $s^2(x,t) = |u(x,t)|^2$ for all $(x,t) \in \Omega \times (0,T]$. Suppose that $s(x_0,t_0) \ne 0$, by (2.26), there is a neighborhood of (x_0,t_0) such that $s(x,t)$ does not vanish. In that neighborhood, using the method in [4], one can prove that (s,u) is a C^2 solution of

$$(2.27)\quad \begin{aligned} s_t &= \Delta s - \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\ s - \frac{V'(s)}{s}\ s \\ u_t &= \Delta u + \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2}\ u - \frac{V'(s)}{s}\ u.\end{aligned}$$

Suppose that $s(x,t)$ has a local maximum at $(x_1,t_1) \in \Omega \times (0,T]$ and $s(x_1,t_1) > \max\{s_2, g_2\}$, where s_2 and g_2 are the constants in (1.3) and (2.18). In a neighborhood of (x_1,t_1), the pair (s,u) is a C^2 solution of (2.27). Since $s^2(x,t) = |u(x,t)|^2$, we have $|\nabla u|^2(x,t) - |\nabla s|^2(x,t) \ge 0$ in a neighborhood near (x_1,t_1). By the maximum principle, it is impossible. Thus, we must have $s(x,t) \le \max\{s_2, g_2\}$ for all $(x,t) \in \Omega \times (0,T]$. Similarly, using maximum principle, one can prove that $s(x,t) \ge \min\{s_1, g_1\}$ for all $(x,t) \in \Omega \times (0,T]$. We see that

$$\sup\{s^2 + |u|^2 :\ (x,t) \in \Omega \times (0,T]\} \le \max\{|s_1+g_1|^2, |s_2+g_2|^2\},$$

and is independent of T. From (2.19), this implies that $V(s(x,t)) = W'(s(x,t))$ in (2.27). Thus, if $s(x_0,t_0) \neq 0$, in a neighborhood of (x_0,t_0), the pair (s,u) is a solution of the system (1.1).

Let $(x_*,t_*) \in \Omega \times (0,T]$ and $B(x_*;4R_*) \times (t_* - (4R_*)^2, t_*] \subset \Omega \times (0,T]$. Since (s_{K_i}, u_{K_i}) converges uniformly to (s,u) on each compact set in $\Omega \times (0,T]$, when i is large enough, we have

$$\begin{aligned}(2.28)\qquad &\sup\{s_{K_i}^2 + |u_{K_i}|^2 :\ (x,t) \in B(x_*;3R_*) \times (t_* - (3R_*)^2, t_*]\}\\ &\quad \le \max\{|s_1+g_1|^2, |s_2+g_2|^2\} + 1.\end{aligned}$$

From (2.24) and (2.28), for each $t \in (0,T]$,

$$\begin{aligned}&\int_0^T \int_\Omega \left(|\partial_t s_K|^2 + |\partial_t u_K|^2\right)\, dx\ dt\\ &\quad \le E_0 + CTM^2(\max\{|s_1+g_1|^2, |s_2+g_2|^2\} + 1).\end{aligned}$$

Thus, we may assume that when $i \to \infty$, $(\partial_t s_{K_i}, \partial_t u_{K_i})$ converges to $(\partial_t s, \partial_t u)$ weakly in $L^2(\Omega \times (0,T]$, and

$$\int_0^T \int_\Omega \left(|\partial_t s_{K_i}| + |\partial_t u_{K_i}|\right)\, dx\ dt \to \int_0^T \int_\Omega \left(|\partial_t s| + |\partial_t u|\right)\, dx\ dt.$$

Since s_{K_i} converges to S uniformly on each compact subset in $\Omega \times (0,T]$ and

$$\left|\frac{V(s_{K_i})}{s_{K_i}}\right|, \quad \left|\frac{V(s)}{s}\right| \le M,$$

by dominated convergence theorem, when $i \to \infty$, for any $t \in (0,T)$,

$$\begin{aligned}&\lim_{i\to\infty} \int_0^t \int_\Omega \frac{V(s_{K_i})}{s_{K_i}}\left(s_{K_i}\partial_t s_{K_i} + u_{K_i}\partial_t u_{K_i}\right)\, dx\ dt\\ &\quad = \int_0^t \int_\Omega \frac{V(s)}{s}\left(s\partial_t s + u\partial_t u\right)\, dx\ dt\\ &\quad = \int_0^t \int_\Omega \frac{W'(s)}{s}\left(s\partial_t s + u\partial_t u\right)\, dx\ dt.\end{aligned}$$

Also, since $s^2 = |u|^2$, we have

$$\int_\Omega \frac{W'(s)}{s}\left(s\partial_t s + u\partial_t u\right)\, dx = \int_\Omega 2W'(s)\partial_t s\ dx = 2\frac{d}{dt}\int_\Omega W'(s)\ dx.$$

Therefore, for any $t \in (0,T]$, as $i \to \infty$,

$$
\begin{aligned}
&-\int_0^t \int_\Omega \frac{V(s_{K_i})}{s_{K_i}} \big(s_{K_i} \partial_t s_{K_i} + u_{K_i} \partial_t u_{K_i} \big) \, dx \, dt \\
&\quad \to 2 \int_\Omega W(g(x)) \, dx - 2 \int_\Omega W(s(x,t)) \, dx.
\end{aligned}
$$

When i is large enough, we have

(2.29)

$$
-\int_0^t \int_\Omega \frac{V(s_{K_i})}{s_{K_i}} \big(s_{K_i} \partial_t s_{K_i} + u_{K_i} \partial_t u_{K_i} \big) \, dx \, dt \le 1 + 2 \int_\Omega W(g(x)) \, dx.
$$

From equations (2.20) and (2.21),

$$
\begin{aligned}
0 &\le \int_\Omega \big(|\partial_t s_{K_i}|^2 + |\partial_t u_{K_i}|^2 \big) \, dx \\
&= \int_\Omega \partial_t s_{K_i} \big(\Delta s_{K_i} - 2K_i (s_{K_i}^2 - |u_{K_i}|^2) s_{K_i} - \frac{V(s_{K_i})}{s_{K_i}} \, s_{K_i} \big) \, dx \\
&\quad + \int_\Omega \partial_t u_{K_i} \big(\Delta u_{K_i} + 2K_i (s_{K_i}^2 - |u_{K_i}|^2) u_{K_i} - \frac{V(s_{K_i})}{s_{K_i}} \, u_{K_i} \big) \, dx \\
&= -\frac{1}{2} \frac{d}{dt} \int_\Omega \big(|\nabla s_{K_i}|^2 + |\nabla u_{K_i}|^2 + K_i (s_{K_i}^2 - |u_{K_i}|^2)^2 \big) \, dx \\
&\quad - \int_\Omega \frac{V(s_{K_i})}{s_{K_i}} \big(s_{K_i} \partial_t s_{K_i} + u_{K_i} \partial_t u_{K_i} \big) \, dx.
\end{aligned}
$$

Using (2.29), we see that, when i is large enough, for $t \in (0,T)$,

$$
\begin{aligned}
&\int_\Omega \big(|\nabla s_{K_i}(x,t)|^2 + |\nabla u_{K_i}(x,t)|^2 + K_i (s_{K_i}^2(x,t) - |u_{K_i}(x,t)|^2)^2 \big) dx \\
&\qquad \le 1 + \int_\Omega \big(|\nabla g|^2 + |\nabla h|^2 \big) \, dx + 2 \int_\Omega W(g(x)) \, dx.
\end{aligned} \tag{2.30}
$$

By Theorem 2.2, using (2.28) and (2.30) instead of (2.23) and (2.25), we may conclude the following: Let $(x_*, t_*) \in \Omega \times (0,T]$ and $B(x_*; 4R_*) \times (t_* - (4R_*)^2, t_*] \subset \Omega \times (0,T]$. Suppose that (x_1,t_1), $(x_2,t_2) \in B(x_*; R_*) \times (t_* - R_*^2, t_*]$, when i is large enough,

$$
\begin{aligned}
&|s_{K_i}(x_1,t_1) - s_{K_i}(x_2,t_2)| + |u_{K_i}(x_1,t_1) - u_{K_i}(x_2,t_2)| \\
&\qquad \le \left(\frac{C}{\ln(4R_*^2/\rho^2)} \right)^{1/2},
\end{aligned} \tag{2.31}
$$

where $\rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$. The positive constant C depends only on m, M, R_* and the initial data, and is independent of K and T. When $i \to \infty$, we have

$$|s(x_1, t_1) - s(x_2, t_2)| + |u(x_1, t_1) - u(x_2, t_2)| \leq \left(\frac{C}{\ln(4R_*^2/\rho^2)} \right)^{1/2}.$$

The positive constant C depends only on m, M, R_* and the initial data, and is independent of T. This completes the proof of Theorem 1.1.

3. Holder continuity

In this section, we will prove that the solution pair (s, u) obtained in Theorem 1.1 is Holder continuous in $\Omega \times (0, \infty)$. Let $V(s)$ be the function defined in (2.19). If $W(s)$ is of C^2 and both (1.5) and (1.8) hold, then $V(s)$ is of C^2 and there is $M > 0$ such that

$$\left| \frac{V'(s)}{s} \right|, \ \left| \left(\frac{V'(s)}{s} \right)' \right| \leq M \quad \text{for} \quad s \in (-\infty, \infty). \tag{3.1}$$

Similar to [3] Theorem 1.2, we first prove that (s, u) satisfies a unique continuation property.

Proposition 3.1. *Let $t_0 > 0$. Either $s(x, t_0) = 0$ for all $x \in \Omega$, or $s(x, t_0)$ cannot vanish of infinite order at any point in Ω.*

Proof. The proof is the same as the proof of [3] Theorem 1.2. Here, we give a sketch of the proof of Proposition 3.1. Let $t_0 > 0$ and $s(x, t_0)$ is not identically zero on Ω. We claim that $s(x, t_0)$ cannot vanish in an open subset in Ω.

If it is not true, there is x_0 such that for some $R > 0$, $B(x_0; 2R) \subset \Omega$, $s(x, t_0) = 0$ when $x \in B(x_0; R/8)$ and

$$\int_{B(x_0, R/4) - B(x_0, R/8)} (s^2 + |u|^2)(x, t_0) \, dx = 4c_0 > 0.$$

After a translation, we assume that $(x_0, t_0) = (0, 0)$. By continuity, there is $r_1 > 0$ such that for $|t - t_0| < (2r_1)^2$, we have

$$\int_{B(x_0, R/4) - B(x_0, R/8)} (s^2 + |u|^2)(x, t) \, dx \geq 2c_0 > 0.$$

Let (s_K, u_K) be a solution pair of (2.20) and (2.21). In the previous section, we proved that there is a sequence K_i such that (s_{K_i}, u_{K_i}) converges to (s, u) uniformly on compact sets in $\Omega \times (0, \infty)$. Since (s_{K_i}, u_{K_i}) converges uniformly to (s, u) on compact subsets, we may assume that for each $i = 1, 2, 3...$, for $|t - t_0| < (2r_1)^2$,

$$\int_{B(x_0,R/4)-B(x_0,R/8)} (s_{K_i}^2 + |u_{K_i}|^2)(x,t)\, dx \geq c_0 > 0. \tag{3.2}$$

Let $E_K(t; x_0, t_0)$ and $I_K(t; x_0, t_0)$ be functions defined in (2.11) and (2.12) and

$$N_K(t; t_0, t_0) = \frac{E_K(t; x_0, t_0)}{I_K(t; x_0, t_0)}. \tag{3.3}$$

Inequality (3.2) implies that there is a positive constant C depending only on m and c_0 only, so that

$$I_{K_i}(t; x_0, t_0) \geq C \exp\left(\frac{1}{20(t - t_0)}\right) \quad \text{for} \quad t_0 - r_1^2 < t < t_0. \tag{3.4}$$

By (2.13), (2.16) and (3.2), one can prove that

$$\frac{d}{dt} N_{K_i}(t; x_0, t_0) \leq C(1 + N_{K_i}(t; x_0, t_0)) \quad \text{for} \quad t_0 - r_1^2 < t < t_0.$$

For detailed computations, see [3] p429-431. Thus, there is a positive constant N_0 such that

$$N_{K_i}(t; x_0, t_0) \leq N_0 \quad \text{for} \quad t_0 - r_1^2 < t < t_0.$$

By (2.13) and (3.2), we have

$$-\frac{I'_{K_i}(t; x_0, t_0)}{I_{K_i}(t; x_0, t_0)} \leq \frac{4N_0 + C}{|t - t_0|} \quad \text{for} \quad t_0 - r_1^2 < t < t_0.$$

After integrating from $t_0 - r_1^2$ to t, we obtain

$$I_{K_i}(t; x_0, t_0) \geq I_{K_i}(t_0 - r_1^2; x_0, t_0)|t - t_0|^{2N_0 + C} \quad \text{for} \quad t_0 - r_1^2 < t < t_0.$$

Thus, using (3.2) again, we see that

$$I_{K_i}(t_0 - r^2; x_0, t_0) \geq D r^{2N_1} \quad \text{for} \quad 0 < r < r_1. \tag{3.5}$$

We may replace t_0 by $t_0 + r^2$ in the above arguments to have

$$I_{K_i}(t_0; x_0, t_0 + r^2) \geq Dr^{2N_1} \quad \text{for} \quad 0 < r < \frac{r_1}{4}.$$

Since (s_i, u_i) converges uniformly to (s, u) on compact subsets, the same is true for (s, u), i.e.,

$$\int_\Omega (s^2(x, t_0) + |u(x, t_0)|^2)\xi^2 G(x, t_0; x_0, t_0 + r^2)\, dx \geq Dr^{2N_1} \quad \text{for} \quad 0 < r < \frac{r_1}{4}. \tag{3.6}$$

It contradicts our assumption that $s(x, t_0) = 0$ when $x \in B(x_0; R/8)$ and the claim is proved.

Finally, by our claim, for any $x_0 \in \Omega$ and $B(x_0; 2R) \subset \Omega$, $s(x, t_0)$ is not zero somewhere inside $B(x_0; R/4)$, i.e., (3.2) always holds for some $c_0 > 0$. By repeating the arguments in the above, we see that for any $x_0 \in \Omega$, the estimate (3.6) holds. This proves the Proposition. □

We may improve Proposition 3.1 to the following form:

Proposition 3.2. *Suppose that the initial-boundary data $(g(x), h(x))$ is not identically zero on $\partial\Omega$. For each $t_0 > 0$, the function $s(x, t_0)$ cannot vanish of infinite order at any point in Ω.*

Proof. If the Proposition is not true, there is $t_1 > 0$ such that $s(x, t_1) = 0$ for all $x \in \Omega$. Let $R_* > 0$ such that $4R_* < t_1$ We claim that there is $t_0 \in (t_1, t_1 + R_*)$ such that $s(x, t_0)$ is not identically zero. If such t_0 does not exist, then $s(x, t) = 0$ for all $(x, t) \in \Omega \times (t_1, t_1 + R_*)$. Since $(s(x, t), u(x, t)) = (g(x), h(x))$ in the sense of trace on $\partial\Omega$, it is impossible. Thus, the claim is true. Let $x_0 \in \Omega$. Choose $R_0 > 0$ such that $B(x_0; 4R_0) \subset \Omega$ and $(4R_0)^2 < t_0$. By (2.15), for each i,

$$I_{K_i}(t; x_0, t_0) \leq CI_{K_i}(t_1; x_0, t_0) + C \exp\left(\frac{1}{6(t - t_0)}\right) \quad \text{for} \quad t_1 < t < t_0.$$

When $i \to \infty$, if $s(x, t_1) = 0$ for all $x \in \Omega$, then

$$\int_\Omega (s^2(x, t) + |u(x, t)|^2)\xi^2 G(x, t; x_0, t_0)\, dx \leq C \exp\left(\frac{1}{6(t - t_0)}\right) \quad \text{for} \quad t_1 < t < t_0.$$

Let r_1 be the constant in (3.4). From (3.5),

$$I_{K_i}(t;x_0,t_0) \geq D|t-t_0|^{2N_1} \quad \text{for} \quad t_0-(r_1/4)^2 < t < t_0.$$

When $i \to \infty$, we have

$$\int_\Omega (s^2(x,t)+|u(x,t)|^2)\xi^2 G(x,t;x_0,t_0)\,dx \\ \geq D|t-t_0|^{2N_1} \quad \text{for} \quad t_0-(r_1/4)^2 < t < t_0.$$

We have a contradiction. □

Let $(x_*,t_*)\in\Omega\times(0,\infty]$ and $B(x_*;4R_*)\times(t_*-(4R_*)^2,t_*]\subset\Omega\times(0,\infty]$. Let $(x_0,t_0)\in B(x_*;R_*)\times(t_*-R_*^2,t_*]$. By Proposition 3.2 and the continuity of (s,u), there is $c_0>0$, for each $(x_0,t_0)\in(t_*-(2R_*)^2,t_*]$,

$$\int_{B(x_0;3R_*)-B(x_0,2R_*)} (s^2(x,t_0)+|u(x,t_0)|^2)\,dx \geq 2c_0 > 0. \tag{3.7}$$

Let $E_K(t;x_0,t_0)$ and $I_K(t;x_0,t_0)$ be functions defined in (2.11) and (2.12). Since (s_{K_i},u_{K_i}) converges to (s,u) uniformly on compact set in $\Omega\times(0,\infty)$, when i is large,

$$\int_{B(x_0;3R_*)-B(x_0,2R_*)} (s_{K_i}^2(x,t_0)+|u_{K_i}(x,t_0)|^2)\,dx \geq c_0 > 0. \tag{3.8}$$

Inequality (3.8) implies that there is a positive constant C depending only on m and c_0 only, so that

$$I_{K_i}(t;x_0,t_0) \geq C\exp\left(\frac{1}{20(t-t_0)}\right) \quad \text{for} \quad t_0-R_*^2 < t < t_0. \tag{3.9}$$

Using (3.9), we can prove that

$$\frac{d}{dt}N_{K_i}(t;x_0,t_0) \leq C(1+N_{K_i}(t;x_0,t_0)) \quad \text{for} \quad t_0-R_*^2 < t < t_0. \tag{3.10}$$

Hence, we obtain

$$N_{K_i}(t;x_0,t_0) = \frac{E_{K_i}(t;x_0,t_0)}{I_{K_i}(t;x_0,t_0)} \leq N_0 \quad \text{for} \quad t_0-R_*^2 < t < t_0, \tag{3.11}$$

where N_0 is a constant depending on m, R_*, M and c_0 and the initial data only. Using (2.13), we see that we have

$$-\frac{I'_{K_i}(t;x_0,t_0)}{I_{K_i}(t;x_0,t_0)} \leq \frac{4N_0+C}{|t-t_0|} \quad \text{for} \quad t_0-R_*^2<t<t_0. \tag{3.12}$$

From (3.12), for any $0<R_1<R_2<R_*$, we have

$$I_{K_i}(t_0-R_1^2;x_0,t_0) \geq D\left(\frac{R_1}{R_2}\right)^{2N_1} I_{K_i}(t_0-R_2^2;x_0,t_0), \tag{3.13}$$

where N_1 and D are positive constants depends on m, R_*, M, c_0 and the initial data only and is independent of i and (x_0,t_0). In particular, for any $0<R<R_*$, we have

$$I_{K_i}(t_0-R^2;x_0,t_0) \geq D\left(\frac{R^2}{R_*^2}\right)^{N_1} I_{K_i}(t_0-R_*^2;x_0,t_0).$$

By (3.8),

$$I_{K_i}(t_0-R_*^2;x_0,t_0) \geq \frac{c_0}{C},$$

where C is a positive constant depending only on m and R_*. It follows that

$$I_{K_i}(t_0-R^2;x_0,t_0) \geq \tilde{D}R^{2N_1} \quad \text{for} \quad t_0-R_*^2<t<t_0. \tag{3.14}$$

The constant $\tilde{D}$ depends on c_0, m, M, R_* and the initial data only and can be chosen independent of i and (x_0,t_0).

Let $(x_*,t_*)\in\Omega\times(0,\infty]$ and $B(x_*;4R_*)\times(t_*-(4R_*)^2,t_*]\subset\Omega\times(0,\infty]$. We wish to show that when i is large enough, then (s_{K_i},u_{K_i}) is uniformly Holder continuous in $B(x_*;R_*)\times(t_*-R_*^2,t_*]$. To simplify the notations, we consider function pairs (s_K,u_K) instead of (s_{K_i},u_{K_i}), and assume that (3.8), (3.10), (3.11), (3.13), (3.14) hold for (s_K,u_K), with constants depending only on c_0, m, M, R_* and the initial data only. For the rest of this section, we will keep all constants depending only on c_0, m, M, R_* and the initial data only. Usually, they will be denoted simply by C or c.

By (2.22) and a mean-value type inequality, we have the following Proposition.

Proposition 3.3. *Let $R_*\in(0,1)$ and $B(x_*;4R_*)\subset\Omega$ and $t_*>(4R_*)^2$. There is a positive constant C, independent of K, such that for any $(x_0,t_0)\in$*

$B(x_*;R_*)\times(t_*-R_*^2,t_*)$ and $R_0\in(0,R_*)$, we have

$$\sup\{s_K^2(x,t)+|u_K(x,t)|^2:\ (x,t)\in B(x_0;2R_0)\times(t_0-4R_0^2,t_0]\} \le \frac{C}{R_0^{m+2}}\int_{t_0-(4R_0)^2}^{t_0}\int_{B(x_0;4R_0)}\left(s_K^2(x,t)+|u_K(x,t)|^2\right)dx\,dt. \tag{3.15}$$

Proposition 3.4. *Let $H\ge 1$. There are positive constants $\bar R\in(0,\frac{1}{2})$ and $\bar\epsilon<1$, such that the following holds: Suppose that $R\in(0,\bar R)$, and (s_K,u_K) is a solution pair of equations (2.20) and (2.21) in $B(4R)\times(-(4R)^2.0]$, and if*

$$\frac{1}{R^{m-2}}\int_{B(x_0;4R)} e_K(x,t)\,dx\le\bar\epsilon \quad \text{for} \quad t\in(-(4R)^2,0],$$

where $e_K(x,t)=|\nabla s_K(x,t)|^2+|\nabla u_K(x,t)|^2+2K(s_K^2(x,t)-|u_K(x,t)|^2)^2$, and

$$\sup\{s_K^2(x,t)+|u_K|^2(x,t):\ (x,t)\in B(4R)\times(-(4R)^2,0]\}\le H,$$

$$\sup\{s_K^2(x,t)+|u_K|^2(x,t):\ (x,t)\in B(2R)\times(-(2R)^2,0]\}=1,$$

then

$$s_K^2(x,t)+|u_K|^2(x,t)\ge 1/3 \quad \text{for} \quad (x,t)\in B(2R)\times(-(2R)^2,0].$$

The choice of $\bar R$ and $\bar\epsilon$ may depend on the constants m, M, and H, but is independent of K.

Proof. Let $\bar R\in(0,\frac{1}{2})$ be a the constant to be determined. If the Proposition is not true, there is a sequence (s_{K_j},u_{K_j}) which are solutions of (2.20) and (2.21) in $B(4R_j)\times(-(4R_j)^2,0]$ with $R_j<\bar R$ and $K=K_j$ in (2.20) and (2.21), and

$$\sup_{t\in(-(4R_j)^2,0]}\frac{1}{R_j^{m-2}}\int_{B(4R_j)} e_{K_j}(x,t)\,dx=\epsilon_j\to 0 \quad \text{as} \quad j\to\infty, \tag{3.16}$$

where $e_{K_j}(x,t)=|\nabla s_{K_j}(x,t)|^2+|\nabla u_{K_j}(x,t)|^2+2K_j(s_{K_j}^2(x,t)-|u_{K_j}(x,t)|^2)^2$,

$$\sup\{s_{K_j}^2(x,t)+|u_{K_j}|^2(x,t):\ (x,t)\in B(4R_j)\times(-(4R_j)^2,0]\}\le H, \tag{3.17}$$

$$\sup\{s_{K_j}^2(x,t)+|u_{K_j}|^2(x,t):\ (x,t)\in B(2R_j)\times(-(2R_j)^2,0]\}=1, \tag{3.18}$$

but

$$\inf\{s_{K_j}^2(x,t) + |u_{K_j}|^2(x,t) : \ (x,t) \in B(2R_j) \times (-(2R_j)^2, 0]\} < \frac{1}{3}. \tag{3.19}$$

However, using rescaling (2.4) we may assume that $R_j = \bar{R}$ for all j. Also, Theorem 2.2 implies that (s_{K_j}, u_{K_j})'s are uniformly continuous in compact subsets in $B(4\bar{R}) \times (-(4\bar{R})^2, 0]$. It follows that there is a subsequence, also called (s_{K_j}, u_{K_j}), which converges uniformly to a function pair (s_0, u_0) in compact subsets in $B(4\bar{R}) \times (-(4\bar{R})^2, 0]$. By (3.18) and (3.19),

$$\sup\{s_0^2(x,t) + |u_0|^2(x,t) : \ (x,t) \in B(2\bar{R}) \times (-(2\bar{R})^2, 0]\} = 1,$$

and

$$\inf\{s_0^2(x,t) + |u_0|^2(x,t) : \ (x,t) \in B(2\bar{R}) \times (-(2\bar{R})^2, 0]\} \le \frac{1}{3}.$$

By (3.16), (s_0, u_0) depends only on t. We may further assume that there are $t_1, t_2 \in [-(2\bar{R})^2, 0]$ such that $s_0^2(t_1) + |u_0(t_1)|^2 = 1$ and $s_0^2(t_2) + |u_0(t_2)|^2 \le 1/3$.

Let $\xi(x)$ be a cutoff function such that $\xi(x) = 0$ when $x \notin B(3\bar{R})$ and $\xi(x) = 1$ when $x \in B(2\bar{R})$. By (2.22), one can compute that

$$\begin{aligned}
&\frac{d}{dt}\int_{B(3\bar{R})} \big(s_{K_j}^2(x,t) + |u_{K_j}|^2(x,t)\big)\xi^2(x)\ dx \\
&\quad = -\int_{B(3\bar{R})} \nabla(s_{K_j}^2 + |u_{K_j}|^2)\cdot\nabla\xi^2\ dx \\
&\qquad - 2\int_{B(3\bar{R})} \big(|\nabla s_{K_j}|^2 + |\nabla u_{K_j}|^2) + 2K_j(s_{K_j}^2 - |u_{K_j}|^2)^2\big)\ dx \\
&\qquad + \int_{B(3\bar{R})} -2\frac{V(s_{K_j})}{s_{K_j}}(s_{K_j}^2 + |u_{K_j}|^2)\xi^2\ dx.
\end{aligned}$$

When $j \to \infty$, we have

$$\left|\frac{d}{dt}\big(s_0^2(t) + |u_0(t)|^2\big)\right| \le 2M\big(s_0^2(t) + |u_0(t)|^2\big)$$

in the weak sense. This implies that

$$\big|\big(s_0^2(t_1) + |u_0(t_2)|^2\big) - \big(s_0^2(t_1) + |u_0(t_2)|^2\big)\big| \le e^{2M|t_1 - t_2|} - 1.$$

If we choose $\bar{R} \in (0, \frac{1}{2})$ so that

$$e^{2M\bar{R}} - 1 < \frac{2}{3},$$

we have a contradiction. □

Proposition 3.5. *Let $B(x_*; 4R_*) \subset \Omega$ and $t_* > (4R_*)^2$. There are positive constants c_1 and c_2, independent of K, such that for any $(x_0, t_0) \in B(x_*; R_*) \times (t_* - R_*^2, t_*]$ and $R \in (0, R_*/4)$, we have*

$$\begin{aligned}(3.20)\qquad &\frac{1}{R^{m+2}} \int_{t_0-6R^2}^{t_0-2R^2} \int_{B(x_0;2R)} |s_K(x,t)|^2 + |u_K(x,t)|^2 \, dx \, dt \\ &\geq c_1 I_K(t_0 - (4R)^2; x_0, t_0).\end{aligned}$$

Proof. If the (3.20) is not true, there are sequences K_j, R_j, (x_j, t_j) such that

$$\begin{aligned}(3.21)\qquad &\frac{1}{R_j^{m+2}} \int_{t_j-6R_j^2}^{t_j-2R_j^2} \int_{B(x_j;2R_j)} |s_{K_j}(x,t)|^2 + |u_{K_j}(x,t)|^2 \, dx \, dt \\ &\leq \frac{1}{j} I_{K_j}(t_j - (4R_j)^2; x_j, t_j).\end{aligned}$$

By choosing subsequences, we may assume that $(x_j, t_j) \to (x_0, t_0)$ and

$$(3.22)\qquad R_j \to R_0 \quad \text{as} \quad j \to \infty.$$

We first assume that $R_0 = 0$, i.e., $R_j \to 0$ as $j \to \infty$. From (2.15), there are positive constants C_1 and C_2, such that if $t \in (t_j - (4R_j)^2, t_j]$, then

$$(3.23)\qquad I_{K_j}(t; x_j, t_j) \leq C_1 I_{K_j}(t_j - (4R_j)^2; x_j, t_j) + C_2 \exp\left(\frac{-1}{24R_j^2}\right).$$

Using (3.14), we see that for each j and $t \in (t_j - (4R_j)^2, t_j]$,

$$(3.24)\qquad I_{K_j}(t; x_j, t_j) \leq 2C_1 I_{K_j}(t_j - (4R_j)^2; x_j, t_j).$$

It is easy to see that

$$
\begin{aligned}
(3.25)\qquad &\frac{1}{R_j^{m+2}}\int_{t_j-(4R_j)^2}^{t_j-R_j^2}\int_{B(x_j,2\sigma_j)} |s_{K_j}(x,t)|^2+|u_{K_j}(x,t)|^2\;dx\;dt\\
&\le \frac{C}{R_j^2}\int_{t_j-(4R_j)^2}^{t_j-R_j^2} I_{K_j}(t;x_j,t_j)\;dt\\
&\le CI_{K_j}(t_j-(4R_j)^2;x_j,t_j).
\end{aligned}
$$

Using the same arguments and equation (3.11), for $t\in(t_j-(4R_j)^2,t_j]$, we have

$$
E_{K_j}(t;x_j,t_j)\le CI_{K_j}(t;x_j,t_j)\le CI_{K_j}(t_j-(4R_j)^2;x_j,t_j).
$$

This implies that for each $t\in(t_j-(4R_j)^2,t_j]$, we have

$$
\begin{aligned}
(3.26)\quad &\frac{1}{R_j^{m-2}}\int_{B(x_j,4R_j)}\big(|\nabla s_{K_j}|^2+|\nabla u_{K_j}|^2+K_j(s_{K_j}^2-|u_{K_j}^2|)^2\big)(x,t)\;dx\\
&\le CI_{K_j}(t_j-4R_j^2;x_j,t_j).
\end{aligned}
$$

Let

$$
s_j(x,t)=\frac{s_{K_j}\Big(x_j+R_jx,t_j+R_j^2t\Big)}{\sqrt{I_{K_j}(t_j-(4R_j)^2;x_j,t_j)}},\quad u_j(x,t)=\frac{u_{K_j}\Big(x_j+R_jx,t_j+R_j^2t\Big)}{\sqrt{I_{K_j}(t_j-(4R_j)^2;x_j,t_j)}}.
$$

For each j, the functions $s_j(x,t)$ and $u_j(x,t)$ are defined for $(x,t)\in B(4)\times(-16,0]$. The pair (s_j,u_j) is a solution to equations (2.1) and (2.2). By Proposition 3.3 and (3.25), we have

$$
(3.27)\qquad |s_j(x,t)|^2+|u_j(x,t)|^2\le C,\quad \text{for}\quad (x,t)\in B(3)\times(-9,-1].
$$

By (3.26),

$$
(3.28)\quad \sup_{t\in(-16,-1]}\int_{B(4)}\big(|\nabla s_j|^2+|\nabla u_j|^2+K_j(s_{K_j}^2-|u_{K_j}^2|)^2\big)(x,t)\;dx\le C.
$$

The function pair (s_j,u_j) is a solution of equations (2.1) and (2.2). Using (3.27) and (3.28), by Theorem 2.2, we see that there is a subsequence, also called (s_j,u_j), which converges to a function pair (s_0,u_0) uniformly on compact sets in $B(3)\times(-9,-1)$. Let $\xi(x)$ be a cutoff function such that $\xi(x)=0$

when $|x| > \frac{5}{2}$ and $\xi(x) = 1$ when $|x| < 2$. Let

$$I_j(t) = \frac{1}{|t|^{n/2}} \int_{B(4)} \big(|s_j(x,t)|^2 + |u_j(x,t)|^2\big)\xi^2(x) \exp\left(\frac{|x|^2}{4t}\right)\, dx.$$

By (3.13), there are positive constants $C > 0$ and N_1, both independent of i, such that

$$I_j(t) \geq C I_j(-4)|t|^{2N_1} = C|t|^{2N_1} \quad \text{for} \quad -9 < t < -1.$$

When $j \to \infty$, we see that

$$I_0(t) \geq C|t|^{2N_1} \quad \text{for} \quad -8 < t < -2.$$

However, by (3.21), $s_0(x,t) = 0$ and $u_0(x,t) = 0$ for $(x,t) \in B(2) \times (-6,-2)$. We have a contradiction.

Next, we assume that $R_0 > 0$ in (3.22). Recall that the function pairs (s_K, u_K) are uniformly continuous. By choosing subsequences, we may assume that as $i \to \infty$, (x_j, t_j) converges to (x_0, t_0) and (s_{K_j}, u_{K_j}) converges uniformly to a pair (s_0, u_0) in $B(x_0; 4R_0) \times (t_0 - (4R_0)^2, t_0)$. By (3.14), there is a positive constant C, independent of j such that

$$I_{K_j}(t; x_j, t_j) \geq C|t - t_j|^{2N_1} \quad \text{for} \quad t \in (t_j - (4R_j)^2, t_j - R_j^2).$$

When $j \to \infty$, we have

$$I_0(t) \geq C|t - t_0|^{2N_1} \quad \text{for} \quad t \in (t_0 - (3R_0)^2, t_0 - R_0^2),$$

where

$$I_0(t) = \frac{1}{|t - t_0|^{n/2}} \int_{B(x_0;4R_*)} \big(|s_0(x,t)|^2 + |u_0(x,t)|^2\big)\xi^2(x) \times \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right)\, dx$$

and $\xi(x)$ is a cutoff function such that $\xi(x) = 1$ when $x \in B(x_*; 2R_*)$. However, by (3.21), $s_0(x,t) = 0$ and $u_0(x,t) = 0$ for $(x,t) \in B(x_0; 2R_0) \times (t_0 - 6R_0^2, t_0 - 2R_0^2)$. We have a contradiction. □

Proposition 3.6. *Let $B(x_*; 4R_*) \subset \Omega$ and $t_* > (4R_*)^2$. There are positive constants $R_0 < R_*$ and $\epsilon_0 < 1$, depending only on m, R_*, M and the initial* *data, such that if* $R_1 \in (0, R_0)$ *and* $(x_1, t_1) \in B(x_*; R_*) \times (t_* - R_*^2, t_*]$ *and* $N_K(t_1 - 4R_1^2; x_1, t_1) \le \epsilon_0$, *then*

$$\sup\{|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 + |u_K|^2)^2 : \\ (x,t) \in B(x_1; R_1^2) \times (t_1 - R_1^4, t_1]\} \le \frac{C}{R_1^4},$$

and $C > 0$ *is constant depending on* c_0, m, M, R_* *and the initial data only.*

Proof. Let $(x_1, t_1) \in B(x_*; 4R_*) \times ((t_* - R_*^2, t_*]$ and $R > 0$. Set $\delta \in (0, \frac{1}{2})$ and $\sigma = \delta R$. For any $x \in \mathbb{R}^m$ and $t_1 - R^2 \le t \le t_1 - 4R^2$, we have

$$G(x,t;x_1,t_1+2\sigma^2) \le \begin{cases} CG(x,t;x_1,t_1) & \text{if } \ |x-x_1| \le R/\delta \\ CR^{-m}\exp\left(-c\delta^{-2}\right) & \text{if } \ |x-x_1| \ge R/\delta \end{cases}.$$

See [4] p491-492. Thus, when $t_1 - R^2 \le t \le t_1 - 4R^2$, we obtain

$$G(x,t;x_1,t_1+2\sigma^2) \le CG(x,t;x_1,t_1) + CR^{-m}\exp\left(-c\delta^{-2}\right), \tag{3.29}$$

where C is a positive constant depending on m only. Also, it is easy to check that if $\delta \in (0, \frac{1}{2})$ and $\sigma = \delta R$, then when $t_1 - R^2 \le t \le t_1 - 4R^2$,

$$G(x,t;x_1,t_1+2\sigma^2) \ge CG(x,t;x_1,t_1), \tag{3.30}$$

where C is a positive constant depending on m only.

Let ϵ_0 and R_0 be positive constants to be determined (see (3.34), (3.35) and (3.41)). Let $0 < R_1 \le R_0$. Let $\delta_1 = (k|\ln R_1|)^{-1/2}$ and k be a constant to be determined. Let $\sigma_1 = \delta_1 R_1$. By (3.29),

$$E_K(t_1 - 4R_1^2; x_1, t_1 + 2\sigma_1^2) \le CE_K(t_1 - 4R_1^2; x_1, t_1) + CR_1^{-m}\exp\left(-c\delta_1^{-2}\right).$$

Also, by (3.30), we have

$$I_K(t_1 - 4R_1^2; x_1, t_1 + 2\sigma_1^2) \ge CI_K(t_1 - 4R_1^2; x_1, t_1).$$

This implies that

$$N_K(t_1 - 4R_1^2; x_1, t_1 + 2\sigma_1^2) \le CN_K(t_1 - 4R_1^2; x_1, t_1) + \frac{CR_1^{-m}\exp\left(-c\delta_1^{-2}\right)}{I_K(t_1 - 4R_1^2; x_1, t_1)}.$$

By (3.14), we have

$$N_K(t_1 - 4R_1^2; x_1, t_1 + 2\sigma_1^2) \le CN_K(t_1 - 4R_1^2; x_1, t_1) + \frac{CR_1^{-m}\exp\left(-c\delta_1^{-2}\right)}{CR_1^{2N_1}}.$$

We choose $\delta_1 = (k|\ln R_1|)^{-1/2}$ with k large enough so that

$$\frac{CR_1^{-m}\exp\left(-c\delta_1^{-2}\right)}{CR_1^{2N_1}} \le CR_1^2 \quad \text{for all} \quad R_1 \in (0, \frac{1}{2}).$$

By (3.10) and (3.11),

$$\begin{aligned} N_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2) &\le N_K(t_1-4R_1^2; x_1, t_1+2\sigma_1^2) + CN_0R_1^2 \\ &\le CN_K(t_1-4R_1^2; x_1, t_1) + CR_1^2. \end{aligned}$$

Let $\bar{R}$ and $\bar{\epsilon}$ be the constants in Proposition 3.4. If $N_K(t_1-4R_1^2; x_1, t_1) \le \epsilon_0$ and $R_1 < R_0$, then

$$N_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2) \le C\epsilon_0 + CR_1^2. \tag{3.31}$$

By Proposition 3.5, we have

$$\begin{aligned} &\sup\{s_K^2(x,t) + |u_K(x,t)|^2 : (x,t) \in B(x_1; 2\sigma_1) \times (t_1-4\sigma_1^2, t_1]\} \\ &\quad \ge \frac{C}{\sigma_1^{m+2}} \int_{t_1-(2\sigma_1)^2}^{t_1} \int_{B(x_1;2\sigma_1)} \left(s_K^2(x,t) + |u_K(x,t)|^2\right) dx\, dt \\ &\quad \ge c_1 I_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2). \end{aligned} \tag{3.32}$$

By (2.17), for $t \in (t_1-4\sigma_1^2, t_1]$,

$$\begin{aligned} &\frac{1}{\sigma_1^{m-2}} \int_{B(x_1;2\sigma_1)} \left(|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2\right)(x,t)\, dx \\ &\quad \le E_K(t; x_1, t_1+2\sigma_1^2) \\ &\quad \le CE_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2) + CR_1^2 I_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2) \\ &\qquad + C\exp\left(\frac{-1}{24\sigma_1^2}\right). \end{aligned}$$

By (3.31), we have

$$E_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2) \le (C\epsilon_0 + CR_1^2) I_K(t_1-(4\sigma_1)^2; x_1, t_1+2\sigma_1^2),$$

and by (3.14), for $t \in (t_1 - 4\sigma_1^2, t_1]$,

$$\begin{aligned}(3.33)\quad & \frac{1}{\sigma_1^{m-2}} \int_{B(x_1;2\sigma_1)} \big(|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2\big)(x,t)\; dx \\ & \le C\epsilon_0 I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2) \\ & \quad + CR_1^2 I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2) \\ & \quad + C \exp\left(\frac{-1}{24\sigma_1^2}\right) \sigma_1^{-2N_1} I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2).\end{aligned}$$

Let $\bar{R}$ and $\bar{\epsilon}$ be the constants in Proposition 3.4. Now, we choose $R_0 \in (0, \frac{1}{2})$ such that

$$(3.34)\qquad 0 < R_0 < \bar{R} \quad \text{and} \quad CR_0^2 + C\exp\left(\frac{-1}{24R_0^2}\right) R_0^{-2N_1} \le \frac{\bar{\epsilon}}{2},$$

and choose $\epsilon_0 \in (0,1)$ such that

$$(3.35)\qquad C\epsilon_0 \le \frac{\bar{\epsilon}}{2}.$$

In conclusion, we have

$$\begin{aligned}(3.36)\quad & \frac{1}{\sigma_1^{m-2}} \int_{B(x_1;2\sigma_1)} \big(|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2\big)(x,t)\; dx \\ & \le \bar{\epsilon}\; I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2) \quad \text{for} \quad t \in (t_1 - 4\sigma_1^2, t_1].\end{aligned}$$

Furthermore, by Proposition 3.3 and (3.13),

$$\begin{aligned}(3.37)\quad & \sup\{s_K^2(x,t) + |u_K(x,t)|^2 :\ (x,t) \in B(x_1; 4\sigma_1) \times (t_1 - (4\sigma_1)^2, t_1]\} \\ & \le C I_K(t_1 - (8\sigma_1)^2; x_1, t_1 + 2\sigma_1^2) \\ & \le C I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2).\end{aligned}$$

Using (3.32), (3.36) and (3.37), by Proposition 3.4, there is a positive constant γ_0 such that

$$\begin{aligned}(3.38)\quad & \sup\{s_K^2(x,t) + |u_K(x,t)|^2 :\ (x,t) \in B(x_1; 2\sigma_1) \times (t_1 - (2\sigma_1)^2, t_1]\} \\ & \ge \gamma_0 I_K(t_1 - (4\sigma_1)^2; x_1, t_1 + 2\sigma_1^2).\end{aligned}$$

From equations (2.20) and (2.21), we have

$$\begin{aligned}
&(-\partial_t+\Delta)(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2)\\
&\quad=2|\nabla^2 s_K|^2+2|\nabla^2 u_K|^2+16K|s_K\nabla s_K-u_K\nabla u_K|^2\\
&\qquad+8K^2(s_K^2-|u_K|^2)^2(s_K^2+|u_K|^2)+8K(s_K^2-|u_K|^2)(|\nabla s_K|^2-|\nabla u_K|^2)\\
&\qquad+2\nabla s_K\cdot(s_K\nabla s_K+u_K\nabla u_K)\left(\frac{V(s)}{s}\right)'+2\frac{V(s)}{s}(|\nabla s_K|^2+|\nabla u_K|^2)\\
&\qquad+4K\frac{V(s)}{s}(s_K^2-|u_K|^2)^2.
\end{aligned}$$

Let

$$\beta=\min\{1,\sqrt{\gamma_0 I_K(t_1-(4\sigma_1)^2;x_1,t_1+2\sigma_1^2)}\}.$$

By (3.38), $s_K^2(x,t)+|u_K(x,t)|^2\geq\beta^2$ for $(x,t)\in B(x_1;2\sigma_1)\times(t_1-4\sigma_1^2,t_1]$. Thus, on $B(x_1;2\sigma_1)\times(t_1-4\sigma_1^2,t_1]$, by (2.28) and (3.1), we have

$$\begin{aligned}
&(-\partial_t+\Delta)(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2)\\
&\quad\geq 8\beta^2K^2(s_K^2-|u_K|^2)^2-4\beta^2K^2(s_K^2-|u_K|^2)^2-\frac{4}{\beta^2}(|\nabla s_K|^2+|\nabla u_K|^2)^2\\
&\qquad-C|\nabla s_K|(|\nabla s_K|+|\nabla u_K|)-C(|\nabla s_K|^2+|\nabla u_K|^2)\\
&\qquad-CK(s_K^2-|u_K|^2)^2\\
&\quad\geq-\frac{D}{\beta^2}(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2)\\
&\qquad-E(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2),
\end{aligned}$$

where C, D, E are positive constants depending only on c_0, m, M, R_* and the initial data only. Let

$$e_K(x,t)=|\nabla s_K(x,t)|^2+|\nabla u_K(x,t)|^2+K(s_K^2(x,t)-|u_K(x,t)|^2)^2.$$

The function e_K satisfies the differential inequality

$$\partial_t e_K\leq\Delta e_K+\frac{D}{\beta^2}e_K^2+Ee_K.$$

For $(x,t)\in B(x_1;2\sigma_1)\times(t_1-4\sigma_1^2,t_1]$, let

$$\tilde{s}_K(x,t)=\frac{s_K(x,t)}{\beta},\quad\tilde{u}_K(x,t)=\frac{u_K(x,t)}{\beta},$$

and $\tilde{e}_K(x,t)=|\nabla\tilde{s}_K(x,t)|^2+|\nabla\tilde{u}_K(x,t)|^2+\beta^2K(\tilde{s}_K^2(x,t)-|\tilde{u}_K(x,t)|^2)^2.$

By (3.37) and (3.38),

$$1 \le |\tilde{s}_K(x,t)|^2 + |\tilde{u}_K(x,t)|^2 \le C \quad \text{for} \quad (x,t) \in B(x_1; 2\sigma_1) \times (t_1 - 4\sigma_1^2, t_1].$$

The function $\tilde{e}_K$ satisfies the differential inequality

$$\partial_t \tilde{e}_K \le \Delta \tilde{e}_K + D\tilde{e}_K^2 + E\tilde{e}_K. \tag{3.39}$$

Let $\tilde{\xi}(x)$ be a cutoff function such that $\tilde{\xi}(x) = 1$ when $|x - x_1| \le \frac{3}{2}\sigma_1$ and $\tilde{\xi}(x) = 0$ when $|x - x_1| \ge 2\sigma_1$. By the small-energy-regularity theory ([1] Lemma 2.4 and Lemma 4.4), there is a positive constant $\tilde{\epsilon}$ such that if

$$\int_{t_0-\sigma_1^2}^{t_0-\sigma_1^2/4} \int_{B(x_1;2\sigma_1)} \tilde{e}_K(x,t)\tilde{\xi}^2(x)G(x,t;x_1,t_1)\, dx\, dt \le \tilde{\epsilon},$$

then

$$\sup\{\tilde{e}_K(x,t) : (x,t) \in B(x_1; \tilde{\delta}\sigma_1) \times (t_1 - (\tilde{\delta}\sigma_1)^2, t_1]\} \le \frac{C}{(\tilde{\delta}\sigma_1)^2}, \tag{3.40}$$

where $\tilde{\delta} \sim (|\ln \sigma_1|)^{-1/2}$. Moreover, the constant $\tilde{\epsilon}$ depends only on the constants D and E in equation (3.39). By (3.33), if we further choose ϵ_0 and R_0 such that

$$C\epsilon_0 \le \frac{\tilde{\epsilon}}{2} \quad \text{and} \quad CR_0^2 + C\exp\left(\frac{-1}{24R_0^2}\right) R_0^{-2N_1} \le \frac{\tilde{\epsilon}}{2}, \tag{3.41}$$

then

$$\begin{aligned} &\int_{t_0-\sigma_1^2}^{t_0-\sigma_1^2/4} \int_{B(x_1;2\sigma_1)} \tilde{e}_K(x,t)\tilde{\xi}^2(x)G(x,t;x_1,t_1)\, dx\, dt \\ &\quad \le \frac{C}{\beta^2\sigma_1^m} \int_{t_1-4\sigma_1^2}^{t_1} \int_{B(x_1;2\sigma_1)} \left(|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2\right) dx\, dt \\ &\quad \le \tilde{\epsilon}. \end{aligned}$$

Since $\sigma_1 = \delta_1 R_1$ and $\delta_1 \sim (|\ln R_1|)^{-1/2}$ and $\tilde{\delta} \sim (|\ln \sigma_1|)^{-1/2}$, we may assume that $\tilde{\delta}\sigma_1 \ge R_1^2$. Thus, (3.40) implies that

$$\sup\{e_K(x,t) : (x,t) \in B(x_1; R_1^2) \times (t_1 - R_1^4, t_1] \le \frac{C}{R_1^4}.$$

The proof is complete. □

Theorem 1.2 follows immediately from the following Holder estimate on (s_K, u_K).

Theorem 3.7. *Let R_* be a positive number such that $B(x_*; 4R_*) \subset \Omega$ and $t_* > (4R_*)^2$. Suppose that (3.10), (3.11), (3.13), (3.14) hold for (s_K, u_K). There are positive constants C and α, independent of K such that for any $(x_1, t_1), (x_2, t_2) \in B(x_*; R_*) \times (t_* - R_*^2, t_*]$, we have*

$$\begin{aligned} |s_K(x_1,t_1) - s_K(x_2,t_2)| &+ |u_K(x_1,t_1) - u_K(x_2,t_2)| \\ &\le C\big(|x_1 - x_2|^2 + |t_1 - t_2|\big)^{\alpha/2}. \end{aligned} \tag{3.42}$$

Proof. Let ϵ_0 and R_0 be the constants in Proposition 3.6. Let $(x_0, t_0) \in B(x_*; R_*) \times (t_* - R_*^2, t_*)$. Suppose that

$$N_K(t_0 - (4R)^2; x_0, t_0) \ge \epsilon_0 \quad \text{for} \quad R \in (0, R_0). \tag{3.43}$$

This implies that

$$E(t; x_0, t_0) \ge \epsilon_0 I_K(t; x_0, t_0) \quad \text{for} \quad t \in (t_0 - 4R_0^2, t_0).$$

By (2.14) and (2.28) and (3.14), if R_0 is chosen small enough and $t \in (t_0 - R_0^2, t_0)$, then

$$\begin{aligned} &-\frac{d}{dt} I_K(t; x_0, t_0) \\ &\ge \frac{2}{|t - t_0|} E_K(t; x_0, t_0) - 2M I_K(t; x_0, t_0) - C \exp\left(\frac{1}{6|t - t_0|}\right) \\ &\ge \frac{2\epsilon_0}{|t - t_0|} I_K(t; x_0, t_0) - 2M I_K(t; x_0, t_0) \\ &\quad - \frac{C}{|t - t_0|^{2N_1}} \exp\left(\frac{1}{6|t - t_0|}\right) I_K(t; x_0, t_0) \\ &\ge \frac{\epsilon_0}{|t - t_0|} I_K(t; x_0, t_0). \end{aligned}$$

Then we see that

$$I_K(t_0 - R^2; x_0, t_0) \le C\left(\frac{R}{R_0}\right)^{2c_2} I_K(t_0 - (4R_0)^2; x_0, t_0), \quad \text{for} \quad R \in (0, R_0),$$

where C is a positive constant independent of K and c_2 is a positive constant depending only on ϵ_0 and is independent of K. By (3.11) and (2.28), when

(3.43) holds, we have

$$E_K(t_0 - R^2; x_0, t_0) \le C\left(\frac{R}{R_0}\right)^{2c_2}, \quad \text{for} \quad R \in (0, R_0). \tag{3.44}$$

If (3.43) is not true, there is $R_1 \in (0, R_0)$ such that

$$N_K(t_0 - (4R)^2; x_0, t_0) \ge \epsilon_0 \quad \text{for} \quad R \in (R_1, R_0),$$

and $N_K(t_0 - (4R_1)^2; x_0, t_0) = \epsilon_0$. Then using the computations in the above, we have

$$I_K(t_0 - R^2; x_0, t_0) \le C\left(\frac{R}{R_0}\right)^{2c_2} I_K(t_0 - (4R_0)^2; x_0, t_0), \quad \text{for } R \in (R_1, R_0).$$

Again, by (3.11) and (2.28), this implies that

$$E_K(t_0 - R^2; x_0, t_0) \le C\left(\frac{R}{R_0}\right)^{2c_2}, \quad \text{for} \quad R \in (R_1, R_0). \tag{3.45}$$

By (2.17) and (3.45), for $R \in (R_1^4, R_1)$, we have

$$\begin{aligned} E_K(t_0 - R^2; x_0, t_0) &\le C E_K(t_0 - R_1^2; x_0, t_0) + C R_1^2 \\ &\le C R_1^{2c_2} + C R_1^2 \\ &\le C R^{2c_3} \quad \text{for} \quad R \in (R_1^4, R_1); \end{aligned} \tag{3.46}$$

where $c_3 = \frac{1}{4}\min\{c_2, 1\}$. When $R \in (0, R_1^4)$, we write

$$\begin{aligned} &E_K(t_0 - R^2; x_0, t_0) \\ &\quad = R^2 \int_{|x-x_0| \le R^{1/2}} e_K(x, t_0 - R^2)\xi^2(x) G(x, t_0 - R^2; x_0, t_0)\, dx \\ &\qquad + R^2 \int_{|x-x_0| \ge R^{1/2}} e_K(x, t_0 - R^2)\xi^2(x) G(x, t_0 - R^2; x_0, t_0)\, dx. \end{aligned}$$

By Proposition 3.6, when $(x, t) \in B(x_0; R_1^2) \times (t_0 - R_1^4; t_0)$,

$$e_K(x, t) = |\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2 \le \frac{C}{R_1^4}.$$

If $R \in (0, R_1^4)$, then $R^{1/2} \le R_1^2$. Thus,

$$R^2 \int_{|x-x_0| \le R^{1/2}} e_K(x, t_0 - R^2)\xi^2(x) G(x, t_0 - R^2; x_0, t_0)\, dx \le \frac{CR^2}{R_1^4} \le CR.$$

By (2.30),

$$R^2 \int_{|x-x_0|\geq R^{1/2}} e_K(x,t_0-R^2)\xi^2(x)G(x,t_0-R^2;x_0,t_0)\,dx \leq CR^{2-m}\exp\left(\frac{-1}{4R}\right).$$

Therefore, we obtain

$$E_K(t_0-R^2;x_0,t_0)\leq CR \quad \text{for} \quad R\in(0,R_1^4). \tag{3.47}$$

By (3.45), (3.46) and (3.47), if (3.43) is not true, then

$$E_K(t_0-R^2;x_0,t_0)\leq CR^{2c_3} \quad \text{for} \quad R\in(0,R_0). \tag{3.48}$$

Let $\alpha = c_3$. By (3.44) and (3.48), at any $(x_0,t_0)\in B(x_*;R_*)\times(t^*-R_*^2,t^*)$ and $R\in(0,R_*)$, we always have

$$E_K(t_0-R^2;x_0,t_0)\leq CR^{2\alpha} \quad \text{for} \quad R\in(0,R_0).$$

We may replace t_0 by t_0+R^2 and have

$$E_K(t_0;x_0,t_0+R^2)\leq CR^{2\alpha} \quad \text{for} \quad R\in(0,R_0).$$

This implies that for each $t_0>(4R_*)^2$ and $B(x_0;4R_*)\subset\Omega$,

$$R^{2-m}\int_{B(x_0;R)} \big(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\big)(x,t_0)\,dx\leq CR^{2\alpha},$$

for all $R\in(0,R_0)$, and R_0, C and α are positive constants depending only on c_0, m, M, R_* and the initial data only. By Proposition 2.1, this proves (3.42). □

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References

[1] Y. Chen and M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989), 83–103.

[2] F.H. Lin, *On nematic liquid crystals with variable degree of orientation*, Comm. Pure Appl. Math. **44** (1991), 453–468.

[3] C.C. Poon, *A heat flow problem from Ericksen's model for nematic liquid crystals with variable degree of orientation*, Comm. Analysis and Geometry **26** (2018), 411–433.

[4] M. Sturwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geometry **28** (1988), 485–502.

Department of Mathematics, National Chung Cheng University
Minghsiung 621, Chiayi, Taiwan
E-mail address: chicheung.poon@gmail.com

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