Communications in
Analysis and Geometry
Volume 30, Number 9, 2095–2122, 2022

K3 surfaces with a pair of commuting non-symplectic involutions

Frank Reidegeld

We study K3 surfaces with a pair of commuting involutions that are non-symplectic with respect to two anti-commuting complex structures that are determined by a hyper-Kähler metric. One motivation for this paper is the role of such $\mathbb{Z}_2^2$-actions for the construction of Riemannian manifolds with holonomy G_2. We find a large class of smooth K3 surfaces with such pairs of involutions. After that, we turn our attention to the case that the K3 surface has ADE-singularities. We introduce a special class of non-symplectic involutions that are suitable for explicit calculations and find 320 examples of pairs of involutions that act on K3 surfaces with a great variety of singularities.

1. Introduction

A K3 surface is a compact, simply connected, complex surface with trivial canonical bundle. There exist many deep and beautiful theorems about K3 surfaces. The main reason for this is that many geometric properties of a K3 surface S can be translated into data that are related to the lattice $H^2(S,\mathbb{Z})$.

A non-symplectic involution of a K3 surface S is a holomorphic involution $\rho : S \to S$ such that ρ acts as -1 on $H^{2,0}(S)$. Any K3 surface with a non-symplectic involution admits a Kähler class that is invariant under ρ. Therefore, there exists a ρ-invariant Ricci-flat Kähler metric in this class. Since any Ricci-flat Kähler metric on a K3 surface is in fact hyper-Kähler, there are three complex structures I, J and K and three Kähler forms ω_I, ω_J and ω_K on S. If ρ is holomorphic with respect to I, we have

$$\rho^*\omega_I = \omega_I \,, \qquad \rho^*\omega_J = -\omega_J \,, \qquad \rho^*\omega_K = -\omega_K$$

In this paper, we search for K3 surfaces that admit two non-symplectic involutions ρ_I and ρ_J. We require that ρ_I and ρ_J commute and that they are non-symplectic with respect to complex structures I and J that satisfy

$IJ = -JI$. This assumption yields the relations

$$(1)\qquad \begin{array}{lll} {\rho_I}^*\omega_I = \omega_I\ , & {\rho_I}^*\omega_J = -\omega_J\ , & {\rho_I}^*\omega_K = -\omega_K \\ {\rho_J}^*\omega_I = -\omega_I\ , & {\rho_J}^*\omega_J = \omega_J\ , & {\rho_J}^*\omega_K = -\omega_K \end{array}$$

A motivation to study these pairs of involutions is their relation to the construction of 7-dimensional Riemannian manifolds with holonomy G_2, or shortly G_2-manifolds. ρ_I and ρ_J generate a group that is isomorphic to $\mathbb{Z}_2^2$ and acts isometrically on S. In the habilitation thesis of the author [17] we have described how such an action can be extended to products $S \times T^3$ of a K3 surface and a 3-torus such that the quotients $(S \times T^3)/\mathbb{Z}_2^2$ carry orbifold metrics whose holonomy is a subgroup of G_2. In a forthcoming paper we resolve the singularities of those quotients by the methods of Joyce and Karigiannis [10] and obtain compact smooth G_2-manifolds.

Moreover, pairs of involutions (ρ_I, ρ_J) with the above properties can be used in Kovalev and Lee's construction of compact G_2-manifolds by twisted connected sums [13]. A crucial step in the twisted connected sum construction is to find a hyper-Kähler rotation, that is an isometry $f : S_1 \to S_2$ such that

$$f^*\omega_{I_2} = \omega_{J_1}\ , \qquad f^*\omega_{J_2} = \omega_{I_1}\ , \qquad f^*\omega_{K_2} = -\omega_{K_1}\ ,$$

where ω_{I_j}, ω_{J_j} and ω_{K_j} are the Kähler forms on S_j. For the particular twisted connected sums in [13] we also need non-symplectic involutions on the S_j. In Remark 6.2, we show how pairs (ρ_I, ρ_J) yield precisely such data.

In this paper, we pay special attention to the case that S has ADE-singularities since in this case S becomes a building block for G_2-orbifolds with ADE-singularities. These orbifolds are studied as compactifications of M-theory since the singularities are needed to explain the presence of non-abelian gauge fields [1, 2]. The construction of G_2-orbifolds with ADE-singularities is the subject of [17] and of a forthcoming paper by the author.

This paper is organized as follows. In Section 2 and 3, we present the necessary background material about smooth and singular K3 surfaces that can be found in the literature. In Section 4, we discuss non-symplectic involutions. A deformation class of K3 surfaces with a non-symplectic involution is determined by the action of the involution on $H^2(S, \mathbb{Z})$, which is always isomorphic to a certain lattice L. Nikulin [14–16] has classified the non-symplectic involutions of K3 surfaces in terms of invariants of their fixed lattices. There are 75 types of non-symplectic involutions.

In Definition 4.3 we introduce a special class of non-symplectic involutions that are suitable for explicit calculations. They act on a certain basis of

L as a permutation and a change of the signs. We call these involutions simple. We show in Theorem 4.4 that 28 out of the 75 types of non-symplectic involutions are simple.

Section 5 deals with K3 surfaces with singularities that admit a simple non-symplectic involution. In Theorem 5.1 it is proven that each deformation class of K3 surfaces with a simple involution contains a K3 surface with 3 A_1- and 2 E_8-singularities. Furthermore, there exist plenty of K3 surfaces with fewer and milder singularities in those deformation classes. This phenomenon is discussed in Theorem 5.2.

In the final Section 6, we study K3 surfaces with a pair (ρ_I, ρ_J) of non-symplectic involutions satisfying (1). At the beginning of that section, we embed certain direct sums of possible fixed lattices into L and find a large class of pairs (ρ_I, ρ_J). To each pair belongs a deformation class of K3 surfaces with hyper-Kähler structures that are invariant under ρ_I and ρ_J. We show in Theorem 6.1 that the hyper-Kähler structure on the K3 surface can be chosen such that it has no singularities.

After that, we turn to K3 surfaces with singularities again. Theorem 6.3 yields 320 pairs of simple involutions (ρ_I, ρ_J). Analogously to Section 5, we prove in Corollary 6.5 that each deformation class that corresponds to a pair (ρ_I, ρ_J) contains a K3 surface with 3 A_1- and 2 E_8-singularities. In Theorem 6.6 we see that again there are many examples of K3 surfaces with milder singularities. Unlike in Section 5, it happens for some pairs (ρ_I, ρ_J) that the K3 surface has a minimal singularity that cannot be resolved without breaking the symmetry of the hyper-Kähler structure with respect to ρ_I and ρ_J. An explicit pair of involutions with this property can be found in Example 6.7.

2. K3 surfaces and their moduli spaces

This section contains a short introduction to the theory of K3 surfaces and their hyper-Kähler moduli space. We refer the reader to [5, Chapter VIII] and references therein for a more detailed account.

We define a K3 surface as at the beginning of the introduction. In this section we restrict ourselves to smooth K3 surfaces. K3 surfaces with ADE-singularities will be studied in Section 3. Since all K3 surfaces are diffeomorphic to each other, their topological invariants are the same. The Hodge numbers of any K3 surface S are determined by $h^{0,0}(S) = h^{2,0}(S) = 1$, $h^{1,0}(S) = 0$ and $h^{1,1}(S) = 20$.

$H^2(S, \mathbb{Z})$ together with the intersection form is a lattice. It is an even lattice, which means that $x^2 \in 2\mathbb{Z}$ for all $x \in H^2(S, \mathbb{Z})$, and it is unimodular, which means that there is a basis $(e_i)_{i=1,\dots,22}$ with $|\det(e_i \cdot e_j)_{i,j=1,\dots,22}| = 1$. Moreover, the signature of $H^2(S,\mathbb{Z})$ is $(3,19)$. Up to isometries, the only lattice with these properties is

$$L := 3H \oplus 2(-E_8)\,,$$

where H is the hyperbolic plane lattice with the bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

and $-E_8$ is the root lattice of E_8 together with the negative of the usual bilinear form. $-E_8$ can also be characterized as the unique negative definite, even, unimodular lattice of rank 8. We call L the *K3 lattice*. Since we have to work with various sublattices of L, we need some results about lattice embeddings.

Definition 2.1. A sublattice N of a lattice M is called *primitive* if the quotient M/N has no torsion. Analogously, a lattice embedding $\imath : K \to M$ is called *primitive* if $\imath(K)$ is a primitive sublattice.

We denote the minimal number of generators of the discriminant group M^*/M, where M^* is the dual lattice, by $\ell(M)$. With this notation we are able to formulate a theorem on primitive embeddings that can be found in [8] or [15].

Theorem 2.2. *Let K be an even non-degenerate lattice of signature (k_+, k_-) and L be an even unimodular lattice of signature (l_+, l_-). We assume that $k_+ \leq l_+$ and $k_- \leq l_-$ and that*

1) $2 \cdot rank(K) \leq rank(L)$ *or*

2) $rank(K) + \ell(K) < rank(L)$.

Then there exists a primitive embedding $\imath : K \to L$. If in addition $k_+ < l_+$ and $k_- < l_-$ and one of the following conditions holds

1) $2 \cdot rank(K) \leq rank(L) - 2$,

2) $rank(K) + \ell(K) \leq rank(L) - 2$,

the embedding $\imath$ is unique up to an automorphism of L.

Any K3 surface S admits a Kähler metric. Since S has trivial canonical bundle, there exists a unique Ricci-flat Kähler metric in each Kähler class.

The holonomy group $SU(2)$ is isomorphic to $Sp(1)$. Therefore, the Ricci-flat Kähler metrics are in fact hyper-Kähler. We need some results about the moduli space of all hyper-Kähler structures on K3 surfaces. These are similar to the usual results about the moduli space of K3 surfaces. In principle, a hyper-Kähler structure on a K3 surface is determined by the cohomology classes $[\omega_I], [\omega_J], [\omega_K] \in H^2(S,\mathbb{R})$ of the three Kähler forms. In order to make this statement precise, we need the following definitions.

Definition 2.3. 1) Let S be a K3 surface. A lattice isometry $\phi : H^2(S,\mathbb{Z}) \to L$ is called a *marking of* S. The pair (S,ϕ) is called a *marked K3 surface.*

2) A *hyper-Kähler structure on a marked K3 surface* is a tuple $(S,\phi,g,\omega_I,\omega_J,\omega_K)$, where g is a hyper-Kähler metric and ω_I, ω_J and ω_K are the Kähler forms with respect to the complex structures I, J and K that satisfy $IJK = -1$. We assume that S has the orientation that makes ω_I^2 a positive 4-form.

3) Two tuples $(S_1,\phi_1,g_1,\omega_I^{(1)},\omega_J^{(1)},\omega_K^{(1)})$ and $(S_2,\phi_2,g_2,\omega_I^{(2)},\omega_J^{(2)},\omega_K^{(2)})$ are *isomorphic* if there exists a map $f : S_1 \to S_2$ with $f^*g_2 = g_1$, $f^*\omega_I^{(2)} = \omega_I^{(1)}$, $f^*\omega_J^{(2)} = \omega_J^{(1)}$, $f^*\omega_K^{(2)} = \omega_K^{(1)}$ and $\phi_1 \circ f^* = \phi_2$. The *moduli space of marked K3 surfaces with a hyper-Kähler structure $\mathscr{K}3$* is the class of all tuples $(S,\phi,g,\omega_I,\omega_J,\omega_K)$ modulo isomorphisms.

4) We denote $L \otimes \mathbb{R}$ by $L_{\mathbb{R}}$ and define the *hyper-Kähler period domain* by

$$\begin{aligned}\Omega := \{(x,y,z) \in L_{\mathbb{R}}^3 | x^2 = y^2 = z^2 > 0,\ x\cdot y = x\cdot z = y\cdot z = 0,\\ \nexists\, d \in L \text{ with } d^2 = -2 \text{ and } x\cdot d = y\cdot d = z\cdot d = 0\}\,.\end{aligned}$$

5) The *hyper-Kähler period map* $p : \mathscr{K}3 \to \Omega$ is defined by

$$p(S,\phi,g,\omega_I,\omega_J,\omega_K) = (\phi([\omega_I]),\phi([\omega_J]),\phi([\omega_K]))$$

Theorem 2.4. *The hyper-Kähler period map $p : \mathscr{K}3 \to \Omega$ is a diffeomorphism.*

Remark 2.5. The above theorem is a consequence of the surjectivity of the usual period map for K3 surfaces, the Torelli theorem and the description of the Kähler cone of a K3 surface. It can also be found in [9, Theorem 7.3.16]. Moreover, the hyper-Kähler moduli space is described in terms of the metric instead of the Kähler forms in [6, pp. 366 - 368].

Finally, we introduce a lemma on isometries of K3 surfaces that will be useful later on.

Lemma 2.6. *Let S_j with $j \in \{1,2\}$ be K3 surfaces together with hyper-Kähler metrics g_j and Kähler forms $\omega_I^{(j)}$, $\omega_J^{(j)}$ and $\omega_K^{(j)}$. Moreover, let $V_j \subseteq H^2(S_j,\mathbb{R})$ be the subspace that is spanned by $[\omega_I^{(j)}]$, $[\omega_J^{(j)}]$ and $[\omega_K^{(j)}]$.*

1) *Let $f : S_1 \to S_2$ be an isometry. The pull-back $f^* : H^2(S_2,\mathbb{Z}) \to H^2(S_1,\mathbb{Z})$ is a lattice isometry. Its $\mathbb{R}$-linear extension maps V_2 to V_1.*
2) *Let $\psi : H^2(S_2,\mathbb{Z}) \to H^2(S_1,\mathbb{Z})$ be a lattice isometry. We denote the maps $\psi \otimes \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ by ψ, too. We assume that $\psi(V_2) = V_1$. Then there exists an isometry $f : S_1 \to S_2$ such that $f^* = \psi$.*
3) *Let $f : S \to S$ be an isometry that acts as the identity on $H^2(S,\mathbb{Z})$. Then, f itself is the identity map. As a consequence, the isometry from (2) is unique.*

Proof. The first claim is obvious and the third one follows from Proposition 11.3 in Chapter VIII in [5]. The second claim is a consequence of the Torelli theorem. More precisely, the fact that $\omega_J^{(j)} + i\omega_K^{(j)}$ is a $(2,0)$-form with respect to the complex structure $I^{(j)}$ determines a splitting of $H^2(S_j,\mathbb{C})$ into $H^{2,0}(S_j) \oplus H^{1,1}(S_j) \oplus H^{0,2}(S_j)$ that is invariant under ψ. The positive cone, which is the connected component of $\{x \in H^{1,1}(S,\mathbb{R}) | x^2 > 0\}$ which contains a Kähler class, is either preserved by ψ or the positive cone of S_2 is mapped to the negative of the positive cone of S_1. In the first case, ψ preserves the Kähler cone, too, ψ is effective [5, Sec. VIII, Proposition 3.11], and the Torelli theorem [5, Sec. VIII, Theorem 11.1] thus yields a biholomorphic map $f : S_1 \to S_2$ with $f^* = \psi$. In the second case, we obtain an anti-holomorphic bijective map $f : S_1 \to S_2$ with $f^* = \psi$. The triples $([\omega_I^{(1)}], [\omega_J^{(1)}], [\omega_K^{(1)}])$ and $(\psi([\omega_I^{(2)}]), \psi([\omega_J^{(2)}]), \psi([\omega_K^{(2)}]))$ determine unique hyper-Kähler metrics on S_1. Since both triples span the same subspace, these metrics are the same and we have $f^* g_2 = g_1$. □

Remark 2.7. The lattice isometry $\psi := -\mathrm{Id}_{H^2(S,\mathbb{Z})}$ satisfies all conditions from the lemma. The corresponding isometry $f : S \to S$ is the identity map, but it has to be interpreted as an anti-holomorphic map between (S, I) and $(S, -I)$.

3. K3 surfaces with ADE-singularities

In this section, we discuss K3 surfaces with ADE-singularities and their relation to smooth ones. The results that we present here were originally proven in [3, 4, 11]. A short overview can also be found in [9, p.161 - 162].

Let S be a K3 surface and let $w \in H^2(S,\mathbb{Z})$ be a class with $w^2 = -2$ that represents a submanifold Z of S. We do not assume that $w \in H^{1,1}(S)$ and thus Z is not necessarily a divisor. S shall carry a hyper-Kähler structure $(g, \omega_I, \omega_J, \omega_K)$. It can be shown that Z can be chosen as a sphere that is minimal with respect to g. Its area A is given by

$$A^2 = ([\omega_I]\cdot w)^2 + ([\omega_J]\cdot w)^2 + ([\omega_K]\cdot w)^2$$

We choose a marking ϕ of S. If we move within the hyper-Kähler period domain towards a triple $(x,y,z) \in L^3_{\mathbb{R}}$ with

$$x\cdot\phi(w) = y\cdot\phi(w) = z\cdot\phi(w) = 0\,,$$

the volume of the sphere shrinks to zero. In other words, we obtain a singularity. This is in fact the geometric meaning of the condition in the definition of Ω^{hk} that there shall be no $d \in L$ with $d^2 = -2$ and $x\cdot d = y\cdot d = z\cdot d = 0$. We assume that w is the only cohomology class with this property. In this situation, we obtain the singularity by collapsing a single sphere with self-intersection -2 to a point. Since this is the reversal of blowing up an A_1-singularity, the K3 surface has an A_1-singularity at a single point. This observation can be generalized to the following theorem.

Theorem 3.1. *Let $\widetilde{\Omega}$ be the set*

$$\{(x,y,z) \in L^3_{\mathbb{R}} | x^2 = y^2 = z^2 > 0\,, x\cdot y = x\cdot z = y\cdot z = 0\} \tag{3}$$

and for any triple $(x,y,z) \in \widetilde{\Omega}$ let

$$D := \{d \in L | d^2 = -2,\ x\cdot d = y\cdot d = z\cdot d = 0\}\,. \tag{4}$$

Moreover, let G be the graph that we obtain by joining $d_1, d_2 \in D$ with $d_1 \neq d_2$ by $d_1\cdot d_2$ edges. G is the disjoint union of simply laced Dynkin diagrams $G_1,\ldots,G_k$, which means that they belong to the A-, D- or E-series. In this situation, there exists a unique K3 surface S with a hyper-Kähler structure and ADE-singularities such that its image with respect to the extension of the hyper-Kähler period map is given by (x,y,z). S has ADE-singularities at k points and the type of the jth singularity is given by G_j.

Remark 3.2. The best reference for the above theorem that the author is aware of is [4]. The fact that $\widetilde{\Omega}$ can be identified with the moduli space of K3 surfaces with a hyper-Kähler structure and ADE-singularities is Theorem IV in [4]. A discussion of the singular set of S together with an example can be found at the beginning of Section 6 in [4]. A detailed description of how the smooth metrics converge to orbifold metrics in the Gromov-Hausdorff limit is provided in Section 3 in [4]. We remark that it is possible to resolve a singularity by several steps by moving from (x, y, z) to a point in $\widetilde{\Omega}$, where D is smaller but non-empty.

4. Non-symplectic involutions

In this section we introduce the most important results about non-symplectic involutions. These results were proven by Nikulin [14–16] and are also summed up in [13]. Moreover, we define a class of non-symplectic involutions that are well suited for explicit calculations and we classify them.

Definition 4.1. Let S be a K3 surface. A *non-symplectic involution* is a biholomorphic map $\rho : S \to S$ such that

1) $\rho^2 = \mathrm{Id}$.
2) The pull-back $\rho^* : H^{2,0}(S) \to H^{2,0}(S)$ is not the identity map, or equivalently $\rho^*(\omega_J + i\omega_K) = -(\omega_J + i\omega_K)$.

From now on, let (S, ϕ) be a marked K3 surface and $\rho : S \to S$ be a non-symplectic involution. We define the *fixed lattice of* ρ by

$$L^\rho := \{x \in L | (\phi \circ \rho^* \circ \phi^{-1})(x) = x\} .$$

It can be shown that L^ρ is a primitive, non-degenerate sublattice of L with signature $(1, t)$. A lattice with that kind of signature is called *hyperbolic.* The rank $r = 1 + t$ is an invariant of L^ρ. Moreover, L^ρ is *2-elementary* which means that $L^{\rho *}/L^\rho$ is isomorphic to a group of type $\mathbb{Z}_2^a$. The number $a \in \mathbb{N}_0$ is a second invariant of L^ρ. We define a third invariant δ by

$$\delta := \begin{cases} 0 & \text{if } x^2 \in \mathbb{Z} \text{ for all } x \in L^{\rho *} \\ 1 & \textit{otherwise} \end{cases}$$

Theorem 4.2. *(Theorem 4.3.2 in [16]) For each triple* $(r, a, \delta) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \{0, 1\}$ *there is up to isometries at most one even, hyperbolic, 2-elementary lattice with invariants* (r, a, δ).

Let N be an even, hyperbolic, 2-elementary lattice such that there exists a primitive embedding of N into L. It is possible to construct a K3 surface with a non-symplectic involution whose fixed lattice is N. Up to isometries of L, there is at most one primitive embedding of a lattice with invariants (r, a, δ) into L and it follows that the deformation classes of K3 surfaces with a non-symplectic involution can be classified in terms of triples (r, a, δ). Nikulin [16] has shown that there exist 75 possible triples that satisfy

$$1 \leq r \leq 20\,, \quad 0 \leq a \leq 11 \quad \text{and} \quad r - a \geq 0\,.$$

A figure with a graphical representation of all possible values of (r, a, δ) and a result about the fixed loci of non-symplectic involutions can be found in [13, 16].

We define a class of non-symplectic involutions whose action on L has a very simple matrix representation. In order to do this, we have to fix a basis of L. We write

$$L = H_1 \oplus H_2 \oplus H_3 \oplus (-E_8)_1 \oplus (-E_8)_2$$

in order to distinguish between the different summands. We choose a basis (u_1^i, u_2^i) of each H_i such that

$$u_1^i \cdot u_1^i = u_2^i \cdot u_2^i = 0\,, \quad u_1^i \cdot u_2^i = 1\,.$$

Moreover, $(v_1^i, \ldots, v_8^i)$ shall be a basis of $(-E_8)_i$ such that the corresponding Gram matrix is the negative of the Cartan matrix of E_8, which means that $v_j^i \cdot v_j^i = -2$ and for $j \neq k$ we have $v_j^i \cdot v_k^i \in \{0, 1\}$. We call

$$(w_1, \ldots, w_{22}) = (u_1^1, u_2^1, u_1^2, u_2^2, u_1^3, u_2^3, v_1^1, \ldots, v_8^1, v_1^2, \ldots, v_8^2)$$

the *standard basis of L*.

Definition 4.3. Let S be a K3 surface and let $\rho : S \to S$ be a non-symplectic involution. We call ρ a *simple non-symplectic involution* if there exists a marking $\phi : H^2(S, \mathbb{Z}) \to L$ such that for all $i \in \{1, \ldots, 22\}$ there exists a $j \in \{1, \ldots, 22\}$ with $\rho(w_i) = \pm w_j$, where ρ is an abbreviation for $\phi \circ \rho^* \circ \phi^{-1}$.

Our next step is to classify the simple non-symplectic involutions in terms of the invariants (r, a, δ). We obtain the following result.

Theorem 4.4. *Let S be a K3 surface and let $\rho : S \to S$ be a non-symplectic involution. ρ is simple if and only if its invariants (r, a, δ) can be found in the table below. Moreover, the action of ρ on the K3 lattice L is conjugate to an involution $\rho_1^i \oplus \rho_2^j$, where the $\rho_1^i : 3H \to 3H$ with $i = 1, \dots, 7$ are defined in the table on page 2105 and the $\rho_2^j : 2(-E_8) \to 2(-E_8)$ with $j = 1, \dots, 4$ are defined by equation (5). The values of i and j that correspond to an involution with invariants (r, a, δ) are included in the table, too.*

(i,j)	(r,a,δ)	(i,j)	(r,a,δ)
$(1,1)$	$(18,0,0)$	$(4,3)$	$(1,1,1)$
$(1,2)$	$(10,0,0)$	$(4,4)$	$(9,9,1)$
$(1,3)$	$(2,0,0)$	$(5,1)$	$(18,2,1)$
$(1,4)$	$(10,8,0)$	$(5,2)$	$(10,2,1)$
$(2,1)$	$(19,1,1)$	$(5,3)$	$(2,2,1)$
$(2,2)$	$(11,1,1)$	$(5,4)$	$(10,10,1)$
$(2,3)$	$(3,1,1)$	$(6,1)$	$(19,3,1)$
$(2,4)$	$(11,9,1)$	$(6,2)$	$(11,3,1)$
$(3,1)$	$(20,2,1)$	$(6,3)$	$(3,3,1)$
$(3,2)$	$(12,2,1)$	$(6,4)$	$(11,11,1)$
$(3,3)$	$(4,2,1)$	$(7,1)$	$(18,2,0)$
$(3,4)$	$(12,10,1)$	$(7,2)$	$(10,2,0)$
$(4,1)$	$(17,1,1)$	$(7,3)$	$(2,2,0)$
$(4,2)$	$(9,1,1)$	$(7,4)$	$(10,10,0)$

Proof. Let ρ be a simple non-symplectic involution. Since $\rho : L \to L$ is a lattice isometry and we have $\rho(w_i) = \pm w_j$, ρ maps any of the sublattices $H_k \subseteq L$ to an H_l. There are four possibilities for the value of $\rho(u_1^k)$ and of $\rho(u_2^k)$. We check for each combination if $\rho|_{H_k} : H_k \to H_l$ is a lattice isometry and see that $\rho|_{H_k}$ has one of the following four matrix representations with respect to the bases (u_1^k, u_2^k) and (u_1^l, u_2^l):

$$M_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$M_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_4 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since ρ is non-symplectic, its fixed lattice is hyperbolic. Therefore, $\rho|_{3H} : 3H \to 3H$ has to preserve exactly one positive vector. By enumerating all possibilities for $\rho|_{3H}$ with this property and comparing the invariants of the fixed lattices, we can conclude that $\rho|_{3H}$ is up to conjugation one of the following maps $\rho_1^i : 3H \to 3H$ with $i = 1, \dots, 7$:

i	Matrix representation	Basis of the fixed lattice	r	a	δ
1	$\begin{pmatrix} M_1 & & \\ & M_2 & \\ & & M_2 \end{pmatrix}$	(u_1^1, u_2^1)	2	0	0
2	$\begin{pmatrix} M_1 & & \\ & M_4 & \\ & & M_2 \end{pmatrix}$	$(u_1^1, u_2^1, u_1^2 - u_2^2)$	3	1	1
3	$\begin{pmatrix} M_1 & & \\ & M_4 & \\ & & M_4 \end{pmatrix}$	$(u_1^1, u_2^1, u_1^2 - u_2^2, u_1^3 - u_2^3)$	4	2	1
4	$\begin{pmatrix} M_3 & & \\ & M_2 & \\ & & M_2 \end{pmatrix}$	$(u_1^1 + u_2^1)$	1	1	1
5	$\begin{pmatrix} M_3 & & \\ & M_4 & \\ & & M_2 \end{pmatrix}$	$(u_1^1 + u_2^1, u_1^2 - u_2^2)$	2	2	1
6	$\begin{pmatrix} M_3 & & \\ & M_4 & \\ & & M_4 \end{pmatrix}$	$(u_1^1 + u_2^1, u_1^2 - u_2^2, u_1^3 - u_2^3)$	3	3	1
7	$\begin{pmatrix} & M_1 & \\ M_1 & & \\ & & M_2 \end{pmatrix}$	$(u_1^1 + u_1^2, u_2^1 + u_2^2)$	2	2	0

Next, we study the restriction of ρ to $2(-E_8)$. Let $i \in \{7, \dots, 22\}$. If $\rho(w_i) = w_j$ with $i \neq j$, we have $\rho(w_j) = w_i$ since ρ is an involution. If $\rho(w_i) =$

$-w_j$ with $i \neq j$, we have $\rho(w_j) = -w_i$ for the same reason. Therefore, there exists a permutation σ of $\{7, \ldots, 22\}$ such that the basis $(w'_1, \ldots, w'_{16}) := (w_{\sigma(7)}, \ldots, w_{\sigma(22)})$ satisfies:

1) $\rho(w'_i) = w'_i$ for $i \in \{1, \ldots, k_1\}$,
2) $\rho(w'_i) = -w'_i$ for $i \in \{k_1 + 1, \ldots, k_2\}$,
3) $\rho(w'_{2i-1}) = w'_{2i}$ and $\rho(w'_{2i}) = w'_{2i-1}$ for $i \in \{\frac{k_2}{2} + 1, \ldots, k_3\}$ and
4) $\rho(w'_{2i-1}) = -w'_{2i}$ and $\rho(w'_{2i}) = -w'_{2i-1}$ for $i \in \{k_3 + 1, \ldots, 8\}$.

for suitable $k_1, k_2, k_3 \in \mathbb{N}_0$. Let $i \in \{k_1 + 1, \ldots, k_2\}$, which means that $\rho(w'_i) = -w'_i$. The number i corresponds to a node of one of the two Dynkin diagrams of type E_8. Let j be a node that is connected to i by an edge. The restriction of the bilinear form to $span(w'_i, w'_j)$ is given by

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

If $\rho(w'_j) = w'_j$, ρ does not preserve the bilinear form. Therefore, we have $\rho(w'_j) = \pm w'_k$ with $k \neq i, j$ or $\rho(w'_j) = -w'_j$. We assume that $\rho(w'_j) = \pm w'_k$. Since $-w'_i \cdot \pm w'_k = \rho(w'_i) \cdot \rho(w'_j) = w'_i \cdot w'_j = 1$ and all off-diagonal coefficients of the Cartan matrix are positive, we have $\rho(w'_j) = -w'_k$ and i and k have to be connected by an edge. Analogously, we can conclude that any node that is connected to j is mapped to a node that is connected to k. By repeating this argument, it follows that ρ acts as a non-trivial graph automorphism on the diagram E_8 to which i belongs. Since E_8 has no symmetries, this is impossible and we have $\rho(w'_j) = -w'_j$. Again, we can repeat this argument and conclude that $\{k_1 + 1, \ldots, k_2\}$ consists of zero, one or both connected components of $2E_8$.

Next, let $i \in \{\frac{k_2}{2} + 1, \ldots, k_3\}$, which means that w'_{2i-1} is mapped to another basis element w'_{2i}. By the same argument as above, we see that all nodes that are connected to the node $2i - 1$ are mapped to nodes that are connected to $2i$. The restriction of ρ to $span(w'_{k_2+1}, \ldots, w'_{2k_3})$ thus maps connected components of $2E_8$ to other connected components. It follows that either $\{\frac{k_2}{2} + 1, \ldots, k_3\}$ is empty or ρ interchanges both copies of E_8. Finally, let $i \in \{k_3 + 1, \ldots, 8\}$. In this case, we have $\rho(w'_{2i-1}) = -w'_{2i}$ and it follows that if $\{k_3 + 1, \ldots, 8\}$ is not empty, the first E_8 is mapped to the second E_8 such that v^1_k is mapped to $-v^2_k$. All in all, the restricted map $\rho|_{2(-E_8)} : 2(-E_8) \to 2(-E_8)$ is up to conjugation one of the following involutions ρ_2^j with $j = 1, \dots, 4$.

$$\begin{aligned} &\rho_2^1(x_1, x_2) := (x_1, x_2)\,, && \rho_2^2(x_1, x_2) := (-x_1, x_2)\,, \\ &\rho_2^3(x_1, x_2) := (-x_1, -x_2)\,, && \rho_2^4(x_1, x_2) := (x_2, x_1)\,, \end{aligned} \tag{5}$$

where $x_1 \in (-E_8)_1$ and $x_2 \in (-E_8)_2$. Moreover, any conjugate $\psi : 2(-E_8) \to 2(-E_8)$ of the ρ_2^j that is still simple is given either by

$$\psi(x_1, x_2) = (x_1, -x_2) \quad \text{or} \quad \psi(x_1, x_2) = (-x_2, -x_1) \tag{6}$$

The fixed lattices and invariants of the ρ_2^j can be found in the following table:

j	Fixed lattice	r	a	δ
1	$2(-E_8)$	16	0	0
2	$-E_8$	8	0	0
3	$\{0\}$	0	0	0
4	$-E_8(2)$	8	8	0

Any $\rho_1^i \oplus \rho_2^j$ with $1 \leq i \leq 7$ and $1 \leq j \leq 4$ is an involution of L. Since the complement of the fixed lattice contains a positive plane, we can conclude with help of the Torelli theorem or with Lemma 2.6 that these lattice involutions are pull-backs of non-symplectic involutions. Finally, we determine the invariants of the involutions that we have found. Let $K = K_1 \oplus K_2$ be a direct sum of even, non-degenerate, 2-elementary lattices. We denote the invariants of K by (r, a, δ) and those of the K_i by (r_i, a_i, δ_i). It is easy to see that $r = r_1 + r_2$, $a = a_1 + a_2$ and $\delta = \max\{\delta_1, \delta_2\}$. By computing (r, a, δ) for all pairs $(i, j) \in \{1, \dots, 7\} \times \{1, \dots, 4\}$, we find the values from the table in the theorem. □

5. K3 surfaces with singularities and a non-symplectic involution

In this section we study which kinds of ADE-singularities a K3 surface with a non-symplectic involution may have. We focus on the case where the involution is simple. Our first theorem guarantees that there exists a K3 surface with an arbitrary simple non-symplectic involution and a particular kind of singular locus.

Theorem 5.1. *Let $(r, a, \delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\}$ be a triple such that there exists a K3 surface with a simple non-symplectic involution with invariants*

(r, a, δ). Then there exists a K3 surface which has 3 singular points with A_1-singularities and 2 singular points with E_8-singularities and carries a hyper-Kähler metric that is invariant with respect to a non-symplectic involution with the same values of (r, a, δ).

Proof. First, we fix some notation. Let (S, ϕ) be a marked K3 surface and let ρ be a simple non-symplectic involution of S with invariants (r, a, δ). The action of ρ on L is described by one of the explicit maps $\rho_1^i \oplus \rho_2^j$ from Section 4. Its fixed lattice will be denoted by L^ρ. We assume that S carries a hyper-Kähler structure such that ρ is biholomorphic with respect to the complex structure I and the hyper-Kähler metric is invariant under ρ. Finally, let $x := \phi([\omega_J])$, $y := \phi([\omega_K])$ and $z := \phi([\omega_I])$. The fact that ρ is non-symplectic implies that

$$\rho(x) = -x\,, \quad \rho(y) = -y\,, \quad \rho(z) = z\,. \tag{7}$$

Moreover, we have as usual

$$x^2 = y^2 = z^2 > 0 \quad \text{and} \quad x \cdot y = y \cdot z = z \cdot x = 0\,.$$

It follows from the description of the hyper-Kähler moduli space that any triple (x, y, z) with the above properties yields a hyper-Kähler structure such that the metric invariant under ρ and ρ is a non-symplectic involution that is holomorphic with respect to I. We recall that the set

$$D = \{d \in L | d^2 = -2,\, x \cdot d = y \cdot d = z \cdot d = 0\}$$

is a root system that determines the number and type of the singular points. The idea behind our theorem is to choose x, y and z in such a way that D is as large as possible. Depending on the index i of $\rho = \rho_1^i \oplus \rho_2^j$ we choose $z \in L_{\mathbb{R}}$ as follows:

$$z := \begin{cases} u_1^1 + u_2^1 & \text{if } \; 1 \leq i \leq 6, \\ u_1^1 + u_2^1 + u_1^2 + u_2^2 & \text{if } \; i = 7. \end{cases}$$

If $i = 7$, we have $z^2 = 4$ and we have $z^2 = 2$ otherwise. We choose x and y as:

$$x := \begin{cases} u_1^2 + u_2^2 & \text{if } 1 \leq i \leq 6, \\ u_1^1 + u_2^1 - u_1^2 - u_2^2 & \text{if } i = 7. \end{cases}$$

$$y := \begin{cases} u_1^3 + u_2^3 & \text{if } 1 \leq i \leq 6, \\ \sqrt{2}(u_1^3 + u_2^3) & \text{if } i = 7. \end{cases}$$

x, y and z satisfy $x^2 = y^2 = z^2$ and the three vectors are pairwise orthogonal. By a short calculation, we see that for any value of i we have $z \in L^\rho$ and x as well as y is orthogonal to L^ρ. Therefore, there exists a non-symplectic involution ρ that satisfies equation (7). The orthogonal complement D of $span(x, y, z)$ is for all values of i given by

$$span_{\mathbb{Z}}(u_1^1 - u_2^1, u_1^2 - u_2^2, u_1^3 - u_2^3) \oplus (-E_8)_1 \oplus (-E_8)_2 \,. \tag{8}$$

The squared length of $u_1^k - u_2^k$ is -2 for $k \in \{1, 2, 3\}$. According to Theorem 3.1, there exists a K3 surface whose hyper-Kähler structure is determined by x, y and z and which has 3 singular points of type A_1 and 2 singular points of type E_8. □

Next, we perturb the hyper-Kähler structure from the above theorem such that we obtain K3 surfaces with a simple non-symplectic involution whose singularities are of a different kind. More precisely, we prove the following theorem.

Theorem 5.2. *Let $(r, a, \delta) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\}$ be a triple such that there exists a K3 surface with a simple non-symplectic involution with invariants (r, a, δ). Moreover, let S be the K3 surface from Theorem 5.1 that has 3 points with A_1-singularities and 2 points with E_8-singularities and let ρ be the non-symplectic involution from the same theorem. Moreover, let $G_1, \ldots, G_{k_1}$ be the connected components of $3A_1 \cup 2E_8$ that are mapped to itself by ρ and let $G_1', \ldots, G_{k_2}'$ be a set of connected components that are not invariant under ρ such that*

$$G_1 \cup \ldots \cup G_{k_1} \cup G_1' \cup \ldots \cup G_{k_2}' \cup \rho(G_1') \cup \ldots \cup \rho(G_{k_2}') = 3A_1 \cup 2E_8$$

Finally, let $\widetilde{G}_1, \ldots, \widetilde{G}_{l_1}$ be connected Dynkin diagrams that can be obtained by deleting some nodes of $G_1 \cup \ldots \cup G_{k_1}$ and let $\widetilde{G}_1', \ldots, \widetilde{G}_{l_2}'$ be connected Dynkin diagrams that can be obtained by deleting some nodes of $G_1' \cup \ldots \cup G_{k_2}'$. Then there exists a K3 surface with a hyper-Kähler metric *that admits an isometric non-symplectic involution with invariants* (r, a, δ) *that has* l_1 *singular points of type* $\widetilde{G}_1, \ldots, \widetilde{G}_{l_1}$ *and* $2l_2$ *singular points of type* $\widetilde{G}'_1, \ldots, \widetilde{G}'_{l_2}$.

Proof. Let G be a Dynkin diagram that can be obtained by deleting some nodes from the union of three Dynkin diagrams of type A_1 and two of type E_8. We investigate under which conditions there exists a K3 surface with a simple non-symplectic involution whose singularities are described by G. We denote the lattice (8) by N and fix a basis

$$(\widetilde{w}_1, \ldots, \widetilde{w}_{19}) := (u_1^1 - u_2^1, u_1^2 - u_2^2, u_1^3 - u_2^3, v_1^1, \ldots, v_8^1, v_1^2, \ldots, v_8^2) \tag{9}$$

of N. Let S be a K3 surface with a simple non-symplectic involution ρ whose invariants are (r, a, δ). We choose a marking such that $\rho(w_i) = \pm w_j$. It is easy to see that $\rho(N) = N$ and that for any $i \in \{1, \ldots, 19\}$ there exists a j such that $\rho(\widetilde{w}_i) = \pm\widetilde{w}_j$. For the same reasons as in Section 4, there exists a permutation σ of $\{1, \ldots, 19\}$ such that $(\widetilde{w}'_1, \ldots, \widetilde{w}'_{19}) := (\widetilde{w}_{\sigma(1)}, \ldots, \widetilde{w}_{\sigma(19)})$ satisfies:

1) $\rho(\widetilde{w}'_i) = \widetilde{w}'_i$ for $i \in \{1, \ldots, k_1\}$,
2) $\rho(\widetilde{w}'_i) = -\widetilde{w}'_i$ for $i \in \{k_1 + 1, \ldots, k_2\}$,
3) $\rho(\widetilde{w}'_{2i}) = \widetilde{w}'_{2i+1}$ and $\rho(\widetilde{w}'_{2i+1}) = \widetilde{w}'_{2i}$ for $i \in \{\frac{k_2+1}{2}, \ldots, k_3\}$ and
4) $\rho(\widetilde{w}'_{2i}) = -\widetilde{w}'_{2i+1}$ and $\rho(\widetilde{w}'_{2i+1}) = -\widetilde{w}'_{2i}$ for $i \in \{k_3 + 1, \ldots, 9\}$.

for suitable $k_1, k_2, k_3 \in \mathbb{N}_0$. We choose four arbitrary subsets $M_1 \subseteq \{1, \ldots, k_1\}$, $M_2 \subseteq \{k_1 + 1, \ldots, k_2\}$, $M_3 \subseteq \{\frac{k_2+1}{2}, \ldots, k_3\}$ and $M_4 \subseteq \{k_3 + 1, \ldots, 9\}$. Moreover, we choose for any $j \in M_i$ an $\alpha_{ij} \in \mathbb{R}$ such that the family

$$(1, \alpha_{1\min M_1}, \ldots, \alpha_{1\max M_1}, \ldots, \alpha_{4\min M_4}, \ldots, \alpha_{4\max M_4})$$

is $\mathbb{Q}$-linearly independent. We replace $x, y, z \in L_{\mathbb{R}}$ that we have defined in the proof of Theorem 5.1 by

$$\begin{aligned} x' &= x + \sum_{j \in M_2} \alpha_{2j}\widetilde{w}'_j + \sum_{j \in M_3} \alpha_{3j}(\widetilde{w}'_{2j} - \widetilde{w}'_{2j+1}) \\ &\quad + \sum_{j \in M_4} \alpha_{4j}(\widetilde{w}'_{2j} + \widetilde{w}'_{2j+1}), \\ y' &= \left(\frac{x'^2}{y^2}\right)^{\frac{1}{2}} y, \qquad z' = z + \sum_{j \in M_1} \alpha_{1j}\widetilde{w}'_j \end{aligned} \tag{10}$$

x' and y' are still in the (-1)-eigenspace of ρ and z' is still ρ-invariant. If the α_{ij} are sufficiently small, x', y' and z' are still positive. We have

$$x'^2 = x^2 - 2\sum_{j\in M_2}\alpha_{2j}^2 - 4\sum_{j\in M_3}\alpha_{3j}^2 - 4\sum_{j\in M_4}\alpha_{4j}^2 = y'^2, \quad z'^2 = z^2 - 2\sum_{j\in M_1}\alpha_{ij}^2,$$

since x, y and z are orthogonal to N. It is possible to choose the α_{ij} such that

$$2\sum_{j\in M_2}\alpha_{2j}^2 + 4\sum_{j\in M_3}\alpha_{3j}^2 + 4\sum_{j\in M_4}\alpha_{4j}^2 = 2\sum_{j\in M_1}\alpha_{ij}^2$$

and thus we can assume that $x'^2 = y'^2 = z'^2 > 0$. Moreover, we have $x'\cdot y' = x'\cdot z' = y'\cdot z' = 0$. If $M_1 = \emptyset$ or $M_2\cup M_3\cup M_4 = \emptyset$, we can define $z' = \lambda z$ or $x' = \mu x$ and $y' = \mu y$ for appropriate $\lambda,\mu\in\mathbb{R}$ such that $x'^2 = y'^2 = z'^2$. Therefore, we obtain a triple (x', y', z') with the same properties as above in that case.

All in all, the triple (x', y', z') defines a new hyper-Kähler structure on S. Since x' and y' remain in the (-1)-eigenspace of ρ, S admits a non-symplectic involution with the same fixed lattice as before. Since z' is ρ-invariant, ρ is the pull-back of an isometry with respect to the new hyper-Kähler metric. The set

$$\begin{aligned} D' &:= \{d\in L \mid d^2 = -2, x'\cdot d = y'\cdot d = z'\cdot d = 0\} \\ &= \left\{\widetilde{w}'_i \,\middle|\, i\notin M_1\cup M_2 \wedge \frac{i}{2}\notin M_3\cup M_4 \wedge \frac{i-1}{2}\notin M_3\cup M_4\right\} \end{aligned}$$

is a root system that describes the number and type of the singular points of the new hyper-Kähler metric. The set of all i with $i\in\{1,\dots,k_2\}$ describes the Dynkin diagrams $\widetilde{G}_1,\dots,\widetilde{G}_{l_1}$ from the statement of the theorem and the set of all i with $2i\in\{k_2+1,\dots,9\}$ describes the Dynkin diagrams $\widetilde{G}'_1,\dots,\widetilde{G}'_{l_2}$. □

Remark 5.3. 1) We interpret D' geometrically. Any $\widetilde{w}'_i\in D'$ corresponds to a sphere S^2 with vanishing area. The isometry $\rho: S\to S$ maps such an S^2 to another S^2 with vanishing area. If $i\in\{1,\dots,k_2\}$, we have $\rho(\widetilde{w}'_i) = \pm\widetilde{w}'_i$. This means that the sphere is mapped to itself and the sign determines if ρ acts orientation-preserving on the sphere. Analogously, the $\widetilde{w}'_{2i}$ and $\widetilde{w}'_{2i+1}$ with $2i\in\{k_2+1,\dots,9\}$ correspond to sets of spheres with area 0 that are mapped to each other. Since the hyper-Kähler metric shall be ρ-invariant, we have to blow up the singularities that are described by the $\widetilde{w}'_{2i+1}$, too, if we blow up the singularities that are described by the $\widetilde{w}'_{2i}$.

2) The singular loci from the above theorem are all that can be obtained by perturbing the hyper-Kähler structure as in equation (10). Nevertheless, it is probable that further singular hyper-Kähler metrics that are invariant under a simple non-symplectic involution exist. In that case, the singular points would not correspond to the $\widetilde{w}_i$ but to other $d \in L$ with $d^2 = -2$.

An example of this phenomenon is the quotient $T^4/\mathbb{Z}_2$ of the 2-dimensional complex torus by ± 1. This quotient, which is a singular Kummer surface, has 16 A_1-singularities. The map ρ that is induced by $(z_1, z_2) \mapsto (-z_1, z_2)$ is a non-symplectic involution. If we blow up the A_1-singularities, ρ acts orientation-reversing on the exceptional divisors. Moreover, it acts as -1 on a 4-dimensional subspace of the 2-forms with constant coefficients on T^4 and therefore we have $r = 2$. There exist 3 non-symplectic involutions with $r = 2$ and all of them can be found in the table of Theorem 4.4. Therefore, ρ is a simple non-symplectic involution, but the case of 16 A_1-singularities is not described by our theorem. A final analysis of these further singular loci is beyond the scope of this paper.

Example 5.4. Let $(r, a, \delta) = (10, 10, 0)$, which is the case where the fixed locus is empty. It is possible to choose the marking such that ρ acts as $\rho_1^7 \oplus \rho_2^4$ on L. More explicitly, we have

$$\rho(u_j^i) = u_j^{3-i}\,, \quad \rho(u_j^3) = -u_j^3\,, \quad \rho(v_k^i) = v_k^{3-i}$$

for all $i, j \in \{1, 2\}$ and $k \in \{1, \ldots, 8\}$. ρ interchanges the two Dynkin diagrams of type E_8 and two of the Dynkin diagrams of type A_1. The third Dynkin diagram of type A_1 is preserved by ρ since $\rho(\widetilde{w}_3) = \rho(u_1^3 - u_2^3) = -\widetilde{w}_3$ and $-\widetilde{w}_3$ is another root of the lattice A_1. We delete the node from E_8 that is connected to three other nodes. The remaining diagram is of type $A_1 \cup A_2 \cup A_4$. Theorem 5.2 guarantees that there exists a singular K3 surface with a non-symplectic involution ρ with invariants $(10, 10, 0)$ that has 5 singular points of type A_1, 2 of type A_2 and 2 of type A_4. Both points of type A_2 and A_4 are mapped by ρ to each other. Moreover, there exist 2 points with A_1-singularities that are mapped to 2 other points with A_1-singularities and one point $p \in S$, that corresponds to the basis element $\widetilde{w}_3$, with an A_1-singularity that is fixed by ρ.

We remove the singularity at p such that $\rho : L \to L$ is still induced by a non-symplectic involution. This is only possible if we add a term $\lambda \widetilde{w}_3$ to x or y. Afterwards, $\widetilde{w}_3$ is not contained in the Picard lattice anymore and thus it does not correspond to a complex curve on the K3 surface. Technically speaking, the family of K3 surfaces with $x_t := x + t\widetilde{w}_3$ defines a one-parameter family of hyper-Kähler metrics that converges to the singular one, but our construction is not a resolution in the sense of algebraic geometry. Since ρ acts orientation-reversing on the 2-sphere that represents $\widetilde{w}_3$, our example does not contradict the fact that an involution with invariants $(10, 10, 0)$ of a smooth K3 surface does not have any fixed points.

6. K3 surfaces with two involutions

In this final section we study K3 surfaces with two commuting involutions that are non-symplectic with respect to two anti-commuting complex structures. As before, let $x, y, z \in L_{\mathbb{R}}$ be the images of the 3 Kähler classes under the marking. We assume that the first involution is holomorphic with respect to I and the second with respect to J. Therefore, we denote them by $\rho_I, \rho_J : S \to S$. ρ_I and ρ_J act on the Kähler classes as

$$\begin{array}{ccc} \rho_I(x) = -x & \rho_I(y) = -y & \rho_I(z) = z \\ \rho_J(x) = x & \rho_J(y) = -y & \rho_J(z) = -z \end{array} \tag{11}$$

The composition $\rho_K := \rho_I \circ \rho_J$ is a third involution that satisfies

$$\rho_K(x) = -x \quad \rho_K(y) = y \quad \rho_K(z) = -z \tag{12}$$

The following theorem describes a straightforward method to construct pairs (ρ_I, ρ_J) with the above properties.

Theorem 6.1. *Let $(r_I, a_I, \delta_I), (r_J, a_J, \delta_J) \in \mathbb{N} \times \mathbb{N}_0 \times \{0, 1\}$ be triples such that there exist non-symplectic involutions with invariants (r_*, a_*, δ_*), where $* \in \{I, J\}$. Moreover, we assume that $r_I + r_J \leq 11$ or $r_I + r_J + a_I + a_J < 22$. In this situation, there exists a smooth K3 surface S with a hyper-Kähler structure that admits two commuting involutions ρ_I and ρ_J that are non-symplectic with respect to complex structures I and J with $IJ = -JI$ and have invariants (r_I, a_I, δ_I) and (r_J, a_J, δ_J).*

Proof. Let L_* with $* \in \{I, J\}$ be the even, hyperbolic, 2-elementary lattice with invariants (r_*, a_*, δ_*). Our idea is to embed $L_I \oplus L_J$ primitively into L. Theorem 2.2 guarantees that this is possible if

$$r_I + r_J \leq 11 \quad \text{or} \quad r_I + r_J + a_I + a_J < 22$$

Since the embedding is primitive, there exists a basis $(u_1, \ldots, u_{22})$ of L such that $(u_1, \ldots, u_{r_I})$ is a basis of L_I and $(u_{r_I+1}, \ldots, u_{r_I+r_J})$ is a basis of L_J. We warn the reader that the span of $(u_{r_I+r_J+1}, \ldots, u_{22})$ is not necessarily the orthogonal complement N of $L_I \oplus L_J$. There exist lattice isometries $\rho_* : L \to L$ that are induced by non-symplectic involutions with respect to suitable complex structures that act as the identity on L_* and as -1 on $L_*^\perp$. Since we have

$$\begin{array}{ccc} \rho_I|_{L_I} = \mathrm{Id} & \rho_I|_{L_J} = -\mathrm{Id} & \rho_I|_N = -\mathrm{Id} \\ \rho_I|_{L_I} = -\mathrm{Id} & \rho_I|_{L_J} = \mathrm{Id} & \rho_I|_N = -\mathrm{Id} \end{array}$$

ρ_I and ρ_J commute. Since L_I and L_J are hyperbolic lattices and L has signature $(3, 19)$, the lattice N is hyperbolic, too. Therefore, it is possible to choose $z \in L_I \otimes \mathbb{R}$, $x \in L_J \otimes \mathbb{R}$ and $y \in N \otimes \mathbb{R}$ such that $x^2 = y^2 = z^2 > 0$. Since the three lattices are pairwise orthogonal, we have $x \cdot y = x \cdot z = y \cdot z = 0$ automatically. Moreover, x, y and z satisfy the relations (11). All in all, we have constructed a K3 surface with a hyper-Kähler structure and two commuting involutions that are non-symplectic with respect to different complex structures. Since the sets of all positive elements in $L_I \otimes \mathbb{R}$, $L_J \otimes \mathbb{R}$ or $N \otimes \mathbb{R}$ are open, we can choose

$$x = \sum_{i=r_I+1}^{r_I+r_J} \alpha_i u_i \quad y = \sum_{i=1}^{22} \beta_i u_i \quad z = \sum_{i=1}^{r_I} \gamma_i u_i$$

such that the coefficients in the above three sums are $\mathbb{Q}$-linearly independent. Since any $d \in L$ has integer coefficients with respect to the basis $(u_1, \ldots, u_{22})$, this condition guarantees that there exists no $d \in L$ with $d^2 = -2$ and $x \cdot d = y \cdot d = z \cdot d = 0$. Therefore, it is possible to choose S as a smooth K3 surface. $\square$

Remark 6.2. An important step in Kovalev's and Lee's construction of G_2-manifolds [13] is to find two K3 surfaces S_1 and S_2 with non-symplectic involutions ρ_1 and ρ_2 and a so called matching. A matching is defined as an isometry $f : S_1 \to S_2$ such that

$$f^*\omega_{I_2} = \omega_{J_1}\,, \qquad f^*\omega_{J_2} = \omega_{I_1}\,, \qquad f^*\omega_{K_2} = -\omega_{K_1}\,,$$

where ω_{I_k}, ω_{J_k} and ω_{K_k} are the three Kähler forms on S_k. Let S be a K3 surface with two involutions that satisfy (1). If we choose the triple of complex structures on S first as (I, J, K) and then as $(J, I, -K)$, the identity map becomes a matching. Moreover, ρ_I and ρ_J are holomorphic with respect to the first complex structure from the triples. Therefore, the above theorem shows that a matching exists if the invariants of ρ_1 and ρ_2 satisfy $r_1 + r_2 \leq 11$ or $r_1 + r_2 + a_1 + a_2 < 22$. This fact is also shown by other methods in [13].

An important aim of this paper is to construct K3 surfaces with ADE-singularities that admit a pair (ρ_I, ρ_J) of commuting involutions that satisfy (1). If ρ_I and ρ_J are simple, it is possible to choose the hyper-Kähler structure such that we can describe the set D from page 2101, that determines the singular locus, explicitly. Under the simplifying assumption that the markings $\phi_I, \phi_J : H^2(S, \mathbb{Z}) \to L$ coincide we are able to prove the following classification theorem.

Theorem 6.3. *Let $(i_I, j_I), (i_J, j_J) \in \{1, \dots, 7\} \times \{1, \dots, 4\}$ be such that $(j_I, j_J) \notin \{(2,4), (4,2)\}$ and $(i_I, i_J) \notin \{(2,7), (5,7), (7,2), (7,5), (7,7)\}$. Moreover, let (r_*, a_*, δ_*) with $* \in \{I, J\}$ be the triples of invariants that characterise the non-symplectic involutions that act as $\rho_1^{i_*} \oplus \rho_2^{j_*}$ on L. In this situation, there exists a possibly singular K3 surface S that admits two commuting involutions ρ_I and ρ_J that are non-symplectic with respect to complex structures I and J with $IJ = -JI$ and have invariants (r_I, a_I, δ_I) and (r_J, a_J, δ_J).*

Proof. We construct maps $\rho_* : L \to L$ with $* \in \{I, J\}$ that can be written up to conjugation as $\rho_1^{i_*} \oplus \rho_2^{j_*}$, where $\rho_1^{i_*} : 3H \to 3H$ and $\rho_2^{j_*} : 2(-E_8) \to 2(-E_8)$ are two of the maps that we have defined in Section 4. By a direct calculation we see that $\rho_2^{j_I}$ and $\rho_2^{j_J}$ commute if and only if $(j_I, j_J) \notin \{(2,4), (4,2)\}$. By adjusting the marking, we can assume that the restriction of ρ_I to $2(-E_8)$ actually is one of the maps $\rho_2^{j_I}$ with $j_I \in \{1, \dots, 4\}$. Nevertheless, the restriction of ρ_J may be a conjugate of a map $\rho_2^{j_J}$ such that we still have $\rho_J(w_i) = \pm w_j$ for $i \in \{7, \dots, 22\}$. As we have remarked in Section 4, the only additional possibilities for $\rho_J|_{2(-E_8)}$ are

$$\rho_J|_{2(-E_8)}(x_1, x_2) = (x_1, -x_2)$$

if $j_J = 2$ or

$$\rho_J|_{2(-E_8)}(x_1, x_2) = (-x_2, -x_1)$$

if $j_J = 4$. If we take account of these additional possibilities, it is still not possible that $\rho_I|_{2(-E_8)}$ and $\rho_J|_{2(-E_8)}$ commute if $(j_I, j_J) \in \{(2,4), (4,2)\}$. Nevertheless, this idea will be helpful in the next case. Let $i_I, i_J \in \{1, \dots, 7\}$.

First, we assume that $i_I, i_J \neq 7$. We see that $\rho_1^{i_I}$ and $\rho_1^{i_J}$ always commute, since the smaller matrix blocks

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

commute pairwisely. In Section 5 we have defined a hyper-Kähler structure by

$$x := u_1^2 + u_2^2\,, \quad y := u_1^3 + u_2^3\,, \quad z := u_1^1 + u_2^1\,.$$

The involution ρ_I preserves z and acts as -1 on x and y. Unfortunately, the same is true for ρ_J, although ρ_J should preserve x and act as -1 on y and z. In order to solve this problem, we conjugate $\rho_1^{i_J}$ by the map $\tau : 3H \to 3H$ that is defined by

$$\tau(u_k^l) := u_k^{3-l} \quad \forall k \in \{1,2\}, l \in \{1,2\} \quad \text{and} \quad \tau(u_k^3) := u_k^3 \quad \forall k \in \{1,2\}\,.$$

In other words, we permute the first and the second block of the matrices that define $\rho_1^{i_J}$. We obtain a map that is still an isometry of $3H$ and maps any w_i to a $\pm w_j$. After this conjugation, $\rho_1^{i_I}$ and $\rho_1^{i_J}$ still commute and the maps $\rho_I, \rho_J : L \to L$ satisfy the relations (11).

If $i_I = 7$ and $i_J \in \{1, \ldots, 6\}$, $\rho_1^{i_I}$ has a 4×4-block in the upper left corner that interchanges H_1 and H_2. We conjugate $\rho_1^{i_J}$ by a map that is analogous to τ but interchanges H_1 and H_3. After that, $\rho_1^{i_I}$ and $\rho_1^{i_J}$ commute if and only if the last two 2×2-blocks of $\rho_1^{i_J}$ are the same. This is the case for all values of i_J except 2 and 5. We consider the second hyper-Kähler structure from Section 5 that is defined by

$$x := u_1^1 + u_2^1 - u_1^2 - u_2^2\,, \quad y := \sqrt{2}(u_1^3 + u_2^3)\,, \quad z := u_1^1 + u_2^1 + u_1^2 + u_2^2\,.$$

After a short calculation, we see that ρ_I and ρ_J satisfy the relations

$$\begin{array}{lll} \rho_I(x) = -x & \rho_I(y) = -y & \rho_I(z) = z \\ \rho_J(x) = -x & \rho_J(y) = y & \rho_J(z) = -z \end{array} \tag{13}$$

Although this is not the same as (11), those relation are satisfied after replacing ρ_J by ρ_K. All in all, we have proven the existence of (ρ_I, ρ_J) in all cases from the theorem. $\square$

Remark 6.4. The above theorem yields 320 pairs (ρ_I, ρ_J) with the desired properties. This number can be calculated as follows. Up to a permutation of ρ_I and ρ_J, there are $\frac{28 \cdot 29}{2} = 406$ possibilities to choose two simple non-symplectic involutions. We assume that $j_I \leq j_J$. After removing the 49 pairs $((i_I, j_I), (i_J, j_J))$ with $(j_I, j_J) = (2, 4)$, 357 pairs remain. There are 5 possibilities for (j_I, j_J) with $j_I < j_J$ left. For each of them we have to substract the 5 excluded values of (i_I, i_J). If $j_I = j_J$, we have to substract 3 since (i_I, i_J) and (i_J, i_I) yield up to permutation the same (ρ_I, ρ_J). All in all, there are $357 - 5 \cdot 5 - 4 \cdot 3 = 320$ pairs left. We shall investigate in a forthcoming paper whether some of these 320 pairs yield twisted connected sums that were not previously given in [13]. Moreover, we analyse if they are diffeomorphic to the examples in other articles on twisted connected sums such as [7, 12].

We remark that our 320 examples are not exhaustive. If we choose for example $\rho_2^{j_*}$ as one of the maps (6) or modify $\rho_1^{i_*}$ by permuting the three summands H_1, H_2 and H_3, we could easily obtain further examples with the same invariants (r_*, a_*, δ_*) but a non-equivalent action on L. Since we have restricted ourselves to the case where the ρ_* are simple and both markings $\phi_* : H^2(S, \mathbb{Z}) \to L$ are the same, it is even possible that examples with further triples of invariants exist. The complete classification of pairs (ρ_I, ρ_J) of commuting involutions that satisfy (1) is beyond the scope of this paper.

Since the hyper-Kähler structure on S that we have introduced in the proof of Theorem 6.3 is the same as in Section 5, we immediately obtain the following corollary.

Corollary 6.5. *In the situation of the above theorem, S can be chosen as a K3 surface that has 3 singular points with A_1-singularities and 2 singular points with E_8-singularities.*

The next step is to investigate if there exist K3 surfaces with further kinds of singularities that admit involutions ρ_I and ρ_J with the same properties as in Theorem 6.3. Our idea is to perturb the hyper-Kähler structure that is determined by (x, y, z) in such a way that it is still invariant under ρ_I and ρ_J. We obtain the following theorem.

Theorem 6.6. *Let S be one of the K3 surfaces from Theorem 6.3 that*

1) *admits a pair (ρ_I, ρ_J) of commuting simple involutions that are non-symplectic with respect to two complex structures I and J with $IJ = -JI$ and*

2) *has 3 points with A_1-singularities and 2 points with E_8-singularities.*

ρ_I and ρ_J generate a group that is isomorphic to $\mathbb{Z}_2^2$ and acts on the Dynkin diagram $3A_1 \cup 2E_8$. Let M be a $\mathbb{Z}_2^2$-invariant subset of the nodes of $3A_1 \cup 2E_8$ such that no node from M corresponds to a $\widetilde{w}_i \in L$ that is fixed by $\mathbb{Z}_2^2$. In this situation, there exists a K3 surface S' that

1) *admits a pair of commuting simple involutions that are non-symplectic with respect to two complex structures I' and J' with $I'J' = -I'J'$ and whose invariants (r_*, a_*, δ_*) are the same as of ρ_* and*
2) *whose singular set is described by the Dynkin diagram that we obtain by deleting the set M of nodes from $3A_1 \cup 2E_8$.*

In particular, S' can be chosen as a smooth K3 surface if there is no $\widetilde{w}_i$ that is fixed by $\mathbb{Z}_2^2$.

Proof. Let $(\widetilde{w}_i)_{i=1,\ldots,19}$ be the basis (9) of the lattice that we have introduced in (8). We recall that $\widetilde{w}_i^2 = -2$ for all i and that the $\widetilde{w}_i$ correspond to the nodes of the Dynkin diagram $3A_1 \cup 2E_8$. We denote the linear span of the orbit of $\widetilde{w}_i$ with respect to the group $\mathbb{Z}_2^2$ that is generated by ρ_I and ρ_J by W_i. The dimension of W_i is either 1, 2 or 4. For the same reasons as in Section 5, $\mathbb{Z}_2^2$ acts on $3A_1 \cup 2E_8$ and maps connected components to connected components. Since $3A_1 \cup 2E_8$ does not contain 4 components of the same type, the dimension of W_i has to be 1 or 2. We call a $\widetilde{w}_i$ of type

- $(1,1)$ if $\rho_I(\widetilde{w}_i) = \rho_J(\widetilde{w}_i) = \widetilde{w}_i$,
- $(1,-1)$ if $\rho_I(\widetilde{w}_i) = \widetilde{w}_i$ and $\rho_J(\widetilde{w}_i) \neq \widetilde{w}_i$,
- $(-1,1)$ if $\rho_I(\widetilde{w}_i) \neq \widetilde{w}_i$ and $\rho_J(\widetilde{w}_i) = \widetilde{w}_i$,
- $(-1,-1)$ if $\rho_I(\widetilde{w}_i) \neq \widetilde{w}_i$ and $\rho_J(\widetilde{w}_i) \neq \widetilde{w}_i$.

Since ρ_I and ρ_J are involutions, their eigenvalues are 1 and -1. Moreover, they commute and therefore we have a decomposition

$$L_{\mathbb{R}} = V_{1,1} \oplus V_{1,-1} \oplus V_{-1,1} \oplus V_{-1,-1}$$

where

$$V_{\epsilon_1,\epsilon_2} = \{v \in L_{\mathbb{R}} | \rho_I(v) = \epsilon_1 v, \rho_J(v) = \epsilon_2 v\} .$$

We have $x \in V_{-1,1}$, $y \in V_{-1,-1}$ and $z \in V_{1,-1}$. If $\widetilde{w}_i$ is of type $(1,-1)$, we define a $\widetilde{w}_i' \in L_{\mathbb{R}}$ by

$$\widetilde{w}_i' = \begin{cases} \widetilde{w}_i & \text{if}\,, \rho_J(\widetilde{w}_i) = -\widetilde{w}_i \\ \widetilde{w}_i - \widetilde{w}_j & \text{if}\,, \rho_J(\widetilde{w}_i) = \widetilde{w}_j \text{ with } i \neq j \\ \widetilde{w}_i + \widetilde{w}_j & \text{if}\,, \rho_J(\widetilde{w}_i) = -\widetilde{w}_j \text{ with } i \neq j \end{cases}$$

If $\widetilde{w}_i$ is of type $(-1,1)$, we define $\widetilde{w}_i'$ analogously but replace ρ_J by ρ_I. Finally, if $\widetilde{w}_i$ is of type $(-1,-1)$, we define

$$\widetilde{w}_i' = \begin{cases} \widetilde{w}_i & \text{if}\,, \rho_I(\widetilde{w}_i) = \rho_J(\widetilde{w}_i) = -\widetilde{w}_i \\ \widetilde{w}_i - \widetilde{w}_j & \text{if}\,, \rho_I(\widetilde{w}_i) = -\widetilde{w}_i \text{ and } \rho_J(\widetilde{w}_i) = \widetilde{w}_j \text{ with } i \neq j \\ \widetilde{w}_i + \widetilde{w}_j & \text{if}\,, \rho_I(\widetilde{w}_i) = -\widetilde{w}_i \text{ and } \rho_J(\widetilde{w}_i) = -\widetilde{w}_j \text{ with } i \neq j \\ \widetilde{w}_i - \widetilde{w}_j & \text{if}\,, \rho_J(\widetilde{w}_i) = -\widetilde{w}_i \text{ and } \rho_I(\widetilde{w}_i) = \widetilde{w}_j \text{ with } i \neq j \\ \widetilde{w}_i + \widetilde{w}_j & \text{if}\,, \rho_J(\widetilde{w}_i) = -\widetilde{w}_i \text{ and } \rho_I(\widetilde{w}_i) = -\widetilde{w}_j \text{ with } i \neq j \end{cases}$$

Since $\dim W_i \neq 4$, these are the only possibilities that can happen for a $\widetilde{w}_i$ of type $(-1,-1)$. By our construction $\widetilde{w}_i' \in V_{\epsilon_1,\epsilon_2}$ if $\widetilde{w}_i'$ is of type (ϵ_1,ϵ_2). We choose arbitrary subsets

$$\begin{aligned} P &\subseteq \{1 \leq i \leq 19 | \widetilde{w}_i \text{ is of type } (-1,1)\} \\ Q &\subseteq \{1 \leq i \leq 19 | \widetilde{w}_i \text{ is of type } (-1,-1)\} \\ R &\subseteq \{1 \leq i \leq 19 | \widetilde{w}_i \text{ is of type } (1,-1)\} \end{aligned}$$

such that for any pair (i,j) with $i \neq j$ from one the three sets we have $W_i \cap W_j = \{0\}$. Let (x,y,z) be the triple of Kähler classes that determines the hyper-Kähler structure with 3 A_1- and 2 E_8-singularities. We define a new hyper-Kähler structure by

$$\begin{aligned} x' &= \mu x + \sum_{i \in P} \alpha_i \widetilde{w}_i' \\ y' &= \nu y + \sum_{i \in Q} \beta_i \widetilde{w}_i' \\ z' &= \lambda z + \sum_{i \in R} \gamma_i \widetilde{w}_i' \end{aligned}$$

The coefficients in the above definition are chosen such that

1) the family that consists of 1, the α_i, the β_i and the γ_i is $\mathbb{Q}$-linearly independent,

2) $x'^2 = y'^2 = z'^2 > 0$.

The hyper-Kähler structure that is defined by x', y' and z' still satisfies the equation (11). Moreover, the set D that determines the number and type of the singular points can be obtained from $3A_1 \cup 2E_8$ by deleting all nodes that correspond to an element of the $\mathbb{Z}_2^2$-orbit of an $i \in P \cup Q \cup R$, as we have stated in the theorem. $\square$

In the proof of the above theorem, we have constructed a (partial) resolution of the singularities that is still invariant under $\mathbb{Z}_2^2$. We remark that in general there is a minimal singularity that cannot be resolved without destroying the $\mathbb{Z}_2^2$-symmetry. Its Dynkin diagram is given by all i such that $\widetilde{w}_i$ is invariant under $\mathbb{Z}_2^2$. If we add a multiple of such an $\widetilde{w}_i$ to x, y or z, we obtain a new hyper-Kähler structure that no longer satisfies (11).

Example 6.7. Let $\rho_I, \rho_J : L \to L$ be the lattice isometries such that ρ_I acts as the identity on $H_1 \oplus 2(-E_8)$ and as -1 on the other two summands that are isometric to H and ρ_J acts as the identity on $H_2 \oplus 2(-E_8)$ and as -1 on the complement. ρ_I and ρ_J commute and are both of type $\rho_1^1 \oplus \rho_2^1$. Corollary 6.5 guarantees that there exists a K3 surface with two E_8- and three A_1-singularities and two non-symplectic involutions that correspond to ρ_I and ρ_J. Theorem 6.6 allows us to resolve one or two of the A_1-singularities, but the two E_8-singularities and the third of the A_1-singularities cannot be resolved without destroying the invariance of the hyper-Kähler metric with respect to ρ_I and ρ_J.

References

[1] Acharya, B.S.: *M-Theory, G_2-manifolds and four dimensional physics*, Classical and Quantum Gravity **19** (2002), no. 22, 5619–5653.

[2] Acharya, B.S.; Gukov, S.: *M-theory and singularities of exceptional holonomy manifolds*, Phys. Rep. **392** (2004), no. 3, 121–189.

[3] Anderson, M.T.: *Moduli spaces of Einstein metrics on 4-manifolds*, Bulletin of the AMS **21** (1989), no. 2, 275–279.

[4] Anderson, M.T.: *The L^2-structure of moduli spaces of Einstein metrics on* 4*-manifolds*, Geometric and Functional Analysis **2** (1992), no. 1, 29–89.

[5] Barth, W.; Hulek, K.; Peters, C.; van de Ven, A.: Compact Complex Surfaces, Second Enlarged Edition. Springer-Verlag, Berlin, Heidelberg, New York, 2004.

[6] Besse, A.L.: Einstein Manifolds. Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[7] Corti, A.; Haskins, M.; Nordström, J.; Pacini, T.: *G_2-manifolds and associative submanifolds via semi-Fano 3-folds*, Duke Math. J. 164 (2015), no. 10, 1971–2092.

[8] Dolgachev, I.: *Integral quadratic forms: Applications to algebraic geometry (after V. Nikulin)*, Bourbaki Seminar Vol. 1982/83. Astérisque 105–106 (1983), 251–278.

[9] Joyce, D.: Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[10] Joyce, D; Karigiannis, S.: *A new construction of compact G_2-manifolds by gluing families of Eguchi-Hanson spaces*, J. Differential Geom. **117** (2021), no. 2, 255–343.

[11] Kobayashi, R.; Todorov, A.N.: *Polarized period map for generalized K3 surfaces and the moduli of Einstein metrics*, Tohoku Math. Journ. **39** (1987), 341–363.

[12] Kovalev, A.: *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.

[13] Kovalev, A.; Lee, N.-H.: *K3 surfaces with non-symplectic involution and compact irreducible G_2-manifolds*, Math. Proc. Cambridge Philos. Soc. **151** (2011), no. 2, 193–218.

[14] Nikulin, V.V.: *Finite groups of automorphisms of Kählerian K3 surfaces*, Trans. Moscow Math. Soc. **2** (1980), 71–135.

[15] Nikulin, V.V.: *Integer symmetric bilinear forms and some of their applications*, Math. USSR Izvestia **14** (1980), 103–167.

[16] Nikulin, V. V.: *Factor groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications*, J. Soviet Math. **22** (1983), 1401–1476.

[17] Reidegeld, Frank: *G_2-orbifolds with ADE-singularities*, Habilitation thesis. Fakultät für Mathematik, TU Dortmund, 2017. Online available: `http://dx.doi.org/10.17877/DE290R-18940`.

Faculty for Mathematics, TU Dortmund University
Vogelpothsweg 87, 44227 Dortmund, Germany
E-mail address: frank.reidegeld@math.tu-dortmund.de

Received October 9, 2018
Accepted March 25, 2020