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Guan-Li type mean curvature flow for free boundary hypersurfaces in a ball

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In this paper we introduce a Guan-Li type volume preserving mean curvature flow for free boundary hypersurfaces in a ball. We give a concept of star-shaped free boundary hypersurfaces in a ball and show that the Guan-Li type mean curvature flow has long time existence and converges to a free boundary spherical cap, provided the initial data is star-shaped.

1. Introduction

Let $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$ be the open unit Euclidean ball centered at the origin and $\mathbb{S}^n = \partial\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$ the unit sphere. In this paper, we shall consider a mean curvature type flow for compact hypersurfaces in $\mathbb{B}^{n+1}$ with free boundary on $\mathbb{S}^n$. Let $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a properly embedded compact hypersurface with boundary, which is given by

$$x : M \to \bar{\mathbb{B}}^{n+1},$$

where M is a compact Riemannian manifold with boundary ∂M. Here properly embedded means that

$$\mathrm{int}(\Sigma) = x(\mathrm{int}(M)) \subset \mathbb{B}^{n+1} \quad \text{and} \quad \partial\Sigma = x(\partial M) \subset \partial\mathbb{B}^{n+1}.$$

We further assume that Σ has free boundary, in the sense that Σ intersects $\partial\mathbb{B}^{n+1} = \mathbb{S}^n$ orthogonally, that is,

$$\langle \nu, \mu \circ x \rangle = 0 \quad \text{on } \partial M,$$

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where ν is a unit normal vector field of x, which will be specified later, and μ is the outward unit normal vector field of $\mathbb{S}^n$, i.e., $\mu \circ x = x$ along ∂M.

Let $e \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a fixed unit vector field. Consider a family of properly embedded compact hypersurfaces $\{\Sigma_t\}_{t\in[0,T)}$ with free boundary, given by embeddings

$$x : M \times [0,T) \to \bar{\mathbb{B}}^{n+1},$$

satisfying

$$\begin{cases} \partial_t x = (n\langle x, e\rangle - H\langle X_e, \nu\rangle)\nu & \text{in } M \times [0,T), \\ \langle \nu, \mu \circ x\rangle = 0 & \text{on } \partial M \times [0,T). \end{cases} \tag{1}$$

with an initial surface $x(\cdot, 0) = x_0$. Here ν and H are a unit normal vector field and the mean curvature of $x(\cdot, t)$ respectively, X_e is a fixed vector field in $\mathbb{R}^{n+1}$ given by

$$X_e = X_e(x) = \langle x, e\rangle x - \frac{1}{2}(|x|^2 + 1)e,$$

for a fixed unit vector e. This vector field plays an important role in our recent paper [11]. We choose ν in the following way. Let Ω_t be the component of the enclosed domain by Σ_t and $\mathbb{S}^n$ which contains e in its interior. Then ν is chosen to be the outward normal of Σ_t with respect to Ω_t. Also, throughout this paper, we make the convention that the enclosed domain Ω_t of Σ_t and $\mathbb{S}^n$ is the one e in its interior. The volume of the enclosed domain Ω_t of Σ is called the enclosed volume of Σ_t.

The flow is designed in this way so that the enclosed volume of Σ_t is preserved along the flow (1). We will discuss it later. Such kinds of flow was first considered by Guan-Li [5] in the setting of closed hypersurfaces in space forms and by Guan-Li-Wang [6] in the setting of closed hypersurfaces in warped product spaces.

The main objective of this paper is to study the existence and the convergence of the flow (1). For this aim we introduce a concept of *star-shaped hypersurfaces with free boundary in* $\bar{\mathbb{B}}^{n+1}$. To arrive at this, we should first make some comments on the vector field X_e above. X_e is a conformal Killing vector field with

$$\langle X_e(x), x\rangle = 0, \forall x \in \partial\mathbb{B}^{n+1}.$$

More precisely, denoting the Euclidean metric by δ, we have

$$\mathcal{L}_{X_e}\delta = \langle x, e\rangle \delta.$$

Let $\phi_t : \bar{\mathbb{B}}^{n+1} \to \bar{\mathbb{B}}^{n+1}$ be the one-parameter family of conformal transformations generated by X_e. Let Π_e be the hyperplane which passes through the origin and is orthogonal to e. For each point $p \in \Pi_e$, there exists a unique planar circle passing through p and $\pm e$. One can check that the integral curves of X_e are given by the intersection of all such planar circles with $\mathbb{B}^{n+1}$. We introduce star-shaped hypersurfaces with free boundary in $\bar{\mathbb{B}}^{n+1}$.

Definition 1.1. *1). A proper embedded hypersurface $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ is called* star-shaped (with respect to e) *if Σ intersects each integral curve of X_e exactly once.*

2). A proper embedded hypersurface $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ is called strictly star-shaped (with respect to e) *if*

$$\langle X_e, \nu \rangle > 0. \tag{2}$$

For our purpose we will consider strictly star-shaped hypersurfaces in $\bar{\mathbb{B}}^{n+1}$ in this paper. This condition is slightly stronger than the condition of star-shapedness, but clearly much weaker than the convexity. For the simplicity in this paper we call hypersurfaces satisfying (2) star-shaped hypersurfaces.

From now on we consider star-shaped hypersurfaces. Being such a hypersurface, it is necessary that M is of ball type. Therefore we use $M = \bar{\mathbb{S}}^n_+$, the closed hemisphere.

Our main result is the following

Theorem 1.1. *Let $\Sigma \subset \bar{\mathbb{B}}^{n+1} (n \geq 2)$ be a properly embedded compact hypersurface with free boundary, given by $x_0 : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{B}}^{n+1}$, which is star-shaped with respect to e. Then there exists a unique solution $x : \bar{\mathbb{S}}^n_+ \times [0, \infty) \to \bar{\mathbb{B}}^{n+1}$ to (1). Moreover, $x(\cdot, t)$ converges smoothly to a spherical cap or the totally geodesic n-ball, whose enclosed domain has the same volume as Σ. When $n \geq 3$, or $n = 2$ and the enclosed volume of x_0 is not that of a half ball, $x(\cdot, t)$ converges exponentially fast.*

The family of spherical caps is given by

$$C_r^{\pm}(e) = \{x \in \bar{\mathbb{B}}^{n+1} : |x \pm \sqrt{r^2+1}e| = r\}, r > 0$$

and the totally geodesic n-ball is given by

$$C_\infty(e) = \{x \in \bar{\mathbb{B}}^{n+1} : \langle x, e \rangle = 0\}.$$

It is clear that either each spherical cap $C_r^{\pm}(e)$ or the totally geodesic n-ball $C_\infty(e)$ has free boundary, that is, it intersects the support $\mathbb{S}^n$ orthogonally.

As a direct consequence, we give a flow proof of the isoperimetric problem for free boundary hypersurfaces in $\mathbb{B}^{n+1}$.

Corollary 1.1. *Among star-shaped free boundary hypersurfaces with fixed enclosed volume, the totally geodesic n-ball or the spherical caps have minimal area.*

For general hypersurfaces it is a classical result proved by Burago-Mazaya [3], Bokowsky-Sperner [2] and Almgren [1], by using the method of symmetrization.

The introduction of flow (1) is motivated by the paper of Guan-Li [5], in which they used at the first time the Minkowski formula for closed hypersurfaces to define a geometric flow for isoperimetric problems. In the same spirit, the flow (1) is based on the following two Minkowski formulas obtained in [11] for free boundary hypersurfaces

$$n\int_\Sigma \langle x, e\rangle = \int_\Sigma \langle X_e, \nu\rangle H, \tag{3}$$

$$\int_\Sigma \langle x, e\rangle H = \frac{2}{n-1}\int \langle X_e, \nu\rangle \sigma_2(\kappa). \tag{4}$$

Here $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ are principal curvatures of Σ and $\sigma_2(\kappa)$ is the 2nd order mean curvature. From these formulas, one can show that flow (1) preserves the volume of Ω_t and decreases the area of Σ_t. See Proposition 4.1. These are crucial properties of this flow.

To prove Theorem 1.1, we first transform the flow equation to a scalar flow (19) on $\mathbb{S}^n_+$ by using star-shapedness. By using the Möbius transformation between the half space $\bar{\mathbb{R}}^{n+1}_+$ and the unit ball $\bar{\mathbb{B}}^{n+1}$, a star-shaped hypersurface in $\bar{\mathbb{B}}^{n+1}$ is equivalent to a classical star-shaped hypersurface in $\bar{\mathbb{R}}^{n+1}_+$ with a conformal flat metric. We remark that a different reparametrization based on Möbius transformation between round cylinder and $\mathbb{B}^{n+1}$ was used by Lambert-Scheuer [7]. For the scalar flow (19), the C^0 estimate follows directly from the barrier argument. We then show the gradient estimate for (19).

Finally we mention some previous results on curvature flows with free boundary in $\mathbb{B}^{n+1}$. The classical mean curvature flow was considered by Stahl [9, 10], where it was shown that strictly convex initial data are driven to a round point in a finite time. The classical inverse mean curvature flow was treated by Lambert-Scheuer [7], where it was shown that strictly convex initial data are driven to a flat perpendicular n-ball in a finite time. Following a similar idea of this paper, a fully nonlinear inverse curvature type flow was considered by Scheuer and the authors [8] to show a class of new Alexandrov-Fenchel's inequalities for convex free boundary hypersurfaces in $\mathbb{B}^{n+1}$.

The rest of this paper is organized as follows. In Section 2 we introduce the Möbius transformation between $\bar{\mathbb{R}}^n_+$ and $\bar{\mathbb{B}}^n$, and reduce flow (1) to a scalar flow (19), provided that all evolving hypersurfaces are star-shaped. In Section 3, we show that C^0 and C^1 estimates of (1). As consequence, we prove in Section 4 that the global convergence of (1), Theorem 1.1 and its consequence, Corollary 1.1.

2. A scalar flow

In this section we reduce (1) to a scalar flow, provided that all evolving hypersurfaces are star-shaped.

Without loss of generality, from now on, we assume $e = E_{n+1}$, the $(n+1)$-coordinate vector. Let

$$\mathbb{R}^{n+1}_+ = \{z = (z_1, \cdots, z_{n+1}) \in \mathbb{R}^{n+1} : z_{n+1} > 0\}$$

be the half space. Define

$$f : \bar{\mathbb{R}}^{n+1}_+ \to \bar{\mathbb{B}}^{n+1}, \tag{5}$$

$$(z', z_{n+1}) \mapsto \left(\frac{2z'}{|z'|^2 + (1 + z_{n+1})^2}, \quad \frac{|z|^2 - 1}{|z'|^2 + (1 + z_{n+1})^2} \right). \tag{6}$$

Here $z' = (z_1, \cdots, z_n) \in \mathbb{R}^n$. f is bijective and

$$f(\mathbb{R}^{n+1}_+) = \mathbb{B}^{n+1}, \tag{7}$$

$$f(\partial\mathbb{R}^{n+1}_+) = \partial\mathbb{B}^{n+1}, \tag{8}$$

$$f(\{|z| = 1\}) = \{x_{n+1} = 0\}. \tag{9}$$

Moreover, f is a conformal diffeomorphism between $(\bar{\mathbb{R}}^{n+1}_+, \delta_{\bar{\mathbb{R}}^{n+1}_+})$ and $(\bar{\mathbb{B}}^{n+1}, \delta_{\bar{\mathbb{B}}})$. Here $\delta_{\bar{\mathbb{R}}^{n+1}_+}$ and $\delta_{\bar{\mathbb{B}}}$ denote the restriction of the Euclidean metric to $\bar{\mathbb{R}}^{n+1}_+$ and $\bar{\mathbb{B}}^{n+1}$ respectively. Precisely,

$$f^*\delta_{\bar{\mathbb{B}}} = e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+} = \frac{4}{(|z'|^2 + (1 + z_{n+1})^2)^2}\delta_{\bar{\mathbb{R}}^{n+1}_+}.$$

In other words, $(\bar{\mathbb{B}}^{n+1}, \delta_{\bar{\mathbb{B}}})$ and $(\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$ are isometric.

In $\bar{\mathbb{R}}^{n+1}_+$, we use the polar coordinates $(\rho, \varphi, \theta) \in [0, \infty) \times [0, \frac{\pi}{2}] \times \mathbb{S}^{n-1}$, where

$$\rho^2 = |z'|^2 + z_{n+1}^2, \quad z_{n+1} = \rho\cos\varphi$$

and $\theta \in \mathbb{S}^{n-1}$ is the spherical coordinate.

By using (ρ, φ, θ) in $\bar{\mathbb{R}}^{n+1}_+$, the mapping f can be rewritten as

$$f(\rho, \varphi, \theta) = \left(\frac{2\rho\sin\varphi\vec{\theta}}{1+\rho^2+2\rho\cos\varphi}, \frac{\rho^2-1}{1+\rho^2+2\rho\cos\varphi} \right). \tag{10}$$

Here $\vec{\theta}$ denotes the position vector of the point $\frac{z'}{|z'|} \in \mathbb{S}^{n-1}$. We also have

$$f^*\delta_{\bar{\mathbb{B}}} = e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+} = \frac{4}{(1+\rho^2+2\rho\cos\varphi)^2}(d\rho^2 + \rho^2 d\varphi^2 + \rho^2\sin^2\varphi g_{\mathbb{S}^{n-1}}),$$

where

$$w = w(\rho, \varphi, \theta) = \log 2 - \log(1+\rho^2+2\rho\cos\varphi).$$

One may also check that the conformal Killing vector field X_{n+1} on $\bar{\mathbb{B}}_+$ is transformed to

$$\tilde{X} = (f^{-1})_*(X_{n+1}) = -\rho\partial_\rho \text{ on } \bar{\mathbb{R}}^{n+1}_+. \tag{11}$$

The integral curves of $\tilde{X}$ are clearly the rays in $\mathbb{R}^{n+1}_+$ initiating from the origin.

Let $\Sigma \subset \bar{\mathbb{B}}^{n+1}$ be a properly embedded compact hypersurface with boundary, given by an embedding $x : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{B}}^{n+1}$. We associate Σ with a corresponding hypersurface $\tilde{\Sigma} \subset \bar{\mathbb{R}}^{n+1}_+$ given by the embedding

$$\tilde{x} = f^{-1} \circ x : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{R}}^{n+1}_+.$$

In view of (11), Σ is star-shaped with respect to E_{n+1} if and only if $\tilde{\Sigma}$ is star-shaped (with respect to the origin) in $\bar{\mathbb{R}}^{n+1}_+$, that is, $\tilde{\Sigma}$ intersects each of the rays in $\mathbb{R}^{n+1}_+$ initiating from the origin exactly once, or in other words, $\tilde{\Sigma}$ is a graph over $\bar{\mathbb{S}}^n_+$.

Since $(\bar{\mathbb{B}}^{n+1}, \delta_{\bar{\mathbb{B}}})$ and $(\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$ are isometric, a proper embedding $x : \bar{\mathbb{S}}^n_+ \to \bar{\mathbb{B}}^{n+1}$ can be identified as an embedding $\tilde{x} : \bar{\mathbb{S}}^n_+ \to (\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$. In the following, we use $\tilde{}$ to indicate the corresponding quantity for $\tilde{x} : \bar{\mathbb{S}}^n_+ \to (\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$.

Given a star-shaped hyersurface $\tilde{\Sigma}$ in $(\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$, by using the polar coordinate $(\rho, \varphi, \theta) \in \bar{\mathbb{R}}^{n+1}_+$, we may write

$$\tilde{x} = \rho(y)y = \rho(\varphi, \theta)y, y = (\varphi, \theta) \in \bar{\mathbb{S}}^n_+.$$

We use $\sigma = d\varphi^2 + \sin^2\varphi d\theta^2$ and ∇^σ to denote the round metric and the covariant derivative on $\bar{\mathbb{S}}^n_+$. Set

$$\gamma = \log\rho, \text{ and } v = \sqrt{1 + |\nabla^\sigma\gamma|^2}.$$

We have the following correspondence for several geometric quantities.

Proposition 2.1.

(i)

$$x_{n+1} = \langle f(\tilde{x}), E_{n+1}\rangle = \frac{1}{2}(\rho^2 - 1)e^w.$$

(ii)

$$|X_{n+1}| = e^w|-\rho\partial_\rho| = \rho e^w.$$

(iii)

$$\langle X_{n+1}, \nu\rangle = e^{2w}\langle -\rho\partial_\rho, \tilde{\nu}\rangle = \frac{\rho e^w}{v}.$$

(iv) *The Weingarten transformation* $h_i^j = g^{jk}h_{ik}$ *satisfies*

$$h_i^j = \tilde{h}_i^j = \frac{1}{\rho v e^w}(\sigma^{kj} - \frac{\gamma^k\gamma^j}{v^2})\gamma_{ik} + \left[\frac{\sin\varphi\gamma_\varphi}{v} + \frac{(\rho^2-1)}{2\rho v}\right]\delta_i^j.$$

(v)

$$H = \tilde{H} = \frac{1}{\rho v e^w}(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2})\gamma_{ij} + \frac{n\sin\varphi\gamma_\varphi}{v} + \frac{n(\rho^2-1)}{2\rho v}.$$

Remark 2.1. *We see from (iii) that in case we have C^0 estimate, a positive lower bound for $\langle X_{n+1}, \nu\rangle$ is equivalent to the gradient estimate for γ.*

Proof. (i) follows from (10) and (ii) follows from (11).

It is clear that the unit outward normal is given by

$$\tilde{\nu} = e^{-w}\nu_\delta = e^{-w}\frac{\rho^{-1}\nabla^\sigma\gamma - \partial_\rho}{v}, \tag{12}$$

where ν_δ is the unit outward normal of $\tilde{\Sigma} \subset (\bar{\mathbb{R}}^{n+1}_+, \delta_{\bar{\mathbb{R}}^{n+1}_+})$. Then (iii) follows from (11) and (12).

By a well-known transformation law for the Weingarten transformation under a conformal change, we know that $\tilde{h}_i^j$ of $\Sigma \subset (\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$ with respect to $-\tilde{\nu}$ is given by

$$\tilde{h}_i^j = e^{-w}((h_\delta)_i^j + \nabla^\delta_{\nu_\delta} w \delta_i^j), \tag{13}$$

where $(h_\delta)_i^j$ is the Weingarten transformation with respect to $-\nu_\delta$ of $\tilde{\Sigma} \subset (\bar{\mathbb{R}}^{n+1}_+, \delta_{\bar{\mathbb{R}}^{n+1}_+})$ and ∇^δ is the Euclidean derivative.

It is known that

$$(h_\delta)_i^j = -\frac{1}{\rho v}\delta_i^j + \frac{1}{\rho v}(\sigma^{kj} - \frac{\gamma^k\gamma^j}{v^2})\gamma_{ik}, \tag{14}$$

On the other hand, using $e^{-w} = \frac{1}{2}(1+\rho^2+2\rho\cos\varphi)$, we have

$$\begin{aligned}\nabla^\delta_{\nu_\delta}(e^{-w}) &= \left\langle (\rho+\cos\varphi)\partial_\rho - \rho^{-1}\sin\varphi\partial_\varphi, \frac{\rho^{-1}\nabla^\sigma\gamma - \partial_\rho}{v}\right\rangle \\ &= -\frac{1}{v}(\rho + \cos\varphi + \sin\varphi\gamma_\varphi).\end{aligned} \tag{15}$$

(iv) follows from (13), (14) and (15). (v) follows from (iv) by taking trace. □

We return to the flow problem (1) in $(\bar{\mathbb{B}}^{n+1}, \delta_{\bar{\mathbb{B}}})$. By the identification using f, the corresponding family of embeddings $\tilde{x} : \mathbb{S}^n_+ \to (\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$ satisfies

$$\begin{cases}\partial_t\tilde{x} = (n\langle f(\tilde{x}), E_{n+1}\rangle - \tilde{H}e^{2w}\langle -\rho\partial_\rho, \tilde{\nu}\rangle)\tilde{\nu} & \text{in } \mathbb{S}^n_+ \times [0,T), \\ \langle \tilde{\nu}, \tilde{\mu}\circ\tilde{x}\rangle = 0, & \text{on } \partial\mathbb{S}^n_+ \times [0,T),\end{cases} \tag{16}$$

with an initial surface $\tilde{x}(\cdot,0) = \tilde{x}_0$. Here $\tilde{\mu}$ is the downward unit normal of $(\bar{\mathbb{R}}^{n+1}_+, e^{2w}\delta_{\bar{\mathbb{R}}^{n+1}_+})$. As long as $\tilde{x}(\cdot,t)$ is star-shaped in $\bar{\mathbb{R}}^{n+1}_+$, we may reduce (16) to a scalar flow.

Using a standard argument (see [4], Eq. (2.4.21)) and Proposition 2.1, we see that

$$\begin{aligned}\partial_t\gamma &= -\frac{v}{\rho e^w}\left(\frac{n}{2}(\rho^2-1)e^w - \tilde{H}\frac{\rho e^w}{v}\right) \\ &= \frac{1}{\rho v e^w}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} + \frac{n\sin\varphi\gamma_\varphi}{v} - \frac{n(\rho^2-1)|\nabla^\sigma\gamma|^2}{2\rho v} \\ &= \operatorname{div}_\sigma\left(\frac{\nabla^\sigma\gamma}{\rho v e^w}\right) - \frac{n+1}{v}\sigma\left(\nabla^\sigma\gamma, \nabla^\sigma\left(\frac{1}{\rho e^w}\right)\right).\end{aligned} \tag{17}$$

The last line above follows from the fact

$$\sigma\left(\nabla^\sigma\gamma, \nabla^\sigma\left(\frac{1}{\rho e^w}\right)\right) = \frac{\rho^2-1}{2\rho}|\nabla^\sigma\gamma|^2 - \sin\varphi\gamma_\varphi.$$

Next we examine the boundary condition. Note that $\mu \perp \partial\mathbb{B}^{n+1}$. Since the conformal change f preserves angles, we have $\tilde{\mu} \perp \partial\mathbb{R}^{n+1}_+$ and in turn

$$\tilde{\mu} = -e^{-w}\partial_\varphi.$$

In view of (12), the boundary condition in (16) reduces to

$$\nabla^\sigma_{\partial_\varphi}\gamma = 0 \text{ on } \partial\mathbb{S}^n_+. \tag{18}$$

In summary, the flow problem (16) reduces to solve the scalar PDE

$$\begin{aligned}\partial_t\gamma = {} & \frac{1}{\rho v e^w}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} \\ & + \frac{n\sin\varphi\gamma_\varphi}{v} - \frac{n(\rho^2-1)|\nabla^\sigma\gamma|^2}{2\rho v}, \quad \text{in } \mathbb{S}^n_+ \times [0,T),\end{aligned} \tag{19}$$

with the initial and the boundary conditions

$$\begin{aligned}\gamma(\cdot,0) &= \gamma_0, && \text{in } \mathbb{S}^n_+,\\ \nabla^\sigma_{\partial_\varphi}\gamma &= 0, && \text{on } \partial\mathbb{S}^n_+ \times [0,T).\end{aligned}$$

where γ_0 is the corresponding function for x_0.

3. A priori estimates

The short time existence of the scalar flow (19) follows by the standard parabolic PDE theory. Next we show the C^0 and C^1 estimates for (19). The a priori C^0 estimate follows directly from the maximum principle.

Proposition 3.1. *Let* $\gamma : \mathbb{S}^n_+ \times [0,T) \to \mathbb{R}$ *solve* (19). *Then*

$$\min_{\mathbb{S}^n_+}\gamma_0 \le \gamma \le \max_{\mathbb{S}^n_+}\gamma_0.$$

The key point is the following gradient estimate for γ.

Proposition 3.2. *Let* $\gamma : \mathbb{S}^n_+ \times [0,T) \to \mathbb{R}$ *solve* (19)*. Then there exists a constant* C*, depending on* $\|\gamma_0\|_{C^1}$ *and* $\min_{\mathbb{S}^n_+}\gamma_0$ *such that*

$$|\nabla^\sigma \gamma|^2 \le C.$$

Moreover, if $n \ge 3$*, we have*

$$|\nabla^\sigma \gamma|^2 \le C_1 e^{-C_2 t}.$$

Proof. For notation simplicity, we use $\nabla = \nabla^\sigma$ in the proof. Denote

$$F(\nabla^2\gamma, \nabla\gamma, \rho, \varphi) = \frac{1}{\rho v e^w}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} + \frac{n\sin\varphi\gamma_\varphi}{v} - \frac{n(\rho^2-1)|\nabla\gamma|^2}{2\rho v},$$

and

$$F^{ij} = \frac{\partial F}{\partial \gamma_{ij}}, \quad F^p = \frac{\partial F}{\partial \gamma_p}, \quad F^\rho = \frac{\partial F}{\partial \rho}, \quad F^\varphi = \frac{\partial F}{\partial \varphi}.$$

Then

$$\partial_t|\nabla\gamma|^2 = 2\gamma_k(\gamma_t)_k = 2F^{ij}\gamma_k\gamma_{ijk} + F^p\nabla_p|\nabla\gamma|^2 + 2F^\rho\rho|\nabla\gamma|^2 + 2F^\varphi\gamma_\varphi. \tag{20}$$

By a direct computation, we have

$$F^{ij} = \frac{1}{\rho v e^w}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right), \tag{21}$$

$$F^\rho = \frac{\rho^2-1}{2\rho^2 v}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} - \frac{n(\rho^2+1)}{2\rho^2 v}|\nabla\gamma|^2, \tag{22}$$

$$F^\varphi = -\sin\varphi\frac{1}{v}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} + \frac{n\cos\varphi}{v}\gamma_\varphi. \tag{23}$$

Using the Ricci identity

$$\gamma_{ijk} = \gamma_{kij} + \gamma_j\sigma_{ki} - \gamma_k\sigma_{ij}$$

and (21), we have

$$\begin{aligned} 2F^{ij}\gamma_k\gamma_{ijk} &= F^{ij}\nabla^2_{ij}|\nabla\gamma|^2 - 2\frac{1}{\rho v e^w}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ik}\gamma_{jk} - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2 \\ &= F^{ij}\nabla^2_{ij}|\nabla\gamma|^2 - \frac{2}{\rho v e^w}|\nabla^2\gamma|^2 + \frac{1}{2\rho v^3 e^w}\left|\nabla|\nabla\gamma|^2\right|^2 - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2. \qquad (24) \end{aligned}$$

Replacing (22), (23) and (24) into (20), we get

$$
\begin{aligned}
\partial_t|\nabla\gamma|^2 = {} & F^{ij}\nabla^2_{ij}|\nabla\gamma|^2 + F^p\nabla_p|\nabla\gamma|^2 \\
& - \frac{2}{\rho v e^w}|\nabla^2\gamma|^2 + \frac{1}{2\rho v^3 e^w}\left|\nabla|\nabla\gamma|^2\right|^2 - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2 \\
& + 2\left[\frac{\rho^2-1}{2\rho^2 v}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} - \frac{n(\rho^2+1)}{2\rho^2 v}|\nabla\gamma|^2\right]\rho|\nabla\gamma|^2 \\
& + 2\left[-\sin\varphi\frac{1}{v}\left(\sigma^{ij} - \frac{\gamma^i\gamma^j}{v^2}\right)\gamma_{ij} + \frac{n\cos\varphi}{v}\gamma_\varphi\right]\gamma_\varphi \\
= {} & F^{ij}\nabla^2_{ij}|\nabla\gamma|^2 + F^p\nabla_p|\nabla\gamma|^2 \\
& + \left(\sin\varphi - \frac{\rho^2-1}{2\rho}|\nabla\gamma|^2\right)\frac{\langle\nabla\gamma, \nabla|\nabla\gamma|^2\rangle}{v^3} \\
& - \frac{2}{\rho v e^w}|\nabla^2\gamma|^2 + \frac{1}{2\rho v^3 e^w}\left|\nabla|\nabla\gamma|^2\right|^2 - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2 \\
& + \frac{\rho^2-1}{\rho v}\Delta\gamma|\nabla\gamma|^2 - \frac{n(\rho^2+1)}{\rho v}|\nabla\gamma|^4 \\
& + \frac{2n\cos\varphi}{v}\gamma_\varphi^2 - \frac{2\sin\varphi}{v}\Delta\gamma\gamma_\varphi.
\end{aligned}
\tag{25}
$$

Now we examine the boundary normal derivative of $|\nabla\gamma|^2$ and have

$$
\nabla_{\partial_\varphi}|\nabla\gamma|^2 = 2(\gamma_{\theta_\alpha}\gamma_{\theta_\alpha\varphi} + \gamma_\varphi\gamma_{\varphi\varphi}) = \gamma_{\theta_\alpha}[\nabla_{\partial_{\theta_\alpha}}(\gamma_\varphi) - (\nabla_{\partial_{\theta_\alpha}}\partial_\varphi)\gamma] = 0. \tag{26}
$$

Here we used $\gamma_\varphi = 0$ along $\partial\mathbb{S}^n_+$ and the fact that $\nabla_{\partial_{\theta_\alpha}}\partial_\varphi = 0$.

Assume for $t \in [0, T)$, $\max_{\mathbb{S}^n_+}|\nabla\gamma|^2(\cdot, t) = |\nabla\gamma|^2(x_t, t)$. If $x_t \in \mathbb{S}^n_+$, it follows from the maximum point condition that

$$
\nabla|\nabla\gamma|^2 = 0, \quad \nabla^2|\nabla\gamma|^2 \le 0. \tag{27}
$$

If $x_t \in \partial\mathbb{S}^n_+$, we see from (26) that $\nabla_{\partial_\varphi}|\nabla\gamma|^2 = 0$, and in turn we also have (27). Thus, for each $t \in [0, T)$, at x_t, we have (27). We choose at x_t local coordinates $x^1, \cdots x^n$ such that $\gamma_1 = |\nabla\gamma|$. One has $\gamma_{1i} = 0$ for all i by (27). By further rotating the $\{x^2, \cdots, x^n\}$ coordinate, we can assume $\nabla^2\gamma$ is diagonal. Then

$$
|\nabla^2\gamma|^2 \ge \frac{1}{n-1}(\Delta\gamma)^2.
$$

It follows from (25) that at x_t,

$$
\begin{aligned}
0 &\le \partial_t|\nabla\gamma|^2(x_t,t)\\
&\le -\frac{2}{\rho v e^w}|\nabla^2\gamma|^2 - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2\\
&\quad + \frac{\rho^2-1}{\rho v}\Delta\gamma|\nabla\gamma|^2 - \frac{n(\rho^2+1)}{\rho v}|\nabla\gamma|^4 + \frac{2n\cos\varphi}{v}\gamma_\varphi^2 - \frac{2\sin\varphi}{v}\Delta\gamma\gamma_\varphi\\
&\le -\frac{2(1-\epsilon)}{(n-1)\rho v e^w}\left(\Delta\gamma - \frac{(n-1)(\rho^2-1)e^w}{4(1-\epsilon)}|\nabla\gamma|^2\right)^2\\
&\quad - \frac{2\epsilon}{(n-1)\rho v e^w}\left(\Delta\gamma + \frac{(n-1)\rho e^w\sin\varphi}{2\epsilon}\gamma_\varphi\right)^2\\
&\quad + \frac{1}{\rho v}\left(\frac{(n-1)(\rho^2-1)^2e^w}{8(1-\epsilon)} - n(\rho^2+1)\right)|\nabla\gamma|^4\\
&\quad + \frac{1}{v}\left(-\frac{2(n-1)}{\rho e^w}|\nabla\gamma|^2 + 2n\cos\varphi\gamma_\varphi^2 + \frac{(n-1)\rho e^w\sin^2\varphi}{2\epsilon}\gamma_\varphi^2\right). \qquad (28)
\end{aligned}
$$

Choosing $\epsilon = \frac{3}{4}$, we have

$$
\begin{aligned}
&\frac{(n-1)(\rho^2-1)^2e^w}{8(1-\epsilon)} - n(\rho^2+1)\\
&\qquad < \frac{ne^w}{2}[(\rho^2-1)^2 - (\rho^2+1)(1+\rho^2+2\rho\cos\varphi)] \le -n\rho^2e^w
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{2(n-1)}{\rho e^w}|\nabla\gamma|^2 + 2n\cos\varphi\gamma_\varphi^2 + \frac{(n-1)\rho e^w\sin^2\varphi}{2\epsilon}\gamma_\varphi^2\\
&\le \Bigg(-\frac{(n-1)(1+\rho^2+2\rho\cos\varphi)}{\rho}\\
&\qquad + 2n\cos\varphi + \frac{4(n-1)}{3}\frac{\rho}{1+\rho^2+2\rho\cos\varphi}\Bigg)|\nabla\gamma|^2\\
&\le (-2(n-1) + 2\cos\varphi + \frac{2(n-1)}{3})|\nabla\gamma|^2\\
&\le (-\frac{4}{3}n + \frac{10}{3})|\nabla\gamma|^2.
\end{aligned}
$$

Thus

$$
0 \le \partial_t|\nabla\gamma|^2 \le -\frac{n\rho e^w}{v}|\nabla\gamma|^4 + (-\frac{4}{3}n + \frac{10}{3})\frac{1}{\rho v}|\nabla\gamma|^2. \qquad (29)
$$

It follows from (29) that $|\nabla\gamma|^2 \leq C$. Moreover, when $n \geq 3$, one sees from (29) that $|\nabla\gamma|^2 \leq C_1 e^{-C_2 t}$. □

4. Global convergence

We first prove the nice properties of (1), mentioned in the Introduction.

Proposition 4.1. *Flow* (1) *satisfies*

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = 0 \tag{30}$$

and

$$\frac{d}{dt}\mathrm{Area}(\Sigma_t) = -\frac{1}{n-1}\int_\Sigma \sum_{i<j}(\kappa_i - \kappa_j)^2 \langle X_{n+1}, \nu\rangle dA_t \leq 0. \tag{31}$$

Proof. From (3), we get

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = \int_\Sigma (n x_{n+1} - H\langle X_{n+1}, \nu\rangle) dA_t = 0.$$

The first variational formula gives

$$\frac{d}{dt}\mathrm{Area}(\Sigma_t) = \int_\Sigma H(n x_{n+1} - H\langle X_{n+1}, \nu\rangle) dA_t.$$

Using the Minkowski formula (4)

$$\int_\Sigma H x_{n+1} - \frac{2}{n-1}\sigma_2(\kappa)\langle X_{n+1}, \nu\rangle dA_t = 0,$$

we get

$$\begin{aligned}\frac{d}{dt}\mathrm{Area}(\Sigma_t) &= -\int_\Sigma \left(H^2 - \frac{2n}{n-1}\sigma_2(\kappa)\right)\langle X_{n+1}, \nu\rangle dA_t \\ &= -\frac{1}{n-1}\int_\Sigma \sum_{i<j}(\kappa_i - \kappa_j)^2 \langle X_{n+1}, \nu\rangle) dA_t \leq 0.\end{aligned}$$

□

Now we prove the global convergence.

Proof of Theorem 1.1. In view of Proposition 2.1 (iii), the C^0 and C^1 estimates in Propositions 3.1 and 3.2 imply that $\langle X_{n+1}, \nu\rangle \geq c > 0$, that is, the star-shapedness of Σ_t is preserved under the flow (1).

Now we are ready to prove the long time existence in Theorem 1.1. Since equation (19) is a quasilinear parabolic PDE of divergent form, the higher order a priori estimates follows from the standard parabolic PDE theory, once we have the C^0 and C^1 estimates in Propositions 3.1 and 3.2. Hence we prove that (19) has a smooth solution for all time. The exponential convergence for $n \geq 3$ follows directly from Proposition 3.2.

For the convergence part in two dimensions, we examine the monotonicity of the area functional along the flow. In the following we restrict to $n = 2$. By integrating (4.1) over $t \in [0, \infty)$ and using the uniform estimate, we get

$$\int_0^\infty \int_{\mathbb{S}^n_+} |\kappa_1(y,t) - \kappa_2(y,t)|^2 \langle X_{n+1}, \nu\rangle dA_t dt \leq C.$$

where $\kappa_i(y,t)$, $i = 1, 2$ are the principal curvatures of the radial graph at (y,t). It follows from the uniform bound for $\langle X_{n+1}, \nu\rangle$ and dA_t that

$$\max_{y \in \bar{\mathbb{S}}^n_+} |\kappa_1 - \kappa_2|(y,t) = o_t(1), \tag{32}$$

where $o_t(1)$ denotes a quantity which goes to zero as $t \to \infty$. See the proof of Proposition 5.5 in [5]. With the help of the property (32), we can show the smooth convergence of flow (1) when $n = 2$. This idea was used first by Guan-Li in [5].

Let us go back to the estimate at x_t, where $\max_{\bar{\mathbb{S}}^n_+} |\nabla\gamma|^2(\cdot,t) = |\nabla\gamma|^2(x_t,t)$. Again we choose the local coordinate around x_t such that at x_t,

$$\gamma_1 = |\nabla\gamma|, \quad \gamma_{11} = 0.$$

In view of Proposition 2.1 (iv), the Weingarten transformation h_i^j is diagonal in this coordinate which means the coordinate directions are the principal directions of $x(\cdot,t)$ at x_t. Thus the principal curvature κ_i at x_t is given by

$$\kappa_i = \frac{\gamma_{ii}}{\rho v e^w} + \frac{\sin\varphi\gamma_\varphi}{v} + \frac{(\rho^2-1)}{2\rho v}, \quad i = 1, 2.$$

It follows that at x_t,

$$|\Delta\gamma| = |\gamma_{22} + \gamma_{11}| = |\gamma_{22} - \gamma_{11}| = \rho v e^w |\kappa_2 - \kappa_1| = o_t(1). \tag{33}$$

Using (33) and the C^1 estimate, we get at (x_t, t),

$$\begin{aligned}
\partial_t|\nabla\gamma|^2 \le& -\frac{2}{\rho v e^w}|\nabla^2\gamma|^2 - \frac{2(n-1)}{\rho v e^w}|\nabla\gamma|^2 \\
&+ \frac{\rho^2-1}{\rho v}\Delta\gamma|\nabla\gamma|^2 - \frac{n(\rho^2+1)}{\rho v}|\nabla\gamma|^4 + \frac{2n\cos\varphi}{v}\gamma_\varphi^2 - \frac{2\sin\varphi}{v}\Delta\gamma\gamma_\varphi \\
\le& -\frac{n(\rho^2+1)}{\rho v}|\nabla\gamma|^4 + \frac{1}{v}\left(-\frac{2}{\rho e^w}|\nabla\gamma|^2 + 4\cos\varphi\gamma_\varphi^2\right) + o_t(1) \\
\le& -C|\nabla\gamma|^4 + o_t(1). \qquad (34)
\end{aligned}$$

Here we have used

$$-\frac{2}{\rho e^w}|\nabla\gamma|^2 + 4\cos\varphi\gamma_\varphi^2 \le \left(-\frac{1+\rho^2+2\rho\cos\varphi}{\rho} + 4\cos\varphi\right)|\nabla\gamma|^2 \le 0.$$

Now we claim that

$$|\nabla\gamma|^2 = o_t(1).$$

The smooth convergence follows from this claim and the interpolation theorem. We show the claim in two steps.

First, we show that there exists a sequence $\{t_i\}$ with $t_i \to \infty$ such that

$$\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma(\cdot,t_i)|^2 \to 0 \quad \text{as } i\to\infty.$$

Assume this is not true. Then there exists $\epsilon_0 > 0$ and $T_0 > 0$ such that

$$\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma(\cdot,t)|^2 \ge \epsilon_0, \quad \text{for } t > T_0.$$

From (34) we have that for a large $T_1 > 0$ and for any $t > T_1$, we have

$$\frac{d}{dt}\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma|^2 \le -C\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma|^4 + \frac{1}{2}C\epsilon_0^4 = -\frac{1}{2}C\epsilon_0^4,$$

which is impossible.

Second, we show that for any sequence $\{s_i\}$ with $s_i\to\infty$, we have

$$\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma(\cdot,s_i)|^2 \to 0 \quad \text{as } i\to\infty.$$

If not, there exists a sequence $\{s_i\}$ with $s_i\to\infty$ such that

$$\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma(\cdot,s_i)|^2 \ge \epsilon_1$$

for any s_i and for some positive constant ϵ_1. Without loss of generality, we may assume that $t_i < s_i$. We consider the interval $I_i := [t_i, s_i]$ for sufficiently large i, such that we have from (34) at a maximum point $x_t \in \bar{\mathbb{S}}^n_+$

$$\frac{d}{dt}\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma|^2 \le -C\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma|^4 + \frac{1}{2}C\epsilon_1^4 \tag{35}$$

for any $t \ge t_i$. Let $y_i \in \bar{\mathbb{S}}^n_+$ and $\bar{t}_i \in [t_i, s_i]$ such that

$$|\nabla\gamma(y_i, \bar{t}_i)|^2 = \max_{t\in[t_i,s_i]}\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma(\cdot, t)|^2 \ge \epsilon_1.$$

By the first step, we may assume that $\bar{t}_i \ne t_i$ for i large. It follows that

$$\frac{d}{dt}\max_{\bar{\mathbb{S}}^n_+}|\nabla\gamma|^2(\bar{t}_i) \ge 0.$$

Together with (35), implies that

$$|\nabla\gamma(y_i, \bar{t}_i)|^2 < \epsilon_1,$$

a contradiction. This proves the claim.

From the claim, it follows easily that $\gamma(t)$ converges smoothly to a constant γ_0 and $\rho \to \rho_0$ smoothly for some constant $\rho_0 > 0$, depending on the initial enclosed volume of x_0.

Next we show the exponential convergence in the case $n = 2$ and the enclosed volume of x_0 is not that of a half ball. In this case, $\rho_0 \ne 1$. We return to (28). By choosing $\epsilon < 1$ close to 1, we have

$$\begin{aligned}
\partial_t|\nabla\gamma|^2(x_t, t) &\le \frac{1}{\rho v}\left(\frac{(\rho^2-1)^2 e^w}{8(1-\epsilon)} - n(\rho^2+1)\right)|\nabla\gamma|^4 \\
&\quad + \frac{1}{v}\left(-\frac{2}{\rho e^w} + 4\cos\varphi + \frac{\rho e^w \sin^2\varphi}{2\epsilon}\right)|\nabla\gamma|^2 \\
&\le \frac{1}{\rho v}\left(\frac{(\rho^2-1)^2 e^w}{8(1-\epsilon)} - n(\rho^2+1)\right)|\nabla\gamma|^4 \\
&\quad + \frac{1}{v}\Bigg(-\frac{(1-\rho\cos\varphi)^2}{\rho} \\
&\qquad + \rho\sin^2\varphi\left(\frac{1}{\epsilon(1+\rho^2+2\rho\cos\varphi)} - 1\right)\Bigg)|\nabla\gamma|^2 \\
&\le C|\nabla\gamma|^4 - \left(\frac{(1-\rho\cos\varphi)^2}{\rho} + C\rho\sin^2\varphi\right)|\nabla\gamma|^2.
\end{aligned}$$

As ρ converges to $\rho_0 \neq 1$,

$$-\left(\frac{(1-\rho\cos\varphi)^2}{\rho}+C\rho\sin^2\varphi\right)\leq -C_1$$

for some $C_1 > 0$ and t large. Then the exponential convergence follows. □

Proof of Corollary 1.1. It follows from Theorem 1.1 and Proposition 4.1. □

References

[1] F. J. Almgren, *Spherical symmetrization,* Proc. International workshop on integral functions in the calculus of variations, Trieste, 1985, Red. Circ. Mat. Palermo 2 Supple. (1987), 11–25.

[2] J. Bokowsky and E. Sperner, *Zerlegung konvexer Körper durch minimale Trennflächen,* J. Reine Angew. Math. 311/312 (1979), 80–100.

[3] Yu. D. Burago and V. G. Maz'ya, *Certain Questions of Potential Theory and Function Theory for Regions with Irregular Boundaries,* (Russian) Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 3 1967 152 pp; English translation: Potential Theory and Function Theory for Irregular Regions. Seminars in Mathematics, V. A. Steklov Mathematical Institute, Leningrad, Vol. 3 Consultants Bureau, New York 1969 vii+68 pp.

[4] C. Gerhardt, *Curvature Problems*, Series in Geometry and Topology, vol. 39, International Press of Boston Inc., Sommerville, 2006,

[5] P. Guan and J. Li, *A mean curvature type flow in space forms*, Intern. Math. Res. Not. **2015** (2015), no. 13, 4716–4740.

[6] P. Guan, J. Li, and M.-T. Wang, *A volume preserving flow and the isoperimetric problem in warped product spaces*, Trans. Am. Math. Soc. **372** (2019), 2777–2798.

[7] B. Lambert and J. Scheuer, *The inverse mean curvature flow perpendicular to the sphere*, Math. Ann. **364** (2016), no. 3, 1069–1093.

[8] J. Scheuer, G. Wang and C. Xia, *Alexandrov-Fenchel inequalities for convex hypersurfaces with free boundary in a ball,* J. Differ. Geom. **120** (2022), no. 2, 345–373.

[9] A. Stahl, *Convergence of solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 5, 421–441.

[10] A. Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 4, 385–407.

[11] G. Wang and C. Xia, *Uniqueness of stable capillary hypersurfaces in a ball*, Math. Ann. **374** (2019), no. 3-4, 1845–1882.

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